FROM COHERENT SHEAVES TO CURVES IN $\mathbb{P}^3$

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Dedicated to Silvio Greco in occasion of his 60-th birthday.

0. Introduction.

Let k be an algebraically closed field, and let $\mathbb{P} = \mathbb{P}^3_k$ be the projective 3-space over k. We denote by R the associated polynomial ring $R = k[X, Y, Z, T]$.

The aim of this paper is to precise the correspondences between equivalence classes of some objects related to $\mathbb{P}$: *vector bundles* (i.e. locally free coherent sheaves), *coherent sheaves*, *finite length graded R-modules* and *locally Cohen-Macaulay curves* (i.e. closed subscheme of pure dimension one with no embedded points). Some of these correspondences are well-known, others are new.

The notion of *pseudo-isomorphism* was introduced in [5]:

Definition 0.1. *Let A be a ring, let $\mathcal{N}$ and $\mathcal{N}'$ be two coherent sheaves on $\mathbb{P}^3_A$, flat over A and let f be a morphism from $\mathcal{N}$ to $\mathcal{N}'$. We say that f is a pseudo-isomorphism if it induces:*

i) *an isomorphism of functors (from the category of finite type A-modules to itself) $H^0(\mathcal{N}(n) \otimes_A \,.\,) \to H^0(\mathcal{N}'(n) \otimes_A \,.\,)$ for all $n \ll 0$,*

ii) *an isomorphism of functors (from the category of finite type A-modules to the category of the graded $A[X, Y, Z, T]$-modules) $H^1_*(\mathcal{N} \otimes_A \,.\,) \to H^1_*(\mathcal{N}' \otimes_A \,.\,)$,*

iii) a monomorphism of functors $H^2_*(\mathcal{N} \otimes_A \bullet) \to H^2_*(\mathcal{N}' \otimes_A \bullet)$.

Two coherent sheaves on $\mathbb{P}^3_A$, flat over A, are pseudo-isomorphic if there exists a chain of pseudo-isomorphisms connecting them:

$$\mathcal{N} = \mathcal{N}_0 \to \mathcal{N}_1 \leftarrow \mathcal{N}_2 \to \mathcal{N}_3 \leftarrow \cdots \to \mathcal{N}_{2p-1} \leftarrow \mathcal{N}_{2p} = \mathcal{N}'.$$

This defines an equivalence relation called psi-equivalence. It turns out that when A is a field the psi-equivalence is an extension to the set of coherent sheaves of (local) projective dimension ≤ 1 of the stable equivalence for vector bundles with $H^2_* = 0$ (cf. 2.4).

Now, using the bijective correspondence between the set of isomorphism classes of finite length graded R-modules and the set of stable equivalence classes of vector bundles with $H^2_* = 0$ ([4], see also 2.3), and a well-known theorem of Rao ([10]), we get a new correspondence:

Corollary 2.8. *The application which maps a curve C to its sheaf of ideals $\mathcal{J}_C$ induces a bijective correspondence between the set of biliaison classes of curves and the set of coherent sheaves of (local) projective dimension ≤ 1 up to psi-equivalence and up to a twist.*

In the third paragraph, we show that there exist rank 2 reflexive sheaves in every class of psi-equivalence, and among them, *minimal* rank 2 reflexive sheaves, in the sense that their first Chern class is minimal. Moreover Buraggina (cf. [2]) points out that if there are rank 2 vector bundles in the class, they are minimal elements and all the minimal elements are vector bundles.

Then two very natural questions arise, concerning the possible relationships between the minimal curves in a biliaison class and the minimal rank 2 reflexive sheaves in a class of psi-equivalence (recall that the curves obtained as the zero locus of a section of a twist of a rank 2 vector bundle are exactly the subcanonical curves (cf. [3])):

Question I. *Is a minimal curve of a biliaison class the zero locus of a section of a twist of a (minimal in the corresponding class of psi-equivalence) rank 2 reflexive sheaf?*

Question II. *If a biliaison class contains subcanonical curves, is a minimal curve subcanonical?*

It is known since [2] that the answer to Question I is NO (we give in 3.9 an explicit example of a minimal curve which is not the zero locus of a section of

a twist of a rank 2 reflexive sheaf) and we have recently proved that the answer to Question II is YES (cf. [6]).

1. The main equivalence relations.

In this section we will discuss some more or less well-known equivalence relations.

We say that a vector bundle which is isomorphic to a finite direct sum $\bigoplus \mathcal{O}_{\mathbb{P}}(-n_i)$ is *dissocié*.

If $\mathcal{N}$ is a sheaf on $\mathbb{P}^3_k$, we denote by $H^i_*(\mathcal{N})$ the graded R-module $\bigoplus_{n \in \mathbb{Z}} H^i(\mathbb{P}, \mathcal{N}(n))$, and by $h^i(\mathcal{N})$ the dimension of the vector space $H^i(\mathbb{P}, \mathcal{N}(n))$.

Definition 1.1. *Two vector bundles $\mathcal{F}$ and $\mathcal{F}'$ on $\mathbb{P}^3_k$ are stably isomorphic if there exist two vector bundles $\mathcal{L}$ et $\mathcal{L}'$ which are dissocié and an isomorphism $\mathcal{F} \oplus \mathcal{L} \simeq \mathcal{F}' \oplus \mathcal{L}'$.*

We will denote by $\mathcal{S}tab$ the set of vector bundles $\mathcal{F}$ such that $H^2_*(\mathcal{F}) = 0$ up to stable equivalence.

The notion of *pseudo-isomorphism* was introduced in [5]:

Definition 1.2. *Let $\mathcal{N}$ and $\mathcal{N}'$ be two coherent sheaves on $\mathbb{P}^3_k$ and let f be a morphism from $\mathcal{N}$ to $\mathcal{N}'$. We say that f is a pseudo-isomorphism (shortly a psi) if it induces:*

  i) *an isomorphism $H^0 \mathcal{N}(n) \simeq H^0 \mathcal{N}'(n)$ for all $n \ll 0$,*
  ii) *an isomorphism $H^1_* \mathcal{N} \simeq H^1_* \mathcal{N}'$,*
  iii) *an injective homomorphism $H^2_* \mathcal{N} \to H^2_* \mathcal{N}'$.*

Two coherent sheaves $\mathcal{N}$ and $\mathcal{N}'$ are pseudo-isomorphic if there exists a chain of psi connecting them:

$$\mathcal{N} = \mathcal{N}_0 \to \mathcal{N}_1 \leftarrow \mathcal{N}_2 \to \mathcal{N}_3 \leftarrow \cdots \to \mathcal{N}_{2p-1} \leftarrow \mathcal{N}_{2p} = \mathcal{N}'.$$

This defines an equivalence relation called psi-equivalence. (In fact, this definition was given in a more general context, i.e. for families of coherent sheaves in the case when k is not a field but a ring.)

Example 1.3. Let $\mathcal{N}$ be a reflexive sheaf of rank $r + 1$, let $\mathcal{P}$ be a dissocié sheaf of rank r and let C be a curve such that there exists an exact sequence: $0 \to \mathcal{P} \to \mathcal{N} \to \mathcal{J}_C \to 0$, where $\mathcal{J}_C$ is the ideal sheaf of C. Then the homomorphism $\mathcal{N} \to \mathcal{J}_C$ arising from the exact sequence is a *psi* of coherent sheaves of projective dimension ≤ 1.

The following definition is well-known:

Definition 1.4. *Two (locally Cohen-Macaulay) curves C and C' are linked if there exists a complete intersection curve X containing C and C' and such that $\mathcal{H}om_{\mathcal{O}_{\mathbb{P}}}(\mathcal{O}_C, \mathcal{O}_X) \simeq \mathcal{I}_{C',X}$ and $\mathcal{H}om_{\mathcal{O}_{\mathbb{P}}}(\mathcal{O}_{C'}, \mathcal{O}_X) \simeq \mathcal{I}_{C,X}$. Two curves C and C' are in the same biliaison class if they can be connected by a chain of an even number of linkages.*

We will denote by $\mathcal{B}il$ the set of biliaison classes of locally Cohen-Macaulay curves.

We denote by $\mathcal{M}_f$ the set of finite length graded R-modules, modulo isomorphism.

2. Correspondences between quotient sets.

We will begin by proving that the psi-equivalence retains some kind of regularity of coherent sheaves.

Lemma 2.1. *Let $f : \mathcal{F} \to \mathcal{F}'$ be a psi of coherent sheaves. Then there exist two dissocié sheaves $\mathcal{L}$ et $\mathcal{L}'$ and an exact sequence*

$$0 \to \mathcal{L}' \to \mathcal{L} \oplus \mathcal{F} \to \mathcal{F}' \to 0.$$

Moreover if $\mathcal{F}'$ is locally free and satisfies $H_*^2(\mathcal{F}') = 0$, then $\mathcal{F}$ is also locally free and $\mathcal{F}$ and $\mathcal{F}'$ are stably isomorphic.

Proof. There exists $n_0 \in \mathbb{N}$ such that $H^0\mathcal{F}(n) \simeq H^0\mathcal{F}'(n)$ is an isomorphism for all $n < n_0$. Let L be a free R-module of finite type, such that there exists a surjective homomorphism : $L \to \oplus_{n \geq n_0} H^0\mathcal{F}'(n)$. Denoting by $\mathcal{L}$ the dissocié sheaf associated to L and combining the associated surjective homomorphism of sheaves: $\mathcal{L} \to \mathcal{N}'$ with f, we get a surjective homomorphism: $\mathcal{L} \oplus \mathcal{F} \to \mathcal{F}'$ whose kernel is denoted by $\mathcal{L}'$. So we have an exact sequence

$$0 \to \mathcal{L}' \to \mathcal{L} \oplus \mathcal{F} \to \mathcal{F}' \to 0.$$

Taking the associated cohomology exact sequence we get:

$$L \oplus H_*^0\mathcal{F} \to H_*^0\mathcal{F}' \to H_*^1\mathcal{L}' \to H_*^1\mathcal{F} \to H_*^1\mathcal{F}' \to H_*^2\mathcal{L}' \to H_*^2\mathcal{F} \to H_*^2\mathcal{F}'$$

Since f is a psi and $L \to H_*^0\mathcal{F}'$ is surjective, we have $H_*^1\mathcal{L}' = H_*^2\mathcal{L}' = 0$. Thus it follows by a theorem of Horrocks [4] that $\mathcal{L}'$ is dissocié.

Now by local duality, $H_*^2(\mathcal{F}') = 0$ implies that $\text{Ext}^1(\mathcal{F}', \mathcal{L}') = 0$ so that the exact sequence splits and we are done.

Corollary 2.2. *Let $\mathcal{F}$ and $\mathcal{F}'$ two coherent sheaves which are pseudo- isomorphic. Then the sheaves $\mathcal{E}xt^i_{\mathcal{O}_{\mathbb{P}}}(\mathcal{F}, \mathcal{O}_{\mathbb{P}})$ and $\mathcal{E}xt^i_{\mathcal{O}_{\mathbb{P}}}(\mathcal{F}', \mathcal{O}_{\mathbb{P}})$ are isomorphic for $i \geq 2$.*

Proof. Suppose that we have a psi $f : \mathcal{F} \to \mathcal{F}'$. Then by 2.1, there exist two dissocié sheaves $\mathcal{L}$ et $\mathcal{L}'$ and an exact sequence

$$0 \to \mathcal{L}' \to \mathcal{L} \oplus \mathcal{F} \to \mathcal{F}' \to 0,$$

from wich we deduce isomorphisms $\mathcal{E}xt^i_{\mathcal{O}_{\mathbb{P}}}(\mathcal{F}', \mathcal{O}_{\mathbb{P}}) \to \mathcal{E}xt^i_{\mathcal{O}_{\mathbb{P}}}(\mathcal{F}, \mathcal{O}_{\mathbb{P}})$ for $i \geq 2$. The same holds if there is a chain of *psi* between $\mathcal{F}$ and $\mathcal{F}'$.

Remark 2.3. Let $\mathcal{F}$ and $\mathcal{F}'$ two coherent sheaves which are pseudo-isomorphic. Then the (local) projective dimension of $\mathcal{F}$ is ≤ 1 if and only if the same is true for $\mathcal{F}'$. Therefore we will restrict ourselves to the set of coherent sheaves of projective dimension ≤ 1, and we will denote by $\mathcal{P}si$ the quotient of this set by the psi-equivalence relation. For such a sheaf $\mathcal{N}$, $H^0\mathcal{N}(n)$ vanishes for $n \ll 0$, and the first condition of 1.2 is always satisfied.

With this notation we have the following result (se also [5] 2.4 and 2.8):

Proposition 2.4. *The injection of the set of vector bundles into the set of coherent sheaves of projective dimension ≤ 1 induces a natural map from $\mathcal{S}tab$ to $\mathcal{P}si$ which is bijective.*

Proof. Suppose that $\mathcal{F}$ and $\mathcal{F}'$ are two pseudo-isomorphic vector bundles such that $H^2_*(\mathcal{F}) = H^2_*(\mathcal{F}') = 0$. By a "Verdier-lemma" ([5] 2.11), there exist a coherent sheaf $\mathcal{F}''$ and two psi $f : \mathcal{F}'' \to \mathcal{F}$ and $f' : \mathcal{F}'' \to \mathcal{F}'$. By lemma 2.1, we conclude that $\mathcal{F}''$ is locally free and that $\mathcal{F}''$ and $\mathcal{F}$ (resp. $\mathcal{F}''$ and $\mathcal{F}'$) are stably isomorphic. Therefore the natural map from $\mathcal{S}tab$ to $\mathcal{P}si$ is injective.

Now let $\mathcal{N}$ be a coherent sheaf of projective dimension ≤ 1. The graded R-modules $N = H^0_*(\mathcal{N})$ and $Ext^1_R(N, R)$ are of finite type. Let L be a free R-module of finite type, such that there exists a surjective homomorphism: $L^\vee \to \mathrm{Ext}^1_R(N, R)$. The composed map:

$$R \to L^\vee \otimes L \to \mathrm{Ext}^1_R(N, R) \otimes L \to \mathrm{Ext}^1_R(N, L)$$

gives an extension of N by L, that is an exact sequence

$$0 \to L \to F \to N \to 0.$$

If we dualize, we obtain a coboundary homomorphism: $L^\vee \to \mathrm{Ext}^1_R(N, R)$, which is the original surjective homomorphism, so that we get $\mathrm{Ext}^1_R(F, R) = 0$.

Denoting by $\mathcal{F}$ (resp. $\mathcal{L}$) the sheaf associated to F (resp. L), we have also $\mathrm{Ext}^1_{\mathcal{O}_{\mathbb{P}}}(\mathcal{F}, \mathcal{O}_{\mathbb{P}}(n)) = 0$ for all n. Using the spectral sequence of Ext we have also $\mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}}}(\mathcal{F}, \mathcal{O}_{\mathbb{P}}) = 0$

From the exact sequence:

$$0 \to \mathcal{L} \to \mathcal{F} \to \mathcal{N} \to 0$$

we deduce isomorphisms $\mathcal{E}xt^i_{\mathcal{O}_{\mathbb{P}}}(\mathcal{F}, \mathcal{O}_{\mathbb{P}}) \simeq \mathcal{E}xt^i_{\mathcal{O}_{\mathbb{P}}}(\mathcal{N}, \mathcal{O}_{\mathbb{P}})$ for $i \geq 2$. Since the projective dimension of $\mathcal{N}$ is ≤ 1, we have $\mathcal{E}xt^i_{\mathcal{O}_{\mathbb{P}}}(\mathcal{N}, \mathcal{O}_{\mathbb{P}}) = 0$ for $i \geq 2$. Hence $\mathcal{F}$ is a vector bundle, and by the same argument as in the proof of 2.1, we have also $H^2_* \mathcal{F} = 0$. Therefore the map from Stab to $\mathcal{P}si$ is also surjective.

Proposition 2.4 proves that there are vector bundles in every class of $\mathcal{P}si$ and that the psi-equivalence is an extension to the set of coherent sheaves of (local) projective dimension ≤ 1 of the stable equivalence for vector bundles with $H^2_* = 0$.

The following result is an easy consequence of [4]. We recall the proof for the convenience of the reader.

Proposition 2.5. *The application which maps a finite length graded R-module to the sheaf associated to its second module of syzygies induces a bijective correspondence between $\mathcal{M}_f$ and Stab. The application which maps a vector bundle $\mathcal{F}$ to its first cohomology module $H^1_*(\mathcal{F})$ induces the inverse correspondence between Stab and $\mathcal{M}_f$.*

Proof. Let M be a finite length graded R-module, and F be its second module of syzygies, so that we have an exact sequence:

$$0 \to F \to L_1 \to L_0 \to M \to 0$$

where L_1 and L_0 are free R-modules, and an exact sequence of (locally free) sheaves: $0 \to \mathcal{F} \to \mathcal{L}_1 \to \mathcal{L}_0 \to 0$. Taking the associated cohomology exact sequence, we get $M \simeq H^1_*(\mathcal{F})$ and $H^2_*(\mathcal{F}) = 0$.

Conversely, let $\mathcal{F}$ be a vector bundle such that $H^2_*(\mathcal{F}) = 0$, and let L_1 be a free R-module of finite type, such that there exists a surjective homomorphism: $L_1^\vee \to H^0_*(\mathcal{F}^\vee)$, whose kernel is denoted by L_0'. Let $\mathcal{L}_0'$ be the associated sheaf. By construction we have $H^1_*(\mathcal{L}_0') = 0$, and by Serre duality $H^1_*(\mathcal{F}^\vee) = 0$, so we get also $H^2_*(\mathcal{L}_0') = 0$. Therefore by Horrocks theorem $\mathcal{L}_0'$ is dissocié; if we set $\mathcal{L}_0'^\vee = \mathcal{L}_0$, we have an exact sequence:

$$0 \to \mathcal{F} \to \mathcal{L}_1 \to \mathcal{L}_0 \to 0$$

and $\mathcal{F}$ is the sheaf associated to the second module of syzygies of $H^1_*(\mathcal{F})$.

As an immediate consequence of 2.5 we obtain the following result:

Corollary 2.6. *The application which maps a coherent sheaf $\mathcal{N}$ to its first cohomology module $H^1_*(\mathcal{N})$ induces a bijective correspondence between $\mathcal{P}\!si$ and $\mathcal{M}_f$.*

Notation 2.7. *Tensorising a sheaf by $\mathcal{O}_{\mathbb{P}}(n)$ defines an action of $\mathbb{Z}$ on $\mathcal{P}\!si$. We denote by $\mathcal{P}\!si'$ the quotient set.*

Corollary 2.8. *The application which maps a curve C to its sheaf of ideals $\mathcal{J}_C$ induces a bijective correspondence between $\mathcal{B}il$ and $\mathcal{P}\!si'$.*

Proof. By the Rao theorem ([10]), there is a one-to-one correspondence between the set of curves, up to biliaison equivalence, and the set of finite length graded R-modules, modulo isomorphism, up to shift in degrees. This correspondence maps the class of a curve C to the class of its Rao-module $H^1_*(\mathcal{J}_C)$. We conclude by using 2.6.

3. Minimal elements.

In the previous paragraph we proved that there are vector bundles in every class of $\mathcal{P}\!si$. We will see now that there are other outstanding elements, namely the reflexive sheaves of rank 2. Moreover, we can characterize them.

Recall that for a function $f : \mathbb{Z} \to \mathbb{N}$ which vanishes for $m \ll 0$, we denote by $f^\sharp$ the "primitive" function defined by $f^\sharp(n) = \sum_{m \leq n} f(n)$ ([7] I.1).

Proposition 3.1. ([8]). *Let $\mathcal{F}$ be a vector bundle of rank r. There exists a function $q'_{\mathcal{F}} : \mathbb{Z} \to \mathbb{N}$ with finite support such that the following properties are equivalent for a dissocié sheaf $\mathcal{L} = \bigoplus \mathcal{O}_{\mathbb{P}}(-n)^{l(n)}$ of rank $r - 2$:*

  i) *a general homomorphism $u : \mathcal{L} \to \mathcal{F}$ is injective, and its cokernel is reflexive,*
  ii) *for all n, $l^\sharp(n) \leq (q'_{\mathcal{F}})^\sharp(n)$, and the condition of the "obligatory direct summand" holds (see [8] for an exact definition).*

Remark 3.2.

  1) The result is still true if we suppose that $\mathcal{F}$ is only reflexive. In [8], we give an explicit construction of the function $q'_{\mathcal{F}}$. In particular, we have $q'_{\mathcal{F} \oplus \mathcal{L}'} = q'_{\mathcal{F}} + l'$ for a dissocié sheaf $\mathcal{L}' = \bigoplus \mathcal{O}_{\mathbb{P}}(-n)^{l'(n)}$.
  2) It is easy to see that the conditions of 3.1 are satisfied in the case when $l = q'_{\mathcal{F}}$.

Proposition 3.3. *There exist rank 2 reflexive sheaves in every class of $\mathcal{P}si$. The set of first Chern classes of these reflexive sheaves has a lower bound. All the rank 2 reflexive sheaves of the class with minimal c_1 (we say that they are minimal rank 2 reflexive sheaves) have the same cohomology.*

Proof. Let $\mathcal{E}$ be a rank 2 reflexive sheaf in the class. In the last part of the proof of 2.4, we saw that there exists an exact sequence: $0 \to \mathcal{L} \to \mathcal{F} \to \mathcal{E} \to 0$, where $\mathcal{F}$ is a vector bundle such that $H^2_*(\mathcal{F}) = 0$ and where $\mathcal{L}$ is dissocié.

Let $\mathcal{F}$ be a vector bundle of rank r, $\mathcal{L} = \bigoplus \mathcal{O}_{\mathbb{P}}(-n)^{l(n)}$ be a a dissocié sheaf of rank $r - 2$ and $0 \to \mathcal{L} \to \mathcal{F} \to \mathcal{E} \to 0$ be an exact sequence where $\mathcal{E}$ is reflexive. Let us set $q'_{\mathcal{F}} = q'$. We have $c_1(\mathcal{E}) = c_1(\mathcal{F}) + \sum_{n \in \mathbb{Z}} n l(n)$ and

$$\sum_{n \in \mathbb{Z}} q'(n) = \sum_{n \in \mathbb{Z}} l(n) = r - 2.$$

For $m \gg 0$ we have:

$$\sum_{n \in \mathbb{Z}} n q'(n) = (m + 1) \sum_{n \in \mathbb{Z}} q'(n) - q'^{\sharp\sharp}(m)$$

$$\sum_{n \in \mathbb{Z}} n l(n) = (m + 1) \sum_{n \in \mathbb{Z}} l(n) - l^{\sharp\sharp}(m).$$

By 3.1, the function $q'^{\sharp} - l^{\sharp}$ is positive, so the function $q'^{\sharp\sharp} - l^{\sharp\sharp}$ is positive and increasing. Since it is positive, we obtain the inequality $\sum_{n \in \mathbb{Z}} n q'(n) \le \sum_{n \in \mathbb{Z}} n l(n)$ and $c_1(\mathcal{E}) \ge c_1(\mathcal{F}) + \sum_{n \in \mathbb{Z}} n q'(n)$.

Since it is increasing and vanishes for $m \ll 0$, if it vanishes also for $m \gg 0$, it vanishes everywhere, and then $l = q'$.

So the set of first Chern classes of rank 2 reflexive sheaves which are a quotient of $\mathcal{F}$ by a dissocié sheaf has a lower bound, which corresponds to the case when $l = q'$. We say that the corresponding quotients are *minimal rank 2 reflexive quotients* of $\mathcal{F}$.

From the equality $q'_{\mathcal{F} \oplus \mathcal{L}'} = q'_{\mathcal{F}} + l'$ for a dissocié sheaf $\mathcal{L}'$, we deduce that the minimal rank 2 reflexive quotients of $\mathcal{F}$ are also the minimal rank 2 reflexive quotients of $\mathcal{F} \oplus \mathcal{L}'$. Since all the vector bundles of a given class of $\mathcal{P}si$ are stably equivalent, they all have the same minimal rank 2 reflexive quotients. The dimensions of their cohomology groups are all the same, as follows from the lemma:

Lemma 3.4. *Let $\mathcal{F}$ be a vector bundle of rank r, $\mathcal{P}$ be a dissocié sheaf of rank $r - 2$, u and u' be two injective homomorphisms $\mathcal{P} \to \mathcal{F}$ whose cokernels $\mathcal{E}$ and $\mathcal{E}'$ are reflexive. Then $\mathcal{E}$ and $\mathcal{E}'$ have the same cohomology, that is $h^i \mathcal{E}(n) = h^i \mathcal{E}'(n)$ for all i and all n.*

Proof. For all n, we have $h^0\mathcal{E}(n) = h^0\mathcal{F}(n) - h^0\mathcal{P}(n) = h^0\mathcal{E}'(n)$ and $h^1\mathcal{E}(n) = h^1\mathcal{F}(n)) = h^1\mathcal{E}'(n)$. Moreover, $\mathcal{E}$ and $\mathcal{E}'$ have the same first Chern class c_1 and $\mathcal{E}(n)$ and $\mathcal{E}'(n)$ have the same Euler characteristic. Then by Serre duality, $H^3\mathcal{E}(n)$ is the dual of $\mathrm{Hom}\,(\mathcal{E}, \mathcal{O}_\mathbb{P}(-n-4) = H^0(\mathcal{E}^\vee(-n-4)) = H^0(\mathcal{E}(-c_1-n-4))$. Therefore for all n $h^3\mathcal{E}(n) = h^3\mathcal{E}'(n)$, and since their Euler characteristic are equal, $h^2\mathcal{E}(n) = h^2\mathcal{E}'(n)$.

Remarks 3.5.

1) By 2.6 the notion of biliaison of rank 2 reflexive sheaves introduced by [2] is the restriction of the psi-equivalence to the set of rank 2 reflexive sheaves.

2) [2] proves that the minimal rank 2 reflexive sheaves in a class of psi-equivalence have also the minimal third Chern class. Hence she points out that if there are rank 2 vector bundles in the class ($c_3 = 0$), they are minimal elements and all the minimal elements are vector bundles.

Now if we want to give an explicit description of the inverse correspondence between $\mathscr{P}si'$ and $\mathscr{B}il$ (cf. 2.8), there are two possibilities, using the two kinds of outstanding elements in a class of psi-equivalence.

The first choice (a vector bundle) will lead to the theory of minimal curves ([9], [7] V, [1]) and give a characterization of the curves in the biliaison class. We use the following result, which is analogous to 3.1.

Proposition 3.6. ([8]). *Let $\mathcal{F}$ be a vector bundle of rank r. There exists a function $q_\mathcal{F} : \mathbb{Z} \to \mathbb{N}$ with finite support such that the following properties are equivalent for a dissocié sheaf $\mathcal{L} = \bigoplus \mathcal{O}_\mathbb{P}(-n)^{l(n)}$ of rank $r-1$:*

i) *a general homomorphism $u : \mathcal{L} \to \mathcal{F}$ is injective, and its cokernel is the twisted ideal of a curve,*

ii) *for all n, $l^\sharp(n) \leq q_\mathcal{F}^\sharp(n)$, and the condition of the "obligatory direct summand" holds (see [8] for an exact definition).*

Remarks 3.7.

1) The function $q_\mathcal{F}$ satisfies also 3.2.1 and 3.2.2, as the function $q'_\mathcal{F}$.

2) For every curve C, there exists an exact sequence: $0 \to \mathcal{L} \to \mathcal{F} \to \mathcal{J}_C \to 0$, where $\mathcal{F}$ is a vector bundle such that $H^2_*(\mathcal{F}) = 0$ and where $\mathcal{L}$ is dissocié. One can obtain this result as a consequence of the last part of the proof of 2.1, but it has been already proved in [7] II (N resolution of the curve).

3) As in 3.3, we can prove that the curves corresponding to the case when $l = q_\mathcal{F}$ have the minimal shift of their ideal, or of their Rao-module. These are the *minimal curves* of the class.

With the second choice (a rank 2 reflexive sheaf $\mathcal{E}$) we obtain curves which are the zero locus of a section of $\mathcal{E}(n)$:

$$0 \to \mathcal{O}_{\mathbb{P}}(-n) \to \mathcal{E} \to \mathcal{J}_C(c_1(\mathcal{E}) + n) \to 0.$$

It is then very natural to ask the following question:

Question I. *Is a minimal curve of a biliaison class the zero locus of a section of a twist of a (minimal in the corresponding class of $\mathcal{P}si$) rank 2 reflexive sheaf?*

Proposition 3.8. ([2]). *The following properties are equivalent:*

i) *the general minimal curve is the zero locus of a section of a twist of a rank 2 reflexive sheaf,*

ii) *the general minimal curve is the zero locus of a section of a twist of a minimal rank 2 reflexive sheaf,*

iii) *let $\mathcal{E}$ be a minimal rank 2 reflexive sheaf and let n_0 its minimal twist which has non-zero sections, i.e. $n_0 = \inf\{ n \mid H^0\mathcal{E}(n) \neq 0 \}$, then a non-zero section of $\mathcal{E}(n_0)$ vanishes along a minimal curve,*

iv) *for all vector bundle $\mathcal{F}$ in the corresponding class, for all $n \in \mathbb{Z}$, $q'_{\mathcal{F}}(n) \leq q_{\mathcal{F}}(n)$.*

It turns out that in general the answer to question I is negative. One example was given in [2]. We will give a particular case of this example, where it is possible to give a description of a corresponding minimal curve.

Example 3.9. Let f_1, f_2, f'_3, f'_4 (resp. f_1, f_2, f''_3, f''_4) a regular sequence of homogeneous polynomials of degrees $n_i = deg(f_i)$ with $n_1 = n_2 < n_3 = n_4$ and consider the finite length graded R-modules $M' = R/(f_1, f_2, f'_3, f'_4)$, $M'' = R/(f_1, f_2, f''_3, f''_4)$ and $M = M' \oplus M''$. Let $\mathcal{F}$ be the sheaf associated to the second module of syzygies of M so that we have an exact sequence:

$$0 \to \mathcal{F} \to \mathcal{O}_{\mathbb{P}}(-n_1)^4 \oplus \mathcal{O}_{\mathbb{P}}(-n_3)^4 \to \mathcal{O}_{\mathbb{P}}^2 \to 0.$$

Using the machinery of [8] we can compute the corresponding functions $q_{\mathcal{F}}$ and $q'_{\mathcal{F}}$ and we obtain:

$$q_{\mathcal{F}}(2n_1) = 2, \quad q_{\mathcal{F}}(n_1 + n_3) = 3 \quad \text{and} \quad q_{\mathcal{F}}(n) = 0 \quad \text{elsewhere,}$$

$$q'_{\mathcal{F}}(n_1 + n_3) = 4 \quad \text{and} \quad q'_{\mathcal{F}}(n) = 0 \quad \text{elsewhere,}$$

which shows that the condition 3.8.iv is not satisfied for $n = n_1 + n_3$.

Moreover, any minimal curve Γ of the biliaison class corresponding to M has a resolution:

$$0 \to \mathcal{O}_\mathbb{P}(-2n_1)^2 \oplus \mathcal{O}_\mathbb{P}(-n_1 - n_3)^3 \to \mathcal{F} \to \mathcal{J}_\Gamma(3n_1 - n_3) \to 0$$

and a Rao-module $H^1_*(\mathcal{J}_\Gamma) \simeq M(n_3 - 3n_1)$.

Let C' (resp. C'') be a general minimal curve of the biliaison class corresponding to M' (resp. M''), which has degree $2n_1^2$. The saturated homogeneous ideal of C' and C'' are given by:

$$I_{C'} = (f_1, f_2)^2 + (g'_3 f'_3 + g'_4 f'_4) \quad I_{C''} = (f_1, f_2)^2 + (g''_3 f''_3 + g''_4 f''_4),$$

where g'_3, g'_4 (resp. g''_3, g''_4) are two independant linear forms in f_1, f_2. We have $H^1_*(\mathcal{J}_{C'}) = M'(n_3 - n_1)$ and $H^1_*(\mathcal{J}_{C''}) = M''(n_3 - n_1)$.

Now we will use the "liaison addition" of Schwartau ([11]). Let P' and P'' be two homogeneous elements of degree $2n_1$ of the ideal $(f_1, f_2)^2$, without common divisor. The ideal $I = P'' I_{C'} + P' I_{C''}$ is saturated and defines a curve C, which is, as a set, the union of C', C'' and the complete intersection defined by (P', P'').

From the exact sequence:

$$0 \to R(-4n_1) \xrightarrow{i} I_{C'}(-2n_1) \oplus I_{C''}(-2n_1) \xrightarrow{p} I \to 0$$

where the maps i and p are given by: $i(a) = (aP', -aP'')$, $p(b', b'') = b'P'' + b''P'$, and the associated exact sequence of sheaves, we deduce that C has degree $8n_1^2$ and that its Rao-module is isomorphic to:

$$H^1_*(\mathcal{J}_{C'}(-2n_1)) \oplus H^1_*(\mathcal{J}_{C''}(-2n_1)) \simeq$$

$$\simeq M'(n_3 - 3n_1) \oplus M''(n_3 - 3n_1) \simeq M(n_3 - 3n_1).$$

Therefore C is a minimal curve of the biliaison class corresponding to M, and it is not the zero locus of a section of a twist of a rank 2 reflexive sheaf. In fact, by 3.8, the general minimal curve is not the zero locus of a section of a twist of a rank 2 reflexive sheaf ; this is an open property, so no minimal curve can be the zero locus of a section of a twist of a rank 2 reflexive sheaf.

By 3.5.2, if there are rank 2 vector bundles in a class, they are the minimal elements (among the rank 2 reflexive sheaves). Recall that the curves obtained as the zero locus of a section of a twist of a rank 2 vector bundle are exactly the so-called *subcanonical* curves (cf. [3]):

Définition 3.10. *A curve C is subcanonical if there exists $a \in \mathbb{Z}$ and an isomorphism: $\omega_C \simeq \mathcal{O}_C(a)$, where ω_C is the dualizing sheaf of C.*

In this particular case, the question takes the following form, under which it was asked to me by R. Hartshorne and P. Ellia:

Question II. *If a biliaison class contains subcanonical curves, is a minimal curve subcanonical ?*

There is a result analogous to 3.8:

Proposition 3.11. ([2]). *The following properties are equivalent:*

i) *if a biliaison class contains subcanonical curves, every minimal curve is subcanonical,*

ii) *let $\mathcal{E}$ be a rank 2 vector bundle and let n_0 be its minimal twist which has non-zero sections, i.e. $n_0 = \inf\{\, n \mid H^0\mathcal{E}(n) \neq 0 \,\}$, then a non-zero section of $\mathcal{E}(n_0)$ vanishes along a minimal curve.*

In a recent work ([6]) I gave a positive answer to question II by proving 3.11. ii.

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