KINEMATIC VARIETIES FOR MASSLESS PARTICLES

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We study algebraic varieties that encode the kinematic data for n massless particles in d-dimensional spacetime subject to momentum conservation. Their coordinates are spinor brackets, which we derive from the Clifford algebra associated to the Lorentz group. This was proposed for $d = 5$ in the recent physics literature. Our kinematic varieties are given by polynomial constraints on tensors with both symmetric and skew symmetric slices.

1. Introduction

The real vector space $\mathbb{R}^d$, when endowed with the Lorentzian inner product

$$x \cdot y = -x_1 y_1 + x_2 y_2 + \cdots + x_n y_n,$$

is known as *d-dimensional spacetime*. The *Lorentz group* $\mathrm{SO}(1, d-1)$ consists of all $d \times d$ matrices g such that $\det g = 1$ and $(gx) \cdot (gy) = x \cdot y$ for all $x, y \in \mathbb{R}^d$. The world we live in, with its three space dimensions and one time dimension, is the case $d = 4$. But, also higher dimensions $d \geq 5$ appear frequently in physics.

   The present paper was inspired by the article [14] on amplitudes for $d = 5$. We here address the problem which was raised in the footnote on page 8 in [14], namely to find the *non-linear identities between spinor helicity variables*.

We consider a configuration of n particles in d-dimensional spacetime after complexification. The ith particle is represented by its momentum vector $p_i = (p_{i1}, p_{i2}, \ldots, p_{id}) \in \mathbb{C}^d$. We assume that each particle is *massless*, which means

$$p_i \cdot p_i = -p_{i1}^2 + p_{i2}^2 + p_{i3}^2 + \cdots + p_{id}^2 = 0 \quad \text{for} \quad i = 1, 2, \ldots, n. \tag{1}$$

We also assume that momentum conservation $\sum_{i=1}^{n} p_i = 0$ holds. In coordinates,

$$p_{1j} + p_{2j} + \cdots + p_{nj} = 0 \quad \text{for} \quad j = 1, 2, \ldots, d. \tag{2}$$

Thus, our parameter space consists of all solutions to the $n + d$ equations in (1) and (2). The pairwise inner products $s_{ij} = p_i \cdot p_j$ are known as *Mandelstam invariants*. They are invariant under the action of $\mathrm{SO}(1, d-1)$ on $(p_1, p_2, \ldots, p_n)$.

In Section 2 we view the parameter space through the lens of commutative algebra. We prove that the ideal generated by (1) and (2), denoted $I_{d,n}$, is a prime complete intersection, and we mention a Gröbner basis. The *Mandelstam variety* consists of symmetric $n \times n$ matrices (s_{ij}) of rank $\leq d$, with zeros on the diagonal, and rows and columns summing to zero. These constraints also define a prime ideal. This result generalizes recent findings for $d = 4$ in [7, Section 4].

In Section 3 we turn to the Clifford algebra $\mathrm{Cl}(1, d-1)$, and we review how this gives rise to the spinor representation of the Lie algebra $\mathfrak{so}(1, d-1)$. We construct the momentum space Dirac matrix P and the charge conjugation matrix C. These matrices have format $2^k \times 2^k$, for $k = \lfloor d/2 \rfloor$, as seen in Examples 3.2 and 3.4. Their symmetry properties are derived in (15) and Proposition 3.5.

In Section 4 we introduce the spinor brackets $\langle ij \rangle$ and $\langle ijk \rangle$. These are polynomials in the momentum coordinates p_{ij} and auxiliary parameters z_{ij}; see Example 4.2. These quantities are understood modulo the ideal $I_{d,n}$ in Section 2.

Section 5 is devoted to kinematic varieties given by order two brackets $\langle ij \rangle$. The Grassmannian $\mathrm{Gr}(2, n)$ and its first secant variety arise for $d \leq 5$. For $d \geq 6$, we obtain subvarieties of determinantal varieties. For d even, the structure of P leads to the spinor helicity variety of [7], with two types of brackets $\langle ij \rangle$ and $[ij]$.

In Section 6 we study the varieties of $n \times (n+1) \times n$ tensors with entries $\langle ij \rangle$ and $\langle ijk \rangle$. These tensors have the symmetry properties in Theorem 4.5. We present some theoretical results, lots of computations, and various conjectures.

2.   Massless Particles with Momentum Conservation

We fix the polynomial ring $\mathbb{C}[p]$ in the nd variables p_{ij}, and we write $I_{d,n} \subset \mathbb{C}[p]$ for the ideal generated by the n quadrics in (1) and the d linear forms in (2). The variety $V(I_{d,n})$ is the parameter space for n massless particles with momentum conservation. We show that this variety is irreducible of the expected dimension.

Theorem 2.1. $I_{d,n}$ is prime and is a complete intersection, if $\max(n,d) \geq 4$.

Proof. We first give the proof for $d \geq 5$. For $d = 4$ see [7, Theorem 4.5]. We eliminate p_n by solving the linear equations for $p_{n1}, \ldots, p_{nd}$. This leaves the quadric $f = (\sum_{i=1}^{n-1} p_i) \cdot (\sum_{i=1}^{n-1} p_i)$ and the ideal $J = \langle p_1 \cdot p_1, \ldots, p_{n-1} \cdot p_{n-1} \rangle$ in the polynomial ring $\mathbb{C}[p_1, \ldots, p_{n-1}]$. Our aim is to show that $\langle f \rangle$ is a prime ideal in

$$R = \mathbb{C}[p_1, \ldots, p_{n-1}]/J \simeq \frac{\mathbb{C}[p_1]}{\langle p_1 \cdot p_1 \rangle} \otimes_{\mathbb{C}} \frac{\mathbb{C}[p_2]}{\langle p_2 \cdot p_2 \rangle} \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \frac{\mathbb{C}[p_{n-1}]}{\langle p_{n-1} \cdot p_{n-1} \rangle}.$$

By [10, Exercise II.6.5], $\mathrm{Spec}(\mathbb{C}[p_i]/\langle p_i \cdot p_i \rangle)$ has trivial divisor class group for $d \geq 5$, and is normal for $d \geq 3$. Then, [10, Proposition II.6.2] implies that $\mathbb{C}[p_i]/\langle p_i \cdot p_i \rangle$ is a UFD for $d \geq 5$. A tensor product of UFDs over $\mathbb{C}$ is a UFD. Hence R is a UFD. It therefore suffices to show that f is irreducible in R. This holds because f is equivalent to $x_1^2 + \cdots + x_{n-1}^2$ by a linear change of coordinates.

The UFD R has Krull dimension $(n-1)(d-1)$. Its principal ideal $\langle f \rangle$ has height one. So, the ring $\mathbb{C}[p]/I_{d,n} \simeq R/\langle f \rangle$ has Krull dimension $(n-1)(d-1) - 1 = nd - (n+d)$. We conclude that the ideal $I_{d,n}$ is a complete intersection.

We now give the proof for $d = 3$ and $n \geq 4$. We claim that $S = \mathbb{C}[p]/I_{3,n}$ is a normal ring, i.e. the localization $S_{\mathfrak{p}}$ is an integrally closed domain for each prime ideal $\mathfrak{p} \subset S$. To do so we apply Serre's criterion for normality in two steps.

1. The ring $R = \mathbb{C}[p_1, \ldots, p_{n-1}]/J$ is normal and has dimension $2n - 2$. The associated primes of the principal ideal $\langle f \rangle$ all have height 1. Therefore, $S \simeq R/\langle f \rangle$ has dimension $2n - 3$. So, $I_{3,n}$ is a complete intersection ideal. Hence S is Cohen-Macaulay and satisfies Serre's depth condition, i.e. $S_{\mathfrak{q}}$ has depth at least 2 for every minimal prime ideal $\mathfrak{q} \subset S$ of height at least 2.

2. We claim that $S = \mathbb{C}[p]/(J + \langle f \rangle)$ is regular in codimension 1. To prove this, it suffices to show that the singular locus of S has codimension at least 2. Up to scaling the columns, the Jacobian matrix of $J + \langle f \rangle$ equals

$$\begin{bmatrix} p_{11} & p_{12} & p_{13} & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & 0 & p_{21} & p_{22} & p_{23} & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & p_{n-1,1} & p_{n-1,2} & p_{n-1,3} \\ \sum_\ell p_{\ell 1} & \sum_\ell p_{\ell 2} & \sum_\ell p_{\ell 3} & \sum_\ell p_{\ell 1} & \sum_\ell p_{\ell 2} & \sum_\ell p_{\ell 3} & \cdots & \sum_\ell p_{\ell 1} & \sum_\ell p_{\ell 2} & \sum_\ell p_{\ell 3} \end{bmatrix}.$$

The singular locus of S is cut out by the maximal minors of this matrix. Pick $i_1, i_2, \ldots, i_{n-1}, j_k \in \{1, 2, 3\}$ with $j_k \neq i_k$. Then we get the $n \times n$ minor

$$\left(\prod_{t=1, t \neq k}^{n-1} p_{t i_t} \right) \left(p_{k i_k} \sum_{\ell=1}^{n-1} p_{\ell j_k} - p_{k j_k} \sum_{\ell=1}^{n-1} p_{\ell i_k} \right).$$

Since $n \geq 4$, there are enough such products to cut out a locus of codimension ≥ 2 in $\mathrm{Spec}(S)$. This fails for $n = 3$. So, the ring S satisfies Serre's regularity condition, that is, $S_{\mathfrak{p}}$ is a DVR for any prime $\mathfrak{p} \subset S$ of height ≤ 1.

Serre's depth and regularity conditions imply that S is a normal ring. Noetherianity implies S is a finite product of normal domains: $S \simeq S_1 \times \cdots \times S_r$. Since S is standard graded ring, it has no non-trivial idempotents, and therefore $r = 1$. In conclusion, we have shown that S is a domain, and $I_{3,n}$ is a prime ideal. $\square$

Example 2.2. The hypothesis $\max(n,d) \geq 4$ is needed in Theorem 2.1. The following session in Macaulay2 [9] shows that $I_{3,3}$ is primary but not prime:

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i1 : R = QQ[p11,p12,p13,p21,p22,p23,p31,p32,p33];
i2 : I = ideal(p11+p21+p31, p12+p22+p32, p13+p23+p33,
     p11^2-p12^2-p13^2,p21^2-p22^2-p23^2,p31^2-p32^2-p33^2);
i3 : codim I, degree I
o3 = (6, 8)
i4 : isPrime I, isPrimary I
o4 = (false, true)
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Remark 2.3. Fix the reverse lexicographic order on $\mathbb{C}[p]$ where the entries of the $n \times d$ matrix (p_{ij}) are sorted row-wise. We find that the reduced Gröbner basis of $I_{d,n}$ stabilizes for $n,d \geq 3$. Namely, the initial monomial ideal equals

$$\mathrm{in}(I_{d,n}) \quad = \quad \langle p_{1j} : j = 1, \ldots, d \rangle + \langle p_{i1}^2 : i = 2, \ldots, n \rangle$$
$$+ \langle p_{21}p_{31}, p_{22}p_{31}p_{32}, p_{22}^2p_{31}, p_{23}^2p_{32}^2, p_{23}^2p_{31}p_{32} \rangle.$$

Remarkably, the last five monomial generators are independent of d and n.

Remark 2.3 suggests that it might be easy to parametrize the variety $V(I_n)$. We found that this is not the case at all. A naive idea is to express the variables p_{1j} and p_{i1} in terms of the entries of the $(n-1) \times (d-1)$ matrix $p' = (p_{ij})_{i,j \geq 2}$.

Remark 2.4. The elimination ideal $I_{d,n} \cap \mathbb{C}[p']$ is principal. Its generator is a large polynomial of degree 2^{n-1}. For instance, for $n = 4, d = 5$, this octic has 4671 terms. This hypersurface is a notable obstruction to any naive parametrization.

For $d = 4$, the momentum space twistors introduced by Hodges in [11] yield a beautiful parametrization. It would be very interesting to extend this to $d \geq 5$.

Physical properties of our n particles are expressions in $p_1, \ldots, p_n$ that are invariant under the action of the orthogonal group $G = \mathrm{O}(1, d-1)$. The ring of G-invariants in $\mathbb{C}[p]$ is generated by the *Mandelstam invariants* $s_{ij} = p_i \cdot p_j$. The ideal $I_{d,n}$ is fixed under this action, and we are interested in the invariant ring

$$(\mathbb{C}[p]/I_{d,n})^G = \mathbb{C}[S]/M_{d,n}. \tag{3}$$

Here $S = (s_{ij})$ is a symmetric $n \times n$ matrix. Its entries are the variables in the polynomial ring $\mathbb{C}[S]$. We shall characterize the prime ideal $M_{d,n}$. The variety

$$V(M_{d,n}) = \mathrm{Spec}\left((\mathbb{C}[p]/I_{d,n})^G\right) = V(I_{d,n})//G \qquad (4)$$

is the *GIT quotient*, whose points are the G-orbits of configurations $(p_1, \ldots, p_n)$. We call (4) the *Mandelstam variety*. The case $d = 4$ was studied in [7, Section 4].

Theorem 2.5. Let $n \geq 2$ and $d \geq 4$. The Mandelstam ideal $M_{d,n}$ is equal to

$$\langle s_{11}, s_{22}, \ldots, s_{nn} \rangle + \left\langle (d+1) \times (d+1) \text{ minors of } S \right\rangle + \left\langle \sum_{j=1}^{n} s_{ij} \; : \; i = 1, \ldots, n \right\rangle. \quad (5)$$

In particular, this ideal is prime. The dimension of the Mandelstam variety equals

$$\dim(V(M_{d,n})) = nd - n - d - \binom{d}{2} = \dim(V(I_{d,n})) - \dim(O(1, d-1)). \quad (6)$$

Proof. The dimension formula (6) holds because the orthogonal group acts faithfully on the complete intersection $V(I_{d,n})$. Consider the polynomials that generate the three ideals in (5). These polynomials vanish for massless particles in d dimensions subject to momentum conservation. It hence suffices to show that the sum in (5) is a prime ideal and that it has the correct dimension. This proof rests on the results for $I_{d,n}$ in Theorem 2.1. The details are presented for $d = 4$ in the proof of [7, Theorem 4.5]. The general case $d \geq 5$ is analogous. $\square$

Remark 2.6. If $n \leq 3$ then (5) is the maximal ideal, i.e. $M_{d,n} = \langle s_{ij} : 1 \leq i < j \leq n \rangle$.

3. Clifford Algebras and Spinors

Our aim is to represent the kinematic data for n particles in terms of spinors. This encoding rests on the Clifford algebra $\mathrm{Cl}(1, d-1)$, which is associated with the matrix $\eta = \mathrm{diag}(-1, 1, \ldots, 1)$. This section offers an introduction to Clifford algebras and the spin representation of $SO(1, d-1)$. For a systematic account see Chevalley's book [5]. Our exposition is inspired by the article [15].

The *Clifford algebra* $\mathrm{Cl}(1, d-1)$ is the free associative algebra $\mathbb{C}\langle \gamma_1, \ldots, \gamma_d \rangle$ modulo the two-sided ideal generated by $\gamma_i \gamma_j + \gamma_j \gamma_i - 2\eta_{ij}$ for $1 \leq i, j \leq d$. Its associated graded algebra is the exterior algebra $\wedge^* \mathbb{C}^d$, so $\dim_{\mathbb{C}} \mathrm{Cl}(1, d-1) = 2^d$. A basis is given by the square-free words $\gamma_{i_1} \gamma_{i_2} \cdots \gamma_{i_k}$ for $1 \leq i_1 < i_2 < \cdots < i_k \leq d$.

For applications in physics, one uses the representation of the Clifford algebra $\mathrm{Cl}(1, d-1)$ by *Dirac matrices* [15, Section 4]. These matrices have size

$2^k \times 2^k$, where $d = 2k$ if d is even and $d = 2k + 1$ if d is odd. We shall define the Dirac matrices $\Gamma_1, \Gamma_2, \ldots, \Gamma_d$ recursively. Starting with $d = 2$ and $k = 1$, we set

$$\Gamma_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad \Gamma_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{7}$$

For $d = 2k \geq 4$, we take tensor products of smaller Γ matrices with Pauli matrices. Namely, if $\Gamma_{k-1,i}$ is the ith Dirac matrix for $d = 2k - 2$ then we define

$$\Gamma_i = \Gamma_{k-1,i} \otimes \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{for } 1 \leq i \leq d - 2, \tag{8}$$

$$\Gamma_{d-1} = \text{Id}_{2^{k-1}} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \Gamma_d = \text{Id}_{2^{k-1}} \otimes \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}. \tag{9}$$

When $d = 2k + 1$ is odd, we construct the first $d - 1$ Dirac matrices as above, and we then add one additional Dirac matrix as follows:

$$\Gamma_d = -i^{k-1} \cdot \Gamma_1 \Gamma_2 \cdots \Gamma_{d-1}. \tag{10}$$

One checks by induction on k that this yields a representation of $\text{Cl}(1, d - 1)$.

Proposition 3.1. The Dirac matrices satisfy the Clifford algebra relations, i.e. we have $\Gamma_1^2 = -\text{Id}_{2^k}$, $\Gamma_j^2 = \text{Id}_{2^k}$ for $j \geq 2$, and $\Gamma_i \Gamma_j + \Gamma_j \Gamma_i = 0_{2^k}$ for $i \neq j$.

In what follows we work with a variant of the matrices above where the rows and columns have been permuted to achieve a desirable block structure. Namely, for $d = 2k$ even, each Dirac matrix Γ_i is anti-block diagonal of size $2^k \times 2^k$ with two blocks each of size $2^{k-1} \times 2^{k-1}$. This corresponds to the fact that the representation of $\text{Cl}(1, d - 1)$ is reducible when d is even. In this case, it splits into two Weyl representations, namely the left handed and right handed spinors [15, Section 4.2.1]. For $d = 2k + 1$ odd, the representation is irreducible, and we augment our basis by the diagonal matrix $\Gamma_d = \text{diag}(1, \ldots, 1, -1, \ldots, -1)$.

Consider a particle with momentum vector $p \in \mathbb{C}^d$. We define its *momentum space Dirac matrix* to be the following linear combination of the Dirac matrices:

$$P = -p_1 \Gamma_1 + p_2 \Gamma_2 + p_3 \Gamma_3 + \cdots + p_d \Gamma_d. \tag{11}$$

We next write P explicitly. This illustrates the block structure mentioned above.

Example 3.2 ($k = 1, 2, 3$). We take a look at the momentum space Dirac matrices $P = P^{(d)}$ for $d \leq 7$. When $d = 2k$ is even, we get the anti-block diagonal matrices

$$P^{(2)} = \begin{bmatrix} 0 & -p_1 + p_2 \\ p_1 + p_2 & 0 \end{bmatrix}, \quad P^{(4)} = \begin{bmatrix} 0 & 0 & p_1 - p_2 & p_3 - ip_4 \\ 0 & 0 & p_3 + ip_4 & p_1 + p_2 \\ -p_1 - p_2 & p_3 - ip_4 & 0 & 0 \\ p_3 + ip_4 & -p_1 + p_2 & 0 & 0 \end{bmatrix},$$

$$P^{(6)} = \begin{bmatrix} 0 & 0 & 0 & 0 & -p_1+p_2 & 0 & -p_3+ip_4 & p_5-ip_6 \\ 0 & 0 & 0 & 0 & 0 & -p_1+p_2 & p_5+ip_6 & p_3+ip_4 \\ 0 & 0 & 0 & 0 & -p_3-ip_4 & p_5-ip_6 & -p_1-p_2 & 0 \\ 0 & 0 & 0 & 0 & p_5+ip_6 & p_3-ip_4 & 0 & -p_1-p_2 \\ p_1+p_2 & 0 & -p_3+ip_4 & p_5-ip_6 & 0 & 0 & 0 & 0 \\ 0 & p_1+p_2 & p_5+ip_6 & p_3+ip_4 & 0 & 0 & 0 & 0 \\ -p_3-ip_4 & p_5-ip_6 & p_1-p_2 & 0 & 0 & 0 & 0 & 0 \\ p_5+ip_6 & p_3-ip_4 & 0 & p_1-p_2 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The matrix $P^{(d)}$ for $d = 2k+1$ odd is obtained from the matrix $P^{(d-1)}$ by adding the diagonal matrix $p_d \cdot \mathrm{diag}(1,\dots,1,-1,\dots,-1)$. For example, we have

$$P^{(5)} = \begin{bmatrix} p_5 & 0 & p_1-p_2 & p_3-ip_4 \\ 0 & p_5 & p_3+ip_4 & p_1+p_2 \\ -p_1-p_2 & p_3-ip_4 & -p_5 & 0 \\ p_3+ip_4 & -p_1+p_2 & 0 & -p_5 \end{bmatrix}.$$

We now return to arbitrary spacetime dimension $d \geq 2$. It follows from Proposition 3.1 that the momentum space Dirac matrix P satisfies the identity

$$P^2 = (-p_1^2 + p_2^2 + \cdots + p_d^2)\,\mathrm{Id}_{2^k}. \tag{12}$$

This implies the formula $\det P = (p_1^2 - p_2^2 - \cdots - p_d^2)^{2^{k-1}}$ for the determinant of P.

Corollary 3.3. For massless particles in d dimensions, the momentum space Dirac matrix P squares to 0 and its rank equals half of its size, i.e. rank $P = 2^{k-1}$.

The Dirac representation of the Clifford algebra $\mathrm{Cl}(1, d-1)$ gives rise to the spin representation of the Lie algebra $\mathfrak{so}(1, d-1)$. We consider the commutators

$$\Sigma_{jk} = \frac{1}{4}[\Gamma_j, \Gamma_k]. \tag{13}$$

These matrices Σ_{jk} define a representation of $\mathfrak{so}(1, d-1)$ because they satisfy

$$[\Sigma_{ij}, \Sigma_{kl}] = \eta_{jk}\Sigma_{il} + \eta_{il}\Sigma_{jk} - \eta_{jl}\Sigma_{ik} - \eta_{ik}\Sigma_{jl}. \tag{14}$$

Indeed, these are the commutation relations satisfied by the matrices in the standard basis of the Lie algebra $\mathfrak{so}(1, d-1)$. One obtains the spin representation of the Lie group $\mathrm{SO}(1, d-1)$ on $\mathbb{C}^{2^k}$ by taking the matrix exponentials $\exp(\Sigma_{jk})$.

Another important player in our story is the *charge conjugation matrix* C. This is an invertible $2^k \times 2^k$ matrix with entries in $\mathbb{Z}[i]$. It represents an equivariant linear map from the spinor representation of $\mathfrak{so}(1, d-1)$ to its dual representation. The characteristic properties of the charge conjugation matrix C are:

$$CP = -P^T C \ \text{ if } d = 2k \text{ is even}, \quad CP = (-1)^k P^T C \ \text{ if } d = 2k+1 \text{ is odd}. \tag{15}$$

We explicitly realize $C = C^{(d)}$ as a matrix whose nonzero entries are $1, -1, i, -i$.

Example 3.4 ($k = 1, 2, 3$)**.** For $d = 2, 3$ we take the same skew symmetric matrix:

$$C^{(2)} = C^{(3)} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

For $d = 4, 5$ we obtain skew symmetric matrices with a block diagonal structure:

$$C^{(4)} = \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{bmatrix}, \quad C^{(5)} = \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{bmatrix}.$$

Finally, for $d = 6, 7$ we have the same charge conjugation matrix:

$$C^{(6)} = C^{(7)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

This is a symmetric 8×8 matrix with an anti-block diagonal structure.

The following result explains the matrix structures we saw in Example 3.4.

Proposition 3.5. The following properties hold for the charge conjugation matrix C associated with particles in spacetimes of dimension $d = 2k$ and $d = 2k + 1$:

1. C is symmetric when $k \equiv 0, 3 \mod 4$; otherwise it is skew symmetric.

2. C is block diagonal when $k \equiv 0 \mod 2$; otherwise it is anti-block diagonal.

3. The $2^{k-1} \times 2^{k-1}$ blocks of C are skew symmetric when $k \equiv 2, 3 \mod 4$; otherwise the blocks are symmetric.

Proof. The charge conjugation matrix can be written as follows:

$$\begin{aligned} C &= \Gamma_{d+1} \Gamma_4 \Gamma_6 \cdots \Gamma_{d-2} \Gamma_d \Gamma_1 && \text{if } d \equiv 0 \mod 4, \\ C &= \Gamma_4 \Gamma_6 \cdots \Gamma_{2k-2} \Gamma_{2k} \Gamma_1 && \text{otherwise.} \end{aligned} \tag{16}$$

Here, we set $\Gamma_{d+1} = -i^{k-1} \Gamma_1 \cdots \Gamma_d$, which is analogous to (10). One can check that this C defines an equivariant map to the dual representation and (15) holds.

The Dirac matrices are either symmetric or skew symmetric. Namely, we have

$$\Gamma_1^T = -\Gamma_1, \quad \Gamma_2^T = \Gamma_2, \quad \text{and} \quad \Gamma_{2i-1}^T = \Gamma_{2i-1}, \ \Gamma_{2i}^T = -\Gamma_{2i} \quad \text{for } i \geq 2. \tag{17}$$

First suppose $d \not\equiv 0 \mod 4$. Then (17) and the Clifford algebra relations imply

$$C^T = (-1)^k \Gamma_1 \Gamma_{2k} \Gamma_{2k-2} \cdots \Gamma_6 \Gamma_4 = (-1)^k (-1)^{\frac{k(k-1)}{2}} C = (-1)^{\frac{(k+1)k}{2}} C.$$

We conclude that the matrix C is symmetric if $k \equiv 3 \mod 4$ and it is skew symmetric if $k \equiv 1, 2 \mod 4$. Now suppose that $d \equiv 0 \mod 4$. In this case, we find

$$C^T = (-1)^k \Gamma_1 \Gamma_d \Gamma_{d-2} \cdots \Gamma_6 \Gamma_4 \Gamma_{d+1} = (-1)^k (-1)^{\frac{(k+1)k}{2}} C = (-1)^{\frac{(k+3)k}{2}} C.$$

Hence C is symmetric if $k \equiv 0 \mod 4$; otherwise it is skew symmetric. Bearing in mind that $d \in \{2k, 2k+1\}$, this completes the proof of part 1 in Proposition 3.5.

All Dirac matrices Γ_i are anti-block diagonal, except the last one when d is odd. Since C always has k anti-block diagonal terms, we find that C is block diagonal if $k \equiv 0 \mod 2$. Otherwise it is anti-block diagonal. This proves part 2.

The verification of part 3 is similar, but it is a bit more technical. $\qquad \square$

4. Spinor Brackets

We now return to our primary goal, namely to model interactions among n massless particles in d-dimensional spacetime. This is based on the Dirac matrices in Section 3. Recall that the ith particle is the vector $p_i = (p_{i1}, p_{i2}, \ldots, p_{id})$. As in Section 2, we assume that the tuple $(p_1, \ldots, p_n) \in \mathbb{C}^{nd}$ lies in the variety $V(I_{d,n})$.

The *momentum space Dirac matrix* for the ith particle is defined as

$$P_i = -p_{i1} \Gamma_1 + p_{i2} \Gamma_2 + p_{i3} \Gamma_3 + \cdots + p_{id} \Gamma_d.$$

This matrix has format $2^k \times 2^k$, where $k = \lfloor d/2 \rfloor$, and its entries are linear forms in p_i. The rank of P_i equals 2^{k-1}, since the particle is massless, by Corollary 3.3. The Clifford algebra relations imply the following anti-commutator identities:

$$P_i P_j + P_j P_i = 2 p_i \cdot p_j \operatorname{Id}_{2^k} = 2 s_{ij} \operatorname{Id}_{2^k}. \tag{18}$$

For each $i \in \{1, 2, \ldots, n\}$, we now introduce 2^{k-1} additional variables z_{ij}. These variables parameterize the column space of P_i, using the basis consisting of the first 2^{k-2} and last 2^{k-2} columns of P_i. We fix the parameter vector

$$z_i = \left(z_{i1}, z_{i2}, \ldots, z_{i,2^{k-2}}, 0, 0, \ldots, 0, z_{i,2^{k-2}+1}, z_{i,2^{k-2}+2}, \ldots, z_{i,2^{k-1}} \right)^T. \tag{19}$$

Here it is assumed that $k \geq 2$. In the small special case $k = 1$ we set $z_i = (z_{i1}, 0)^T$.

We use Dirac's ket-notation for a general vector in the column space of P_i:

$$|i\rangle = P_i z_i. \tag{20}$$

The bra-notation $\langle i |$ is used for the corresponding row vector $|i\rangle^T$. Thus $|i\rangle$ and $\langle i |$ are vectors that depend on $d + 2^{k-1}$ parameters. They represent particle i.

Remark 4.1. In this paper we restrict ourselves to massless particles. If particle i is massive then its momentum vector p_i satisfies the inhomogeneous equation

$$p_{i1}^2 - p_{i2}^2 - \cdots - p_{id}^2 = m_i^2 \qquad \text{for some constant } m_i.$$

Hence the determinant of its matrix P_i equals $\det P_i = m_i^{2^k}$. The vector $|i\rangle$ is now an eigenvector of P_i, by equation (12). The algebraic relations among the spinor brackets, defined below, would involve the masses $m_1, \ldots, m_n$ as parameters. The study of such affine varieties for massive particles is left for future work.

We write $\mathbb{C}[p, z]$ for the polynomial ring in nd variables p_{ij} and $n2^{k-1}$ variables z_{ij}. We view $I_{d,n}$ as an ideal in $\mathbb{C}[p, z]$. The quantities that represent interactions among n massless particles are certain elements in the quotient ring

$$R_{d,n} = \mathbb{C}[p, z]/I_{d,n}.$$

We know from Theorem 2.1 that $R_{d,n}$ is an integral domain for $\max(d, n) \geq 4$.

The following elements of $R_{d,n}$ are invariant under the action of the Lorentz group $SO(1, d-1)$. We define the *spinor brackets* of order two and three to be

$$\langle ij \rangle = \langle i|C|j \rangle \quad \text{and} \quad \langle ijk \rangle = \langle i|CP_j|k \rangle. \tag{21}$$

Here $i, j, k \in \{1, 2, \ldots, n\}$. Similarly, we define the ℓ-*th order spinor brackets*:

$$\langle i_1 i_2 \cdots i_\ell \rangle = \langle i_1|CP_{i_2} \cdots P_{i_{\ell-1}}|i_\ell \rangle. \tag{22}$$

Here C is the charge conjugation matrix from Example 3.4 and Proposition 3.5.

Example 4.2 ($d = 3$). Explicitly, the spinor brackets of order two and three are

$$\langle ij \rangle = \begin{bmatrix} z_{i1} & 0 \end{bmatrix} \begin{bmatrix} p_{i3} & p_{i1} + p_{i2} \\ -p_{i1} + p_{i2} & -p_{i3} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} p_{j3} & -p_{j1} + p_{j2} \\ p_{j1} + p_{j2} & -p_{j3} \end{bmatrix} \begin{bmatrix} z_{j1} \\ 0 \end{bmatrix}$$

$$= -p_{i1}p_{j3}z_{i1}z_{j1} - p_{i2}p_{j3}z_{i1}z_{j1} + p_{i3}p_{j1}z_{i1}z_{j1} + p_{i3}p_{j2}z_{i1}z_{j1},$$

$$\begin{aligned}
\langle ijk \rangle = \ & p_{i1}p_{j1}p_{k1}z_{i1}z_{k1} + p_{i1}p_{j1}p_{k2}z_{i1}z_{k1} - p_{i1}p_{j2}p_{k1}z_{i1}z_{k1} - p_{i1}p_{j2}p_{k2}z_{i1}z_{k1} \\
& -p_{i1}p_{j3}p_{k3}z_{i1}z_{k1} + p_{i2}p_{j1}p_{k1}z_{i1}z_{k1} + p_{i2}p_{j1}p_{k2}z_{i1}z_{k1} - p_{i2}p_{j2}p_{k1}z_{i1}z_{k1} \\
& -p_{i2}p_{j2}p_{k2}z_{i1}z_{k1} - p_{i2}p_{j3}p_{k3}z_{i1}z_{k1} + p_{i3}p_{j1}p_{k3}z_{i1}z_{k1} + p_{i3}p_{j2}p_{k3}z_{i1}z_{k1} \\
& -p_{i3}p_{j3}p_{k1}z_{i1}z_{k1} - p_{i3}p_{j3}p_{k2}z_{i1}z_{k1}.
\end{aligned}$$

These are elements of the quotient ring $R_{3,n}$, so the constraints (1) and (2) hold.

Remark 4.3. The spinor brackets $\langle ij \rangle$ and $\langle ijk \rangle$ are invariant under the action of $SO(1, d-1)$ by left multiplication on the z-vectors and conjugation on the P-matrices. For instance, given $\Sigma \in \mathfrak{so}(1, d-1)$, the matrix $g = \exp(\Sigma)$ transforms

$$\langle ij \rangle = z_i^T P_i^T C P_j z_j \tag{23}$$

as desired: $(gz_i)^T (gP_ig^{-1})^T C (gP_jg^{-1}) gz_j = z_i^T P_i^T g^T C g P_j z_j = \langle ij \rangle$. Here $g^T C g = C$ holds because $C\Sigma = -\Sigma^T C$ implies $C \exp(\Sigma) = \exp(-\Sigma^T) C = \exp(\Sigma^T)^{-1} C$.

Remark 4.4. In what follows we consider only spinor brackets with $\ell = 2, 3$. Their varieties are already quite intriguing, as we shall see in Sections 5 and 6. We do not claim that the brackets for $\ell \geq 4$ can be reduced to $\ell = 2, 3$. It would be worthwhile to study such reductions from the perspective of invariant theory.

We next examine symmetries of the spinor brackets. To this end, we define

$$S = \big(\langle ij \rangle \big)_{1 \leq i,j \leq n} = \big(|1\rangle, |2\rangle, \ldots, |n\rangle \big)^T \cdot C \cdot \big(|1\rangle, |2\rangle, \ldots, |n\rangle \big). \qquad (24)$$

This is an $n \times n$ matrix, obtained by multiplying matrices of formats $n \times 2^k$, $2^k \times 2^k$ and $2^k \times n$. Similarly, for each index $j \in \{1, 2, \ldots, n\}$ we define the $n \times n$ matrix

$$T_j = \big(\langle ijk \rangle \big)_{1 \leq i,k \leq n} = \big(|1\rangle, |2\rangle, \ldots, |n\rangle \big)^T \cdot C \cdot P_j \cdot \big(|1\rangle, |2\rangle, \ldots, |n\rangle \big). \qquad (25)$$

Theorem 4.5. The $n \times n$ matrix S has rank $\leq 2^k$ with zeros on the diagonal. If $k \equiv 0, 3 \bmod 4$ then S is symmetric; otherwise S is skew symmetric. In symbols,

$$\langle ii \rangle = 0 \quad \text{and} \quad \langle ij \rangle = \pm \langle ji \rangle \qquad \text{for } 1 \leq i < j \leq n.$$

The $n \times n$ matrix T_j has rank $\leq 2^{k-1}$ with zeros in the jth row and column. If $d \equiv 1, 2, 3, 4 \bmod 8$ then T_j is symmetric; otherwise T_j is skew symmetric. Thus,

$$\langle jjk \rangle = \langle ijj \rangle = 0 \quad \text{and} \quad \langle ijk \rangle = \pm \langle kji \rangle \qquad \text{for } 1 \leq i, j, k \leq n.$$

The sum of the n matrices T_j is the zero matrix, i.e. $T_1 + T_2 + \cdots + T_n = 0$.

Proof. Since C is of rank 2^k, equation (24) implies that S has rank at most 2^k. Now since $P_i^2 = 0$, the vector $|i\rangle$ lies in the kernel of P_i, so $\langle ii \rangle = \pm z_i^T C P_i |i\rangle = 0$. The (skew) symmetry property of S follows from part (1) in Proposition 3.5.

We now turn to T_j. By Corollary 3.3, the $2^k \times 2^k$ matrix P_j has rank 2^{k-1}. It follows from (25) that T_j has rank at most 2^{k-1}. Since $|j\rangle$ lies in the kernel of P_j, we conclude $\langle ijj \rangle = \langle i | C P_j | j \rangle = 0$, and similar for $\langle jjk \rangle$. The (skew) symmetry property of T_j follows from Proposition 3.5 and equation (15). Finally, momentum conservation $\sum_{j=1}^n P_j = 0$ implies the matrix identity $\sum_{j=1}^n T_j = 0$. $\quad\square$

We regard the tuple $T = (T_1, \ldots, T_n)$ as a tensor of format $n \times n \times n$. By augmenting this with the matrix S, we obtain a tensor ST of format $n \times (n+1) \times n$.

Remark 4.6. For the ℓ-th order spinor brackets we can introduce the tensor $T^{(\ell)} = (\langle i_1 \cdots i_\ell \rangle)_{i_1, \ldots, i_\ell}$ of size n^ℓ. In general, the following symmetries will hold:

$$\langle i_1 i_1 i_3 \cdots i_\ell \rangle = \langle i_1 \cdots i_j i_j \cdots i_\ell \rangle = \langle i_1 \cdots i_{\ell-2} i_\ell i_\ell \rangle = 0 \quad \text{and} \quad \langle i_1 \cdots i_\ell \rangle = \pm \langle i_\ell \cdots i_1 \rangle.$$

In the latter equation, the sign is positive when ℓ is even and $d \equiv 0, 1, 6, 7 \bmod 8$, or when ℓ is odd and $d \equiv 1, 2, 3, 4 \bmod 8$. Otherwise we get a negative sign.

The next two sections are concerned with polynomial relations satisfied by the spinor brackets. These relations define the kinematic varieties for massless particles which are promised in our title. We offer a preview for the case in Example 4.2. Our computations were performed with the software `Macaulay2` [9].

Example 4.7 ($d = 3, k = 1, n = 4$)**.** We consider four particles in spacetime for flatlanders. The six spinor brackets of order two form a skew symmetric matrix:

$$S = \begin{bmatrix} 0 & \langle 12 \rangle & \langle 13 \rangle & \langle 14 \rangle \\ -\langle 12 \rangle & 0 & \langle 23 \rangle & \langle 24 \rangle \\ -\langle 13 \rangle & -\langle 23 \rangle & 0 & \langle 34 \rangle \\ -\langle 14 \rangle & -\langle 24 \rangle & -\langle 34 \rangle & 0 \end{bmatrix}.$$

The 24 spinor brackets of order three are the entries of four symmetric matrices:

$$T_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \langle 212 \rangle & \langle 213 \rangle & \langle 214 \rangle \\ 0 & \langle 213 \rangle & \langle 313 \rangle & \langle 314 \rangle \\ 0 & \langle 214 \rangle & \langle 314 \rangle & \langle 414 \rangle \end{bmatrix}, \qquad T_2 = \begin{bmatrix} \langle 121 \rangle & 0 & \langle 123 \rangle & \langle 124 \rangle \\ 0 & 0 & 0 & 0 \\ \langle 123 \rangle & 0 & \langle 323 \rangle & \langle 324 \rangle \\ \langle 124 \rangle & 0 & \langle 324 \rangle & \langle 424 \rangle \end{bmatrix},$$

$$T_3 = \begin{bmatrix} \langle 131 \rangle & \langle 132 \rangle & 0 & \langle 134 \rangle \\ \langle 132 \rangle & \langle 232 \rangle & 0 & \langle 234 \rangle \\ 0 & 0 & 0 & 0 \\ \langle 134 \rangle & \langle 234 \rangle & 0 & \langle 434 \rangle \end{bmatrix}, \qquad T_4 = \begin{bmatrix} \langle 141 \rangle & \langle 142 \rangle & \langle 143 \rangle & 0 \\ \langle 142 \rangle & \langle 242 \rangle & \langle 243 \rangle & 0 \\ \langle 143 \rangle & \langle 243 \rangle & \langle 343 \rangle & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We are interested in the subvariety of $\mathbb{P}^5 \times \mathbb{P}^{23}$ parametrized by the 30 brackets. This *kinematic variety* is irreducible of dimension 4 and its *multidegree* equals

$$5s^5t^{19} + 28s^4t^{20} + 24s^3t^{21} + 10s^2t^{22} + 2st^{23} \in H^*(\mathbb{P}^5 \times \mathbb{P}^{23}, \mathbb{Z}). \qquad (26)$$

The prime ideal of our variety is minimally generated by the 10 linear forms in

$$T_1 + T_2 + T_3 + T_4 = 0, \qquad (27)$$

together with $54 = 1 + 24 + 29$ quadrics. First, there is the Plücker quadric

$$\langle 12 \rangle \langle 34 \rangle - \langle 13 \rangle \langle 24 \rangle + \langle 14 \rangle \langle 23 \rangle \quad = \quad \mathrm{Pfaffian}(S), \qquad (28)$$

which ensures that S has rank two. Next, we have 24 binomial quadrics like

$$\langle ijk \rangle \langle ljm \rangle - \langle ijm \rangle \langle ljk \rangle. \qquad (29)$$

For each j, we have six such binomials. They are the 2×2 minors which ensure that the matrix T_j has rank ≤ 1. Finally, we have 29 bilinear relations, such as

$$\langle 12 \rangle \langle 324 \rangle - \langle 34 \rangle \langle 142 \rangle \quad \text{and} \quad \langle 12 \rangle \langle 243 \rangle - \langle 13 \rangle \langle 242 \rangle + \langle 23 \rangle \langle 142 \rangle. \qquad (30)$$

These ensure that the 4×20 matrix (S, T_1, T_2, T_3, T_4) has rank ≤ 2. We obtain (30) from the 3×3 minors of this flattening of the $4 \times 5 \times 4$ tensor ST. Our prime ideal is generated by (27), (28), (29) and quadrics like (30). See also Conjecture 6.2.

5.   Varieties in Matrix Spaces

The order two spinor brackets $\langle ij \rangle$ are the $\binom{n}{2}$ entries of an $n \times n$ matrix S which is either symmetric or skew symmetric. The *kinematic variety* $\mathcal{K}_{d,n}^{(2)}$ consists of such matrices S. This is an irreducible variety in $\mathbb{P}^{\binom{n}{2}-1}$. We seek its prime ideal.

Theorem 5.1. For $d = 3$, the ideal of $\mathcal{K}_{3,n}^{(2)}$ is generated by the 4×4 Pfaffians of a skew symmetric $n \times n$ matrix, so this kinematic variety is the Grassmannian $\mathrm{Gr}(2,n)$. For $d = 4,5$, the ideal is generated by the 6×6 Pfaffians of a skew symmetric $n \times n$ matrix, and hence $\mathcal{K}_{4,n}^{(2)} = \mathcal{K}_{5,n}^{(2)}$ is the secant variety of $\mathrm{Gr}(2,n)$.

Proof. For $d = 3,4,5$ we have $k \equiv 1,2 \mod 4$. Theorem 4.5 tells us that S is skew symmetric of rank $\leq 2^k$. Hence $\mathcal{K}_{d,n}^{(2)}$ is contained in $\mathrm{Gr}(2,n)$ and its secant variety respectively. To see that they are equal, we checked (using `Macaulay2`) that the entries of the column vector $|i\rangle$ are algebraically independent. Thus (24) is a generic skew symmetric $n \times n$ matrix of rank $\leq 2^k$. This might fail for $d \geq 6$. $\square$

Remark 5.2. The dimensions of Grassmannians and their secant varieties are well known; see e.g. [3]. For the kinematic varieties in Theorem 5.1, we have

$$\dim \mathcal{K}_{3,n}^{(2)} = 2n - 4 \quad \text{and} \quad \dim \mathcal{K}_{4,n}^{(2)} = \dim \mathcal{K}_{5,n}^{(2)} = 4n - 11 \qquad \text{for } n \geq 4. \quad (31)$$

Next we consider spacetime dimensions $d = 6,7,8,9$. Here $k = 3,4$, so S is a symmetric $n \times n$ matrix. It still has zeros on the diagonal and rank $S \leq 2^k$, by Theorem 4.5. The variety of such matrices is irreducible, and its prime ideal is generated by the $(2^k + 1) \times (2^k + 1)$ minors. This is proved for 4×4 minors in [6, Theorem 3.5], but the proof is the same for matrix of fixed size larger than 4.

Conjecture 5.3. For $d \equiv 0,1,6,7 \mod 8$, the kinematic variety $\mathcal{K}_{d,n}^{(2)}$ consists of all symmetric $n \times n$ matrices with zero diagonal and rank $\leq 2^{\lfloor d/2 \rfloor}$. For $d \equiv 2,3,4,5 \mod 8$, $\mathcal{K}_{d,n}^{(2)}$ is the variety of skew symmetric $n \times n$ matrices of rank $\leq 2^{\lfloor d/2 \rfloor}$.

The determinantal varieties in Conjecture 5.3 are irreducible and their dimensions are known. For instance, going one step beyond (31), we would have

$$\dim \mathcal{K}_{d,n}^{(2)} = 7n - 29 \quad \text{for } d = 6,7 \text{ and } n \geq 7. \qquad (32)$$

To prove Conjecture 5.3, it suffices to show that the dimension $\mathcal{K}_{d,n}^{(2)}$ equals that of the determinantal variety. Both varieties are irreducible, and the former is contained in the latter. For instance, we verified (32) numerically for $n = 7,8,9$.

The difficulty in extending the proof of Theorem 5.1 to higher dimensions $d \geq 6$ is that the entries of the vector $|i\rangle$ are no longer algebraically independent.

Remark 5.4. For $d \geq 6$, the spinors $|i\rangle$ live in a proper subvariety of $\mathbb{P}^{2^{k-1}}$. For instance, for $d = 6$, the eight coordinates of $|i\rangle$ satisfy the algebraic relation

$$|i\rangle_{\varnothing} \cdot |i\rangle_{123} - |i\rangle_1 \cdot |i\rangle_{23} + |i\rangle_2 \cdot |i\rangle_{13} - |i\rangle_3 \cdot |i\rangle_{12} = 0. \qquad (33)$$

For $k \geq 4$, we used `HomotopyContinuation.jl` [1] to compute the codimension of the variety parametrized by $|i\rangle$. Based on this, we conjecture the formulas

$$\text{codim}(2k) = 2^{k-1} - 2(k-1) \quad \text{and} \quad \text{codim}(2k+1) = \text{codim}(2k) - 1. \qquad (34)$$

We now relate $\mathcal{K}_{d,n}^{(2)}$ to the *spinor-helicity formalism* in [8, Section 2.2]. For this, we assume that $d = 2k$ is even. As seen in Example 3.2, the momentum space Dirac matrix P is anti-block diagonal with two blocks of size $2^{k-1} \times 2^{k-1}$:

$$P = \begin{bmatrix} 0 & P' \\ P'' & 0 \end{bmatrix}. \qquad (35)$$

This reflects the fact that the action of $\text{SO}(1, d-1)$ decomposes into two irreducible representations. We consider the blocks P' and P'' separately, and define both angle and square spinor brackets. Let $\tilde{x}_i$ be the vector consisting of the first 2^{k-1} entries of z_i and x_i the vector consisting of the last 2^{k-1} entries of z_i. We set

$$|i\rangle = P_i' x_i, \quad \langle i| = |i\rangle^T \qquad \text{and} \qquad |i] = P_i'' \tilde{x}_i, \quad [i| = |i]^T.$$

By Proposition 3.5, the C matrix is either block diagonal or anti-block diagonal:

$$C = \begin{bmatrix} C' & 0 \\ 0 & C'' \end{bmatrix} \text{ if } k \text{ is even} \quad \text{and} \quad C = \begin{bmatrix} 0 & C'' \\ C' & 0 \end{bmatrix} \text{ if } k \text{ is odd}.$$

In analogy to (22), we define two types of elements in the ring $R_{d,n}$ as follows:

$$\langle i_1 i_2 \cdots i_\ell \rangle = \langle i_1 | C' P_{i_2}' \cdots P_{i_{\ell-1}}' | i_\ell \rangle \quad \text{and} \quad [i_1 i_2 \cdots i_\ell] = [i_1 | C'' P_{i_2}'' \cdots P_{i_{\ell-1}}'' | i_\ell]. \qquad (36)$$

Moreover, it makes sense to mix and match the brackets, so we can also define

$$\langle i_1 i_2 \cdots i_\ell] = \langle i_1 | C'' P_{i_2}'' \cdots P_{i_{\ell-1}}'' | i_\ell] \quad \text{and} \quad [i_1 i_2 \cdots i_\ell \rangle = [i_1 | C' P_{i_2}' \cdots P_{i_{\ell-1}}' | i_\ell \rangle. \qquad (37)$$

If k is even, then the brackets in (36) are Lorentz invariant, in the sense of Remark 4.3. Likewise, if k is odd, then the brackets in (37) are Lorentz invariant.

We now focus on $d = 4$ and $\ell = 2$, and we erase the z parameters by setting

$$x_i = \tilde{x}_i = (p_{i1} + p_{i2})^{-1/2} \quad \text{for } i = 1, 2, \ldots, n. \qquad (38)$$

The $\langle ij \rangle$ and $[ij]$ form skew symmetric $n \times n$ matrices of rank 2 whose product is the zero matrix. This recovers the *spinor helicity variety* in [7, Example 1.1]:

$$\text{SH}(2, n, 0) \subset \text{Gr}(2, n) \times \text{Gr}(2, n) \subset \mathbb{P}^{\binom{n}{2}-1} \times \mathbb{P}^{\binom{n}{2}-1}.$$

It would be interesting to study the varieties given by (36) and (37) for $d = 6, 8, \ldots$. In the present paper we limit ourselves to the angle brackets in (21).

6. Varieties in Tensor Spaces

Let $\mathcal{K}_{d,n}^{(3)}$ denote the kinematic variety of second and third order spinor brackets. This is an irreducible subvariety of $\mathbb{P}^{\binom{n}{2}-1} \times \mathbb{P}^{K-1}$, where $K = n \cdot \binom{n}{2}$ when the slices T_j are symmetric, i.e. $d \equiv 1, 2, 3, 4 \mod 8$, and $K = n \cdot \binom{n-1}{2}$ when the slices T_j are skew symmetric; see Theorem 4.5. The prime ideal of $\mathcal{K}_{d,n}^{(3)}$ consists of all polynomial relations among the spinor brackets $\langle ij \rangle$ and $\langle ijk \rangle$. It is $\mathbb{Z}^2$-graded because $\langle ij \rangle$ and $\langle ijk \rangle$ are homogeneous elements in the $\mathbb{Z}^2$-graded ring $R_{d,n} = \mathbb{C}[p, z]/I_{d,n}$, of degrees $(2, 2)$ and $(3, 2)$ respectively; see Example 4.2.

Points in the kinematic variety $\mathcal{K}_{d,n}^{(3)}$ are $n \times (n+1) \times n$ tensors ST. The tensor slices $S, T_1, \ldots, T_n$ are $n \times n$ matrices which are symmetric or skew symmetric, depending on the residue classes of $k = \lfloor d/2 \rfloor$ modulo 4 and d modulo 8. We refer to [12, 13] for basics on tensors, such as rank, slices and flattenings.

Remark 6.1 (Three particles)**.** Let $n = 3$ and $d \geq 4$. The kinematic variety $\mathcal{K}_{d,3}^{(3)}$ is empty since $T_1 = T_2 = T_3 = 0$. To see this, we first note that $\langle ijk \rangle = 0$ for i, j, k distinct, since $T_1 + T_2 + T_3 = 0$. Remark 2.6 implies $P_i P_j + P_j P_i = 0_{2^k}$, and hence $\langle iji \rangle = z_i^T P_i^T C P_j P_i z_i = \pm z_i^T C P_j P_i^2 z_i = 0$. Therefore, $\langle ijk \rangle = 0$ for all i, j, k, showing the variety is empty. The matrix S is unconstrained, and hence $\mathcal{K}_{d,3}^{(2)} = \mathbb{P}^2$.

From now on we assume $n \geq 4$. We start with the case $d = 3$. Here, the matrix S is skew symmetric and the slices T_j are symmetric. Our kinematic variety $\mathcal{K}_{3,n}^{(3)}$ lives in the tensor space $\mathbb{P}^{\binom{n}{2}-1} \times \mathbb{P}^{n\binom{n}{2}-1}$. See Example 4.7 for the case $n = 4$.

To study $\mathcal{K}_{3,n}^{(3)}$ for $n \geq 5$, we introduce some terminology for third order tensors. Recall from [12, Chapter 2] that the *multilinear rank* of a tensor is the triple of the ranks of its flattenings. A *Tucker decomposition* of an $n_1 \times n_2 \times n_3$ tensor T with multilinear rank (r_1, r_2, r_3) is a factorization of T into an $r_1 \times r_2 \times r_3$ *core tensor* K and matrices U_1, U_2, U_3 of sizes $n_1 \times r_1$, $n_2 \times r_2$ and $n_3 \times r_3$ respectively.

Conjecture 6.2. The variety $\mathcal{K}_{3,n}^{(3)}$ has dimension $3n - 8$. Its points are tensors ST of multilinear rank $\leq (2, 4, 2)$. Here S is skew symmetric of rank 2, and the T_j are symmetric of rank ≤ 1, summing to 0, with zeros in the j-th row and column. The prime ideal of $\mathcal{K}_{3,n}^{(3)}$ is generated by polynomials of degree ≤ 3, namely the entries of $T_1 + T_2 + \cdots + T_n$, 4×4 Pfaffians of S, 2×2 minors of the T_j's, 3×3 minors of the flattening $(S, T_1, \ldots, T_n)$, and quadrics that are mixed Pfaffians.

We now present evidence for this conjecture. First and foremost, every $n \times (n+1) \times n$ tensor ST in the kinematic variety $\mathcal{K}_{3,n}^{(3)}$ has a Tucker decomposition whose core tensor K has format $2 \times 4 \times 2$. The four 2×2 slices of the core K are

$$K_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad K_3 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad K_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

The corresponding factor matrices of format $2 \times n$ and $4 \times (n+1)$ are

$$U_1 = U_3 = \big(|1\rangle, |2\rangle, \ldots, |n\rangle\big), \quad U_2 = \begin{bmatrix} 1 & 0 \\ 0 & \overline{U}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & -p_{13} & \cdots & -p_{n3} \\ 0 & p_{11}+p_{12} & \cdots & p_{n1}+p_{n2} \\ 0 & p_{11}-p_{12} & \cdots & p_{n1}-p_{n2} \end{bmatrix}.$$

The matrices whose columns are the spinors (20) can be written as follows:

$$U_1 = U_3 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \cdot \overline{U}_2 \cdot \begin{bmatrix} z_{11} & 0 & \cdots & 0 \\ 0 & z_{21} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & z_{n1} \end{bmatrix}.$$

From the Tucker decomposition we see that the $n \times n$ slices of ST are

$$S = U_1^T K_1 U_1 \quad \text{and} \quad T_j = U_1^T \big(-p_{j3}K_2 + (p_{j1}+p_{j2})K_3 + (p_{j1}-p_{j2})K_4\big)U_1. \quad (39)$$

The $2 \times n$ matrix U_1 is generic, as in the proof of Theorem 5.1. This implies that S is a generic skew symmetric $n \times n$ matrix of rank 2. The space of symmetric 2×2 matrices has the basis $\{K_2, K_3, K_4\}$. The parenthesized matrix in (39) has determinant $p_{j1}^2 - p_{j2}^2 - p_{j3}^2$, and hence T_j is a symmetric $n \times n$ matrix of rank ≤ 1.

More precisely, T_j is the outer product of the following vector with itself:

$$v_j = (p_{j1}+p_{j2})^{-\frac{1}{2}} \big[p_{j1}+p_{j2} \quad -p_{j1}+p_{j2} \big] U_1 = (p_{j1}+p_{j2})^{-\frac{1}{2}} u_j U_1.$$

Since u_j is in the kernel of the 2×2 matrix P_j, the j-th coordinate of v_j is zero. Therefore the j-th row and column of T_j are zero. In summary, we have shown that the $n \times (n+1) \times n$ tensor ST has all the properties stated in Conjecture 6.2.

We now return to arbitrary spacetime dimension $d \geq 3$. Our ultimate goal is to understand $\mathcal{K}_{d,n}^{(3)}$ for arbitrary d and n. A first step is to determine the dimension of this kinematic variety. For small cases, we computed this using the numerical software `HomotopyContinuation.jl` [1]. This computation is non-trivial because the momenta p_i must satisfy the constraints studied in Section 2.

Proposition 6.3. The dimensions of the kinematic varieties $\mathcal{K}_{d,n}^{(3)}$ are as follows:

$d\backslash n$	4	5	6	7	8	9	10	11	12
4	8	13	18	23	28	33	38	43	48
5	7	13	19	25	31	37	43	49	55
6	9	20	30	40	49	58	67	76	85
7	9	20	30	40	50	60	70	80	90
8	10	28	51	67	82	97	112	127	142
9	15	33	49	65	81	97	113	129	145

Proposition 6.3 suggests that $\dim \mathcal{K}_{d,n}^{(3)}$ is a linear function in n when d is fixed. We note that the slope is $5, 6, 9, 10, 15, 16$ for $d = 4, 5, 6, 7, 8, 9$. Following [14], we are especially interested in the case $d = 5$. Proposition 6.3 suggests that $\mathcal{K}_{5,n}^{(3)}$ has dimension $6n - 17$. We next characterize this variety for five particles.

Example 6.4. Fix $d = 5, k = 2, n = 5$. The variety $\mathcal{K}_{5,5}^{(3)} \subset \mathbb{P}^9 \times \mathbb{P}^{29}$ is irreducible of dimension 13 and degree 1761. Its class in $H^*(\mathbb{P}^9 \times \mathbb{P}^{29}, \mathbb{Z})$ is the multidegree

$$80s^8t^{17} + 265s^7t^{18} + 430s^6t^{19} + 450s^5t^{20} + 320s^4t^{21} + 155s^3t^{22} + 50s^2t^{23} + 10st^{24} + t^{25}.$$

Modulo the 10 linear forms in $T_1 + T_2 + T_3 + T_4 + T_5$, the ideal is generated by 25 quadrics, 15 cubics and 5 quartics. Each T_j is a skew symmetric 5×5 matrix with one zero row, so it contributes one Pfaffian $\langle ijk \rangle \langle \ell jm \rangle - \langle ij\ell \rangle \langle kjm \rangle + \langle ijm \rangle \langle kj\ell \rangle$. The other 20 quadrics are bilinear Pfaffians $\langle ij \rangle \langle ki\ell \rangle - \langle ik \rangle \langle ji\ell \rangle + \langle i\ell \rangle \langle jik \rangle$, e.g.

$$4 \times 4 \text{ Pfaffians of} \quad \begin{bmatrix} 0 & \langle 12 \rangle & \langle 13 \rangle & \langle 14 \rangle & \langle 15 \rangle \\ -\langle 12 \rangle & 0 & \langle 213 \rangle & \langle 214 \rangle & \langle 215 \rangle \\ -\langle 13 \rangle & -\langle 213 \rangle & 0 & \langle 314 \rangle & \langle 315 \rangle \\ -\langle 14 \rangle & -\langle 214 \rangle & -\langle 314 \rangle & 0 & \langle 415 \rangle \\ -\langle 15 \rangle & -\langle 215 \rangle & -\langle 315 \rangle & \langle -415 \rangle & 0 \end{bmatrix}. \quad (40)$$

Next, we have 15 cubics which ensure that the flattening $(S, T_1, T_2, T_3, T_4, T_5)$ has rank ≤ 4. One of them is $\langle 213 \rangle \langle 123 \rangle \langle 435 \rangle - \langle 213 \rangle \langle 325 \rangle \langle 134 \rangle + \langle 213 \rangle \langle 324 \rangle \langle 135 \rangle + \langle 314 \rangle \langle 123 \rangle \langle 235 \rangle - \langle 314 \rangle \langle 325 \rangle \langle 132 \rangle - \langle 315 \rangle \langle 123 \rangle \langle 234 \rangle + \langle 315 \rangle \langle 324 \rangle \langle 132 \rangle$. Finally, the 5 quartic generators come from the 4×4 minors of the following slices:

$$\begin{bmatrix} 0 & \langle 12 \rangle & \langle 13 \rangle & \langle 14 \rangle & \langle 15 \rangle \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \langle 123 \rangle & \langle 124 \rangle & \langle 125 \rangle \\ 0 & \langle 132 \rangle & 0 & \langle 134 \rangle & \langle 135 \rangle \\ 0 & \langle 142 \rangle & \langle 143 \rangle & 0 & \langle 145 \rangle \\ 0 & \langle 152 \rangle & \langle 153 \rangle & \langle 154 \rangle & 0 \end{bmatrix}, \begin{bmatrix} -\langle 12 \rangle & 0 & \langle 23 \rangle & \langle 24 \rangle & \langle 25 \rangle \\ 0 & 0 & \langle 213 \rangle & \langle 214 \rangle & \langle 215 \rangle \\ 0 & 0 & 0 & 0 & 0 \\ -\langle 132 \rangle & 0 & 0 & \langle 234 \rangle & \langle 235 \rangle \\ -\langle 142 \rangle & 0 & \langle 243 \rangle & 0 & \langle 245 \rangle \\ -\langle 152 \rangle & 0 & \langle 253 \rangle & \langle 254 \rangle & 0 \end{bmatrix}, \text{ etc} \ldots \quad (41)$$

We conclude this paper by summarizing what we know in general about the kinematic variety $\mathcal{K}_{5,n}^{(3)}$ for $n \geq 4$ massless particles in spacetime dimension $d = 5$. The points in $\mathcal{K}_{5,n}^{(3)}$ are $n \times (n+1) \times n$ tensors ST. The matrix S is skew symmetric of rank ≤ 4 and the slices T_j are skew symmetric of rank ≤ 2, summing to 0, and the j-th row and column of T_j is zero. We saw this in Theorem 4.5. From these constraints we see that the 6×6 Pfaffians of S and the 4×4 Pfaffians of T_j vanish.

Our next proposition explains the mixed relations we found in (40) and the cubics arising from the flattening $(S, T_1, T_2, T_3, T_4, T_5)$ of the $5 \times 6 \times 5$ tensor ST.

Proposition 6.5. For each index $j \in \{1, \ldots, n\}$, the skew symmetric $n \times n$ matrix

$$\big(|1\rangle, \ldots, |j-1\rangle, z_j, |j+1\rangle, \ldots, |n\rangle \big)^T \cdot C \cdot P_j \cdot \big(|1\rangle, \ldots, |j-1\rangle, z_j, |j+1\rangle, \ldots, |n\rangle \big) \quad (42)$$

has the spinor brackets $\langle ij \rangle$ and $\langle ijk \rangle$ for its non-zero entries. It has rank ≤ 2 on the kinematic variety $\mathcal{K}_{5,n}^{(3)}$, so the 4×4 Pfaffians give bilinear ideal generators. Furthermore, the $n \times (n^2 + n)$ matrix flattening $(S, T_1, \ldots, T_n)$ of the tensor ST has rank ≤ 4 on $\mathcal{K}_{5,n}^{(3)}$. It contributes 6×6 Pfaffians to the ideal generators.

Proof. The statement about (42) holds because the 4×4 matrix $C \cdot P_j$ is skew symmetric and its rank is 2. See (15) and Example 3.4. The bound on the rank of $(S, T_1, \ldots, T_n)$ follows from the matrix factorizations in (24) and (25). $\qquad\square$

We also consider the $n \times (n+1)$ slices of the tensor ST obtained by fixing the the first or last index. Numerical computations show that these matrices have rank ≤ 3 on $\mathcal{K}_{5,n}^{(3)}$. We also discovered that the multilinear rank of ST equals $(4, 6, 4)$. Our final question concerns the tensor rank of ST on the variety $\mathcal{K}_{5,n}^{(3)}$. We can prove that *the tensor rank of ST is at least* 5. We show this by evaluating the Strassen invariant [13, Example 9.21] on subtensors of format $3 \times 3 \times 3$. At present we do not have a conjecture for the tensor rank on $\mathcal{K}_{5,n}^{(3)}$ for general n.

We conclude by expressing the hope that our tensor varieties for arbitrary d will be of interest to physicists. Potential points of connection in the physics literature are the studies of spinor helicity for $d = 6$ in [4] and for $d = 10$ in [2]. The case $d = 10$ is especially relevant for $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. Perhaps some of the insights gained there can help in proving our conjectures?

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