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ON ALL-PATH CONVEX, GATED AND CHEBYSHEV SETS IN GRAPHS

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We present new characterizations for trees, block graphs, and geodetic graphs using all-path convex, gated and Chebyshev sets. Specifically, we prove that trees are exactly the graphs in which all-path convexity is a convex geometry. Block graphs are characterized as graphs in which all balls are all-path convex (equivalently, gated), and geodetic graphs are exactly those graphs where all balls (equivalently, closed neighborhoods) are Chebyshev. Additionally, we prove that almost all graphs have geodesically convex Chebyshev sets, provide a characterization of bipartite graphs with connected Chebyshev sets, and establish a criterion for graphs with trivial Chebyshev sets in the class of graph joins. Finally, we show that graphs of odd order with maximal number of edges under the Seidel switching operation always have trivial Chebyshev sets.

1. Introduction

An abstract convexity on a set X is a family $\mathcal{C}$ of its subsets, called convex sets, which satisfies three axioms (resembling axioms for closed sets in topology): the empty set and the whole X are convex sets; arbitrary intersection of convex sets is a convex set; any nested union of convex sets is a convex set.

A common approach to defining a convexity on a set X involves the use of the so-called interval operators. These are the maps of the form $I\colon X \times X \to 2^X$

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satisfying two conditions: for all $a,b \in X$, we have $a,b \in I(a,b)$ and $I(a,b) = I(b,a)$. The set $I(a,b)$ is interpreted as the "interval" between a,b. A subset $A \subset X$ is I-convex provided $I(a,b) \subset A$ for all $a,b \in A$. It is straightforward to verify that the family of I-convex subsets of X forms an abstract convexity on X. Conversely, any convexity $\mathcal{C}$ on X naturally induces an interval operator on X: here, $I(a,b)$ is defined as the convex hull (the intersection of all convex sets containing a given set) of the set $\{a,b\}$. Furthermore, in [11, Proposition 2.2.1] it was proved that these two correspondences form an antitone Galois connection between the poset of all convexities (ordered by inclusion) and the poset of all interval operators (ordered by: $I_1 \leq I_2$ if $I_1(a,b) \subset I_2(a,b)$ for all $a,b \in X$).

The two most well-known interval operators used to define a convexity are the following: the metric interval in a metric space (X,d), which is the set $[a,b] = \{x \in X : d(a,x) + d(x,b) = d(a,b)\}$; the linear interval in a normed space, defined as $[a,b] = \{ta + (1-t)b : t \in [0,1]\}$.

Every connected graph G naturally induces a metric on its vertex set $V(G)$, which gives rise to a corresponding convexity on $V(G)$ called the geodesic convexity. However, in graphs, the metric interval $[a,b]$ is exactly the set of vertices lying on all shortest paths between two vertices $a,b \in V(G)$. Replacing "shortest" with any other class of paths (which must connect any pair of vertices in G), we obtain various other interval operators. In addition to shortest paths (geodesic convexity), notable examples include induced paths (monophonic convexity), simple paths (all paths convexity), and other families of paths that define new graph convexities (see [5]).

In this work, we study properties of the classes of all-path convex, gated and Chebyshev sets in connected graphs. The paper is organized as follows. In Section 2, we provide the necessary definitions and preliminary results from metric graph theory and abstract convexity theory.

In Section 3.1, we present new characterizations for trees, block graphs, and geodetic graphs in terms of the aforementioned classes of vertex subsets. In particular, we show that trees are exactly the graphs in which all-path convexity forms a convex geometry (Theorem 3.3), prove that block graphs can be characterized as graphs in which every connected set of vertices is geodesically convex (Proposition 3.5), and provide new criterion for geodetic graphs using Chebyshev sets (Theorem 3.9). Additionally, Proposition 3.7 provides another characterization of trees in terms of Chebyshev sets, and Theorem 3.11 establishes a similar characterization for block graphs in terms of all-path convex and gated sets.

In Section 3.2, we study the properties of graphs with trivial Chebyshev sets, drawing an analogy with the so-called g-minimal graphs considered in [3]. We show that almost every graph have g-convex Chebyshev sets (Corollary 3.13), characterize bipartite graphs with connected Chebyshev sets in Proposition 3.14, and provide a criterion for graphs with trivial Chebyshev sets in the class of graph joins in Theorem 3.15 and Proposition 3.16. As a corollary of a more general result, it is established that graphs of odd order with a maximal number of edges under the Seidel switching operation always have trivial Chebyshev sets (Corollary 3.19).

2. Definitions and preliminary results

2.1. Basic definitions

In this paper, all graphs under consideration are simple, undirected, and finite. Thus, a *graph* G is a pair (V,E), where $V = V(G)$ is the *vertex set* and $E = E(G)$ is the *edge set* of G. Instead of $\{u,v\}$, edges will be denoted simply as uv. Two vertices $u,v \in V(G)$ are *adjacent* provided $uv \in E(G)$. The *neighborhood* of u is the set $N_G(u) = \{v \in V(G) : uv \in E(G)\}$. The set $N_G[u] = N_G(u) \cup \{u\}$ is called the *closed neighborhood* of u. The number $d_G(u) = |N_G(u)|$ is called the *degree* of a vertex $u \in V(G)$. A vertex u is a *leaf* provided $d_G(u) = 1$. A vertex u is *simplicial* if any two vertices from $N_G(u)$ are adjacent. Clearly, each leaf vertex is simplicial.

A graph is *complete* if every two its vertices are adjacent. By K_n we denote the complete graph with n vertices. The graph $K_4 - e$, obtained from the complete graph K_4 by deleting an edge, is called the *diamond-graph*.

A graph H is a *subgraph* of G if $V(H) \subset V(G)$ and $E(H) \subset E(G)$. For a set of vertices $A \subset V(G)$, by $E_G(A)$ we denote the set of all edges in G with endpoints in A. The *subgraph induced by* A is denoted by $G[A]$: here $V(G[A]) = A$ and $E(G[A]) = E_G(A)$. A graph G is said to be *H-free* provided G has no subgraphs isomorphic to H. Note that K_3-free graphs are called *triangle-free*.

Given a pair of graphs G and H, their *join* if the graph $G_1 + G_2$, having vertex set $V(G_1) \sqcup V(G_2)$ and the edge set $E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1), v \in V(G_2)\}$. In other words, $G_1 + G_2$ is obtained from the disjoint union of G_1 and G_2 by adding all possible edges between them.

A set of vertices $A \subset V(G)$ is *independent* if no two vertices from A are adjacent. A graph is called *bipartite* if its vertex set can be partitioned into two independent subsets, which are called *parts*. A bipartite graph is called *complete bipartite* if it contains all the possible edges between the corresponding parts.

2.2. Metric graph theory

A graph is *connected* if there is a path between every pair of its vertices. A set $A \subset V(G)$ is *connected* provided the induced subgraph $G[A]$ is connected.

The vertex set $V(G)$ of a connected graph G is endowed with the natural metric d_G, where $d_G(x,y)$ equals the length of a shortest path between u and v. It is clear that for any connected set $A \subset V(G)$, we have $d_G(x,y) \leq d_{G[A]}(x,y)$ for all $x,y \in A$. A connected subgraph $H \subset G$ is called *isometric* if $d_H(x,y) = d_G(x,y)$ for all $x,y \in V(H)$.

The *diameter* of a connected graph G is the number

$$\mathrm{diam}(G) = \max\{d_G(x,y) : x,y \in V(G)\}.$$

Let G be a connected graph, $u \in V(G)$, and $k \in \mathbb{N}$. By $N_G^k(u)$ and $N_G^k[u]$ we denote the *sphere* and the *(closed) ball* of radius k centered at u. In other words, $N_G^k(u) = \{x \in V(G) : d_G(u,x) = k\}$ and $N_G^k[u] = \{x \in V(G) : d_G(u,x) \leq k\}$. Note that $N_G^1(u) = N_G(u)$ and $N_G^1[u] = N_G[u]$.

The following simple characterization of connected bipartite graphs will be used later in the paper.

Lemma 2.1. *For a connected graph G, the following statements are equivalent:*

1. *G is bipartite;*
2. *all spheres in G are independent;*
3. *there is a vertex $u \in V(G)$ with all spheres $N_G^k(u)$, $k \geq 1$, being independent.*

Proof. Since in a connected bipartite graph, every sphere lies in one side of the bipartition, the first statement implies the second. Further, the second statement trivially implies the third. Finally, suppose that the third statement holds. Put $A = \{v \in V(G) : v \in N_G^k(u), k \text{ is even}\}$ and $B = \{v \in V(G) : v \in N_G^k(u), k \text{ is odd}\}$. Then $V(G) = A \sqcup B$ is a bipartition of G. □

The *metric interval* between a pair of vertices $u,v \in V(G)$ in a connected graph G is the set $[u,v]_G = \{x \in V(G) : d_G(u,x) + d_G(x,v) = d_G(u,v)\}$. Hence, $[u,v]_G$ equals the union of vertex sets of all shortest paths between u and v.

A set $A \subset V(G)$ is *geodesically convex*, or simply *g-convex*, if $[u,v]_G \subset A$ for all $u,v \in A$. It is clear that any g-convex set induces an isometric subgraph.

For a vertex $u \in V(G)$ and a set $A \subset V(G)$, by $d_G(u,A) := \min\{d_G(u,a) : a \in A\}$ we denote the *distance from u to A*. The set $\mathrm{pr}_A(u) = \{a \in A : d_G(u,a) = d_G(u,A)\}$ is called the *projection set* of u in A. A set A is called *Chebyshev* if $|\mathrm{pr}_A(u)| = 1$ for all $u \in V(G)$. A set A is called *gated* provided for every

$u \in V(G)$, there exists $a_u \in A$ such that for all $a \in A$, it holds $a_u \in [u,a]_G$. The vertex a_u is called the *gate* for u in A, and it is unique if it exists. Note that any gated set is non-empty. Moreover, one can observe that any gated set is Chebyshev (since the gate a_u is the unique projection of u in A). It is also can be proved that each gated set is g-convex. Furthermore, the graphs in which these classes of vertex sets coincide are called *fully gated graphs* [7].

A connected graph without cycles is called a *tree*. A vertex in a graph is called a *cut vertex* if its deletion increases the number of connected components. Clearly, in trees, every non-leaf vertex is a cut vertex. A graph is called *biconnected* provided it has no cut vertices. A *block* in a graph is its maximal biconnected subgraph. When dealing with blocks in graphs, we will frequently use the following technical result.

Lemma 2.2 ([1])**.** *Let G be a non-complete biconnected graph. Then G contains a cycle of length at least four, and for each cycle C of minimal length, we have: C is an isometric subgraph in G, or $G[V(C)]$ is a diamond-graph.*

The *block graph* $B(G)$ of a given graph G is the intersection graph of all blocks in G. In other words, the vertices of $B(G)$ correspond to the blocks in G, and there is an edge in $B(G)$ provided two blocks share a common cut vertex. Abstractly, a graph H is called a *block graph* if it is isomorphic to $B(G)$ for some G. In his seminal paper [12], Harary proved that a graph is a block graph if and only if each of its blocks is complete. In particular, any tree is a block graph.

A connected graph is called *geodetic* if there is a unique shortest path between every pair of its vertices. For example, connected block graphs (hence, trees) and odd cycles are geodetic graphs.

Another generalization of trees is provided by the so-called median graphs. Here, a *median* for a triple of vertices $u,v,w \in V(G)$ is any vertex from the set $[u,v]_G \cap [u,w]_G \cap [v,w]_G$. Note that a given triple of vertices can have several medians or no median at all. A graph is called a *median graph* if every triple of vertices has a unique median. It is easy to prove that each median graph is bipartite. Moreover, each median graph is fully gated (in fact, a graph is median if and only if it is a fully gated partial cube [7, Lemma 2.1]). Additionally, note that trees are exactly the median graphs that are block graphs.

2.3. Convexity spaces on graphs

A *convexity* on a given set X is a family of its subsets $\mathcal{C}$ that satisfies the next three conditions (see [16, p. 3]):

1. $\emptyset, X \in \mathcal{C}$;

2. for all subfamilies $\mathcal{C}' \subset \mathcal{C}$, it holds $\bigcap_{A \in \mathcal{C}'} A \in \mathcal{C}$;

3. for any subfamily $\mathcal{C}' \subset \mathcal{C}$ that is totally ordered by inclusion, we have $\bigcup_{A\in\mathcal{C}'} A \in \mathcal{C}$.

The elements of $\mathcal{C}$ are called *convex sets*, and the pair $(X,\mathcal{C})$ is called *convexity space*. For example, the family of all g-convex sets in a connected graph G forms a convexity on $V(G)$. Also, all gated sets in a (finite) connected graph together with the empty set, forms a convexity (the empty set must be included, as every gated set is non-empty).

Let $(X,\mathcal{C})$ be a convexity space and $A \subset X$. The *convex hull* of A is the smallest convex set containing A. Formally, the convex hull equals the intersection of all convex sets containing A.

A closely related concept to convexities is provided by the so-called interval operators. Here, an *interval operator* on a set X is any map of the form $I : X \times X \to 2^X$ which satisfies the next two conditions (see [16, p. 71]):

1. $a,b \in I(a,b)$ for all $a,b \in X$;
2. $I(a,b) = I(b,a)$ for all $a,b \in X$.

Convexities and interval operators are related by the following constructions. Given a convexity $\mathcal{C}$ on X and $a,b \in X$, we define $I_{\mathcal{C}}(a,b)$ to be the convex hull of the set $\{a,b\}$. It is clear that $I_{\mathcal{C}}$ is an interval operator on X. Conversely, each interval operator I on X induces a convexity by declaring a set $A \subset X$ to be I-convex if $I(a,b) \subset A$ for all $a,b \in A$. It can be easily proved that the family of I-convex sets indeed forms a convexity on X. Moreover, in [11, Proposition 2.2.1], it was shown that these two constructions establish an antitone Galois connection between the posets of all convexities (ordered by inclusion) and all interval operators (ordered by: $I_1 \leq I_2$ if $I_1(a,b) \subset I_2(a,b)$ for all $a,b \in X$) on X.

In graph theory, the majority of studied convexities arise from interval operators which are constructed based on different path classes. Namely, let $\mathcal{P}$ be a collection of paths in a connected graph G such that for every pair of vertices in G, there exists an element of $\mathcal{P}$ that joins them. For $a,b \in V(G)$, define

$$I_{\mathcal{P}}(a,b) = \{x \in V(G) : x \text{ lies on some path } P \in \mathcal{P} \text{ joining } a,b\}.$$

Then $I_{\mathcal{P}}$ is an interval operator on $V(G)$, which induces the corresponding convexity $\mathcal{C}$. Prominent examples of graph convexities that arise in this way are the following: if $\mathcal{P}$ is the family of all shortest paths, then $\mathcal{C}$ is the family of g-convex sets; if $\mathcal{P}$ is the family of all induced paths, then $\mathcal{C}$ is the family of *monophonically convex*, or shortly, *m-convex sets*; if $\mathcal{P}$ is the family of all simple paths, then the elements of $\mathcal{C}$ are called *all-path convex* (shortly, *AP-convex*) sets. Note that since each shortest path is necessarily induced, each m-convex set is g-convex. Similarly, it can be shown that each AP-convex set is gated (in fact, see [11, Theorem 3.3.1] for another criterion for AP-convex sets in terms of the existence of the so-called strong gates). To summarize, the scheme of inclusions between these classes of vertex subsets is depicted in Figure 1.

$$
\begin{array}{ccccccc}
 & & \text{Chebyshev} & & & & \\
 & & \cup & & & & \\
\text{AP-convex} & \subset & \text{gated} & \subset & \text{g-convex} & \subset & \text{connected} \\
 & & & & \cup & & \\
 & & & & \text{m-convex} & &
\end{array}
$$

Figure 1: The scheme of inclusions between classes of vertex sets.

Using the inclusions between these classes, one can characterize several known graph families. For example, the corresponding criterion for trees was obtained in [15].

Theorem 2.3 ([15, Theorem 5])**.** *A connected graph is a tree if and only if each connected set of its vertices is AP-convex.*

The characterization of AP-convex sets was obtained in [11].

Theorem 2.4 ([11, Theorem 3.1.1])**.** *Let G be a connected graph, $A \subset V(G)$ and $|A| \geq 2$. The set A is AP-convex if and only if the induced subgraph $G[A]$ is a connected union of blocks in G.*

Hence, in a tree, the classes of connected, g-convex, gated, and AP-convex sets coincide. In fact, they form a lattice under the inclusion. This lattice was studied by Zelinka in [17] for finite and infinite trees. In particular, this lattice determines the tree up to isomorphism [17, Theorem 11].

Also, note that connected bipartite graphs can be characterized as graphs in which each edge is gated (equivalently, Chebyshev). Similarly, a graph is median if and only if each of its metric intervals is gated (equivalently, Chebyshev); see [2].

Let $(X, \mathcal{C})$ be a convexity space, and let $S \in \mathcal{C}$ be a convex set. An element $x \in S$ is called an *extremal point* for S provided $S \setminus \{x\} \in \mathcal{C}$. The convexity space satisfies the *Minkowski-Krien-Milman property* (shorly, *MKM-property*) if every convex set is the convex hull of its extremal points. In this case, the corresponding convexity is called a *convex geometry*. For example, in the case of g-convexity, a vertex $x \in S$ is extremal for a g-convex set $S \subset V(G)$ if and only if x is a simplicial vertex in the induced subgraph $G[S]$.

It turns out that g-convexity in a graph G is a convex geometry if and only if G is a Ptolemaic graph (see [9, Theorem 4.1]). Similarly, the m-convexity in

G is a convex geometry if and only if G is a chordal graph (see [9, Corollary 3.3]). For more similar characterizations of graph classes in terms of various graph convexities being convex geometries, we refer to the survey paper [8].

3. Main results

3.1. Characterizations of trees, block graphs, and geodetic graphs

Recall that every AP-convex set of vertices is gated. The cycle C_5 serves as an example of a graph in which each gated set is AP-convex, as C_5 does not have non-trivial gated sets. In the following two propositions, we provide necessary and sufficient conditions for graphs where AP-convexity and gated convexity coincide.

Proposition 3.1. *Let G be a connected graph in which every gated set is AP-convex. Then every edge in G is either a bridge or lies on an odd cycle in G.*

Proof. Assume an edge $e = uv \in E(G)$ is not a bridge. Then e lies in a block larger than itself in G, which implies that e is not an AP-convex set and therefore not gated. Consequently, there exists a vertex $x \in V(G)$ with $d_G(u,x) = d_G(v,x)$. This implies that the edge e lies on an odd closed walk. Hence, e lies on an odd cycle in G. □

Proposition 3.2. *In connected block graphs, every gated set is AP-convex.*

Proof. Let G be a connected block graph, and $A \subset V(G)$ be a gated set with $|A| \geq 2$. Since each block in G is complete, for every edge $e \in E_G(A)$, the set A must contain the entire vertex set of the block in G that contains e (otherwise, A will be non-Chebyshev). Thus, $G[A]$ is the union of blocks in G. However, A is g-convex, implying that A is connected. Hence, $G[A]$ is a connected union of blocks. Therefore, A is AP-convex by Theorem 2.4. □

We proceed by showing that trees are the only graphs with AP-convex sets forming a convex geometry.

Theorem 3.3. *The AP-convexity on a connected graph G forms a convex geometry if and only if G is a tree.*

Proof. We first observe that a vertex $u \in A$ is an extremal point for an AP-convex set $A \subset V(G)$ if and only if u is a leaf vertex in $G[A]$.

Indeed, let $A \setminus \{u\}$ be an AP-convex set. If there exist two different vertices $x, y \in N_G(u) \cap A$, then since $A \setminus \{u\}$ is connected, u lies on a cycle with x, y. This contradicts the fact that $A \setminus \{u\}$ is AP-convex. Thus, $|N_G(u) \cap A| \leq 1$. However, A is connected, which implies that u is a leaf vertex in $G[A]$.

Conversely, if u is a leaf vertex in $G[A]$, then as A is AP-convex, $A \setminus \{u\}$ also must be AP-convex (as simple paths between vertices from $A \setminus \{u\}$ cannot pass through u).

Thus, if G is a tree, then every AP-convex set $A \subset V(G)$ induces a subtree. Therefore, A is the convex hull of the set of all leaves in $G[A]$ (which are exactly its extremal vertices in AP-convexity). This establishes the sufficiency of the condition. To prove its necessity, to the contrary, assume that G has a cycle C. Then the vertex set of a block B in G containing C is an AP-convex set. However, B clearly has no leaf vertices. This means that $V(B)$ is not the convex hull of its extremal points, contradicting the definition of a convex geometry. □

By combining Proposition 3.2 and Theorem 3.3, we immediately derive the following known result.

Corollary 3.4 ([6, Theorem 5.1])**.** *The gated convexity on a connected graph G forms a convex geometry if and only if G is a tree.*

Recall that each g-convex set is trivially connected. Interestingly, the graphs for which the classes of AP-convex and g-convex subsets coincide are precisely the block graphs.

Proposition 3.5. *A connected graph G is a block graph if and only if every connected set of vertices in G is g-convex.*

Proof. **Necessity.** To the contrary, suppose $A \subset V(G)$ is a connected, but not a g-convex set. Then there exist two vertices $x, y \in A$ with $[x,y]_G \setminus A \neq \emptyset$. In particular, $xy \notin E(G)$. Assume that $x, y \in A$ are such vertices with the smallest possible distance $d_G(x,y)$. Since A is connected, there exists a path P in $G[A]$ joining x and y. Furthermore, there is a shortest path Q between x, y such that $V(Q) \cap A = \{x,y\}$. The concatenation of P and Q forms a simple cycle in G. However, the vertex set $V(C)$ induces a biconnected non-complete subgraph in G, contradicting the assumption that G is a block graph.

Sufficiency. Again, towards the contradiction, let us assume that G is not a block graph. Then it contains a non-complete block B. Fix a pair of non-adjacent vertices $u, v \in V(B)$ with distance $d_G(u,v) = d_B(u,v) = 2$. Fix their common neighbor $w \in N_B(u) \cap N_B(v)$. As B is a biconnected subgraph, the edges uw and wv lie on a common cycle C in B. Now take A to be the set of vertices of the path from u to v on C that does not pass through w. Then A is a connected but not a convex set. This contradiction establishes that G must be a block graph. □

Corollary 3.6. *A connected graph is a tree if and only if it is bipartite and every connected set of vertices is g-convex.*

In addition to Theorem 2.3 and Corollary 3.6, trees can also be characterized as graphs in which all metric intervals (equivalently, all edges) are AP-convex. Moreover, Chebyshev sets provide yet another criterion for identifying trees.

Proposition 3.7. *A connected graph is a tree if and only if every connected set of its vertices is Chebyshev.*

Proof. Let $A \subset V(T)$ be a connected set of vertices in a tree T. If A is not Chebyshev, then there is $x \in V(T)$ with $|\operatorname{pr}_A(x)| \geq 2$. Fix such an x with the smallest distance $d_T(x,A)$. Choose two distinct vertices $u, v \in \operatorname{pr}_A(x)$ with two shortest paths from x to u, and from x to v. Since A is connected, there is a shortest path between u and v in A. The concatenation of these three paths forms a cycle in T, which is a contradiction.

Conversely, assume that in a connected graph G, every connected set of vertices is Chebyshev, but G is not a tree. Hence, G contains a cycle C. Fix a vertex $x \in V(C)$, and consider the set $A = V(C) \setminus \{x\}$. Then A is connected, but non-Chebyshev as $|\operatorname{pr}_A(x)| \geq 2$. This contradiction completes the proof. □

Recall that a connected graph is called fully gated if each of its g-convex sets is gated. By relaxing this condition to require that every g-convex set is Chebyshev, we obtain a broader class of graphs. The following result demonstrates that the existence of a subgraph isomorphic to C_4 is fundamental for this class of graphs.

Proposition 3.8. *Let G be a C_4-free graph. Then every g-convex set of vertices in G is Chebyshev if and only if G is a tree.*

Proof. The sufficiency of this condition follows directly from Proposition 3.7. Thus, we only need to prove the necessity. Assume that G is a C_4-free graph in which all g-convex sets are Chebyshev. Then G is triangle-free, since any edge that lies on a triangle forms a g-convex set that is not Chebyshev. Consequently, G is also diamond-free.

If G is not a tree, then using Lemma 2.2, we can conclude that G contains an isometric cycle C of length $m \geq 5$. Let $V(C) = \{x_1, \ldots, x_m\}$ and $E(C) = \{x_i x_{i+1} : 1 \leq i \leq m-1\} \cup \{x_1 x_m\}$. If m is odd, then the set $A = \{x_1, x_m\}$ is a g-convex but not a Chebyshev set as

$$d_G(x_{\frac{m+1}{2}}, A) = d_G(x_{\frac{m+1}{2}}, x_1) = d_G(x_{\frac{m+1}{2}}, x_n) = d_C(x_{\frac{m+1}{2}}, x_n) = \frac{m-1}{2}.$$

If m is even, then the set $B = \{x_1, x_2, x_m\}$ is g-convex. However, B is not Chebyshev as

$$d_G(x_{\frac{m+2}{2}}, B) = d_G(x_{\frac{m+2}{2}}, x_2) = d_G(x_{\frac{m+2}{2}}, x_m) = d_C(x_{\frac{m+2}{2}}, x_m) = \frac{m}{2}.$$

The obtained contradiction implies that G must be a tree. □

Proposition 3.5 immediately asserts that connected block graphs have g-convex balls, as every ball in such graphs is a connected set. Graphs with this property were characterized in [10, Theorem 2.2] (see also [10, Theorem 2.1]) and have been further extensively studied in [4]. Next, we characterize graphs in which all balls are Chebyshev.

Theorem 3.9. *For a connected graph G, the following statements are equivalent:*

1. *G is a geodetic graph;*
2. *$N_G[u]$ is Chebyshev for all $u \in V(G)$;*
3. *all balls in G are Chebyshev.*

Proof. If G contains a non-Chebyshev ball, then for some vertex $x \in V(G)$ and some $k \geq 1$, there exists $y \in V(G)$ such that $|\mathrm{pr}_{N_G^k[x]}(y)| \geq 2$. Fix two distinct projections $u, v \in \mathrm{pr}_{N_G^k[x]}(y)$. It is clear that these projections must lie on the corresponding sphere, i.e., $u, v \in N_G^k(x)$. Thus, $d_G(x,u) = d_G(x,v) = k$.

Concatenating a shortest path from x to u with a shortest path from u to y produces a shortest path from x to y. Similarly, concatenating a shortest path from x to v with a shortest path from v to y gives a different shortest path between x and y. This implies that G is not geodetic, leading to a contradiction. Therefore, the first statement implies the third.

It is clear that the third statement impleis the second.

Hence, we now prove that the second statement implies the first. Assume, for contradiction, that G is not geodetic. Then there exist two vertices $u, v \in V(G)$ with at least two distinct shortest paths between them. Moreover, we can assume that there exist such shortest paths P and Q that have no common internal vertices, i.e., $V(P) \cap V(Q) = \{u, v\}$.

Consider the set $N_G[u]$. This set is not Chebyshev because v has at least two distinct projections in $N_G[u]$. This contradiction completes the proof of the theorem. □

Corollary 3.10. *In block graphs, all balls are Chebyshev.*

Proof. The result follows directly from Theorem 3.9, as block graphs are geodetic. □

Note that C_5 is a geodetic graph that contains a non-gated ball—specifically, the closed neighborhood of any vertex. However, it was shown in [11] that block graphs are precisely the graphs in which $N_G[u]$ is gated for all $u \in V(G)$. Using this result, we derive another characterization of block graphs.

Theorem 3.11. *For a connected graph G, the following statements are equivalent:*

1. *G is a block graph;*

2. *all balls in G are AP-convex;*

3. *all balls in G are gated.*

Proof. Let G be a block graph and let $u \in V(G)$ be a vertex. We will show that the induced subgraph $G[N_G^k[u]]$ is the connected union of blocks. We use induction on $k \geq 1$. If $k = 1$, then $G[N_G[u]]$ is either a block (if u is a non-cut vertex) or a collection of blocks having a common cut vertex. In each case, $G[N_G[u]]$ is a connected union of blocks. Now let us consider the ball $N_G^{k+1}[u]$ for $k \geq 1$. We have

$$N_G^{k+1}[u] = N_G^k[u] \cup \left(\bigcup_{x \in N_G^k(u)} N_G[x] \right).$$

By the induction assumption, $G[N_G^k[u]]$ is the connected union of blocks. The same holds for any ball $G[N_G[x]]$, where $x \in N_G^k(u)$. Thus, $G[N_G^{k+1}[u]]$ is also the connected union of these blocks. Consequently, Theorem 2.4 implies that $N_G^k[u]$ is AP-convex. This shows that the first statement implies the second one.

It is trivial that the second statement implies the first, since each AP-convex set is gated. The fact that the third statement implies the first follows directly from [11, Theorem 3.1.2]. □

3.2. Chebyshev-minimal graphs

It is clear from the definition that a Chebyshev set can be disconnected. For example, in the path P_4, the set of its leaf vertices is Chebyshev. Moreover, paths P_{2k} are the only trees with at least two vertices whose sets of leaf vertices are Chebyshev. Indeed, if T is such a tree with at least three distinct vertices $a, b, c \in V(T)$, then for their median $m = m_T(a,b,c)$, two of the three distances $d_T(a,m)$, $d_T(b,m)$, $d_T(c,m)$ must coincide. Hence, T contains exactly two leaf vertices, which makes it a path. Finally, it is clear that T must have an even number of vertices.

One might expect that studying the properties of graphs in which all Chebyshev sets are connected or g-convex would be of interest. However, the next proposition shows that a satisfactory criterion for such graphs is unlikely to exist.

Proposition 3.12. *If* $\operatorname{diam}(G) \leq 2$*, then in G each Chebyshev set is g-convex.*

Proof. To the contrary, suppose $A \subset V(G)$ is not a g-convex Chebyshev set. Then there exists two vertices $u, v \in A$ with $[u,v]_G \setminus A \neq \emptyset$. Fix a vertex $w \in [u,v]_G \setminus A \neq \emptyset$. Since $w \neq u, v$, then $uw, wv \in E(G)$ as $d_G(u,v) \leq \mathrm{diam}(G) \leq 2$. Hence, $u, v \in \mathrm{pr}_A(w)$. This means that A is not Chebyshev, which is a contradiction. □

It is well-known that almost all graphs are connected and have diameter at most two (see [14]). Combining this observation with Proposition 3.12, we obtain the following corollary.

Corollary 3.13. *In almost all graphs, every Chebyshev set is g-convex.*

Despite the result in Corollary 3.13, graphs with connected Chebyshev sets in the class of bipartite graphs are rare in the class of bipartite graphs.

Proposition 3.14. *Let G be a bipartite graph. Then in G, every Chebyshev set is connected if and only if G is a complete bipartite graph.*

Proof. Let $V(G) = A \cup B$ be a bipartition of the complete bipartite graph G. If $F \subset V(G)$ is a disconnected set of vertices, then trivially, $F \subset A$ or $F \subset B$. Without loss of generality, assume that $F \subset A$. For every $u \in B$, we have $F = \mathrm{pr}_F(u)$ and $|F| \geq 2$, which implies that $|\mathrm{pr}_F(u)| \geq 2$. This proves the sufficiency of the condition.

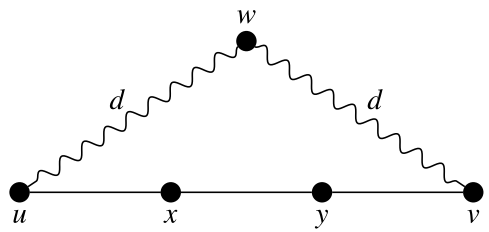

Figure 2: The existence of a vertex w with $d_G(w,u) = d_G(w,v) = d$.

Now, assume that G is a bipartite graph, but not a complete bipartite graph. This implies that $\mathrm{diam}(G) \geq 3$. Fix two vertices $u, v \in V(G)$ with $d_G(u,v) = 3$, and let $P = u - x - y - v$ be a shortest path. Since the set $\{u, v\}$ is disconnected, it is non-Chebyshev. Therefore, there exists a vertex $w \in V(G)$ such that $d_G(w,u) = d_G(w,v) =: d$ (see Figure 2). Consider the set $A = \mathrm{pr}_{V(P)}(w)$. Without loss of generality, we can assume that $[w,a_1]_G \cap [w,a_2]_G = w$ for all $a_1, a_2 \in A$. We now have the following cases.

Case 1: $u \in A$ (similarly, $v \in A$).

Since $d_G(w,u) = d_G(w,v)$, we have $v \in A$. If also $x \in A$, then $u,x \in N_G^d(w)$, meaning that $N_G^d(w)$ is not an independent set (similarly, if $y \in A$). Therefore, let $x,y \notin A$. In this case, $d_G(w,x) = d_G(w,y) = d+1$, so $N_G^{d+1}(w)$ is not independent.

Case 2: $u,v \notin A$, $x \in A$ and $y \notin A$ (similarly, $u,v \notin A$, $x \notin A$ and $y \in A$).

In this case, $d_G(w,x) = d-1$ and $d_G(w,y) = d$. This means that $N_G^d(w)$ is not independent.

Case 3: $u,v \notin A$ and $x,y \in A$.

In this case, $d_G(w,x) = d_G(w,y) = d-1$, and thus $N_G^{d-1}(w)$ is not independent.

In all of the cases above, the graph G contains a non-independent sphere. From Lemma 2.1, it follows that G is not bipartite, which is a contradiction. □

It should be noted that graphs with trivial g-convex sets (such as singletons, edges, and the entire vertex set) were studied in [3] under the name of *g-minimal graphs*. In particular, it was shown that a graph G is g-minimal if and only if it is triangle-free and the convex hull (in the g-convexity) of every pair of its non-adjacent vertices equals $V(G)$ (see [3, Proposition 1]). Furthermore, every triangle-free graph with at least four vertices is an induced subgraph of a g-minimal graph (see [3, Theorem 1]).

In analogy with g-minimal graphs, let us call a graph G *Chebyshev minimal* or *Ch-minimal* if singletons and the entire vertex set $V(G)$ are the only Chebyshev sets in G. Clearly, in a Ch-minimal graph, each Chebyshev set is g-convex. Additionally, it is easy to see that every Ch-minimal graph is biconnected (since the vertex set of each block is Chebyshev). In particular, Ch-minimal graphs provide an example of graphs in which every gated set is AP-convex.

The next theorem provides a characterization of Ch-minimal graphs within the class of graph joins.

Theorem 3.15. *Let G_1 and G_2 be a pair of (not necessarily connected) graphs, each with at least two vertices. Then their join $G_1 + G_2$ is Ch-minimal if and only if at least one of G_1 or G_2 does not contain an isolated vertex.*

Proof. If $x_1 \in V(G_1)$ and $x_2 \in V(G_2)$ are two isolated vertices, then the set $M = \{x_1, x_2\}$ is Chebyshev in $G_1 + G_2$. Clearly, M is not a singleton. Since $|V(G_i)| \geq 2, i = 1,2$, we also have $M \neq V(G_1 + G_2)$. Thus, $G_1 + G_2$ is not a Ch-minimal graph. This proves the necessity of the condition.

To prove its sufficiency, fix a set $M \subset V(G_1 + G_2)$ with $2 \leq |M| \leq n-1$, where $n = |V(G_1 + G_2)| = |V(G_1)| + |V(G_2)|$.

We now consider the following four cases.

Case 1: $M \cap V(G_1) = \emptyset$.

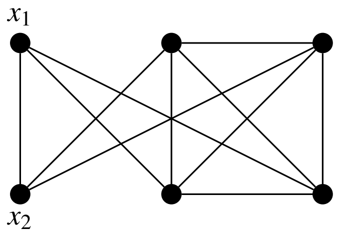

Figure 3: The join of $K_1 \cup K_2$ with itself contains a non-trivial Chebyshev set.

In this case, for every $y \in V(G_1)$, we have $|N_{G_1+G_2}(y) \cap M| = |M| \geq 2$, which implies that M is not a Chebyshev set.

Case 2: $|M \cap V(G_1)| \geq 2$ and $|M \cap V(G_2)| < |V(G_2)|$.

Here for all $y \in V(G_2) \setminus M$, it holds $|N_{G_1+G_2}(y) \cap M| \geq |M \cap V(G_1)| \geq 2$, and thus, M is not a Chebyshev set.

Case 3: $|M \cap V(G_1)| \geq 2$ and $M \cap V(G_2) = V(G_2)$.

Since $|M| \leq n-1$, there exists a vertex $y \in V(G_1) \setminus M$. Clearly, $|N_{G_1+G_2}(y) \cap M| \geq |M \cap V(G_2)| = |V(G_2)| \geq 2$, and again, M is not Chebyshev.

Case 4: $|M \cap V(G_1)| = 1$.

By symmetry, we can assume that $|M \cap V(G_2)| = 1$. Thus, let $M \cap V(G_1) = \{x\}$ and $M \cap V(G_2) = \{y\}$. Without loss of generality, suppose G_1 does not have isolated vertices. Then there is a vertex $z \in V(G_1) \cap N_{G_1+G_2}(x)$. Hence, $|N_{G_1+G_2}(z) \cap M| = |\{x,y\}| = 2$, and therefore, M is not Chebyshev. □

To illustrate the statement in Theorem 3.15, consider the graph $G_1 = G_2 = K_1 \cup K_2$. The join $G_1 + G_2$ is depicted in Figure 3. It is clear that $\{x_1, x_2\}$ is a non-trivial Chebyshev set in $G_1 + G_2$.

The case of joins $G_1 + G_2$ where $\min |V(G_1)|, |V(G_2)| = 1$ is addressed in the following proposition.

Proposition 3.16. *Let G be a (not necessarily connected) graph with $n \geq 2$ vertices. Then $G + K_1$ is Ch-minimal if and only if G is connected.*

Proof. Let the vertex $v \in V(G + K_1)$ correspond to the vertex of K_1. First, we prove the necessity of the condition. For every connected component G_1 of G, the set $M = V(G_1) \cup \{v\}$ is a Chebyshev set in $G + K_1$. Since M is not a singleton, it follows that $M = V(G) \cup \{v\}$. Hence, $V(G_1) = V(G)$, which implies that G is connected.

Now we prove the sufficiency of this condition. Fix a set $M \subset V(G + K_1)$ with $2 \leq |M| \leq n$, where $n = |V(G)|$. If $v \notin M$, then $|N_{G+K_1}(v) \cap M| = |M| \geq 2$, which implies that M is not Chebyshev. If $v \in M$, then taking into account the inequality $|M| \leq n$, we obtain $V(G) \setminus M \neq \emptyset$. Since G is connected, there exists a vertex $x \in V(G) \setminus M$ with $|N_G(x) \cap M| \geq 1$. But $v \in N_{G+K_1}(x)$. This means that $|N_{G+K_1}(x) \cap M| \geq 2$, and thus, M is not a Chebyshev set. □

For a set of vertices $U \subset V(G)$ in a graph G, define $S(G,U)$ to be the graph with the vertex set $V(S(G,U)) = V(G)$ and the edge set $E(S(G,U)) = E_G(U) \cup E_G(V(G) \setminus U) \cup \{uv : u \in U, v \in V(G) \setminus U \text{ and } uv \notin E(G)\}$. We say that $S(G,U)$ is obtained from G by *switching* the set U.

Two graphs G_1 and G_2 are said to be *s-equivalent* (writing $G_1 \sim_s G_2$) if there is $U \subset V(G_1)$ such that $G_2 \simeq S(G_1,U)$. Clearly, the relation $\sim_s$ is an equivalence of graphs. A graph G is *s-maximal* if it has the maximum number of edges in its s-class. In other words, G is s-maximal if $|E(S(G,U))| \leq |E(G)|$ for all $U \subset V(G)$. Put $l_G(U)$ for the number of edges between U and $V(G) \setminus U$. The next lemma gives a simple technical criterion for s-maximal graphs.

Lemma 3.17 ([13]). *A graph G is s-maximal if and only if*

$$2 \cdot l_G(U) \geq |U| \cdot (|V(G)| - |U|)$$

for all $U \subset V(G)$.

In particular, Lemma 3.17 asserts that for each vertex $u \in V(G)$ in an n-vertex s-maximal graph G, it holds $d_G(u) \geq \frac{n-1}{2}$. The following theorem examines the structure of Chebyshev sets in s-maximal graphs.

Theorem 3.18. *Let G be an s-maximal graph with $n \geq 2$ vertices. If $M \subset V(G)$ is a Chebyshev set, then $|M| \in \{1,2,|V(G)|\}$. Additionally, if $M = \{u,v\}$ with $u \neq v$, then $uv \in E(G)$, $d_G(u) = d_G(v) = \frac{n}{2}$, and the graph $G - \{u,v\}$ is also s-maximal.*

Proof. Put $|M| = m$. Since M is Chebyshev, for all vertices $u \in V(G) \setminus M$, it holds $|N_G(u) \cap M| \leq 1$. Therefore,

$$l_G(U) = l_G(V(G) \setminus M) = \sum_{u \in V(G) \setminus M} |N_G(u) \cap M| \leq n - m.$$

But G is also an s-maximal graph, which by Lemma 3.17, asserts that $2l_G(U) \geq m(n-m)$. This means that $m(n-m) \leq 2(n-m)$, implying $m = n$ or $m \leq 2$.

Next, suppose that $m = 2$ and $M = \{u,v\}$. Assume that $uv \notin E(G)$. Since M is Chebyshev, we have $N_G(u) \cap N_G(v) = \emptyset$. Therefore, $d_G(u) + d_G(v) \leq n-2$. This implies the inequality $\min\{d_G(u), d_G(v)\} \leq \frac{n-2}{2}$, which is a contradiction. Thus, $uv \in E(G)$. Moreover, $d_G(u) = d_G(v) = \frac{n}{2}$. Finally, let us show that the graph $H := G - \{u, v\}$ is also s-maximal. To do so, fix a set $U \subset V(H)$ and observe that $l_H(U) = l_G(U) - |U|$. By Lemma 3.17,

$$\begin{aligned} 2l_H(U) &= 2l_G(U) - 2|U| \geq |U|(n - |U|) - 2|U| \\ &= |U|(n - 2 - |U|) = |U|(|V(H)| - |U|), \end{aligned}$$

and the desired result is proved. □

Corollary 3.19. *Each s-maximal graph with an odd number of vertices is Ch-minimal.*

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