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MULTIPLE SOLUTIONS FOR A DYNAMIC STURM-LIOUVILLE BOUNDARY VALUE PROBLEM ON TIME SCALES WITH IMPULSIVE EFFECTS

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In this study, we explore the existence of multiple solutions for a dynamic Sturm-Liouville boundary value problem on time scales that incorporate impulsive effects. Utilizing variational methods and applying certain critical point theorems from Ricceri for smooth functionals, we demonstrate the existence of at least three solutions to the problem. To illustrate the practical relevance of our findings, we provide an example at the end.

1. Introduction

Let $\mathbb{T}$ be a time scale, defined as a nonempty closed subset of $\mathbb{R}$. Notably, examples of time scales include $\mathbb{T} = \mathbb{R}$, which relates to differential equations, and $\mathbb{T} = \mathbb{Z}$, which pertains to difference equations. Assume a fixed value $T > 0$ with the condition that $0, T \in \mathbb{T}$. This paper aims to explore the existence of

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three solutions for the following problem:

$$\begin{cases} -(p(\varsigma)z^{\Delta}(\varsigma))^{\Delta}+q(\varsigma)z^{\rho}(\varsigma)=\lambda f(\varsigma,z^{\rho}(\varsigma))+\mu g(\varsigma,z^{\rho}(\varsigma)), \varsigma\neq\varsigma_j, \varsigma\in[0,T]_{\mathbb{T}},\\ \Delta(p(\varsigma_j)z^{\Delta}(\varsigma_j))=I_j(z(\varsigma_j)), \qquad j=1,\ldots,m,\\ \iota_1 z(0)-\iota_2 z^{\Delta}(0)=0, \qquad \iota_3 z(\rho^2(T))+\iota_4 z^{\Delta}(\rho(T))=0 \end{cases} \tag{P_{λ}^{f}}$$

where $p\in C^1([0,\rho(T)]_{\mathbb{T}},(0,+\infty))$, $q\in C([0,T]_{\mathbb{T}},[0,+\infty))$, $f,g\in C([0,T]_{\mathbb{T}}\times\mathbb{R},\mathbb{R})$, $\lambda>0$, $\mu\geq 0$, $\iota_i\geq 0$, for $i=1,2,3,4$ and $\iota_1+\iota_2\geq 0$, $\iota_3+\iota_4>0$, $\iota_1+\iota_3>0$. $0=\varsigma_0<\varsigma_1<\cdots<\varsigma_m<\varsigma_{m+1}=T$, $\Delta(p(\varsigma_j)z^{\Delta}(\varsigma_j))=p(\varsigma_j)(z^{\Delta}(\varsigma_j^{+})-z^{\Delta}(\varsigma_j^{-}))$, $z^{\Delta}(\varsigma_j^{+})=\lim_{h\to 0^+}z^{\Delta}(\varsigma_j+h)$ and $z^{\Delta}(\varsigma_j^{-})=\lim_{h\to 0^+}z^{\Delta}(\varsigma_j-h)$ denote the right and the left limits of $z^{\Delta}(\varsigma)$ at $\varsigma=\varsigma_j$, respectively, in the sense of the time scale, that is, in terms of $h>0$ for which $\varsigma_j+h,\varsigma_j-h\in[0,T]_{\mathbb{T}}$, whereas if ς_j is left-scattered, we interpret $z^{\Delta}(\varsigma_j^{-})=z^{\Delta}(\varsigma_j)$ and $z(\varsigma_j^{-})=z(\varsigma_j)$ and $I_j:\mathbb{R}\to\mathbb{R}$, $j=1,\ldots,m$ are continuous.

The theory of dynamic equations on time scales was proposed by Stefan Hilger in his PhD thesis in 1988 [23]. This approach has garnered significant interest from researchers because it unifies both continuous and discrete dynamic equations on a theoretical level, while also offering substantial practical applications. By leveraging this theory, researchers can investigate various fields, including insect population models, phytoremediation of metals, wound healing, and epidemic modeling [4, 10, 11, 33].

A time scale is defined as a nonempty closed subset of real numbers. In the realm of time scale calculus, various notations and theorems have been well established. To better understand the application of variational methods and critical point theory in dynamic equations on time scales, we briefly review relevant findings from the existing literature [1–3, 5, 7–9, 12, 14, 17, 19, 36, 37].

For instance, in [5], the authors employed variational methods to demonstrate the existence of at least one nontrivial solution for problem (P_{λ}^{f}) when $I_j=0$, $j=1,\ldots,m$. Additionally, [19] utilized variational methods to show the existence of infinitely many solutions for the perturbed problem (P_{λ}^{f}) under the same conditions. Furthermore, [17] discussed conditions that ensure the existence of at least three solutions for the perturbed problem (P_{λ}^{f}) when $I_j=0$, $j=1,\ldots,m$ employing critical point theory alongside variational methods.

Conversely, impulsive differential equations have gained significance in recent years, particularly in mathematical models that describe real processes and phenomena in fields such as physics, chemical technology, population dynamics, biotechnology, and economics. Recently, numerous researchers have focused on impulsive differential equations, employing variational methods, fixed-point theorems, and critical point theory. For further information, we direct the reader to [16, 22, 31] and the associated references.

Furthermore, impulsive and periodic boundary value problems on time scales have been extensively explored in the literature. Various methods have been employed to investigate periodic solutions of impulsive differential equations on time scales, including the methods of lower and upper solutions, fixed-point theory, and coincidence degree theory. However, the application of variational methods to study solutions for impulsive differential equations on time scales has received significantly less attention. To our knowledge, the variational method is an effective tool for addressing nonlinear problems on time scales that involve certain types of discontinuities, such as impulses (see [6, 13, 27, 39, 40] and their references).

For instance, in [13] the authors have studed the existence of weak solutions for second-order boundary value problem of impulsive dynamic equations on time scales by employing critical point theory. In [40] the authors have presented a approach via variational methods and critical point theory to obtain the existence of solutions for the nonautonomous second-order system on time scales with impulsive effects. In [27] the authors have proven some conditions for the existence of solutions to a nonlinear impulsive dynamic equation with homogeneous Dirichlet boundary conditions employing variational techniques and critical point theory.

Inspired by the aforementioned studies, this paper presents new criteria to ensure that the problem (P_λ^f) has at least three weak solutions for suitable values of the parameters λ and μ within specific real intervals. Notably, we only require g to be a continuous function, which allows us to apply variational methods effectively. Additionally, we derive multiplicity results for two scenarios: when the nonlinearity f is asymptotically quadratic, as well as when it is subquadratic as $|z| \to \infty$. Our approach leverages variational methods alongside a three critical points theorem established by Ricceri [28].

2. Preliminaries

Our main tool is a theorem due to Ricceri, who is recalled below in Lemma 2.1 and has been obtained in [28, Theorem 2]. Let X be a real Banach space, and as in [28], we denote by $\mathcal{W}_X$ the class of all functionals $\Theta : X \to \mathbb{R}$ possessing the following property: If $\{z_n\}$ is a sequence in X converging weakly to $z \in X$ with $\liminf_{n\to\infty} \Theta(z_n) \leq \Theta(z)$, then $\{z_n\}$ has a subsequence converging strongly to z. For example, if X is uniformly convex and $g : [0,\infty) \to \mathbb{R}$ is a continuous and strictly increasing function, then the functional $z \to g(\|z\|)$ belongs to the class $\mathcal{W}_X$.

Lemma 2.1. *Let X be a separable and reflexive real Banach space, let Θ : $X \to \mathbb{R}$ be a coercive, sequentially weakly lower semicontinuous C^1-functional,* *belonging to $\mathcal{W}_X$, bounded on each bounded subset of X and whose derivative admits a continuous inverse on X^*, and let $J : X \to \mathbb{R}$ be a C^1-functional with compact derivative. Assume that Θ has a strict local minimum z_0 with $\Theta(z_0) = J(z_0) = 0$. Finally, setting*

$$\rho = \max\left\{0, \limsup_{\|z\|\to\infty} \frac{J(z)}{\Theta(z)}, \limsup_{u\to z_0} \frac{J(z)}{\Theta(z)}\right\},$$

$$\chi = \sup_{z\in\Theta^{-1}((0,\infty))} \frac{J(z)}{\Theta(z)},$$

and assume that $\rho < \chi$. Then for each compact interval $[c,d] \subset (1/\chi, 1/\rho)$ (with the conventions that $1/0 = \infty$ and $1/\infty = 0$), there exists $R > 0$ with the following property: for each $\lambda \in [c,d]$ and every C^1-functional $\Psi : X \to \mathbb{R}$ with compact derivative, there exists $\gamma > 0$ such that for each $\mu \in [0,\gamma]$, the equation

$$\Theta'(z) = \lambda J'(z) + \mu\Psi'(z)$$

has at least three solutions in X whose norms are less than R.

We direct the reader to the papers [15, 20, 21, 24, 32, 34], where Lemma 2.1 was effectively utilized to demonstrate the existence of at least three solutions for boundary value problems.

The subsequent two results by Ricceri are sourced from [29, Theorem 1] and [30, Proposition 3.1], respectively.

Lemma 2.2. *Let X be a reflexive real Banach space, $I \subseteq \mathbb{R}$ an interval, let $\Theta : X \to \mathbb{R}$ be a sequentially weakly lower semicontinuous C^1 functional, bounded on each bounded subset of X, whose derivative admits a continuous inverse on X^*, $J : X \to \mathbb{R}$ functional with compact derivative. Assume that*

$$\lim_{\|z\|\to\infty} (\Theta(z) - \lambda J(z)) = \infty \quad \textit{for all } \lambda \in I,$$

and that there exists $\rho \in \mathbb{R}$ such that

$$\sup_{\lambda\in I} \inf_{z\in X} (\Theta(z) - \lambda(\rho - J(x))) < \inf_{z\in X} \sup_{\lambda\in I} (\Theta(z) - \lambda(\rho - J(z))).$$

Then there exists a nonempty open set $A \subseteq I$ and a positive number R with the following property: for each $\lambda \in A$ and every C^1 functional $J : X \to \mathbb{R}$ with compact derivative, there exists $\delta > 0$ such that, for each $\mu \in [0,\delta]$, the equation

$$\Theta'(z) - \lambda J'(z) - \mu\Psi'(z) = 0$$

has at least three solutions in X whose norms are less than R.

Lemma 2.3. *Let X be a nonempty set and Θ, J two real functions on X. Assume that there are $s>0$ and $z_0, z_1 \in X$ such that*

$$\Theta(z_0)=J(z_0)=0,\ \Theta(z_1)>s, \quad \sup_{z\in\Theta^{-1}(-\infty,s]} J(z) < s\frac{J(z_1)}{\Theta(z_1)}.$$

Then for each ρ satisfying

$$\sup_{z\in\Theta^{-1}(-\infty,s]} J(z) < \rho < s\frac{J(z_1)}{\Theta(z_1)},$$

one has

$$\sup_{\lambda\geq 0}\inf_{z\in X}(\Theta(z)-\lambda(\rho-J(z))) < \inf_{z\in X}\sup_{\lambda\geq 0}(\Theta(z)-\lambda(\rho-J(z))).$$

We direct the reader to the paper [32], where Lemma 2.2 was effectively utilized to guarantee the existence of at least three solutions for boundary value problems.

To establish suitable function spaces and apply critical point theory, we introduce the following notations and results, which will be instrumental in the proof of our main findings.

For $f\in L^1_\Delta([\varsigma_1,\varsigma_2)_\mathbb{T})$, we denote for convenience

$$\int_{\varsigma_1}^{\varsigma_2} f(c)\Delta c = \int_{[\varsigma_1,\varsigma_2)\cap\mathbb{T}} f(c)\Delta c.$$

To examine the problem (P^f_λ), we will use a variational framework within the space

$$H^1_\Delta([0,\rho^2(T)]_\mathbb{T}) = \left\{z:[0,\rho^2(T)]_\mathbb{T}\to\mathbb{R}:\ z\in AC[0,\rho^2(T)]_\mathbb{T} \text{ and } z^\Delta\in L^2_\Delta([0,\rho^2(T))_\mathbb{T})\right\}.$$

Then $H^1_\Delta([0,\rho^2(T)]_\mathbb{T})$ is a Hilbert space with the inner product,

$$(z,v)_{H^1_\Delta} = \int_0^{\rho^2(T)} z(\varsigma)v(\varsigma)\Delta\varsigma + \int_0^{\rho^2(T)} z^\Delta(\varsigma)v^\Delta(\varsigma)\Delta\varsigma$$

(see [38]), and let $||\cdot||_{H^1_\Delta}$ be the norm induced by the inner product $(\cdot,\cdot)_{H^1_\Delta}$. For every $z,v\in H^1_\Delta([0,\rho^2(T)]_\mathbb{T})$, we define

$$(z,v)_0 = \int_0^{\rho^2(T)} p(\varsigma)z^\Delta(\varsigma)v^\Delta(\varsigma)\Delta\varsigma + \int_0^{\rho(T)} q(\varsigma)z^\rho(\varsigma)v^\rho(\varsigma)\Delta\varsigma$$
$$+\upsilon_1 p(0)z(0)v(0) + \upsilon_2 p(\rho(T))z(\rho^2(T))v(\rho^2(T))$$

where

$$\upsilon_1 = \begin{cases} \dfrac{\iota_1}{\iota_2}, & \text{if } \iota_2 > 0,\\ 0, & \text{if } \iota_2 = 0 \end{cases}$$

and

$$\upsilon_2 = \begin{cases} \dfrac{\iota_3}{\iota_4}, & \text{if } \iota_4 > 0,\\ 0, & \text{if } \iota_4 = 0. \end{cases}$$

For every $z \in H^1_\Delta([0,\rho^2(T)]_{\mathbb{T}})$, we define

$$\|z\|_0 = \Bigg(\int_0^{\rho^2(T)} p(\varsigma)|z^\Delta(\varsigma)|^2\Delta\varsigma + \int_0^{\rho(T)} q(\varsigma)|z^\rho(\varsigma)|^2\Delta\varsigma + \upsilon_1 p(0)z^2(0) + \upsilon_2 p(\rho(T))z^2(\rho^2(T)) \Bigg)^{\frac{1}{2}}.$$

Lemma 2.4. *[37, Lemmas 2.1, 2.2 and 4.2] The immersion $H^1_\Delta([0,\rho^2(T)]_{\mathbb{T}}) \hookrightarrow C([0,\rho^2(T)]_{\mathbb{T}})$ is compact. If $z \in H^1_\Delta([0,\rho^2(T)]_{\mathbb{T}})$, then*

$$|z(\varsigma)| \le \sqrt{2}\max\{(\rho^2(T))^{\frac{1}{2}}, (\rho^2(T))^{-\frac{1}{2}}\}\|z\|_{H^1_\Delta} \textit{for each}\, \varsigma \in [0,\rho^2(T)]_{\mathbb{T}}.$$

If $\iota_2,\ \iota_4 > 0$ or $q(\varsigma) > 0$ for $\varsigma \in [0,T]_{\mathbb{T}}$, then for $z \in H^1_\Delta([0,\rho^2(T)]_{\mathbb{T}})$, $|z(\varsigma)| \le C\|z\|_0$ for each every $\varsigma \in [0,\rho^2(T)]_{\mathbb{T}}$ where $C = \min\{M_1, M_2, M_3\}$ and

$$M_1 = \sqrt{2}\max\left\{ \frac{1}{\sqrt{\upsilon_1 p(0)}}, \frac{\sqrt{\rho^2(T)}}{\min_{\varsigma\in[0,\rho(T)]_{\mathbb{T}}} p(\varsigma)} \right\},$$

$$M_2 = \sqrt{2}\max\left\{ \frac{1}{\sqrt{\upsilon_2 p(0)}}, \frac{\sqrt{\rho^2(T)}}{\min_{\varsigma\in[0,\rho(T)]_{\mathbb{T}}} p(\varsigma)} \right\},$$

$$M_3 = \sqrt{2}\max\left\{ \frac{\sqrt{\rho(T)}}{\min_{\varsigma\in[0,T]_{\mathbb{T}}} q(\varsigma)}, \frac{\sqrt{\rho^2(T)}}{\min_{\varsigma\in[0,\rho(T)]_{\mathbb{T}}} p(\varsigma)} \right\},$$

and where $\dfrac{1}{0} = +\infty$.

Put

$$F(\varsigma,\vartheta) = \int_0^\vartheta f(\varsigma,s)\mathrm{d}s \qquad \text{for all } (\varsigma,\vartheta) \in [0,T]_{\mathbb{T}} \times \mathbb{R}$$

and

$$G(\varsigma,\vartheta) = \int_0^\vartheta g(\varsigma,s)\mathrm{d}s \qquad \text{for all } (\varsigma,\vartheta) \in [0,T]_{\mathbb{T}} \times \mathbb{R}.$$

We will now present the following assumption regarding the impulsive terms:

($\mathcal{I}$) Assume that I_j for each $j = 1, \dots, m$ is a increasing function such that $I_j(0) = 0$ and $I_j(\vartheta)\vartheta \geq 0$ for each $\vartheta \in \mathbb{R}$.

We need the following lemma to aid in the proof of the main result.

Lemma 2.5. *Let $S : H^1_\Delta([0,\rho^2(T)]_\mathbb{T}) \to (H^1_\Delta([0,\rho^2(T)]_\mathbb{T}))^*$ be the operator defined by*

$$S(z)(v) = \int_0^{\rho^2(T)} p(\varsigma)z^\Delta(\varsigma)v^\Delta(\varsigma)\Delta\varsigma + \int_0^{\rho(T)} q(\varsigma)z^\rho(\varsigma)v^\rho(\varsigma)\Delta\varsigma$$
$$+\upsilon_1 p(0)z(0)v(0) + \upsilon_2 p(\rho(T))z(\rho^2(T))v(\rho^2(T)) + \sum_{j=1}^m I_j(z(\varsigma_j))v(\varsigma_j)$$

for each $z, v \in H^1_\Delta([0,\rho^2(T)]_\mathbb{T})$. Then S admits a continuous inverse on $(H^1_\Delta([0,\rho^2(T)]_\mathbb{T}))^*$.

Proof. By the similar way as in the proof of [18, Lemma 4] it is obvious that

$$S(u)(u) = \int_0^{\sigma^2(T)} p(t)|u^\Delta(t)|^2\Delta t + \int_0^{\sigma(T)} q(t)|u^\sigma(t)|^2\Delta t$$
$$+\beta_1 p(0)u^2(0) + \beta_2 p(\sigma(T))u^2(\sigma^2(T)) = \|u\|_0^2,$$

which follows that S is coercive. Owing to our assumptions on the data, one has

$$\langle S(u) - S(v), u - v\rangle = \|u_n - u\|_0^2 > 0$$

for every $u, v \in H^1_\Delta([0,\sigma^2(T)]_\mathbb{T})$, which means that S is strictly monotone. Moreover, since $H^1_\Delta([0,\sigma^2(T)]_\mathbb{T})$ is reflexive, for $u_n \longrightarrow u$ strongly in $H^1_\Delta([0,\sigma^2(T)]_\mathbb{T})$ as $n \to +\infty$, one has $S(u_n) \to S(u)$ weakly in $(H^1_\Delta([0,\sigma^2(T)]_\mathbb{T}))^*$ as $n \to \infty$. Hence, S is demicontinuous, so by [35, Theorem 26.A(d)], the inverse operator S^{-1} of S exists and it is continuous. Indeed, let e_n be a sequence of $(H^1_\Delta([0,\sigma^2(T)]_\mathbb{T}))^*$ such that $e_n \to e$ strongly in $(H^1_\Delta([0,\sigma^2(T)]_\mathbb{T}))^*$ as $n \to \infty$. Let u_n and u in $H^1_\Delta([0,\sigma^2(T)]_\mathbb{T})$ such that $S^{-1}(e_n) = u_n$ and $S^{-1}(e) = u$. Taking into account that S is coercive, one has that the sequence u_n is bounded in the reflexive space $H^1_\Delta([0,\sigma^2(T)]_\mathbb{T})$. For a suitable subsequence, we have $u_n \to \hat{u}$ weakly in $H^1_\Delta([0,\sigma^2(T)]_\mathbb{T})$ as $n \to \infty$, which concludes

$$\langle S(u_n) - S(u), u_n - \hat{u}\rangle = \langle e_n - e, u_n - \hat{u}\rangle = 0.$$

Note that if $u_n \to \hat{u}$ weakly in $H^1_\Delta([0,\sigma^2(T)]_\mathbb{T})$ as $n \to +\infty$ and $S(u_n) \to S(\hat{u})$ strongly in $(H^1_\Delta([0,\sigma^2(T)]_\mathbb{T}))^*$ as $n \to +\infty$, one has $u_n \to \hat{u}$ strongly in $H^1_\Delta([0,\sigma^2(T)]_\mathbb{T})$ as $n \to +\infty$, and since S is continuous, we have $u_n \to \hat{u}$ weakly in $H^1_\Delta([0,\sigma^2(T)]_\mathbb{T})$ as $n \to +\infty$ and $S(u_n) \to S(\hat{u}) = S(u)$ strongly in $(H^1_\Delta([0,\sigma^2(T)]_\mathbb{T}))^*$ as $n \to +\infty$. Hence, taking into account that S is an injection, we have $u = \hat{u}$. □

3. Main results

In this section, we present and prove our main results. Let

$$\lambda_1 = \inf_{z\in H^1_\Delta([0,\rho^2(T)]_{\mathbb{T}})\setminus\{0\}} \left\{ \frac{\|z\|_0^2 + \sum_{j=1}^m \int_0^{z(\varsigma_j)} I_j(\zeta)\mathrm{d}\zeta}{\int_0^{\rho(\varsigma)} F(\varsigma, z^\rho(\varsigma))\Delta\varsigma} : \int_0^{\rho(\varsigma)} F(\varsigma, z^\rho(\varsigma))\Delta\varsigma > 0 \right\}$$

and

$$\lambda_2 = \frac{1}{\max\{0, \lambda_0, \lambda_\infty\}},$$

where

$$\lambda_0 = \limsup_{z\to 0} \left(\frac{\int_0^{\rho(\varsigma)} F(\varsigma, z^\rho(\varsigma))\Delta\varsigma}{\|z\|_0^2 + \sum_{j=1}^m \int_0^{z(\varsigma_j)} I_j(\zeta)\mathrm{d}\zeta} \right)^{-1}$$

and

$$\lambda_\infty = \limsup_{\|z\|_0\to\infty} \left(\frac{\int_0^{\rho(\varsigma)} F(\varsigma, z^\rho(\varsigma))\Delta\varsigma}{\|z\|_0^2 + \sum_{j=1}^m \int_0^{z(\varsigma_j)} I_j(\zeta)\mathrm{d}\zeta} \right)^{-1}.$$

Theorem 3.1. *Assume that*

$(\mathcal{A}_1)$ *there exists a constant* $\varepsilon > 0$ *such that*

$$\max\left\{ \limsup_{z\to 0} \frac{\max_{\varsigma\in[0,T]_{\mathbb{T}}} F(\varsigma, z)}{|z|^2},\ \limsup_{|z|\to\infty} \frac{\max_{\varsigma\in[0,T]_{\mathbb{T}}} F(\varsigma, z)}{|z|^2} \right\} < \varepsilon,$$

$(\mathcal{A}_2)$ *there exists a function* $w \in H^1_\Delta([0,\rho^2(T)]_{\mathbb{T}})$ *such that*

$$\|w\|_0^2 + \sum_{j=1}^m \int_0^{w(\varsigma_j)} I_j(\zeta)\mathrm{d}\zeta \neq 0$$

and

$$2C^2\rho(\varsigma)\varepsilon < \frac{\int_0^{\rho(\varsigma)} F(\varsigma, w^\rho(\varsigma))\Delta\varsigma}{\|w\|_0^2 + \sum_{j=1}^m \int_0^{w(\varsigma_j)} I_j(\zeta)\mathrm{d}\zeta}.$$

Then for each compact interval $[c,d] \subset (\lambda_1, \lambda_2)$*, there exists* $R > 0$ *such that for each* $\lambda \in [c,d]$ *and every continuous function* $g : [0,T]_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$*, there exists* $\gamma > 0$ *such that for each* $\mu \in [0,\gamma]$*, the problem* (P_λ^f) *has at least three weak solutions whose norms in* $H_\Delta^1([0,\rho^2(T)]_{\mathbb{T}})$ *are less than R.*

Remark 3.2. Under conditions $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$, it is true that $\lambda_1 < \lambda_2$ as shown in the proof of Theorem 3.1 given below.

Proof. We aim to apply Lemma 2.1 to the problem (P_λ^f). Take $X = H_\Delta^1([0,\rho^2(T)]_{\mathbb{T}})$ and let the functionals Θ, Ψ and J for each $z \in X$, defined by

$$\Theta(z) = \frac{1}{2}\|z\|_0^2 + \sum_{j=1}^{m} \int_0^{z(\varsigma_j)} I_j(\zeta)\mathrm{d}\zeta, \tag{3.1}$$

$$\Psi(z) = \int_0^{\rho(T)} G(\varsigma, z^\rho(\varsigma))\Delta\varsigma$$

and

$$J(z) = \int_0^{\rho(T)} F(\varsigma, z^\rho(\varsigma))\Delta\varsigma.$$

Let us prove that the functionals Θ, Ψ and J satisfy the required conditions in Lemma 2.1. Standard arguments show that $\Theta - \lambda J - \mu\Psi$ is a Gâteaux differentiable functional whose Gâteaux derivative at

$$\begin{aligned}(\Theta' - \lambda J' - \mu\Psi')(z)(v) &= \int_0^{\rho^2(T)} p(\varsigma)z^\Delta(\varsigma)v^\Delta(\varsigma)\Delta\varsigma + \int_0^{\rho(T)} q(\varsigma)z^\rho(\varsigma)v^\rho(\varsigma)\Delta\varsigma \\ &+ \upsilon_1 p(0)z(0)v(0) + \upsilon_2 p(\rho(T))z(\rho^2(T))v(\rho^2(T)) + \sum_{j=1}^{m} I_j(z(\varsigma_j))v(\varsigma_j) \\ &- \lambda\int_0^{\rho(T)} f(\varsigma, z^\rho(\varsigma))v^\rho(\varsigma)\Delta\varsigma - \mu\int_0^{\rho(T)} g(\varsigma, z^\rho(\varsigma))v^\rho(\varsigma)\Delta\varsigma\end{aligned}$$

for each $z, v \in X$. Recalling (3.1), we have

$$\Theta(z) \geq \frac{1}{2}\|z\|_0^2$$

for all $z \in X$. Then, we have

$$\lim_{\|z\|_0 \to +\infty} \Theta(z) = \infty,$$

i.e., Θ is coercive, while Lemma 2.5 gives that Θ' admits a continuous inverse on X^*. Now, let A be a bounded subset of X. Then there exist constants $s > 0$, such that $\|z\|_0 \leq s$ for each $z \in A$. So, $\max_{\varsigma \in [0,T]_{\mathbb{T}}} |z(\varsigma)| \leq Cs$ for all

$z \in A$. Thus, by the continuity of I_j, we see that there exists $K > 0$ such that $\left|\sum_{j=1}^{m}\int_0^{z(\varsigma_j)} I_j(\zeta)\mathrm{d}\zeta\right| < K$. Then, by (3.1), we have

$$\Theta(z) \leq \frac{1}{2}\|s\|_0^2 + K.$$

Hence, Θ is bounded on each bounded subset of X. We now prove $\Theta \in \mathcal{W}_X$. Let the sequence $\{z_k\} \subset X$, $z \subset X$, $z_k \rightharpoonup z$ and $\liminf_{k\to\infty}\Theta(z_k) \leq \Theta(z)$. Below, we show that $\{z_k\}$ has a subsequence converging strongly to z. Assume, to the contrary, that $\{z_k\}$ does not have a subsequence converging strongly to z. Then, there exist $\varepsilon > 0$ and a subsequence of z_k, still denoted by itself, such that

$$\left\|\frac{z_k - z}{2}\right\|_{H^1_\Delta} \geq \varepsilon \quad \text{for all } k \in \mathbb{N}.$$

Note that $\{z_k\}$ converges uniformly to z by [26, Proposition 1.2]. Then, in view of the definition of $\|\cdot\|_0$, there exists $\varepsilon_1 > 0$ such that

$$\left\|\frac{z_k - z}{2}\right\|_0 \geq \varepsilon_1 \quad \text{for all } k \in \mathbb{N}.$$

Thus, one has

$$\Theta\left(\frac{z_k - z}{2}\right) \geq \tilde{\varepsilon_1} \quad \text{for all } k \in \mathbb{N}. \tag{3.2}$$

Since $\{z_k\}$ converges uniformly to z and I_j, $j = 1\ldots,m$, are continuous, we reach that

$$\lim_{k\to\infty}\Theta(z_k) = \Theta(z). \tag{3.3}$$

It is obvious that Θ is convex, continuous, strictly increasing and $\Theta(0) = 0$. Moreover, Θ is sequentially weakly lower semicontinuous and that $(z_k + z)/2 \rightharpoonup z$. Then, we obtain that

$$\Theta(z) \leq \liminf_{k\to\infty}\Theta\left(\frac{z_k + z}{2}\right). \tag{3.4}$$

Thus, from [25, Theorem 2.1], we have

$$\frac{1}{2}\Theta(z_k) + \frac{1}{2}\Theta(z) \geq \Theta\left(\frac{z_k + z}{2}\right) + \Theta\left(\frac{z_k - z}{2}\right) \quad \text{for all } k \in \mathbb{N}.$$

Taking limit superior as $k \to \infty$ and using (3.2) and (3.3) in the above inequality, we get that

$$\Theta(z) - \varepsilon_1 \geq \limsup_{k\to\infty}\Theta\left(\frac{z_k + z}{2}\right),$$

which contradicts (3.4). This shows that $\{z_k\}$ has a subsequence converging strongly to z. Therefore, $\Theta \in \mathcal{W}_X$. The functionals Θ and J are two C^1-functionals with compact derivatives. Moreover, Θ has a strict local minimum 0 with $\Theta(0) = J(0) = 0$. Therefore, the regularity assumptions on Θ and J, as requested in Lemma 2.1, are verified. In view of $(\mathcal{A}_1)$, there exist τ_1, τ_2 with $0 < \tau_1 < \tau_2$ such that

$$F(\varsigma, z) \le \varepsilon |z|^2 \tag{3.5}$$

for each $\varsigma \in [0,T]_{\mathbb{T}}$ and every z with $|z| \in [0, \tau_1) \cup (\tau_2, \infty)$. Since $F(\varsigma, u)$ is continuous on $[0,T]_{\mathbb{T}} \times \mathbb{R}$, $F(\varsigma, z)$ is bounded on $[0,T]_{\mathbb{T}} \times [\tau_1, \tau_2]$. Thus we can choose $b > 0$ and $q > 2$ such that

$$F(\varsigma, z) \le \varepsilon |z|^2 + b|z|^q$$

for all $(\varsigma, z) \in [0,T]_{\mathbb{T}} \times \mathbb{R}$. Then, we have

$$J(z) \le C^2 \rho(\varsigma) \varepsilon \|z\|_0^2 + bC^q \rho(\varsigma) \|z\|_0^q \tag{3.6}$$

for all $z \in X$. Hence, from (3.1) and (3.6), we have

$$\limsup_{|z| \to 0} \frac{J(z)}{\Theta(z)} \le 2C^2 \rho(\varsigma) \varepsilon. \tag{3.7}$$

Moreover, by (3.5), for each $z \in X \setminus \{0\}$, we obtain that

$$\begin{aligned}
\frac{J(z)}{\Theta(z)} &= \frac{\int_{|z| \le \tau_2} F(\varsigma, z^{\rho}(\varsigma)) \Delta\varsigma}{\Theta(z)} + \frac{\int_{|z| > \tau_2} F(\varsigma, z^{\rho}(\varsigma)) \Delta\varsigma}{\Theta(z)} \\
&\le \frac{\rho(\varsigma) \sup_{\varsigma \in [0,T], |z| \in [0,\tau_2]} F(\varsigma, z)}{\Theta(z)} + \frac{C^2 \rho(\varsigma) \varepsilon \|z\|_0^2}{\Theta(z)} \\
&\le \frac{\rho(\varsigma) \sup_{\varsigma \in [0,T], |z| \in [0,\tau_2]} F(\varsigma, z)}{\frac{1}{2}\|z\|_0^2} + 2C^2 \rho(\varsigma) \varepsilon.
\end{aligned}$$

So, we get that

$$\limsup_{\|z\|_0 \to \infty} \frac{J(z)}{\Theta(z)} \le 2C^2 \rho(\varsigma) \varepsilon. \tag{3.8}$$

In view of (3.7) and (3.8), we have

$$\rho = \max \left\{ 0, \limsup_{\|z\|_0 \to \infty} \frac{J(z)}{\Theta(z)}, \limsup_{z \to 0} \frac{J(z)}{\Theta(z)} \right\} \le 2C^2 \rho(\varsigma) \varepsilon. \tag{3.9}$$

The assumption $(\mathcal{A}_2)$ combined with (3.9) results in

$$\chi = \sup_{z \in \Theta^{-1}(0,\infty)} \frac{J(z)}{\Theta(z)} = \sup_{X \setminus \{0\}} \frac{J(z)}{\Theta(z)}$$

$$\geq \frac{\int_0^{\rho(\varsigma)} F(\varsigma, w^\rho(\varsigma))\Delta\varsigma}{\Theta(w(\varsigma))} = \frac{\int_0^{\rho(\varsigma)} F(\varsigma, w^\rho(\varsigma))\Delta\varsigma}{\frac{1}{2}\|z\|_0^2 + \sum_{j=1}^{m}\int_0^{w(\varsigma_j)} I_j(\zeta)d\zeta}$$

$$> 2C^2\rho(\varsigma)\varepsilon \geq \rho.$$

Therefore, all the conditions of Lemma 2.1 are fulfilled. Clearly, $\lambda_1 = 1/\chi$ and $\lambda_2 = 1/\rho$. Then, by Lemmas 2.1, for each compact interval $[c,d] \subset (\lambda_1, \lambda_2)$, there exists $R > 0$ such that for every $\lambda \in [c,d]$ and every continuous function $g : [0,T]_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$, there exists $\gamma > 0$ such that for each $\mu \in [0,\gamma]$, the problem (P_λ^f) has at least three weak solutions whose norms in X are less than R. □

Another application of Lemma 2.1 can be stated as follows.

Theorem 3.3. *Assume that*

$$\max_{z\in H_\Delta^1([0,\rho^2(T)]_{\mathbb{T}})}\left\{\limsup_{z\to 0}\frac{\max_{\varsigma\in[0,T]_{\mathbb{T}}}F(\varsigma,z)}{|z|^2}, \limsup_{|z|\to\infty}\frac{\max_{\varsigma\in[0,T]_{\mathbb{T}}}F(\varsigma,z)}{|z|^2}\right\} \leq 0 \tag{3.10}$$

and

$$\sup_{z\in H_\Delta^1([0,\rho^2(T)]_{\mathbb{T}})}\frac{\int_0^{\rho(\varsigma)} F(\varsigma, z^\rho(\varsigma))\Delta\varsigma}{\frac{1}{2}\|z\|_0^2 + \sum_{j=1}^{m}\int_0^{z(\varsigma_j)} I_j(\zeta)d\zeta} > 0. \tag{3.11}$$

Then for each compact interval $[c,d] \subset (\lambda_1, \infty)$, there exists $R > 0$ such that for each $\lambda \in [c,d]$ and every continuous function $g : [0,T]_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$, there exists $\gamma > 0$ such that for each $\mu \in [0,\gamma]$, the problem (P_λ^f) has at least three weak solutions whose norms in $H_\Delta^1([0,\rho^2(T)]_{\mathbb{T}})$ are less than R.

Proof. For any $\varepsilon > 0$, (3.10) implies that there exist τ_1 and τ_2 with $0 < \tau_1 < \tau_2$ such that

$$F(\varsigma, z) \leq \varepsilon|z|^2$$

for each $\varsigma \in [0,T]_{\mathbb{T}}$ and every z with $|z| \in [0,\tau_1) \cup (\tau_2,\infty)$. Since $F(\varsigma,u)$ is continuous on $[0,T]_{\mathbb{T}} \times \mathbb{R}$, $F(\varsigma,z)$ is bounded on $[0,T] \times [\tau_1,\tau_2]$. Thus, we can choose $\eta > 0$ and $\iota > 2$ so that

$$F(\varsigma, z) \leq \varepsilon|z|^2 + \eta|z|^\iota$$

for all $(\varsigma, z) \in [0,T]_{\mathbb{T}} \times \mathbb{R}$. Then, by following the same process as in the proof of Theorem 3.1, we obtain the results stated in equations (3.7) and (3.8). Since

ε is arbitrary, (3.7) and (3.8) give

$$\max\left\{0,\ \limsup_{\|z\|_0\to+\infty}\frac{J(z)}{\Theta(z)},\ \limsup_{z\to 0}\frac{J(z)}{\Theta(z)}\right\}\le 0.$$

Then, with ρ and χ defined in Lemma 2.1, we have $\rho = 0$. By (3.11), we have $\chi > 0$. In this case, clearly $\lambda_1 = 1/\chi$ and $\lambda_2 = \infty$. Thus, by Lemma 2.1 the result is achieved. □

Remark 3.4. In Assumption $(\mathcal{A}_2)$ of Theorem 3.1, if we choose $w(\varsigma) = \eta > 0$ for each $\varsigma \in [0,T]_{\mathbb{T}}$. Clearly, $w \in H^1_\Delta([0,\rho^2(T)]_{\mathbb{T}})$ and $(\mathcal{A}_2)$ now takes the following form:

$(\mathcal{A}_3)$ there exists a positive constant d such that

$$K_\eta + \sum_{j=1}^{m}\int_0^{w(\varsigma_j)} I_j(\zeta)\mathrm{d}\zeta \neq 0$$

where

$$K_\eta = \frac{\eta^2}{2}\left(\int_0^{\rho(T)} q(\varsigma)\Delta\varsigma + \upsilon_1 p(0) + \upsilon_2 p(\rho(T))\right)$$

and

$$2C^2\rho(\varsigma)\varepsilon < \frac{\displaystyle\int_0^{\rho(\varsigma)} F(\varsigma, w^\rho(\varsigma))\Delta\varsigma}{\displaystyle K_\eta + \sum_{j=1}^{m}\int_0^{w(\varsigma_j)} I_j(\zeta)\mathrm{d}\zeta}.$$

Now, we point out some results in which the function f is separable. To be precise, we consider the problem

$$\begin{cases} -(p(\varsigma)z^\Delta(\varsigma))^\Delta + q(\varsigma)z^\rho(\varsigma) = \lambda\theta(\varsigma)f(z^\rho(\varsigma)) + \mu g(\varsigma, z^\rho(\varsigma)), \varsigma \neq \varsigma_j, \varsigma \in [0,T]_{\mathbb{T}},\\ \Delta(p(\varsigma_j)z^\Delta(\varsigma_j)) = I_j(z(\varsigma_j)), \qquad j = 1,\ldots,m,\\ \iota_1 z(0) - \iota_2 z^\Delta(0) = 0, \qquad \iota_3 z(\rho^2(T)) + \iota_4 z^\Delta(\rho(T)) = 0 \end{cases} \tag{$\phi^\theta_{\lambda,\mu}$}$$

where $\theta : [0,T]_{\mathbb{T}} \to \mathbb{R}$ is a nonzero function such that $\theta \in L^1([0,T]_{\mathbb{T}})$ and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and $g : [0,T]_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$ is as introduced for the problem (P^f_λ) in the Introduction.

Let $F(\varsigma, c) = \theta(\varsigma)F(c)$ for each $(\varsigma, c) \in [0,T]_{\mathbb{T}} \times \mathbb{R}$, where

$$F(c) = \int_0^c f(\vartheta)\mathrm{d}\vartheta \quad \text{for all } c \in \mathbb{R}.$$

The following existence results are consequences of Theorem 3.1.

Theorem 3.5. *Assume that*

$(\mathcal{A}_4)$ *there exists a constant* $\varepsilon > 0$ *such that*

$$\sup_{\varsigma\in[0,T]_{\mathbb{T}}} \theta(\varsigma)\cdot\max\left\{\limsup_{z\to 0}\frac{F(z)}{|z|^2}, \limsup_{|z|\to\infty}\frac{F(z)}{|z|^2}\right\} < \varepsilon,$$

$(\mathcal{A}_5)$ *there exists a positive constant* η *such that*

$$K_\eta + \sum_{j=1}^{m}\int_0^{w(\varsigma_j)} I_j(\zeta)\mathrm{d}\zeta \neq 0$$

and

$$2C^2\rho(\varsigma)\varepsilon < \frac{f(w(\varsigma))\int_0^{\rho(\varsigma)}\theta(\varsigma)\Delta\varsigma}{K_\eta + \sum_{j=1}^{m}\int_0^{w(\varsigma_j)} I_j(\zeta)\mathrm{d}\zeta}$$

where $w(\varsigma) = \eta > 0$ *for each* $\varsigma \in [0,T]_{\mathbb{T}}$.

Then for each compact interval $[c,d] \subset (\lambda_3,\lambda_4)$*, where* λ_3 *and* λ_4 *are the same as* λ_1 *and* λ_2*, but* $\int_0^{\rho(\varsigma)} F(\varsigma,z^\rho(\varsigma))\Delta\varsigma$ *is replaced by* $\int_0^{\rho(\varsigma)}\theta(\varsigma)F(z^\rho(\varsigma))\Delta\varsigma$*, respectively, there exists* $R > 0$ *such that for each* $\lambda \in [c,d]$ *and every continuous function* $g : [0,T]_{\mathbb{T}}\times\mathbb{R}\to\mathbb{R}$*, there exists* $\gamma > 0$ *such that for each* $\mu \in [0,\gamma]$*, the problem* $(\phi^\theta_{\lambda,\mu})$ *has at least three weak solutions whose norms in* $H^1_\Delta([0,\rho^2(T)]_{\mathbb{T}})$ *are less than R.*

Theorem 3.6. *Assume that there exists a positive constant* η *such that*

$$K_\eta + \sum_{j=1}^{m}\int_0^{w(\varsigma_j)} I_j(\zeta)\mathrm{d}\zeta > 0$$

and

$$\int_0^{\rho(\varsigma)}\theta(\varsigma)F(w^\rho(\varsigma))\Delta\varsigma > 0 \tag{3.12}$$

where $w(\varsigma) = \eta > 0$ *for each* $\varsigma \in [0,T]_{\mathbb{T}}$*. Moreover, suppose that*

$$\limsup_{z\to 0}\frac{F(z)}{|z|^2} = \limsup_{|z|\to\infty}\frac{F(z)}{|z|^2} = 0. \tag{3.13}$$

Then for each compact interval $[c,d] \subset (\lambda_3,\infty)$*, where* λ_3 *is the same as* λ_1 *but* $\int_0^{\rho(\varsigma)} F(\varsigma,z^\rho(\varsigma))\Delta\varsigma$ *is replaced by* $\int_0^{\rho(\varsigma)}\theta(\varsigma)F(z^\rho(\varsigma))\Delta\varsigma$*, there exists* $R > 0$

such that for each $\lambda \in [c,d]$ and every continuous function $g : [0,T]_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$, there exists $\gamma > 0$ such that for each $\mu \in [0,\gamma]$, the problem $(\phi^{\theta}_{\lambda,\mu})$ has at least three classical solutions whose norms in $H^1_\Delta([0,\rho^2(T)]_{\mathbb{T}})$ are less than R.

Proof. From (3.13), we can easily observe that the assumption $(\mathcal{A}_4)$ is satisfied for each $\varepsilon > 0$. Furthermore, using (3.12) and by selecting $\varepsilon > 0$ small enough, we can derive the assumption $(\mathcal{A}_5)$. Therefore, the conclusion follows from Theorem 3.5. □

Now, we present an example in which the hypotheses of Theorem 3.6 are satisfied.

Example 3.7. Let the time scale be given by

$$\mathbb{T} = \left\{ \frac{4}{n} : n = 1,2,\ldots \right\} \cup \{0\}$$

and $T = 1$. The elements of $\mathbb{T}$ are

$$\mathbb{T} = \left\{ 4, 2, \frac{4}{3}, 1, \frac{4}{5}, \frac{4}{6}, \ldots \right\} \cup \{0\}$$

At $\varsigma = 1$, the next point in $\mathbb{T}$ is $\frac{4}{5}$, since $\frac{4}{5} < 1$ and is the largest point less than 1 in $\mathbb{T}$. Thus, the forward shift operator gives:

$$\rho(1) = \frac{4}{3} \text{ and } \rho^2(1) = 2.$$

Consider the problem

$$\begin{cases} -z^{\Delta\Delta}(\varsigma) + z^{\rho}(\varsigma) = \lambda\theta(\varsigma) f(z^{\rho}(\varsigma)), & \varsigma \neq \varsigma_1, \qquad \varsigma \in [0,1]_{\mathbb{T}}, \\ \Delta(z^{\Delta}(\varsigma_1)) = I_1(z(\varsigma_1)), & \\ z(0) - 2z^{\Delta}(0) = 0, \qquad z^{\Delta}(1) = 0 & \end{cases} \tag{3.14}$$

where $\theta(\varsigma) = 1$ for each $\varsigma \in [0,1]_{\mathbb{T}}$, $\varsigma_1 = 1/5$, $I_1(\vartheta) = \vartheta^3$ for each $\vartheta \in \mathbb{R}$, and

$$f(\zeta) = \begin{cases} 8\zeta^3, & \zeta \leq 1, \\ 8\zeta, & 1 < \zeta \leq 2, \\ 16, & \zeta \geq 2. \end{cases}$$

It can then be easily verified that

$$F(\zeta) = \begin{cases} 2\zeta^4, & \zeta \leq 1, \\ 4\zeta^2 - 2, & 1 < \zeta \leq 2, \\ 16\zeta - 18, & \zeta > 2. \end{cases}$$

By simple calculations, we obtain

$$\upsilon_1 = \frac{1}{2} \text{ and } \upsilon_2 = 0,$$

$$M_1 = \sqrt{2}\max\left\{\frac{1}{\sqrt{\upsilon_1 p(0)}}, \frac{\sqrt{\rho^2(T)}}{\min_{\varsigma\in[0,\rho(T)]_{\mathbb{T}}} p(\varsigma)}\right\} = \sqrt{2}\max\left\{\frac{1}{\sqrt{\frac{1}{2}}}, \frac{\sqrt{2}}{1}\right\},$$

$$M_2 = \sqrt{2}\max\left\{\frac{1}{\sqrt{\upsilon_2 p(0)}}, \frac{\sqrt{\rho^2(T)}}{\min_{\varsigma\in[0,\rho(T)]_{\mathbb{T}}} p(\varsigma)}\right\} = \sqrt{2}\max\left\{\frac{1}{0}, \frac{\sqrt{2}}{1}\right\},$$

$$M_3 = \sqrt{2}\max\left\{\frac{\sqrt{\rho(T)}}{\min_{\varsigma\in[0,T]_{\mathbb{T}}} q(\varsigma)}, \frac{\sqrt{\rho^2(T)}}{\min_{\varsigma\in[0,\rho(T)]_{\mathbb{T}}} p(\varsigma)}\right\} = \sqrt{2}\max\left\{\frac{\sqrt{\frac{4}{3}}}{0}, \frac{\sqrt{2}}{1}\right\}$$

and $C = \min\{M_1, M_2, M_3\} = 2$. By choosing $\eta = 1$, $w(\varsigma)$ has the form $w(\varsigma) = 1$. It is trivial to verify that

$$K_\eta + \sum_{j=1}^{m}\int_0^{w(\varsigma_j)} I_j(\zeta)\mathrm{d}\zeta = K_1 + \int_0^{w(\frac{1}{5})} I_j(\zeta)\mathrm{d}\zeta > 0,$$

$$\int_0^{\rho(\varsigma)} \theta(\varsigma)F(w^\rho(\varsigma))\Delta\varsigma = \int_0^{\rho(\varsigma)} F(1)\Delta\varsigma > 0,$$

and

$$\lim_{z\to 0}\frac{F(z)}{|z|^2} = \lim_{|z|\to\infty}\frac{F(z)}{|z|^2} = 0.$$

Hence, by Theorem 3.6, for each compact interval $[c,d] \subset (0,\infty)$, there exists $R > 0$ such that for each $\lambda \in [c,d]$ and every continuous function $g : [0,1]_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$, there exists $\gamma > 0$ such that for each $\mu \in [0,\gamma]$, the problem (3.14) has at least three weak solutions whose norms in $H^1_\Delta([0,\rho^2(T)]_{\mathbb{T}})$ are less than R.

The theorem presented below is a result of Lemma 2.3.

Theorem 3.8. *Assume that there exist three positive constants $1 \le \zeta < 2$, θ, and η with*

$$\theta < C\sqrt{2K_\eta} \tag{3.15}$$

such that

$(\mathcal{B}_1)$ $f(\varsigma,c) \ge 0$ *for each* $(\varsigma,c) \in [0,T]_{\mathbb{T}} \times \mathbb{R}$,

$(\mathcal{B}_2)$

$$\frac{\int_0^{\rho(\varsigma)} \max_{|z|\le\theta} F(\varsigma,z)\Delta\varsigma}{\theta^2} < \frac{1}{2C^2}\frac{\int_0^{\rho(\varsigma)} F(\varsigma,\rho)\Delta\varsigma}{K_\eta + \sum_{j=1}^{m}\int_0^{w(\varsigma_j)} I_j(\zeta)\mathrm{d}\zeta},$$

$(\mathcal{B}_3)$ *there exists* $p>0$ *and a positive constant* q *such that*

$$|F(\varsigma,z)| \le p|z|^\zeta + q \quad \textit{for all } (\varsigma,z)\in[0,T]_{\mathbb{T}}\times\mathbb{R},$$

$(\mathcal{B}_4)$ *there exists* $l>0$ *and a function* $p\in\mathbb{R}$ *such that*

$$G(\varsigma,z)\le lz^\zeta + p \quad \textit{for all } (\varsigma,z)\in[0,T]_{\mathbb{T}}\times\mathbb{R}.$$

Then there exist a nonempty open set $A\subset[0,\infty)$ *and a positive number* $R>0$ *such that for each* $\lambda\in A$ *and every continuous function* $g:[0,T]_{\mathbb{T}}\times\mathbb{R}\to\mathbb{R}$, *there exists* $\delta>0$ *such that for each* $\mu\in[0,\delta]$, *the problem* (P_λ^f) *has at least three weak solutions whose norms in* $H_\Delta^1([0,\rho^2(T)]_{\mathbb{T}})$ *are less than R.*

Proof. For any $\lambda\ge0$, $z\in H_\Delta^1([0,\rho^2(T)]_{\mathbb{T}})$, by (Q_3), $(\mathcal{B}_3)$, and $(\mathcal{B}_4)$, we have

$$\begin{aligned}\Theta(z)-\lambda J(z) \ge& \frac{1}{2}\|z\|_0^2 - \lambda\int_0^{\rho(\varsigma)}\left(F(\varsigma,z^\rho(\varsigma)) + \frac{\mu}{\lambda}G(\varsigma,z^\rho(\varsigma))\right)\Delta\varsigma\\ \ge& \frac{1}{2}\|z\|_0^2 - \lambda\left(\int_0^{\rho(\varsigma)}(p|z^\rho(\varsigma)|^\zeta+q)\Delta\varsigma\right) - \mu\left(l\int_0^{\rho(\varsigma)}(|z^\rho(\varsigma)|^\zeta+p)\Delta\varsigma\right)\\ \ge& \frac{1}{2}\|z\|_0^2 - \lambda pC^\zeta\rho(\varsigma)\|z\|_0^\zeta - \mu lC^\zeta\rho(\varsigma)\|z\|_0^\zeta - \lambda\rho(\varsigma)q - \mu\rho(\varsigma)p.\end{aligned}$$

Since $\zeta<2$, one has

$$\lim_{\|z\|_0\to+\infty}\Theta(z)-\lambda J(z)=\infty \quad \text{for all } \lambda>0.$$

Let $w(\varsigma)=\eta>0$ for each $\varsigma\in[0,T]_{\mathbb{T}}$ with χ given in the condition. We have

$$J(w)=\int_0^{\rho(T)}F(\varsigma,w^\rho(\varsigma))\Delta\varsigma=\int_0^{\rho(T)}F(\varsigma,\eta)\Delta\varsigma.$$

Moreover, by simple calculations, we see that $\Theta(w)=K_\eta$. Let $s=\frac{\theta^2}{2C^2}$. Then, from (3.15), we have $\Theta(w)>s$. From the definition of Θ, it can be concluded that

$$\Theta^{-1}(-\infty,s]\subseteq\left\{z\in X:\ \|z\|_0\le\sqrt{2s}\right\}$$

$$\subseteq \left\{ z \in X : |z(\varsigma)| \leq C\sqrt{2s} \text{ for all } \varsigma \in [0, \rho^2(T)]_{\mathbb{T}} \right\}$$
$$= \left\{ z \in X : |z(\varsigma)| \leq \theta \text{ for all } \varsigma \in [0, \rho^2(T)]_{\mathbb{T}} \right\}.$$

Therefore,

$$\sup_{z \in \Theta^{-1}((-\infty, s])} J(z) \leq \int_0^{\rho(T)} \max_{|\vartheta| \leq \theta} F(\varsigma, \vartheta) \Delta\varsigma.$$

Thus, from the assumption $(\mathcal{B}_2)$, we have

$$\begin{aligned} s\frac{J(w)}{\Theta(w)} &= \frac{s}{\Theta(w)} \left(\int_0^{\rho(\varsigma)} F(\varsigma, w^{\rho}(\varsigma)) \Delta\varsigma \right) \\ &\geq \frac{\dfrac{\theta^2}{2C^2} \left(\displaystyle\int_0^{\rho(T)} F(\varsigma, \eta) \Delta\varsigma \right)}{K_\eta + \displaystyle\sum_{j=1}^{m} \int_0^{w(\varsigma_j)} I_j(\zeta) \mathrm{d}\zeta} \\ &> \int_0^{\rho(T)} \max_{|\vartheta| \leq \theta} F(\varsigma, \vartheta) \Delta\varsigma \geq \sup_{z \in \Theta^{-1}((-\infty, s])} J(z). \end{aligned}$$

Thwn, we can fix ρ such that

$$\sup_{z \in \Theta^{-1}((-\infty, s])} J(z) < \rho < s\frac{J(w)}{\Theta(w)}.$$

From Lemma 2.3, we obtain

$$\sup_{\lambda \geq 0} \inf_{z \in H^1_\Delta([0, \rho^2(T)]_{\mathbb{T}})} (\Theta(z) - \lambda(\rho - J(z)) < \inf_{z \in H^1_\Delta([0, \rho^2(T)]_{\mathbb{T}})} \sup_{\lambda \geq 0} (\Theta(z) - \lambda(\rho - J(z)).$$

Hence, by Lemma 2.2, for each compact interval $[c, d] \subset (\lambda_1, \lambda_2)$, there exists $R > 0$ such that for every $\lambda \in [c, d]$, and every continuous function $g : [0, T]_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$ there exists $\delta > 0$ such that, for each $\mu \in [0, \delta]$, $\Theta'(z) - \lambda J'(z) - \mu\Psi' = 0$ has at least three solutions in $H^1_\Delta([0, \rho^2(T)]_{\mathbb{T}})$. Hence, the problem (P^f_λ) has at least three weak solutions whose norms are less than R. □

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REFERENCES

[1] R.P. Agarwal, M. Bohner, P.J.Y. Wong, Sturm-Liouville eigenvalue problems on time scales, *Appl. Math. Comput.* **99** (1999) 153–166.

[2] R.P. Agarwal, V. Otero-Espinar, K. Perera, D.R. Vivero, Existence of multiple positive solutions for second order nonlinear dynamic BVPs by variational methods, *J. Math. Anal. Appl.* **331** (2007) 1263–1274.

[3] R.P. Agarwal, V. Otero-Espinar, K. Perera, D.R. Vivero, Multiple positive solutions of singular Dirichlet problems on time scales via variational methods, *Nonlinear Anal. TMA* **67** (2007) 368–381.

[4] C.D. Ahlbrandt, C. Morian, Partial differential equations on time scales, *J. Comput. Appl. Math.* **141** (2002) 35–55.

[5] D. Barilla, M. Bohner, S. Heidarkhani, S. Moradi, Existence results for dynamic Sturm-Liouville boundary value problems via variational methods, *Appl. Math. Comput.* **409** (2021) 125614.

[6] M. Benchohra, S. K. Ntouyas, A. Ouahab, Existence results for second order boundary value problem of impulsive dynamic equations on time scales, *J. Math. Anal. Appl.* **296** (2004) 65–73.

[7] M. Bohner, G. Gelles, Risk aversion and risk vulnerability in the continuous and discrete case, *Decis. Econ. Finance* **35** (2012) 1–28.

[8] M. Bohner, G. Gelles, J. Heim, Multiplier-accelerator models on time scales, *Int. J. Stat. Econ.* **4** (2010) 1–12.

[9] M. Bohner, J. Heim, A. Liu, Qualitative analysis of a Solow model on time scales, *J. Concr. Appl. Math.* **13** (2015) 183–197.

[10] M. Bohner, A. Peterson, Advances in Dynamic Equations on Time Scales, Birkhäuser Boston, Inc., Boston, MA, (2003).

[11] M. Bohner, A. Peterson, Dynamic Equations on Time Scales, An Introduction with Applications, Birkhäuser Boston, Inc., Boston, MA, (2001).

[12] E. Çetin, F.S. Topal, Symmetric positive solutions of fourth order boundary value problems for an increasing homeomorphism and homomorphism on time-scales, *Appl. Math. Lett.* **24** (2011) 87–92.

[13] H. Duan, H. Fang, Existence of weak solutions for second-order boundary value problem of impulsive dynamic equations on time scales, *Adv. Differ. Equ.* **2009**, Article ID 907368 (2009).

[14] J. Eckhardt, G. Teschl, Sturm-Liouville Operators on time scales, *J. Differ. Equ. Appl.* **18** (2012) 1875–1887.

[15] J.R. Graef, L. Kong, Q. Kong, On a generalized discrete beam equation via variational methods, *Commun. Appl. Anal.* **16** (2012) 293–308.

[16] S. Heidarkhani, G.A. Afrouzi, M. Ferrara, G. Caristi, S. Moradi, Existence results for impulsive damped vibration systems, *Bull. Malays. Math. Sci. Soc.* **41** (2018) 1409–1428.

[17] S. Heidarkhani, M. Bohner, G. Caristi, F. Ayazi, A critical point approach

for a second-order dynamic Sturm-Liouville boundary value problem with *p*-Laplacian, *Appl. Math. Comput.* **409** (2021) 125521.

[18] D. Barilla, M. Bohner, G. Caristi, S. Heidarkhani, S. Moradi, A dynamic Sturm-Liouville equation related to time scales and machine learning algorithms, preprint.

[19] S. Heidarkhani, S. Moradi, G. Caristi, Existence results for a dynamic Sturm-Liouville boundary value problem on time scales, *Optim. Letters* **15** (2021) 2497–2514.

[20] S. Heidarkhani, A. Salari, Existence of three solutions for impulsive perturbed elastic beam fourth-order equations of Kirchhoff-type, *Stud. Sci. Math. Hungarica* **54** (2017) 119–140.

[21] S. Heidarkhani, A. Salari, Existence of three solutions for Kirchhoff-type three-point boundary value problems, *Hacet. J. Math. Stat.* **50** (2021) 304–317.

[22] S. Heidarkhani, Y. Zhao, G. Caristi, G.A. Afrouzi, S. Moradi, Infinitely many solutions for perturbed impulsive fractional differential systems, *Appl. Anal.* **96** (2017) 1401–1424.

[23] S. Hilger, Ein MaSSkettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeiten, Ph.D. Thesis, Universität Würzburg (1988) (in German).

[24] L. Kong, M. Wang, On a second order discrete problem, *Le Matematiche* **LXXVII** (2022) 407–418.

[25] J. Lamperti, On the isometries of certain function-spaces, *Pac. J. Math.* **8** (1958) 459–466.

[26] J. Mawhin, M. Willem, Critical Point Theory and Hamiltonian Systems, Applied Mathematical Sciences, Vol. **74**, Springer-Ver1ag, New York, (1989).

[27] V. Otero-Espinar, T. Pernas-Castaño, Variational approach to second-order impulsive dynamic equations on time scales, *Bound. Value Prob.* **2013** (2013), 119, 1–15.

[28] B. Ricceri, A further three critical points theorem, *Nonlinear Anal. TMA* **71** (2009) 4151–4157.

[29] B. Ricceri, A three critical points theorem revisited, *Nonlinear Anal. TMA* **70** (2009) 3084–3089.

[30] B. Ricceri, Existence of three solutions for a class of elliptic eigenvalue problem, *Math. Comput. Model.* **32** (2000) 1485–1494.

[31] J. Sun, H. Chen, Variational method to the impulsive equation with Neumann boundary conditions, *Bound. Value Prob.* **2009** (2009), 316812, 1–17.

[32] J. Sun, H. Chen, J.J. Nieto, M. Otero-Novoa, The multiplicity of solutions for perturbed second-order Hamiltonian systems with impulsive effects, *AIMS Math.* **12** (2010) 4575–4586.

[33] D.M. Thomas, L. Vandemuelebroeke, K. Yamaguchi, A mathematical evolution model for phytoremediation of metals, *Discrete Contin. Dyn. Syst. Ser. B* **5** (2005) 411–422.

[34] L. Yang, Multiplicity of solutions for perturbed nonhomogeneous Neumann prob-

lem through Orlicz-Sobolev spaces, *Abstr. Appl. Anal.*, Volume 2012, Article ID 236712, 10 pages.

[35] E. Zeidler, *Nonlinear Functional Analysis and its Applications,* Vol. II/B, Springer, Berlin, Heidelberg, New York, (1985).

[36] Q.G. Zhang, X.P. He, H.R. Sun, Positive solutions for Sturm-Liouville BVPs on time scales via sub-supersolution and variational methods, *Bound. Value Prob.* **2013** (2013), 123 1–12.

[37] Q. Zhang, H. Sun, Variational approach for Sturm-Liouville boundary value problems on time scales, *J. Appl. Math. Comput.* **36** (2011) 219–232.

[38] J. Zhou, Y. Li, Sobolev's spaces on time scales and its applications to a class of second order Hamiltonian systems on time scales, *Nonlinear Anal. TMA* **73** (2010) 1375–1388.

[39] J. Zhou, Y. Wang, Y. Li, An application of variational approach to a class of damped vibration problems with impulsive effects on time scales, *Bound. Value Prob.* **2015** (2015), 48 1–25.

[40] J. Zhou, Y. Wang, Y. Li, Existence and multiplicity of solutions for some second-order systems on time scales with impulsive effects, *Bound. Value Probl.* **2012** (2012), 148, 1–26.

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