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A SURVEY ON THE CLASSIFICATION OF CODIMENSION TWO SUBVARIETIES OF $\mathbb{P}^n$

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Dedicated to Silvio Greco in occasion of his 60-th birthday.

1. Introduction.

By classification of space curves, one could mean to enumerate and describe (dimension, singularities, etc...) the irreducible components of $H(d, g)$, the Hilbert scheme of (smooth, irreducible) curves of degree d, genus g of $\mathbb{P}^3$; also one would like to have a fairly complete description (postulation, generators and syzygies, multisecants, etc...) of the generic curve of each irreducible component.

As it is well known, this program is hopeless!
The Hilbert scheme of space curves is highly reducible (see for example [20]) and there are definitely too many components to look out. The only thing we can (hope to) do is to establish general laws and to carry on the complete classification only for some "distinguished" (to be defined) irreducible components. In the last few years there have been many results in these directions. We won't report on these results.

It could be however that our initial program could be carried on for smooth codimension two subvarieties of $\mathbb{P}^n$ if n is big enough. Indeed, according to Hartshorne's conjecture, every smooth, codimension two subvariety of $\mathbb{P}^n$, $n \geq 6$, should be a complete intersection.

There are good reasons to believe that the classification of codimension two subvarieties of $\mathbb{P}^n$ gets simpler and simpler as n grows. In the first section of this survey we will recall some of these good reasons, then we will briefly review the state of the art.

This survey doesn't pretend to be complete and I apologize in advance if, by lack of time or space or just by ignorance, I miss some relevant contribution.

It is a pleasure to thank the organizers for the wonderful atmosphere of the conference.

2. Some general facts..

Here we will review some general facts which tend to indicate that the classification of codimension two subvarieties of $\mathbb{P}^n$ gets simpler as n grows.

We work over an algebraically closed field of characteristic zero. The characteristic zero assumption is important since the basic assumption is that we are dealing with *smooth* subvarieties.

2.1. *Linear normality..*

Since the expected dimension of the secant variety, $Sec(X)$, of $X \subset \mathbb{P}^n$ is $2 \cdot dim(X) + 1$, every curve can be embedded in $\mathbb{P}^3$ and this is what makes the classification of space curves so complex. However, this count of parameters also shows that, a priori, we cannot expect to get, by projection from a higher space, a smooth codimension two $X \subset \mathbb{P}^n$ if $n \geq 4$. In fact a first important result in this direction is Severi's theorem on surfaces in $\mathbb{P}^4$ ([34]):

Theorem 2.1. (Severi). *Let $S \subset \mathbb{P}^4$ be a smooth, non degenerate surface. Then S is linearly normal (i.e. $h^0(\mathcal{O}_S(1)) = 5$) except if S is a (projected) Veronese surface.*

This theorem has many striking consequences and must be regarded as the "first" theorem on surfaces in $\mathbb{P}^4$.
One of the most beautiful result in projective algebraic geometry is Zak's solution of Hartshorne's conjecture on linear normality; it is the ideal and complete generalization of the above theorem. We refer to [36]. Concerning the codimension two case, as a special case of Zak's theorem, we have:

Theorem 2.2. (Zak). *Let $X \subset \mathbb{P}^n$ be a smooth, codimension two subvariety. If $n \geq 5$, then X is linearly normal.*

Clearly this is a strong restriction for the existence of smooth codimension two subvarieties

2.2. *The second Chern class of the normal bundle..*

Another important fact concerning codimension two subvarieties $X \subset \mathbb{P}^n$ is the presence, when $n \geq 4$, of the second Chern class of the normal bundle of X in $\mathbb{P}^n$. Indeed N_X is a rank two vector bundle on X and if $dim(X) \geq 2$, we can consider its second Chern class $c_2(N_X)$. By the "self-intersection formula": $i_*(c_2(N_X)) = X^2$ where $i : X \hookrightarrow \mathbb{P}^n$ is the inclusion map ([23] Appendix A, sec. 3). Now we can compute $c_2(N_X)$ in another way, using the exact sequence:

$$0 \to TX \to T\mathbb{P}^n_{|X} \to N_X \to 0$$

comparing these two expressions of $c_2(N_X)$, we get an important relation among the invariants of X. For instance if $S \subset \mathbb{P}^4$ is a smooth surface, we get the famous "double points formula" (see [23] Appendix A, Ex. 4.1.3).

$$d(d-5) - 10(\pi - 1) + 12\chi = 2K^2$$

This formula can be seen as an analogous of the formula giving the genus of a smooth plane curve. Clearly this formula imposes strong restrictions on the invariants of a smooth surface in $\mathbb{P}^4$.

There is another important reason which brings into play the second Chern class of the normal bundle. Assume $S \subset \Sigma \subset \mathbb{P}^4$, S a smooth surface and Σ a degree s hypersurface. The inclusion $S \subset \Sigma$ yields $\mathcal{O}(-s) \to \mathcal{I}_S$ and by restriction to S we get: $\mathcal{O}_S(-s) \to N_S^*$. In other words the inclusion $S \subset \Sigma$ defines a section σ of $N_S^*(s)$. The zero locus of this section is the intersection of S with $Sing(\Sigma)$, the singular locus of Σ. If σ vanishes in codimension two, we have $\mu := deg(\sigma)_0 = c_2(N_S^*(s))$ and a short computation gives: $\pi - 1 = \frac{d(d+s^2-4s)-\mu}{2s}$. In particular if $\mu = 0$, then π is the genus of a complete intersection $(\frac{d}{s}, s)$. In fact this is the starting point to prove:

Theorem 2.3. *Let $\Sigma \subset \mathbb{P}^n$ be an hypersurface. If $n \geq 4$, then $Pic(\Sigma) \simeq \mathbb{Z}.H$.*

In particular let $X \subset \mathbb{P}^n$, $n \geq 4$ be a codimension two subscheme. If $X \subset \Sigma$ and if X is a Cartier divisor on Σ, then X is the complete intersection of Σ with another hypersurface.

This theorem was first proved by Severi ([35]), a modern version of Severi's proof can be found in [13]; of course this is a special case of Lefschetz's theorem.

It follows for example that if $S \subset \mathbb{P}^4$ is a smooth, non complete intersection surface, then S must meet the singular locus of every hypersurface containing it. This is another big difference with the case of curves in $\mathbb{P}^3$ (for every $C \subset \mathbb{P}^3$, if n is big enough, there exists a smooth surface of degree n containing C).
We will find this type argument (see Lemma 3.2) all along these notes.

2.3. *Topology (Barth-Larsen theorem)..*

In 1970 Barth ([7]) discovered that the topology of low codimension subvarieties in $\mathbb{P}^n$ is similar to the topology of complete intersections. In the codimension two case we have (see [24] Thm. 2.2):

Theorem 2.4. (Barth-Larsen). *Let $X \subset \mathbb{P}^n$ be a smooth, codimension two subvariety.*
1. *If $n \geq 5$, then $h^1(\mathcal{O}_X) = 0$*
2. *If $n \geq 6$, then $Pic(X) \simeq \mathbb{Z}.H$*

This marvelous result has been the main motivation for Hartshorne's conjecture ([24]:

Conjecture 1. *Let $X \subset \mathbb{P}^n$ be a smooth subvariety of dimension m. If $m > \frac{2n}{3}$, then X is a complete intersection.*

In the codimension two case, this gives $m \geq 7$. But one consequence of Theorem 2.4 is the connection between codimension two subvarieties and rank two vector bundles. Indeed, if $Pic(X) \simeq \mathbb{Z}.H$, then $\omega_X \simeq \mathcal{O}_X(e)$ for some integer e and, by Serre's construction, we can associate a rank two vector bundle to X:

$$0 \to \mathcal{O} \to E \to \mathcal{I}_X(e+n+1) \to 0$$

Now, as it is well known, E splits if and only if X is a complete intersection. Conversely, if E is a rank two vector bundle on $\mathbb{P}^n$, then for $k >> 0$, $E(k)$ has a section vanishing along a smooth (irreducible) codimension two subvariety. In conclusion the existence of indecomposable rank two vector bundles on $\mathbb{P}^n$ is equivalent to the existence of smooth, codimension two, subcanonical (i.e. $\omega_X \simeq \mathcal{O}_X(e)$) subvarieties; if $n \geq 6$, thanks to Theorem 2.4, the "subcanonical condition" is automatically satisfied. So we see that there exists a non split rank two vector bundle on $\mathbb{P}^n$, $n \geq 6$, if and only if there exists a smooth, non complete intersection, codimension two subvariety.

So far, attempts to construct indecomposable rank two vector bundles on $\mathbb{P}^n$, $n \geq 5$, have failed (here, as usual, we are assuming $ch(k) = 0$; if $ch(k) > 0$, things are different). Moreover, there is, essentially, only one known, non split, rank two vector bundle on $\mathbb{P}^n$, $n > 3$; it is the Horrocks-Mumford bundle on $\mathbb{P}^4$, arising from abelian surfaces ([27]). Since in the last 30 years, none has been able to construct an indecomposable rank two vector bundle on $\mathbb{P}^n$, $n \geq 5$, some people start thinking that this is due the fact that, simply, such bundles do not exist!

Conjecture 2. *Every rank two vector bundle on $\mathbb{P}^n_k$, $n \geq 5$ (and $ch(k) = 0$) splits.*

Equivalently, every smooth codimension two subcanonical subvariety $X \subset \mathbb{P}^n$ is a complete intersection.

This is a slightly modified version of the original conjecture ([13]). At the moment little is known on this conjecture.

3. Surfaces in $\mathbb{P}^4$.

3.1. *Surfaces of non general type.*

The breakthrough in the classification of surfaces in $\mathbb{P}^4$ is Ellingsrud-Peskine's theorem ([21]):

Theorem 3.1. (Ellingsrud-Peskine). *There are only finitely many irreducible components of the Hilbert scheme containing smooth surfaces of non general type.*

It follows that the degrees of surfaces of non general type are bounded (in fact this is almost equivalent to the theorem): there exists d_0 such that $deg(S) \leq d_0$ for every smooth surface of non general type $S \subset \mathbb{P}^4$. Braun and Fløystad ([11]) refined the proof of Ellingsrud and Peskine to give an effective bound ($d_0 \leq 105$); then many authors gave some further refinements and, it seems, that at the moment the best result (if you accept computer-aided proofs) is: $d_0 \leq 52$ (see [14] and the bibliography therein). However a better bound is conjectured:

Conjecture 3. *If $S \subset \mathbb{P}^4$ is a smooth surface of non general type, then $deg(S) \leq 15$.*

Why 15? Well, on one hand there exist smooth surfaces of non general type of degree 15, on the other hand in the past ten years none has been able to construct such a surface of degree > 15, that's it! (for construction of smooth surfaces in $\mathbb{P}^4$, see [14]). For example, at the time of this writing, no rational surface of degree $d > 12$ is known.

We should also mention that the classification of surfaces of degree at most ten is fairly complete (see [5], [31]).
Observe that curves of non general type are rational and elliptic curves. Of course the classification of rational and elliptic space curves is much more envolved (there are infinitely many irreducible components in the Hilbert scheme $Hilb(\mathbb{P}^3)$).

Just a few words about the proof of Theorem 3.1. First one shows that $d \leq 90$ or $h^0(\mathcal{I}_S(5)) \neq 0$; so we have only to worry about surfaces lying on hypersurfaces of degree ≤ 5. Then, as observed by Braun and Fløystad, to get a bound is a question of Castelnuovo's theory; i.e. to relate the invariants of S, C and Γ (C a general hyperplane section of S, Γ a section of C). To get a better bound is a question to understanding the relationships between the generators of $\mathbb{I}(S)$, $\mathbb{I}(C)$ and $\mathbb{I}(\Gamma)$. In any case, the crucial ingredient is the following ([21], Lemme 1):

Lemma 3.2. *Let $S \subset \mathbb{P}^4$ be a smooth surface with $s(S) = \sigma$. Set $\mu := c_2(N_S(-\sigma)) = d(d + \sigma(\sigma - 4)) - \sigma(2\pi - 2)$. Then: $0 \leq \mu \leq (\sigma - 1)^2 d$.*

This lemma shows again the importance of the second Chern class of the normal bundle, we will meet it again in these notes.

3.2. *Surfaces on low degree hypersurfaces..*

The proof of Theorem 3.1 shows that most of the problem of classifying surfaces of non general type is concentrated on surfaces lying on hypersurfaces of low degree, so we turn to the question (of independant interest) of the classification of such surfaces. The first step is quite classical:

Lemma 3.3. *Let $S \subset \mathbb{P}^4$ be a smooth surface. If $h^0(\mathcal{I}_S(2)) \neq 0$, then S is arithmetically Cohen-Macaulay (a.C.M.); more precisely S is a complete intersection if d is even and is linked to a plane if d is odd.*

Proof. Assume $S \subset Q$, Q an irreducible hyperquadric. If $S \cap Sing(Q) = \emptyset$, then S is a Cartier divisor on Q and, by Theorem 2.3, S is the complete intersection of Q with another hypersurface (and d is even). If $S \cap Sing(Q) \neq \emptyset$, pick $p \in S \cap Sing(Q)$; a general hyperplane through p cut S along a smooth curve, C, which lies on a quadric cone K. Since C passes through the vertex of K, C is linked to a line by the complete intersection of K with another surface (and d is odd). This implies that S is linked to a plane in a complete intersection $Q \cap \Sigma$. $\square$

The case of hypercubics is much more involved. In [4], Aure established the classification of smooth surfaces on cubic hypersurfaces with only isolated singularities, then this result has been extended by Koelblen ([28]) to arbitrary cubic hypersurfaces:

Theorem 3.4. (Koelblen). *Let $S \subset \mathbb{P}^4$ be a smooth surface. Assume $S \subset \Sigma$ where Σ is an irreducible cubic hypersurface. Then one of the following occurs:*

1. *S is a.C.M. and is linked, on Σ, to an a.C.M. surface of degree ≤ 3 which is a cone if Σ is a cone and which is smooth otherwise*
2. *S is linked, on Σ, to a Veronese surface*
3. *S is linked, on Σ, to a quintic elliptic scroll.*

In fact this theorem follows from a more general result, namely the classification of *locally Cohen-Macaulay* surfaces lying on *normal* hypercubics (see [28], Thm. 1.5): one has the same statement as above. The proof is rather long and technical.
Using Theorem 3.4 one easily deduces:

Corollary 3.5. *Let $S \subset \mathbb{P}^4$ be a surface of non general type. If $h^0(\mathcal{I}_S(3)) \neq 0$, then $deg(S) \leq 8$.*

For partial results on surfaces on hyperquartics with isolated singularities, see [18] where it is proved, among others, that a surface of non general type lying on such an hyperquartic has degree ≤ 27.

3.3. *Surfaces of general type..*

Of course there are plenty of smooth surfaces in $\mathbb{P}^4$ (just use liaison), in general they will be of general type (as indicated by the name), what can be said about them? One of the most tantalizing conjecture in projective algebraic geometry concerns precisely these surfaces:

Conjecture 4.

1. *There exists an integer M such that if $S \subset \mathbb{P}^4$ is a smooth surface, then $q(S) \leq M$.*
2. *$M = 2$ should do the job*

At the moment, very little is known about this conjecture. Here are a few remarks:
Part 1 of the conjecture is true for surfaces of non general type (but we are still far from getting $M = 2$ in this case).

Certainly $M \geq 2$ because there exist abelian surfaces (related to the Horrocks-Mumford bundle) with $q = 2$.
The easiest way to construct smooth surfaces is by liaison. If S is a smooth surface and if $\mathcal{I}_S(a)$ and $\mathcal{I}_S(b)$ are generated by global sections, and if F_a, F_b are general hypersurfaces containing S, then $F_a \cap F_b$ will link S to a smooth surface T ([29]). One may hope, starting from a known S to construct surfaces with big irregularity. This doesn't seem to work too well:

Lemma 3.6. *If $\mathcal{I}_S(a)$ is generated by global sections, then $h^1(\mathcal{I}_S(m)) = 0$ if $m \geq 2a - 4$.*
With notations as above, if $b > a$, then $q(T) = 0$.

Proof. If $\mathcal{I}_S(a)$ is generated by global sections, we link S to a smooth surface, S', by a complete intersection, U, of type (a, a). The exact sequence of liaison is: $0 \to \mathcal{I}_U \to \mathcal{I}_S \to \omega_{S'}(5 - 2a) \to 0$. Twisting by $2a - 5 + t$, $t \geq 1$, since $h^1(\omega_{S'}(t)) = 0$ by Kodaira, we get $h^1(\mathcal{I}_S(2a - 5 + t)) = 0$.
By the exact sequence of liaison $q(T) = h^1(\omega_T) = h^1(\mathcal{I}_S(a + b - 5))$, so $q(T) = 0$ if $b > a$. $\square$

So there is little room left ($a = b$) to apply this naive plan. Of course one can have S linked to a smooth surface by a complete intersection of type (a, b) without $\mathcal{I}_S(a)$, nor $\mathcal{I}_S(b)$ being globally generated.

3.4. *Subcanonical surfaces.*.

Definition 3.7. *A smooth surface $S \subset \mathbb{P}^4$ is said to be subcanonical if $\omega_S \simeq \mathcal{O}_S(e)$ for some integer e.*

The interest of these surfaces is that, through Serre's correspondance, they yield rank two vector bundles:

$$0 \to \mathcal{O} \to E \to \mathcal{I}_S(e + 5) \to 0$$

where E is a rank two vector bundle with $c_1(E) = e + 5$, $c_2(E) = d$.

By adjunction, if C is a general hyperplane section of S, $\omega_C \simeq \mathcal{O}_C(e + 1)$, this shows that: $2\pi - 2 = d(e + 1)$. Since $C \subset \mathbb{P}^3$ is non degenerate, $\pi \geq 1$, hence $e \geq -1$.

We have $K^2 = (eH)^2 = e^2 d$ and the double points formula gives: $12_\chi = d(2e^2 + 5e - d + 10)$

Since $\omega_S^{\otimes n} \simeq \mathcal{O}_S(ne)$, if $e \geq 1$, S is of general type (in particular we have Yau's inequality: $K^2 \leq 9_\chi$).

Lemma 3.8. *Let $S \subset \mathbb{P}^4$ be a smooth subcanonical surface, if $h^0(\mathcal{I}_S(3)) \neq 0$, then S is a complete intersection.*

Proof. If $h^0(\mathcal{I}_S(2)) \neq 0$, this follows from Lemma 3.3 (S is a CM and subcanonical hence a complete intersection).

We may assume that $S \subset \Sigma$ where Σ is an irreducible cubic. By Theorem 3.3, S is a CM or linked on Σ to surface T which an elliptic scroll or a Veronese surface. In the first case we are done. Assume S linked to T by a complete intersection, U, of type $(3, b)$, $b \geq 3$. The exact sequence of liaison gives:

$$0 \to \mathcal{I}_U(3) \to \mathcal{I}_T(3) \to \omega_S(5-b) \to 0$$

It follows that $h^0(\mathcal{O}_S(5-b+e)) = h^0(\omega_S(5-b)) = h^0(\mathcal{I}_T(3)) - h^0(\mathcal{I}_U(3))$. We have $h^0(\mathcal{I}_T(3)) = 7$ if T is a Veronese (resp. 5 if T is an elliptic scroll), it follows that: $1 < h^0(\mathcal{O}_S(5-b+e)) < 15$, this implies $5-b+e = 1$ and by Severi's theorem (Thm 2.1), we see that the only possibility is: T is a Veronese surface and $b = 3$. So $e = 1$ and since $d + 4 = 9$ by liaison, $d = 5$, looking at the hyperplane section C of S we see that this is impossible. □

Let's review quickly what is known on the classification of subcanonical surfaces.

Lemma 3.9. *If $e = -1$, then S is a complete intersection (2,2).*

Proof. If $\omega_S \simeq \mathcal{O}_S(-1)$, then $p_g = 0$ and $q = h^1(\mathcal{O}_S) = h^1(\omega_S(1)) = 0$ by Kodaira. So $\chi = 1$ and plugging into the double points formula yields $d = 4$. We conclude by looking at the hyperplane section. □

The case $e = 0$ is more interesting:

Lemma 3.10. *If $e = 0$ then $d = 6$ and S is a complete intersection (2,3), or $d = 10$ and S is an abelian surface ($p_g = 1$ and $q = 2$).*

Proof. If $\omega_S \simeq \mathcal{O}_S$, by Kodaira, $\chi(\mathcal{O}_S(1)) = h^0(\mathcal{O}_S(1))$. By Severi's theorem (Thm 2.1) and Riemann-Roch, we get $\chi = 5 - \frac{d}{2}$. Plugging into the double points formula gives $d = 6$ or $d = 10$. If $d = 6$, we conclude by looking at the hyperplane section C of S. If $d = 10$, then $\chi = 0$ and since $p_g = 1$, we get $q = 2$, so S is an abelian surface. □

As it is well known there exist abelian surfaces of degree 10 in $\mathbb{P}^4$: they arise from the Horrocks-Mumford bundle ([27]) (it seems by the way that Commessatti was aware of the existence of such surfaces):

$$0 \to \mathcal{O} \to \mathcal{E}(3) \to \mathcal{I}_S(5) \to 0$$

where $c_1(\mathcal{E}) = -1$, $c_2(\mathcal{E}) = 4$; all the bundles arising this way are stable and projectively equivalent. The Horrocks-Mumford bundle is (essentially) the only known, non split, rank two vector bundle on $\mathbb{P}^n$, $n \geq 4$. Nowaday there are so many papers on the Horrocks-Mumford bundle, that the interested reader should make a search on the web to get a complete list of references.

The next case, $e = 1$, has been solved by Ballico-Chiantini ([6]):

Theorem 3.11. (Ballico-Chiantini). *Let $S \subset \mathbb{P}^4$ be a smooth surface. If $\omega_S \simeq \mathcal{O}_S(1)$, then S is a complete intersection.*

Proof. We have $h^0(\mathcal{O}_S(1)) = 5$ by Thm 2.1, $q = h^1(\mathcal{O}_S(1))$ and $h^2(\mathcal{O}_S(1)) = h^0(\omega_S(-1)) = 1$, so $\chi(\mathcal{O}_S(1)) = 6 - q$. It follows that $\chi = 6 - q$. By Yau's inequality: $d \leq 9(6 - q)$ which implies $q \leq 5$. By the double points formula: $72 - 12q = d(17 - d)$, so $d < 17$ and, after some short computations, we see that the possible cases are: a) $q = 0$, $\chi = 6$ and $d = 8$ or $d = 9$, b) $q = 1$, $\chi = 5$ and $d = 5$ or $d = 12$.
Since $\chi(\mathcal{O}_S(3)) = h^0(\mathcal{O}_S(3))$ by Kodaira, by Riemann-Roch: $h^0(\mathcal{O}_S(3)) = 3d + \chi$. So if $d \neq 12$, we have $h^0(\mathcal{I}_S(3)) \neq 0$ and we conclude with Lemma 3.8. So we may assume $d = 12$. From Serre's correspondance we have: $0 \to \mathcal{O} \to \mathcal{E}(3) \to \mathcal{I}_S(6) \to 0$, where $\mathcal{E}$ is a rank two bundle with $c_1(\mathcal{E}) = 0$ and $c_2(\mathcal{E}) = 3$. If $h^0(\mathcal{I}_S(3)) \neq 0$ ($\mathcal{E}$ not stable), we conclude with Lemma 3.8. If $h^0(\mathcal{I}_S(3)) = 0$, then $\mathcal{E}$ is stable and we conclude with [8] where it is proved that there exist no stable rank two bundles with $c_1 = 0$ and $c_2 = 3$ on $\mathbb{P}^4$. $\square$

In fact Ballico-Chiantini proved something more, namely ([6], Prop. 3):

Proposition 3.12. *There exists no semi-stable rank two vector bundle on $\mathbb{P}^4$ with $c_1 = 0$ and $c_2 = 3$.*

The existence of non semi-stable rank two vector bundles with $c_1 = 0$ and $c_2 = 0$ is still an open problem.

If $e > 1$, little is known (see [15] for partials results in the case $e = 2$).

4. Threefolds in $\mathbb{P}^5$.

For the classification of low degree threefolds, see [10], [33].

4.1. *Threefolds of non general type..*

The analogous of Ellingsrud-Peskine's theorem holds for threefolds in $\mathbb{P}^5$:

Theorem 4.1. (Braun-Ottaviani-Schneider-Schreyer). *There exists an integer B such that if $X \subset \mathbb{P}^5$ is a threefold of non general type, then $deg(X) \leq B$.*

Proof. See [12]. $\square$

As far as I know, no effective bound is known. Observe that the theorem above doesn't follows from Ellingsrud-Peskine's theorem, since a general hyperplane section of X will be, in most cases, a surface of general type.

By Barth-Larsen's theorem if $X \subset \mathbb{P}^5$ is a smooth threefold, then $h^1(\mathcal{O}_X) = 0$. This implies that if S is a general hyperplane section of X, then $S \subset \mathbb{P}^4$ is a smooth surface with $q(S) = 0$ (look at the exact sequence: $0 \to \mathcal{O}_X(-1) \to \mathcal{O}_X \to \mathcal{O}_S \to 0$ and use Kodaira to get $h^2(\mathcal{O}_X(-1)) = h^1(\omega_X(1)) = 0$).

4.2. *Subcanonical threefolds..*

In this section $X \subset \mathbb{P}^5$ will denote a smooth subcanonical threefold of degree d with $\omega_X \simeq \mathcal{O}_X(e)$. A general hyperplane section of X will be a smooth surface $S \subset \mathbb{P}^4$ with $\omega_S \simeq \mathcal{O}_S(e+1)$ and $q(S) = 0$.

Theorem 4.2. (Ballico-Chiantini). *If $e \leq 2$, then X is a complete intersection.*

Proof. See [6]. $\square$

Theorem 4.3. *If $h^0(\mathcal{I}_X(4)) \neq 0$, then X is a complete intersection.*

Proof. If $h^0(\mathcal{I}_X(3)) \neq 0$, we conclude with Lemma 3.8, if X lies on an irreducible hyperquartic, this follows from [19]. $\square$

Remark 4.4.

1. *We will come back on Theorem 4.3 in the next section. The part $h^0(\mathcal{I}_X(3)) \neq 0$ is also a particular case of a theorem by Ran ([32], see also Theorem 5.5).*
2. *Combining Theorems 4.2 and 4.3, one can prove that a smooth subcanonical threefold in $\mathbb{P}^5$ of degree $d \leq 23$ is a complete intersection. The first unknown case is $d = 24$ and $e = 3$.*

5. Codimension two subvarieties in $\mathbb{P}^n$, $n \geq 6$.

This is the general case of Hartshorne conjecture, since by Barth-Larsen's theorem *every* smooth, codimension two $X \subset \mathbb{P}^n$, $n \geq 6$, has $Pic(X) \simeq \mathbb{Z} \cdot H$, hence is subcanonical, i.e. $\omega_X \simeq \mathcal{O}_X(e)$ for some integer e.

5.1. *k-linear normality.*

As a special case of a theorem of Evans-Griffith ([22]) and Horrocks (if $n = 3$), we have:

Theorem 5.1. *Let $\mathcal{E}$ be a rank two vector bundle on $\mathbb{P}^n$, $n \geq 3$, then $\mathcal{E}$ splits if and only if $H^1_*(\mathcal{E}) = 0$.*

Corollary 5.2. *Let $X \subset \mathbb{P}^n$, $n \geq 6$, be a smooth codimension two subvariety, then X is a complete intersection if and only if X is projectively normal (i.e. $h^1(\mathcal{I}_X(m)) = 0$, $\forall m \in \mathbb{Z}$).*

Remark 5.3. *Recall that X is said to be k-normal if $h^1(\mathcal{I}_X(k)) = 0$. By Zak's theorem, with assumptions as in the corollary, X is 1-normal. By Theorem 5.1, a possible approach to Hartshorne's conjecture is to prove k-normality for every k.*

There are many vanishing theorems for codimension two subvarieties in $\mathbb{P}^n$. The first one is of course Zak's theorem: $h^1(\mathcal{I}_X(1)) = 0$ if $n \geq 5$. Concerning quadratic normality we have: $h^1(\mathcal{I}_X(2)) = 0$ if $n \geq 10$ ([17], [2]); also: $h^r(\mathcal{O}_X(t)) = 0$ if $r \geq 1$ and $n \geq 6t + r$ ($t \geq 1$) (see [2]). For precise statements we refer the interested reader to the following papers (and their references): [30], [17], [2], [1], [3].

5.2. *Rank two vector bundles..*

Another approach to Hartshorne's conjecture is through rank two vector bundles. Observe that althought, at the end, the conclusion from the vector bundles side or from the smooth subvarieties side should be equivalent, one cannot immediately translate results from one side to another. Indeed, given a vector bundle, E, it is hard in general to decide for which k, $E(k)$ will have a section vanishing along a smooth codimension two subvariety.

A first general result was obtained, using vector bundles techniques, by Barth-Van de Ven in [9] where they gave a linear bound $f(n)$ for the degree of a non complete intersection $X \subset \mathbb{P}^n$ of codimension two (i.e. if $deg(X) \leq f(n)$, then X is a complete intersection). A few years later, Z. Ran ([32]) proved what has to be considered, till now, the best general result from the "vector bundles side":

Theorem 5.4. (Ran).

1. *Let E be a rank two vector bundle on $\mathbb{P}^n$ and assume E has a section vanishing in codimension two, then:*

if $c_1(E) \geq \frac{c_2(E)}{\alpha} + \alpha$, for some $\alpha \leq n-2$ or if $c_2(E) \leq n-2$, then E splits.

2. *Let $X \subset \mathbb{P}^n$ be a smooth, codimension two subvariety of degree d, with $\omega_X \simeq \mathcal{O}_X(e)$.*
   If $e \geq \frac{d}{n-2} - 3$ or if $d \leq n-2$, then X is a complete intersection.

Of course 2) is a direct consequence of 1).
The main ingredient in Ran's proof is the following fact: take a general point $P \in \mathbb{P}^n \setminus X$ and set $\Sigma^P_{k+1} = \{r \in G(1,n)/P \in r$ and r is a $k+1$-secant line to $X\}$, then, if $k \leq n-2$, $deg(\Sigma^P_{k+1}) = e(0)\cdots e(k)$, where $e(t) = c_2(E(-t))$, E being the rank two vector bundle associated to X.

Taking this fact for granted, let's outline the proof of the theorem. The assumption $c_1(E) \geq c_2(E)/\alpha + \alpha$, implies $e(\alpha) \leq 0$ and that E is not stable. If $k = min\{l/h^0(E(l-c_1)) \neq 0\}$, then $k \leq c_1(E)/2$ and $E(k-c_1)$ has a section, s, vanishing in codimension two. If $(s)_0 = Z$, then $deg(Z) = e(k-c_1) = e(k)$; so $e(k) \geq 0$. We have $k \leq n-2$, indeed since $k \leq c_1(E)/2$, $e(k) \geq 0$ and $e(\alpha) \leq 0$, by looking at the graph of $e(t)$, we see that $k \leq \alpha$. On the other hand, since E is not stable, $h^0(\mathcal{I}_X(k)) \neq 0$. This implies $\Sigma^P_{k+1} = \emptyset$ (every $k+1$-secant to X is contained in a degree k hypersurface containing X). This implies that there exists $i \leq k$ such that $e(i) = 0$ (recall that $k \leq n-2$). Since $e(k) \geq 0$ and $k \leq c_1(E)/2$, necessarly (look at the graph of $e(t)$) $i = k$; therefore $deg(Z) = e(k) = 0$, and E splits.

Observe that Ran's theorem deals with non stable vector bundles. In fact it seems easier to attack the conjecture for non stable bundles.

As pointed out in [6], Ran's theorem has the following consequence:

Theorem 5.5. *Let $X \subset \mathbb{P}^n$ be a smooth, subcanonical, codimension two subvariety. If $h^0(\mathcal{I}_X(n-2)) \neq 0$, then X is a complete intersection.*

Using Ran's theorem, Ballico and Chiantini proved ([6]) that if $e \leq 0$, then X is a complete intersection; they also gave a quadratic lower bound on the degree of a non complete intersection. Ran's theorem has been refined, in various ways, by Holme and Schneider ([26]) and Holme ([25]), we refer to those papers for precise statements "on the vector bundles side", for smooth codimension two subvarieties we may summarize the main known results as follows:

Theorem 5.6. *Let $X \subset \mathbb{P}^n$, $n \geq 6$, be a smooth codimension two subvariety.*

1. *If $e \leq n+1$, then X is a complete intersection (here, as usual: $\omega_X \simeq \mathcal{O}_X(e)$)*
2. *If $deg(X) < (n-1)(n+5)$, then X is a complete intersection*

3. *If* $h^0(\mathcal{I}_X(n-2)) \neq 0$*, then* X *is a complete intersection*
4. *If* $n = 6$ *and* $deg(X) \leq 62$*, then* X *is a complete intersection.*

The first two items are from [26], the third is Theorem 5.5 and the fourth is from [26], [25].

A new ingredient introduced in [26], [25] is a careful study of the Schwarzenberger conditions, these are conditions that the Chern classes of a topological complex vector bundle have to satisfy. The Schwarzenberger conditions are as follows: write $c_1 = \alpha + \beta$, $c_2 = \alpha\beta$ with $\alpha, \beta \in \mathbb{C}$, then:

$$\binom{\alpha - 1 + m}{m} + \binom{\beta - 1 + m}{m} \in \mathbb{Z}, \ m = 2, \cdots, n$$

This gives strong conditions on the Chern classes and introduces the next topic.

5.3. *Numerically complete intersections varieties..*

As already said, a careful study of the Schwarzenberger conditions will eliminate many (c_1, c_2) for rank two vector bundles on $\mathbb{P}^n$ (especially if n is big enough), but, of course, you want never throw away values of the type $c_1 = a + b$, $c_2 = ab$, a, b integers, since the bundle $\mathcal{O}(a) \oplus \mathcal{O}(b)$ exists! So, it could happen that in some range the only possible values are of this kind, thus to prove the conjecture in that range it will be enough to show that every *numerically split* bundle (i.e. a bundle E with $c_1(E) = a + b$, $c_2(E) = ab$) actually splits. This motivates the following:

Definition 5.7. *A smooth (irreducible) codimension two* $X \subset \mathbb{P}^n$ *is said to be numerically a complete intersection (n.c.i.) of type* (a, b) *if* X *has the same characters as a complete intersection of type* (a, b)*:* $deg(X) = ab$ *and* $\omega_X \simeq \mathcal{O}_X(a + b - n - 1)$*.*

Remark 5.8.

1. *There exist n.c.i. curves in* $\mathbb{P}^3$ *which are not complete intersections.*
2. *From Theorem 5.4 it follows that if* X *is n.c.i. of type* (a, b)*,* $a \leq b$ *with* $a \leq n - 2$*, then* X *is a complete intersection.*

Lemma 5.9. *Let* $X \subset \mathbb{P}^n$*,* $n \geq 4$*, be a smooth codimension two subvariety of degree* d *with* $\omega_X \simeq \mathcal{O}_X(e)$*. Set* $s := min\{m / h^0(\mathcal{I}_X(m)) \neq 0\}$*. Then:* $d \leq s(n - 1 + e) + 1$

Proof. 1. Apply Lemma 3.2 to a section of X with a general $\mathbb{P}^4$. □

For a more precise statement, see [19], Lemma 2.2.

Proposition 5.10. *Let $X \subset \mathbb{P}^4$ be a smooth codimension subvariety. Assume X is n.c.i. of type (a, b), $a \leq b$, then:*

1. *If $b > a(a-3)+3$, X is a complete intersection.*
2. *If $a \leq n-1$, X is a complete intersection.*

Proof. See [19], Corollary 2.3. □

5.4. *Further results..*

To prove Hartshorne's conjecture in codimension two it is enough to prove that every rank two bundle on $\mathbb{P}^5$ (or $\mathbb{P}^6$ if you want to work in a more natural range: there you have $Pic(X) \simeq \mathbb{Z} \cdot H$ for free) splits. In spite of many efforts this doesn't seems to be a big simplification. Here we present a new approach to the problem, taken from [19], which at the moment yields only a slight improvement of Theorem 5.5 when $n = 5$ or $n = 6$ ([19], Theorem 1.1):

Theorem 5.11.

1. *Let $X \subset \mathbb{P}^5$ be a smooth, subcanonical threefold, if $h^0(\mathcal{I}_X(4)) \neq 0$, then X is a complete intersection.*
2. *Let $X \subset \mathbb{P}^6$ be a smooth codimension two subvariety of degree d. If $h^0(\mathcal{I}_X(5)) \neq 0$ or if $d \leq 73$, then X is a complete intersection.*

Idea of the proof: To fix ideas take $n = 6$ and assume $X \subset \mathbb{P}^6$ is smooth of codimension two with $e >> s$ and $d > s^2$ (the bundle corresponding to X will be unstable), so $X \subset \Sigma$, where Σ is a reduced, irreducible hypersurface of degree s. The starting point is Theorem 2.3; according to that theorem, if X is not a complete intersection, we must have $X \cap Sing(\Sigma) \neq \emptyset$, we try to investigate this intersection. Consider the rank two bundle associated to X:

$$0 \to \mathcal{O} \to E \to \mathcal{I}_X(e+7) \to 0$$

We see that $E(-e-7+s)$ has a section vanishing in codimension two:

$$0 \to \mathcal{O} \to E(-e-7+s) \to \mathcal{I}_Z(-e-7+2s) \to 0$$

where $Z \subset \mathbb{P}^6$ is a locally complete intersection subscheme of degree $d(Z) = c_2(E(-e-7+s)) = d - se + s^2 - 7s$ and with $\omega_Z \simeq \mathcal{O}_Z(-e-14+2s)$.

We observe three facts:

1. X and Z are bilinked on Σ, in particular $Z \subset \Sigma$
2. Moreover $X \cap Z = Jac(\Sigma) \cap X$ ($Jac(\Sigma)$ is the subscheme defined by the partials of Σ)
3. Z is a very bad guy: every irreducible component of Z_{red} appears with multiplicity in Z.

The first two facts are standard from vector bundles techniques (see [19], Lemma 2.7) and the third follows from the assumption $e >> s$ ([19] Lemma 2.6).

So we wonder if $Z \subset Jac(\Sigma)$? or more simply if $Z_{red} \subset Sing(\Sigma)$?

Assume for a moment that $Z_{red} \subset Sing(\Sigma)$. Consider a section with a general $\mathbb{P}^3$, we have the following situation: $C, Y \subset F \subset \mathbb{P}^3$, where $C := X \cap \mathbb{P}^3$, $Y := Z \cap \mathbb{P}^3$ and $F := \Sigma \cap \mathbb{P}^3$ is an irreducible surface of degree s containing Y_{red} in its singular locus.

Now observe that C is linearly normal ([19] Lemma 2.8).
To simplify further, assume $Y_0 := Y_{red}$ smooth, irreducible. If $deg(Y_0)$ is big enough, we are done. Indeed, in general, a surface $F \subset \mathbb{P}^3$ containing an integral curve of "high" degree in its singular locus won't be "linearly normal", i.e. F will be the projection of a surface $\tilde{F} \subset \mathbb{P}^4$; this contradicts the linear normality of C. The argument applies for example if $s = 4$ and $deg(Y_0) \geq 2$ or $s = 5$ and $deg(Y_0) \geq 4$. So we are left with the cases where $deg(Y_0)$ is small; these cases are handled by ad-hoc arguments (by the way observe that Lemma 5.9 gives a bound on $deg(Z)$).

It remains to show $Z_{red} \subset Sing(\Sigma)$.
If this is not the case, then $Y \subset F$ is a mutiple structure on Y_0 and $dim(Y_0 \cap Sing(F)) \leq 0$. But we have $p_a(Y)$ very negative (because $e >> s$). This sounds strange, because the singularities, which are isolated on Y_0, will increase the degree of the sub-line bundle of N_{Y_0} defined by F, equivalently the genus of the resulting double structure on Y_0 will increase (a double line on a smooth quadric has genus-1, whereas a double line on a quadric cone has genus 0). We wonder if this will be the case also for multiplicity m structures, $m > 2$; in fact we have ([19] Proposition 3.1):

Proposition 5.12. *Let $C \subset S \subset \mathbb{P}^3$ be an integral Gorenstein curve of degree d, arithmetic genus g, lying on the irreducible surface S of degree s. Assume $dim(C \cap Sing(S)) \leq 0$ and let C_m be the unique loc. C.M. multiplicity m structure on C contained in S. Then $p_a(C_m) \geq \mu(d, g, s, m)$, where $\mu(d, g, s, m) := 1 + m^2(g-1) - (s-4)d\frac{m(m-1)}{2}$.*

Observe, that if $C \subset S$, S a smooth surface of degree s, then the arithmetic genus of $mC \subset S$ is $\mu(d, g, s, m)$.

So, if $p_a(Y) < \mu(d, g, s, m)$, it must be $Y_{red} \subset Sing(F)$; since $p_a(Y) <<$ 0 in our case, we are done. $\square$

This approach, as it stands, seems difficult to generalize, some new ingredients are needed, however, this type of considerations could be of some help.

REFERENCES

[1] A. Alzati, *A new Castelnuovo bound for two-codimensional subvarieties of* $\mathbb{P}^n$*,* Proc. A. M. S., 114 (1992), pp. 607–611.

[2] A. Alzati - G. Ottaviani, *A linear bound on the* t*-normality of codimension two subvarieties of* $\mathbb{P}^n$*,* J. reine u. angew. Math., 409 (1990), pp. 35–40.

[3] A. Alzati - G. Ottaviani, *A vanishing theorem for the ideal sheaf of codimension two subvarieties of* $\mathbb{P}^n$*,* Rend. Ist. Mat. Univ. Trieste, 22 (1990), pp. 136–139.

[4] A.B. Aure, *The smooth surfaces on cubic hypersurfaces in* $\mathbb{P}^5$ *with isolated singularities,* Math. Scand., 67 (1990), pp. 215–222.

[5] A.B. Aure - K. Ranestad, *The smooth surfaces of degree* 9 *in* $\mathbb{P}^4$*,* London Math. Soc. Lect. Notes, 179 (1992), pp. 32–46.

[6] E. Ballico - L. Chiantini, *On smooth subcanonical varieties of codimension 2 in* $\mathbb{P}^n$*,* $n \geq 4$*,* Annal. Mat. Pura Appl., 135 (1983), pp. 99–118.

[7] W. Barth, *Transplanting cohomology classes in complex projective space,* Amer. J. Math., 92 (1970), pp. 951–967.

[8] W. Barth - G. Elencwajg, *Concernant la cohomologie des fibrés algébriques stables sur* $\mathbb{P}^n(\mathbb{C})$*,* Springer L.N.M., 683 (1978), pp. 1–24.

[9] W. Barth - A. Van de Ven, *On the geometry in codimension 2 of Grassmann manifolds,* Springer L.N.M., 412 (1974), pp. 1–35.

[10] M. Beltrametti - M. Schneider - A.J. Sommese, *Threefolds of degree* 9 *and* 10 *in* $\mathbb{P}^5$*,* Math. Ann., 288 (1990), pp. 413–444.

[11] R. Braun - G. Fløystad, *A bound for the degree of smooth surfaces in* $\mathbb{P}^5$ *not of general type,* Compositio Math., 93 (1994), pp. 211–229.

[12] R. Braun - G. Ottaviani - M. Schneider - F.O. Schreyer, *Boundedness for non-general type 3-folds in* $\mathbb{P}^5$*,* in *Complex Analysis and Geometry*, ed. V. Ancona - A. Silva, Plenum Press, New York (1993), pp. 311–338.

[13] J. D'Almeida, *Preuve "élémentaire" du théorème de Grothendieck-Lefschetz pour les hypersurfaces,* Arch. Math., 62 (1994), pp. 408–410.

[14] W. Decker - L. Ein - F.O. Schreyer, *Construction of surfaces in* $\mathbb{P}^5$, J. Algebraic Geometry, 2 (1993), pp. 185–237.

[15] W. Decker - T. Peternell - J. Le Potier - M. Schneider, *Half-canonical surfaces in* $\mathbb{P}^5$, L.N.M., 1417 (1990), pp. 91–110.

[16] W. Decker - F.O. Schreyer, *Non general type surfaces in* $\mathbb{P}^5$*: some remarks on bounds and constructions,* J. Symb. Computation, 29 (2000), pp. 545–582.

[17] L. Ein, *Vanishing theorems for varieties of low codimension,* L.N.M., 1311 (1986), pp. 71–75.

[18] Ph. Ellia - D. Franco, *On smooth surfaces in* $\mathbb{P}^5$ *lying on hyperquartics with isolated singularities,* Comm. in Algebra, 28 (2000), pp. 5703–5713.

[19] Ph. Ellia - D. Franco, *On subvarieties of codimension two in* $\mathbb{P}^5$, $\mathbb{P}^6$, J. Algebraic Geometry (to appear).

[20] Ph. Ellia - A. Hirschowitz - E. Mezzetti, *On the number of irreducible components of the Hilbert scheme of smooth space curves,* International J. of Math., 6 (1992), pp. 799–807.

[21] G. Ellingsrud - Ch. Peskine, *Sur les surfaces lisses de* $\mathbb{P}^4$, Invent. Math., 95 (1989), pp. 1–11.

[22] E.G. Evans - P. Griffith, *The syzygy problem,* Ann. of Math.,114 (1981), pp. 323–333.

[23] R. Hartshorne, *Algebraic Geometry,* G.T.M., 52, Springer (1977).

[24] R. Hartshorne, *Varieties of small codimension in projective space,* Bull. A.M.S., 80 (1974), pp. 1017–1032.

[25] A. Holme, *Codimension 2 subvarieties of projective space,* Manuscripta Math., 65 (1989), pp. 427–446.

[26] A. Holme - M. Schneider, *A computer aided approach to codimension 2 subvarieties in* $\mathbb{P}^n$, $n \geq 6$, J. reine u. angew Math., 357 (1985), pp. 205–220.

[27] G. Horrocks - D. Mumford, *A rank two vector bundle on* $\mathbb{P}^4$ *with 15,000 symmetries,* Topology, 12 (1973), pp. 63–81.

[28] L. Koelblen, *Surfaces de* $\mathbb{P}^4$ *tracées sur une hypersurface cubique,* J. reine angew. Math., 433 (1992), pp. 113–141.

[29] Ch. Peskine - L. Szpiro, *Liaison des variétés algébriques,* Invent. Math., 26 (1974), pp. 271–302.

[30] Th. Peternell - J. Le Potier - M. Schneider, *Vanishing theorems, linear and quadratic normality,* Invent. Math., 87 (1987), pp. 573–586.

[31] S. Popescu - K. Ranestad, *Surfaces of degree* 10 *in the projective fourspace via linear systems and linkage,* J. Algebraic Geometry, 5 (1996), pp. 13–76.

[32] Z. Ran, *On projective varieties of codimension 2,* Invent. Math., 73 (1983), pp. 333–336.

[33] M. Schneider, *3-folds in $\mathbb{P}^5$: classification in low degree and finiteness results,* in *Geometry of complex projective varieties*, Cetraro, 1990, Sem. Conf., 9, Mediterranean, Rende 1993, pp. 275–288.

[34] F. Severi, *Intorno ai punti doppi impropri di una superficie generale dello spazio a quattro dimensioni e ai suoi punti tripli apparenti,* Rend. Circ. Mat. Palermo, II, ser 15, (1901) pp. 33–51.

[35] F. Severi, *Una proprietá delle forme algebriche prive di punti multipli* in *Memorie Scelte,* vol. I, Ed. Cremonese, 1950, pp. 397–404.

[36] F.L. Zak, *Tangents and secants of algebraic varieties,* Translations of Mathematical Monographs, 127 (1993).

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