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ALGEBRAIC APPROACHES TO COSMOLOGICAL INTEGRALS

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Cosmological correlators encode statistical properties of the initial conditions of our universe. Mathematically, they can often be written as Mellin integrals of a certain rational function associated to graphs, namely the flat space wavefunction. The singularities of these cosmological integrals are parameterized by binary hyperplane arrangements. Using different algebraic tools, we shed light on the differential and difference equations satisfied by these integrals. Moreover, we study a multivariate version of partial fractioning of the flat space wavefunction, and propose a graph-based algorithm to compute this decomposition.

1. Introduction

Cosmological correlation functions are central to the study and characterization of the very early universe. They encode the statistical properties of the initial density inhomogeneities, around which structures cluster, giving rise to galaxies, stars, planets, etc. Computing cosmological correlators requires to integrate their time evolution in the early universe. The resulting cosmological integrals are closely analogous to Feynman integrals from particle physics. In this paper, we investigate how various algebraic techniques can shed light on the structure of cosmological integrals and their differential equations.

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The cosmological integral is the multi-dimensional Mellin integral

$$\psi_\varepsilon(X,Y) = \int_{\mathbb{R}^n_{>0}} \psi_{\text{flat}}(X+\alpha,Y)\alpha^\varepsilon \,\mathrm{d}\alpha\,, \tag{1}$$

where the integration variable α parameterizes the time in which a physical process happens. Here, ψ_{flat} is the wavefunction in flat space where there is no cosmological time evolution. It has been studied extensively in the past few years [4]. Interestingly, it can be extracted from the canonical form of a cosmological polytope [4, 18, 20]. For our purposes, it is enough to know that it is a rational function of kinematics, thus resembling the product of propagators that is typical for Feynman integrals. The variables X and Y summarize these kinematics and we will sometimes refer to these variables as "energies."

An example that already illustrates the challenge of computing these integrals, is the single-exchange process, with X_1 and X_2 being the external outgoing energies, while Y is the exchanged energy in the process; the integrand is

$$\psi_{\text{flat}}(X_1,X_2,Y) = \frac{2Y}{(X_1+Y)(X_2+Y)(X_1+X_2)}\,.$$

The numerator $2Y$ normalizes the residues of the poles of the integrand to be ± 1, doing justice to its origin as a canonical form. In this case, as we have two external energies, there will be two integration variables.

Cosmological correlators can be represented by diagrams in spacetime or in kinematic space, some examples of which we show in Figure 1.

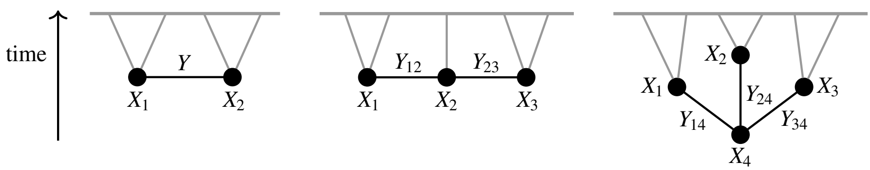


Figure 1: A single and a double exchange process, and the 4-site star graph.

In the figure, time progresses upwards, so the physical picture for the leftmost diagram is of a pair of particles being created with energy Y, which subsequently decay into two new pairs of particles, with energies X_1 and X_2. Their quantum mechanical wavefunction is described by (1), where we associate a different integrand to each diagram. For practical applications, the (X,Y)-variables are real and positive, though it is useful to think of them in larger domains using analytic continuation.

To integrate time, the energies associated to vertices, X, have to be shifted by the α-variables. This way, the flat space wavefunction carves up the integration space along hyperplanes. The hyperplane arrangement is crucial for determining the analytic structure of the integrals. The exponents of α appearing in the integrand are considered to be the same, single variable ε. It sets the cosmology in which the process takes place. For example, $\varepsilon = 0$ corresponds to an accelerated universe (called "de Sitter space" in the physics literature, often a good approximation to cosmic inflation), while $\varepsilon = -1$ recovers the original integrand, as the cosmology will be that of flat space. Other interesting examples are $\varepsilon = -2$ and $\varepsilon = -3$, that parameterize a universe filled with matter and radiation, respectively. Non-integer values of ε are also interesting for physics, so we consider ε to be a generic real variable.

To summarize, cosmological integrals have a structure that is similar to Mellin integrals, with all Mellin variables being equal to ε. The integrand ψ_{flat} is singular only along a hyperplane arrangement that depends on the kinematic variables. This hyperplane arrangement is non-generic, as the coefficients of the integration variables are all zero or one.

Although challenging, the integrals are amenable to many techniques in nonlinear algebra, some of which we will establish and apply in this article. We will focus on differential and difference equations for the cosmological correlators. Our goal here is exploratory: we will present a few connections, and open up directions of investigation, posing new questions and conjectures. These connections are promising, and we hope that with further investigation, they will trigger deeper mathematical insights, importing techniques developed in physics, and also enhance the physicist's mathematical toolkit to tackle these fundamental questions.

Outline. Section 2 presents methods from the algebraic theory of linear differential equations as well as their discrete counterpart: shift operators. In Section 3, we apply our methods to cosmological integrals. For the single exchange diagram, we explain the differential equations constructed in [3], in terms of restricted GKZ systems, and moreover systematically construct all recurrence relations for the correlator. Furthermore, we conjecture a multivariate partial fraction decomposition of the flat space wavefunction for arbitrary graphs, which at the same time recovers the physically relevant singularities. Section 4 summarizes our findings and outlines open questions.

2. Mathematical tools

In this section, we introduce the mathematical toolkit that we will use to analyze cosmological integrals.

2.1. The Weyl algebra

Let D_n (or just D) denote the n-th Weyl algebra $D_n = \mathbb{C}[x_1,\dots,x_n]\langle\partial_{x_1},\dots,\partial_{x_n}\rangle$ in the variables $x = (x_1,\dots,x_n)$. All generators are assumed to commute, except x_i and ∂_{x_i}; they obey Leibniz's rule, i.e., $\partial_{x_i}x_i - x_i\partial_{x_i} = 1$, $i = 1,\dots,n$. Speaking about D-ideals, we will always mean left D-ideals. We will denote by $R_n = \mathbb{C}(x_1,\dots,x_n)\langle\partial_{x_1},\dots,\partial_{x_n}\rangle$ the rational Weyl algebra. D_n gathers linear differential operators with polynomial coefficients, and R_n allows for coefficients in the field of rational functions. We denote the action of a differential operator $P \in D_n$ on a function $f(x_1,\dots,x_n)$ as $P \bullet f$. For instance, $\partial_{x_i} \bullet f = \frac{\partial f}{\partial x_i}$, while $\cdot$ denotes the product of differential operators, i.e., $\partial_{x_i} \cdot x_i = x_i\partial_{x_i} + 1$.

The *holonomic rank* of a D-ideal I is the dimension of the quotient R_n/R_nI as a $\mathbb{C}(x)$-vector space. *Holonomic* D_n-ideals, i.e., ideals whose characteristic variety is of dimension n, have a finite holonomic rank. A function $f(x_1,\dots,x_n)$ is *holonomic* if its annihilating D_n-ideal $\mathrm{Ann}_{D_n}(f) = \{P \in D_n \,|\, P \bullet f = 0\}$ is holonomic as a D_n-ideal, see for instance [30] for a friendly introduction of the topic.

It follows from a theorem of Cauchy, Kovalevskaya, and Kashiwara, that the holonomic rank encodes the dimension of the $\mathbb{C}$-vector space of holomorphic solutions to the system of PDEs encoded by I—whenever outside $\mathrm{Sing}(I)$, the singular locus of I (see [30, Definition 1.12] for the construction of it). D-ideals of finite holonomic rank do not need to be holonomic, but they can be turned into such by taking the *Weyl closure*: The Weyl closure of a D-ideal I is

$$W(I) = R_nI \cap D_n\,.$$

It is again a D_n-ideal, it contains I, and it has the same holonomic rank as I.

Remark 2.1. When studying cosmological integrals, the choice of the most suitable tools to use will depend on which parameters of the integrals we vary. The possible choices are explained in detail in [1]. In Section 2.3, the integrals will be considered as functions of the coefficients of the polynomials defining the integrand, whereas in Section 2.5, they will be functions of the exponent ε. Importantly, the choice of parameters will also determine which Weyl algebra to use. We will specify this choice in each section.

2.2. Connection matrices and gauge transformation

Let I be a D_n-ideal of holonomic rank m. Let $(s_1,s_2,\dots,s_m)$ be a $\mathbb{C}(x)$-basis of R_n/R_nI. The s_i's can be chosen to be monomials in the ∂_{x_i}'s, and w.l.o.g., $s_1 = 1$. By a Gröbner basis reduction of the $\partial_{x_i}s_j$ mod I, one can then read the *connection matrices* of I. This system is sometimes called a "Pfaffian system," cf. [27, p. 38]. For a solution f to I, let $F = (f, s_2 \bullet f, \dots, s_m \bullet f)^\top$. The connection matrices of I then are the unique matrices $M_1, \dots, M_n \in \mathrm{Mat}_{m\times m}(\mathbb{C}(x))$ s.t.

$$\partial_{x_i} \bullet F = M_i \cdot F, \qquad i = 1, \dots, n \tag{2}$$

for any $f \in \mathrm{Sol}(I)$. The left-hand side of (2) denotes the vector $(\partial_{x_i} \bullet f, (\partial_{x_i} \cdot s_2) \bullet f, \dots, (\partial_{x_i} \cdot s_m) \bullet f)^\top$. One may equivalently write (2) as $\mathrm{d}F = MF$, with M an $m \times m$ matrix of rational differential one-forms. Changing basis via $\widetilde{F} = GF$ for some invertible $m \times m$ matrix G acts via the *gauge transformation* on the connection matrices: the transformed system then reads as $\partial_{x_i} \bullet \widetilde{F} = \widetilde{M_i} \cdot \widetilde{F}$ for

$$\widetilde{M_i} = GM_iG^{-1} + \frac{\partial G}{\partial x_i} G^{-1},$$

where entry-wise differentiation of the matrix G is meant. We will use gauge transforms in Section 3.1 in order to change to a basis consisting of integrals against canonical forms of the bounded regions of the hyperplane arrangement.

2.3. Restrictions and GKZ systems

Later on, we will need restrictions of D_n-ideals to subspaces of $\mathbb{C}^n$. For now, let D_n denote the Weyl algebra in variables $x_1, \dots, x_n$. For $m < n$, we will denote by D_m the Weyl algebra $\mathbb{C}[x_1, \dots, x_m]\langle \partial_{x_1}, \dots, \partial_{x_m} \rangle$ in the first m variables.

Definition 2.2. Let I be a D_n-ideal. The D_m-ideal

$$(I + x_{m+1}D_n + \cdots + x_nD_n) \cap D_m \tag{3}$$

is the *restriction ideal* of I to the coordinate subspace $\{x_{m+1} = \cdots = x_n = 0\} \subset \mathbb{C}^n$.

Let $f(x_1, \dots, x_n)$ be a holonomic function and $m < n$. Then the restriction of f to the coordinate subspace $\{x_{m+1} = \cdots = x_n = 0\}$, i.e., $f(x_1, \dots, x_m, 0, \dots, 0)$, is a holonomic function in the variables $x_1, \dots, x_m$, which is annihilated by the restriction ideal (3), cf. [27, Proposition 5.2.4].

In this article, we are interested in restrictions of GKZ systems, which are prominent examples of D-ideals. They are also called "A-hypergeometric systems," and we refer to [26] for the concise connection between these systems and hypergeometric functions. They are holonomic systems, implying that their space of holomorphic solutions is finite-dimensional, and their structure is deeply linked to toric geometry and combinatorics. Their singular locus coincides with the zero locus of the principal A-determinant, see [14, Remark 1.8]. GKZ systems are encoded by an integer matrix A and a parameter vector κ. Since this is what we need in our study, we here directly address the case that A

comes from a family of Laurent polynomials. For that, let $\{A_j\}_{j=1,\dots,k}$ be finite subsets of $\mathbb{Z}^n$ representing the monomial supports of k Laurent polynomials

$$f_j = \sum_{u\in A_j} c_{u,j}\alpha^u \quad \text{with} \quad \alpha^u = \alpha_1^{u_1}\cdots\alpha_n^{u_n}, \tag{4}$$

and parameters $(c_{u,j})_{u\in A_j}$ with values in $\mathbb{C}^{A_j} = \mathbb{C}^{|A_j|}$. We build the matrix

$$A = \left(\begin{array}{ccc|ccc|c|ccc} & A_1 & & & A_2 & & & & A_k & \\ 1 & \cdots & 1 & 0 & \cdots & 0 & & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & \cdots & 1 & & & \vdots & \\ & \vdots & & & \vdots & & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & & 1 & \cdots & 1 \end{array}\right),$$

namely the *Cayley configuration* of the Laurent polynomials $f_1,\dots,f_k$, and consider the Weyl algebra $D_A = \mathbb{C}[c_u\,|\,u\in A]\langle \partial_u\,|\,u\in A\rangle$ with variables indexed by the columns of A. The *toric ideal* of A is defined to be the binomial ideal

$$I_A := \left\langle \partial^a - \partial^b \mid a-b\in \ker(A),\ a,b\in\mathbb{N}^A\right\rangle, \tag{5}$$

where $a=(a_u)_{u\in A}$ and $\partial^a = \prod_{u\in A}\partial_u^{a_u}$, and analogously for b. In addition, given a vector of possibly complex parameters $\kappa\in\mathbb{C}^{n+k}$, we define a second left ideal $J_{A,\kappa}$ to be generated by the entries of $A\theta-\kappa$, where $\theta=(\theta_u)_{u\in A}$ and $\theta_u = c_u\partial_u$. Finally, the GKZ system associated to the pair (A,κ) is the left D_A-ideal

$$H_A(\kappa) = I_A + J_{A,\kappa}\,.$$

A full description of the solutions to such a D_A-ideal is given by generalized Euler integrals. We refer to [1, 13] for a general definition of such integrals and a proof that they provide solutions to GKZ systems, and to [6, 7] for convergence criteria for the integral in terms of the Newton polytopes of the Laurent polynomials f_j in (4).

Here, we observe that cosmological integrals as in (1) are examples of generalized Euler integrals, where the coefficients of the polynomials appearing in the integral are restricted to a linear subspace of the full coefficient space $\mathbb{C}^A = \mathbb{C}^{A_1}\times\cdots\times\mathbb{C}^{A_k}$. This fact motivates the need of considering the restriction ideal of $H_A(\kappa)$ when aiming to compute differential equations for (1). Computing GKZ restrictions is a challenging problem in computer algebra. We also point out that they are, in general, not GKZ systems themselves again. We refer to [10] for detailed formulae. Recent advances in restriction algorithms for D-ideals have been achieved in [8], specifically at the level of their Pfaffian systems [7]. In [12], based on theoretical and computational considerations, it was conjectured that the singular locus of a restricted GKZ system, arising from a

Feynman integral, coincides with the Euler discriminant, cf. (19). In Section 3, we will show that the differential equations for the single-exchange diagram described in [3] can be interpreted in terms of a restricted GKZ system. The connection between cosmological integrals and restrictions of GKZ systems was recently pioneered in [16]. Where our works overlap, we find agreement in the results. In the study of the two-site chain, we constructed the differential operator Δ_3 (10) to obtain the correct holonomic rank of our annihilating D-ideal; this operator was also found in [16, (5.75)].

2.4. Flat space wavefunctions from graphs

Before delving deeper into the mathematical tools required to study cosmological integrals (1), we establish the required notation. Let $G = (V,E)$ (or $G = (V(G),E(G))$ to highlight the underlying graph) be an undirected connected graph, where V is a set of n vertices, and E is a collection of edges, namely pairs ij for some $v_i, v_j \in V$. We denote by $(X,Y) := (X_1,\dots,X_n,\{Y_{ij}\}_{ij\in E})$ the vector of kinematic parameters associated to each vertex and edge of the graph. This will be the general notation for graphs like the ones drawn in black color within diagrams in spacetime, some of which are presented in Figure 1.

To each graph G, we associate the rational function

$$\psi_{\text{flat}} = \psi_G^{\text{flat}} = 2^{n-1}\cdot\Big(\prod_{ij\in E} Y_{ij}\Big)\cdot\frac{P}{\ell_1\ell_2\cdots\ell_k} \in \mathbb{C}(X,Y), \tag{6}$$

where $P,\ell_1,\dots,\ell_k \in \mathbb{C}[X,Y]$, and the normalization factor $2^{n-1}\prod_{ij\in E} Y_{ij}$ arises from the requirement that the residue of a canonical form is ± 1. More precisely, the polynomial P is conjectured to represent the adjoint hypersurface of the cosmological polytope associated to G [4], and $\ell_i = \ell(H_i)$ for $i = 1,\dots,k$ are linear forms that can be computed from the connected subgraphs of G. Here, a subgraph is another graph formed from a subset of the vertices and edges of G where all endpoints of the edges of H are in the vertex set of H. Given a connected subgraph $H = (V(H),E(H))$ of G, the linear form associated to H is

$$\ell(H) = \sum_{v_i\in V(H)} X_i + \sum_{\substack{e=\{i,j\},\\ v_i\in V(H),v_j\notin V(H)}} Y_{ij} + \sum_{\substack{e=\{i,j\}\notin E(H),\\ v_i,v_j\in V(H)}} 2Y_{ij}, \tag{7}$$

see [4, 20] for references. Then, unveiling the use of multi-index notation, the cosmological integral is given as

$$\psi_\varepsilon(X,Y) = \int_{\mathbb{R}^n_{>0}} \psi_{\text{flat}}\big(X_1+\alpha_1,\dots,X_n+\alpha_n,\{Y_{ij}\}_{ij\in E}\big)\cdot\alpha_1^\varepsilon\cdots\alpha_n^\varepsilon\,\mathrm{d}\alpha, \tag{8}$$

where $\mathrm{d}\alpha \coloneqq \mathrm{d}\alpha_1 \wedge \cdots \wedge \mathrm{d}\alpha_n$. In what follows, we denote $L_i \coloneqq \ell_i(X+\alpha, Y)$. We will denote the complement of the hyperplane arrangement $(L_1,\dots,L_k)$ in the algebraic n-dimensional torus with coordinates $\alpha = (\alpha_1,\dots,\alpha_n)$ by

$$\mathcal{H}^G_{(X,Y)} = \{\alpha \in (\mathbb{C}^*)^n \,:\, L_i(X,Y;\alpha) \neq 0,\ i = 1,\dots,k\},$$

with $(X,Y) \in \mathbb{C}^{|V|+|E|}$ generic. Figure 2 shows the hyperplane arrangement arising from the integral associated to the 2-site chain.

2.5. Shift-relations

The Weyl algebra we will be working with here is

$$D = \mathbb{C}(X,Y)[\alpha_1,\dots,\alpha_n]\langle \partial_{\alpha_1},\dots,\partial_{\alpha_n}\rangle .$$

Its elements are linear differential operators, defined globally on affine n-space in the α-variables over the field of rational functions in the X_i's and Y_{ij}'s.

Lemma 2.1. *Let $p \in \mathbb{C}[\alpha_1,\dots,\alpha_n]$ be a polynomial. Let I be the D_n-ideal generated by $p\partial_{\alpha_i} + \frac{\partial p}{\partial \alpha_i}$, $i = 1,\dots,n$. Then the full annihilating D-ideal of $1/p$ equals the Weyl closure of I. As a formula,* $\mathrm{Ann}_{D_n}(1/p) = W(I)$.

This statement is an immediate consequence of the following lemma, which we learned from discussions with Uli Walther.

Lemma 2.2. *Let p, I be as in Lemma 2.1. Let $P \in \mathrm{Ann}_D(1/p)$, and denote by $d = \mathrm{ord}(P)$ its order. Then $p^d P \in I$.*

Proof. We proceed by an inductive argument on the order of P. If $\mathrm{ord}(P) = 0$, then $P \in \mathbb{C}[\alpha_1,\dots,\alpha_n]$. A polynomial annihilates $1/p$ iff it is zero, and so $p^0 P = P \in I$. Now let $P \in \mathrm{Ann}_D(1/p)$ be of order d. Note that $pP = Pp - [P,p]$. Write $P = q + \sum Q_i \partial_{\alpha_i}$, where q is a polynomial, and the Q_i are some elements of D. Then $Pp = qp + \sum Q_i \partial_{\alpha_i} p$. The sum term is in I. So, modulo I, $qp \equiv Pp$. Thus, $pP = Pp - [P,p] \equiv qp - [P,p]$. Since P annihilates $1/p$, so does $Pp - [P,p] = pP$ and hence the same is true for $qp - [P,p]$. This operator is of order at most $d-1$, hence $p^{d-1}(qp - [P,p]) \in I$. So, modulo I, $p^d P \in I$. □

To construct shift-relations among cosmological integrals, we will make use of the Mellin transform. Denote by $\mathcal{S}_n = \mathbb{C}[\varepsilon]\langle \sigma_\varepsilon, \sigma_\varepsilon^{-1}\rangle$ the shift algebra in the variables $\varepsilon = (\varepsilon_1,\dots,\varepsilon_n)$. The shifts are w.r.t. the variables ε, i.e., $\sigma_{\varepsilon_i}^{\pm 1}\colon \varepsilon_i \mapsto \varepsilon_i \pm 1$. The generators obey

$$\sigma_{\varepsilon_i}^{\pm 1}\varepsilon_i = (\varepsilon_i \pm 1)\sigma_{\varepsilon_i}^{\pm 1}.$$

This implies $\sigma_\varepsilon^a \varepsilon^b = (\varepsilon + a)^b \sigma_\varepsilon^a$ for $a \in \mathbb{Z}^n$, $b \in \mathbb{N}^n$. To elements of D_n, one associates linear partial differential equations; to elements of $\mathcal{S}_n$, recurrence relations. We will need to extend the shift algebra to

$$\mathcal{S}_n = \mathbb{C}(X_1,\ldots,X_n,\{Y_{ij}\}_{ij\in E})[\varepsilon]\langle \sigma_\varepsilon^{\pm 1}\rangle .$$

The algebraic Mellin transform of Loeser–Sabbah [22] is the isomorphism of the non-commutative $\mathbb{C}(X_1,\ldots,X_n,\{Y_{ij}\}_{ij\in E})$-algebras

$$\begin{gathered}\mathfrak{M}\{\cdot\}\colon \mathbb{C}(X,Y)[\alpha^{\pm 1}]\langle \partial_{\alpha_1},\ldots,\partial_{\alpha_n}\rangle \xrightarrow{\cong} \mathcal{S}_n , \\ \alpha_i^{\pm 1} \mapsto \sigma_{\varepsilon_i}^{\pm 1},\ \alpha_i\partial_{\alpha_i} \mapsto -\varepsilon_i,\ X_i \mapsto X_i,\ Y_{ij} \mapsto Y_{ij} \, .\, . \end{gathered} \tag{9}$$

Note that $\mathfrak{M}\{\partial_{\alpha_i}\} = -(\varepsilon_i - 1)\sigma_{\varepsilon_i}^{-1}$ and $\mathfrak{M}\{\alpha_i\theta_{\alpha_i}\} = -(\varepsilon_i+1)\sigma_{\varepsilon_i}$, where $\theta_{\alpha_i} = \alpha_i\partial_{\alpha_i}$ denotes the i-th Euler operator. For integration cycles that are suitable for integration by parts with vanishing boundary terms, one has $\mathfrak{M}\{P \bullet f\} = \mathfrak{M}\{P\} \bullet \mathfrak{M}\{f\}$. The isomorphism (9) implies that

$$\mathfrak{M}\{\mathrm{Ann}_{D[\alpha^{\pm1}]}(f)\} = \mathrm{Ann}_{\mathcal{S}_n}(\mathfrak{M}\{f\}) .$$

Hence, the Mellin transform of the full annihilator of ψ_{flat} recovers all shift relations in the ε-variables for our cosmological integrals $\mathfrak{M}\{\psi_{\text{flat}}\}$.

3. Use in cosmology

We here showcase the methods from Section 2 at integrals arising from graphs in cosmology—the path graph on n vertices being called "n-site chain" there.

3.1. Two-site chain

The linear forms appearing in the denominator of ψ_2^{flat}, after shifting the X variables, are the following elements of $\mathbb{C}(X_1,X_2,Y)[\alpha_1,\alpha_2]$:

$$L_1 = \alpha_1+\alpha_2+X_1+X_2 , \quad L_2 = \alpha_1+X_1+Y, \quad L_3 = \alpha_2+X_2+Y .$$

The hyperplane arrangement (L_1,L_2,L_3) is depicted in Figure 2.

The underlying rational function of interest hence is $\frac{2Y}{L_1L_2L_3}$, and the cosmological integral is the Mellin integral

$$\int_{\mathbb{R}^2_{>0}} \frac{2Y}{L_1L_2L_3}\alpha_1^\varepsilon\alpha_2^\varepsilon \,\mathrm{d}\alpha_1\mathrm{d}\alpha_2 = 2Y\cdot\mathfrak{M}\left\{L_1^{-1}L_2^{-1}L_3^{-1}\right\}(\varepsilon+1,\varepsilon+1) .$$

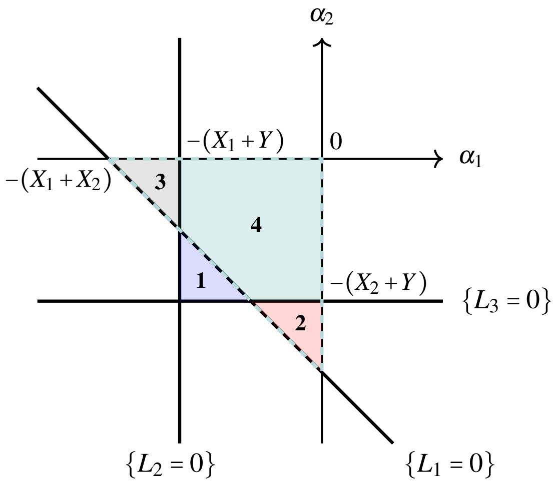


Figure 2: A real picture of the hyperplane arrangement (L_1,L_2,L_3) for the two-site graph, here depicted for generic X_1,X_2,Y with $X_1,X_2>|Y|$. Together with the coordinate axes, it encloses four bounded regions. To the triangle with label i, we have an associated differential operator Q_i as resulting from (13).

Shift relations Denote $L=L_1L_2L_3$. The following four differential operators generate the annihilator of $1/(L_1L_2L_3)$:

$$\begin{aligned}
P_1 &= \alpha_1\alpha_2\partial_{\alpha_2}+\alpha_2^2\partial_{\alpha_2}+(X_2+Y)\alpha_1\partial_{\alpha_2}+(X_1+2X_2+Y)\alpha_2\partial_{\alpha_2}+\alpha_1+2\alpha_2\\
&\quad +(X_1+X_2)(X_2+Y)\partial_{\alpha_2}+(X_1+2X_2+Y),\\
P_2 &= \alpha_1\alpha_2\partial_{\alpha_1}+\alpha_2^2\partial_{\alpha_2}+(X_2+Y)\alpha_1\partial_{\alpha_1}+(X_1+Y)\alpha_2\partial_{\alpha_1}+2X_2\alpha_2\partial_{\alpha_2}+3\alpha_2\\
&\quad +(X_1+Y)(X_2+Y)\partial_{\alpha_1}+(X_2^2-Y^2)\partial_{\alpha_2}+(3X_2+Y),\\
P_3 &= \alpha_1^2\partial_{\alpha_1}-\alpha_2^2\partial_{\alpha_2}+2X_1\alpha_1\partial_{\alpha_1}-2X_2\alpha_2\partial_{\alpha_2}+2\alpha_1-2\alpha_2\\
&\quad +(X_1^2-Y^2)\partial_{\alpha_1}-(X_2^2-Y^2)\partial_{\alpha_2}+2(X_1-X_2),\\
P_4 &= \alpha_2^2\partial_{\alpha_1}\partial_{\alpha_2}-\alpha_2^2\partial_{\alpha_2}^2+2X_2\alpha_2\partial_{\alpha_1}\partial_{\alpha_2}-2X_2\alpha_2\partial_{\alpha_2}^2+2\alpha_2\partial_{\alpha_1}-4\alpha_2\partial_{\alpha_2}\\
&\quad +(X_2^2-Y^2)\partial_{\alpha_1}\partial_{\alpha_2}-(X_2^2-Y^2)\partial_{\alpha_2}^2+2X_2\partial_{\alpha_1}-4X_2\partial_{\alpha_2}-2.
\end{aligned}$$

This D-ideal can be computed with the SINGULAR:PLURAL [9, 15] library `dmod_lib` [21], or, equivalently, by computing the Weyl closure of the D-ideal generated by $L\partial_{\alpha_i}-\partial_{\alpha_i}\bullet L$, where $i=1,2$.

Remark 3.1. The Bernstein–Sato polynomial of $L=L_1L_2L_3$ is $b_L=(s+1)^2$. Since $-1\notin V(b_L)+\mathbb{Z}_{>0}$, one has the following relation to the s-parametric annihilator of L: $\mathrm{Ann}_{D_2[s_1,s_2,s_3]}(L_1^{s_1}L_2^{s_2}L_3^{s_3})|_{(s_1,s_2,s_3)=(-1,-1,-1)}=\mathrm{Ann}_{D_2}(1/L)$, cf. [23, Theorem 2]. A multivariate analogue is given in [24, Proposition 3.6]. If one instead allows individual powers of the hyperplanes, one needs to pass on to Bernstein–Sato ideals. For some well-behaved cases, generators of parametric annihilators can be expressed in terms of logarithmic derivations, cf. [5].

The left ideal in the shift algebra $\mathcal{S}_2$ that is generated by $\mathfrak{M}\{P_1\},\ldots,\mathfrak{M}\{P_4\}$ encodes all shift relations for $\mathfrak{M}\{1/L\}(\varepsilon_1,\varepsilon_2)$. We here display a selection of the obtained recurrence relations:

$$\begin{aligned}
\mathfrak{M}\{P_1\} &= -(\varepsilon_2-1)\big[(X_2+Y)\sigma_1\sigma_2^{-1}+\sigma_1+\sigma_2 \\
&\quad +(X_1+X_2)(X_2+Y)\sigma_2^{-1}+(X_1+2X_2+Y)\big], \\
\mathfrak{M}\{P_2\}+\mathfrak{M}\{P_3\} &= -(\varepsilon_1-1)\big[(X_1+Y)\sigma_1^{-1}\sigma_2+\sigma_1+\sigma_2 \\
&\quad +(X_1+X_2)(X_1+Y)\sigma_1^{-1}+(2X_1+X_2+Y)\big].
\end{aligned}$$

Note that the shift operators mirror the symmetry of the integral when swapping the variables X_1 and X_2. The remaining operators can be computed analogously.

Remark 3.2. To obtain shift relations among master integrals involving only ε and $\varepsilon-1$ as in [3, (3.44)] for $\varepsilon=\varepsilon_1=\varepsilon_2$, one will have to combine our shift operators with the Pfaffian system [3, (3.30)] arising from the canonical forms.

GKZ system We now consider the cosmological integral for the two-site chain as a generalized Euler integral and set up its GKZ system. For that, we leave the coefficients of the linear forms generic, i.e., we consider the integral

$$2Y\cdot\int_\Gamma \frac{1}{(c_1\alpha_1+c_2\alpha_2+c_3)(c_4\alpha_1+c_5)(c_6\alpha_2+c_7)}\,\alpha_1^{\varepsilon+1}\alpha_2^{\varepsilon+1}\,\frac{\mathrm{d}\alpha_1}{\alpha_1}\frac{\mathrm{d}\alpha_2}{\alpha_2}.$$

Note that, for generalized Euler integrals, Γ is taken to be a twisted n-cycle, cf. [1, Section 4]. However, in physics applications, one instead integrates over the positive orthant. The toric ideal corresponding to the integral above is

$$I_A = \langle \partial_{c_2}\partial_{c_7}-\partial_{c_3}\partial_{c_6},\,\partial_{c_1}\partial_{c_5}-\partial_{c_3}\partial_{c_4}\rangle,$$

and the parameter vector $\kappa=(-(\varepsilon+1),-(\varepsilon+1),-1,-1,-1)^\top$, cf. [1, Section 4]. The ideal $J_{A,\kappa}$ is generated by the operators

$$\begin{gathered}
\theta_{c_1}+\theta_{c_4}+(\varepsilon+1),\ \theta_{c_2}+\theta_{c_6}+(\varepsilon+1),\\
\theta_{c_1}+\theta_{c_2}+\theta_{c_3}+1,\ \theta_{c_4}+\theta_{c_5}+1,\ \theta_{c_6}+\theta_{c_7}+1.
\end{gathered}$$

Together, they yield the D-ideal $H_A(\kappa)=I_A+J_{A,\kappa}$. We now restrict $H_A(\kappa)$ to the subspace $\{c\in\mathbb{C}^7\,|\,c_1=c_2=c_4=c_6=1\}$, and moreover change variables from the remaining c's to (X_1, X_2, Y) via $c_3 = X_1 + X_2$, $c_5 = X_1 + Y$, and $c_7 = X_2 + Y$. The resulting D-ideal, $H_A^{\text{res}}(\kappa)$, has holonomic rank 4. This D-ideal was reconstructed purely combinatorially from the line arrangement $\{L_1, L_2, L_3\}$ in [25].

We compare the restricted GKZ system with a D-ideal which originates from operators in the variables X_1, X_2, Y considered in [3]. Let

$$\begin{aligned}\Delta_1 &= (X_1^2 - Y^2)\partial_{X_1}^2 + 2(1-\varepsilon)X_1\partial_{X_1} - \varepsilon(1-\varepsilon)\,,\\ \Delta_2 &= (X_2^2 - Y^2)\partial_{X_2}^2 + 2(1-\varepsilon)X_2\partial_{X_2} - \varepsilon(1-\varepsilon)\,.\end{aligned}$$

These operators are right factors of the operators considered in [3, Section 3.3]. The D-ideal generated by these, however, has infinite holonomic rank. We extend it by using the operator

$$\Delta_3 = (X_1 + Y)(X_2 + Y)\partial_{X_1}\partial_{X_2} - \varepsilon(X_1 + Y)\partial_{X_1} - \varepsilon(X_2 + Y)\partial_{X_2} + \varepsilon^2\,, \qquad (10)$$

and the homogeneity operator

$$H = X_1\partial_{X_1} + X_2\partial_{X_2} + Y\partial_Y - 2\varepsilon\,.$$

Denote by I the D-ideal

$$I = \langle \Delta_1 + \Delta_3, \Delta_2 + \Delta_3, H\rangle\,. \qquad (11)$$

This D-ideal in (11) annihilates the correlation function. Its singular locus is

$$\operatorname{Sing}(I) = V\left(Y(X_1 + X_2)(X_1 + Y)(X_1 - Y)(X_2 + Y)(X_2 - Y)\right),$$

which corresponds exactly to the singular locus of $H_A^{\text{res}}(\kappa)$. The holonomic rank of I is 4.

As a $\mathbb{C}(X_1, X_2, Y)$-basis of R_3/R_3I, we choose $(1, \partial_{X_1}, \partial_{X_2}, \partial_{X_1}\partial_{X_2})$. We have the following relation, which we proved computationally using `Singular` and the package `HolonomicFunctions` [19] in Mathematica. We provide our code via GitLab at https://uva-hva.gitlab.host/universeplus/algebraic-approaches-to-cosmological-integrals.

Proposition 3.3. Let I be the D-ideal in (11). The D-ideal $DIY = \{PY \,|\, P \in I\}$ is contained in the restricted GKZ system, $DIY \subset H_A^{\text{res}}(\kappa)$. Moreover, the two D-ideals coincide when considered as ideals in the rational Weyl algebra.

In particular, the proposition implies that the Weyl closures of the two D-ideals coincide, i.e., $W(H_A^{\text{res}}(\kappa)) = W(DIY)$.

Canonical forms Denote by Ω_i, $i = 1,\dots,4$ the canonical form [2] of the triangle labelled by i in Figure 2. For the vector of integrals

$$\left(\int_{\mathbb{R}^2_{>0}} x_1^{\varepsilon} x_2^{\varepsilon}\,\Omega_1,\,\dots,\,\int_{\mathbb{R}^2_{>0}} x_1^{\varepsilon} x_2^{\varepsilon}\,\Omega_4\right), \tag{12}$$

the resulting matrix differential system is in ε-factorized form, cf. [3, (3.30)]. Denoting the first entry in (12) by ψ, the full vector (12) can then be written as $(Q_1 \bullet \psi,\dots,Q_4 \bullet \psi)^\top$ for $(Q_1,Q_2,Q_3,Q_4) = G\cdot(1,\partial_{X_1},\partial_{X_2},\partial_{X_1}\partial_{X_2})^\top$, where G denotes the matrix

$$\frac{1}{2Y\varepsilon^2}\cdot\begin{pmatrix} 2Y\varepsilon^2 & -\varepsilon^2(X_1-Y) & -\varepsilon^2(X_2-Y) & -\varepsilon^2(X_1+X_2)\\ 0 & \varepsilon\left(X_1^2-Y^2\right) & 0 & \varepsilon(X_1+X_2)(X_1+Y)\\ 0 & 0 & \varepsilon\left(X_2^2-Y^2\right) & \varepsilon(X_1+X_2)(X_2+Y)\\ 0 & 0 & 0 & -(X_1+X_2)(X_1+Y)(X_2+Y)\end{pmatrix}^\top. \tag{13}$$

This is the gauge matrix for changing from the basis $(1,\partial_{X_1},\partial_{X_2},\partial_{X_1}\partial_{X_2})$ of R_3/R_3I to the basis obtained from canonical forms, which yields the ε-factorized form of the connection matrix.

3.2. Partial fractioning of the flat space wavefunction

Partial fraction decomposition is a widely used technique in particle physics and cosmology for simplifying the computation of Feynman integrals, cosmological integrals, and their differential equations. Although standard partial fraction decomposition applies to univariate rational functions, most expressions arising in physics are multivariate. New algorithmic methods for multivariate decompositions were presented in [17]. We introduce a formula for the partial fraction decomposition of the wavefunction in flat space associated to a graph from (6), inspired by the Feynman rules for cosmology as presented in [3, Section 2]. Our approach is entirely combinatorial and can be described in terms of subgraphs of the original graph. We begin by introducing some definitions.

Let H be an oriented connected graph with $m > 0$ vertices. Given a vertex $v \in V(H)$, we define its degree $\deg(v)$ to be the difference between the number of outgoing and incoming edges connected to v. The degree of the graph H is $\deg(H) = \sum_{v\in V(H)} \deg(v)$. When we consider a subgraph of H, we label the vertices in the subgraph by the degree of the vertices in H and, in slight abuse of notation, refer to these labels as "degrees" again. Notice that, in contrast to H, which is always of degree 0, the subgraphs can have a non-zero degree in general. We denote by $(H)^+$ the set of connected subgraphs of H which have degree strictly positive. We will denote the elements of $(H)^+$ by H_i^+. We write $H' \le H$ if the graph H' is a subgraph of H.

Definition 3.4. A tuple of $m-1$ indices $(i_1,\ldots,i_{m-1})$ is said to be *admissible* if for all $i_k,i_j \in \{i_1,\ldots,i_{m-1}\}$, the subgraphs $H^+_{i_j}$ and $H^+_{i_k}$ do not share any vertex or one is contained in the other, i.e., $H^+_{i_k} \leq H^+_{i_j}$ or $H^+_{i_j} \leq H^+_{i_k}$.

To a graph H as above, we associate the rational function

$$R_H(X,Y) := \begin{cases} \frac{1}{\ell(H)} & \text{if } m=1, \\ \frac{1}{\ell(H)} \cdot \sum_{\substack{(i_1,\ldots,i_{m-1}) \\ \text{admissible}}} \frac{1}{\ell(H^+_{i_1})\cdots\ell(H^+_{i_{m-1}})} & \text{if } m>1, \end{cases} \tag{14}$$

and $R_H(X,Y)=0$ if H includes a directed cycle. In other words, $R_H(X,Y)$ is only non-zero if H is a directed acyclic graph. Here, for a subgraph H, the linear form $\ell(H)$ is the one defined in (7). We now have all the necessary tools to describe the decomposition of the flat space wavefunction.

Let $G=(V,E)$ be a graph as in Section 2, where n denotes the number of vertices. We denote by $\mathcal{F}=\mathcal{F}(G)=\{H_1,H_2,\ldots,H_r\}$ the set of spanning subgraphs of G, where we also take orientation into account. Since each H_i may be disconnected, we denote by $H_i^1,\ldots,H_i^{s_i}$ its connected components, i.e., the maximal connected subgraphs of H_i.

Conjecture 3.5. Let $G=(V,E)$ be a Feynman graph as above. Then the flat space wavefunction can be decomposed into partial fractions of the form

$$\psi_{\text{flat}} = \sum_{H_i\in\mathcal{F}} (-1)^{n-1-|E(H_i)|} \cdot \prod_{j=1}^{s_i} R_{H_i^j}(X,Y). \tag{15}$$

We believe that some kind of minimality—and in that sense uniqueness—may be stated for such a decomposition. Specifically, the number of linear factors in the denominator of each term in the decomposition always coincides with the number of variables X_i associated to vertices, and the degree of the numerator is always zero. On that note, we expect (15) to be the finest decomposition that does not introduce any new hyperplanes in the denominators.

Remark 3.6. Note that the condition yielding $R_{H_i}=0$ is never satisfied when G is a tree. To obtain $R_{H_i}=0$ for some H_i, one needs a graph of genus at least 1, i.e., containing at least one "loop" in physics terminology. Some of such graphs are shown in Figure 3.

Example 3.7 (3-site chain)**.** Here, we demonstrate our procedure at the example of the flat space wavefunction of the 3-site chain in detail. An illustration of the graph with the labeling of the kinematic parameters can be found in Figure 1. The linear forms associated to its connected subgraphs are

$$\ell_1 = X_1+Y_{12}, \qquad \ell_2 = X_2+Y_{12}+Y_{23}, \qquad \ell_3 = X_3+Y_{23}, \tag{16}$$

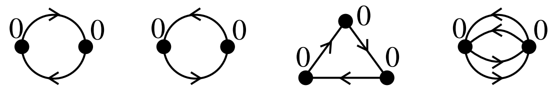


Figure 3: Some examples of graphs with $R_{H_i} = 0$.

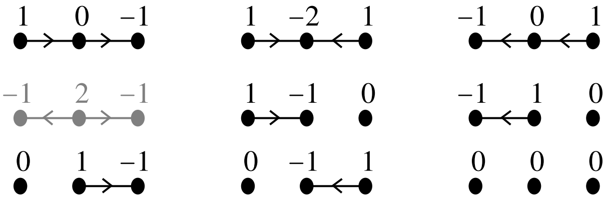
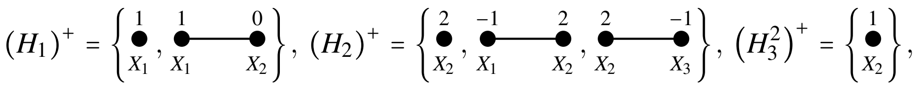


Figure 4: All oriented spanning trees for the 3-site chain. The numbers represent the degree of each vertex. The one in gray is the only one which is not totally time-ordered in the sense of Definition 3.10.

$$\ell_4 = X_1 + X_2 + X_3\,, \qquad \ell_5 = X_1 + X_2 + Y_{23}\,, \qquad \ell_6 = X_2 + X_3 + Y_{12}\,.$$

The set $\mathcal{F} = \{H_1, \ldots, H_9\}$ contains nine elements that we list in Figure 4.
We now construct the rational function in (14) for the three oriented graphs in the left-most column of Figure 4. Let us denote them, from top to bottom, by H_1, H_2, H_3. Note that, as H_3 is disconnected, we denote H_3^1, H_3^2 its connected components. We have



and $(H_3^1)^+ = \varnothing$, which, using the definition in (14), give

$$R_{H_1} = \frac{1}{\ell_4} \cdot \frac{1}{\ell_1 \ell_5}\,, \quad R_{H_2} = \frac{1}{\ell_4}\left(\frac{1}{\ell_2\ell_5} + \frac{1}{\ell_2\ell_6}\right), \quad R_{H_3^1} \cdot R_{H_3^2} = \frac{1}{\ell_1}\left(\frac{1}{\ell_2} \cdot \frac{1}{\ell_6}\right).$$

Note that the two summands in R_{H_2} are given by the two possible choices of admissible sets given by first and third graph or first and third graph in $(H_2)^+$. The flat space wavefunction of the 3-site chain is given by

$$\psi_3^{\text{flat}} = \frac{4Y_{12}Y_{23}(\ell_5 + \ell_6)}{\ell_1\ell_2\ell_3\ell_4\ell_5\ell_6} \in \mathbb{C}(X_1, X_2, X_3, Y_{12}, Y_{23})\,,$$

with $\ell_1, \ldots, \ell_6$ as in (16), arising from Equation (7). The term $\ell_5 + \ell_6$ in the numerator can be obtained via the recursion formula introduced in [4]; see also the appendix of [11] for an explanation of this method.

One can check that ψ_3^{flat} decomposes as

$$\begin{aligned}\psi_3^{\text{flat}} = \; & \frac{1}{\ell_1\ell_4\ell_5} + \frac{1}{\ell_1\ell_3\ell_4} + \frac{1}{\ell_3\ell_4\ell_6} + \left(\frac{1}{\ell_2\ell_4\ell_5} + \frac{1}{\ell_2\ell_4\ell_6}\right) \\ & - \frac{1}{\ell_1\ell_3\ell_5} - \frac{1}{\ell_2\ell_3\ell_5} - \frac{1}{\ell_1\ell_2\ell_6} - \frac{1}{\ell_1\ell_3\ell_6} + \frac{1}{\ell_1\ell_2\ell_3}.\end{aligned} \tag{17}$$

In physics, this sum of ten terms naturally arises when computing the so-called "bulk integrals" constructed from a set of Feynman rules. For our current discussion, it suffices to know that, according to these rules, every vertex is associated to a time integral and physically represents an interaction between particles. Every edge is represented by a "bulk-to-bulk propagator": a Green's function for the wave equation with cosmological boundary conditions. The propagator tells us how to order the two time integrals corresponding to the two vertices connected to the edge. Let v_1 and v_2 be two vertices. There are three options appearing as a sum in the bulk-to-bulk propagator: the vertex v_1 happens before v_2 in time, v_2 before v_1, or the two are unordered, i.e., disconnected. The integrand consists of a product of $n-1$ propagators which, after expanding, results in 3^{n-1} different terms. All of these different time-orderings are in one-to-one correspondence with the diagrams from the combinatorial approach allowing for a physical interpretation. An edge corresponds to a particle travelling between two vertices with the orientation of the edge describing the direction in time. In other words, all the oriented spanning graphs are different time-orderings in a physical process.

Example 3.8 (One-loop bubble)**.** In this example, we work out the partial fraction decomposition of the wavefunction in flat space for the genus-1 graph with two vertices, called "one-loop bubble" in the physics literature. For an illustration of this graph, see the Introduction of [4]. The hyperplanes appearing in $\psi_{\text{1-loop}}^{\text{flat}}$ can be derived from the connected subgraphs by using (7). They are

$$\ell_1 = X_1+Y+Y',\ \ell_2 = X_2+Y+Y',\ \ell_3 = X_1+X_2+2Y',\ \ell_4 = X_1+X_2+2Y,\ \ell_5 = X_1+X_2.$$

The set $\mathcal{F}$ contains nine elements, two of which are the leftmost graphs in Figure 3, which will not contribute in the decomposition, while the other ones are displayed in Figure 5.

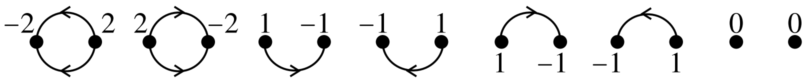
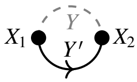


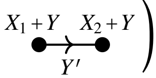
Figure 5: The oriented spanning subgraphs for the one-loop bubble graph.

Hence, the partial fraction decomposition is

$$\psi^{\text{flat}}_{\text{1-loop}} = \frac{1}{\ell_2\ell_5} + \frac{1}{\ell_1\ell_5} - \frac{1}{\ell_1\ell_4} - \frac{1}{\ell_2\ell_4} - \frac{1}{\ell_1\ell_3} - \frac{1}{\ell_2\ell_3} + \frac{1}{\ell_1\ell_2},$$

where the order of the terms matches the one of the spanning subgraphs in Figure 5. The diagrams with at least one edge being deleted can be regarded as oriented spanning trees for the two-site chain after we redefine the variables associated to vertices. In physics, this recurrence is related to the so-called "tree theorem" [29]. It is best illustrated by the example where we delete the upper edge from the bubble:

$R\Big($ 

$\Big) = R\Big($ 

$\Big) = \dfrac{1}{\ell_1\ell_4}.$

The primary motivation for applying partial fraction decomposition to the integrand is to facilitate and streamline the computation of the integral. The cosmological integral in (8) would turn into the linear combination

$$\sum_{H_i \in \mathcal{F}} (-1)^{n-1-|E(H_i)|} \cdot \int_{\mathbb{R}^n_{>0}} \prod_{j=1}^{s_i} R_{H_i^j}(X+\alpha, Y)\alpha^{\varepsilon}\,\mathrm{d}\alpha. \tag{18}$$

The singularities of generalized Euler integrals with integrands involving only linear forms were studied in [11]. Specifically, the singularities characterize the locus where the Euler characteristic of the complement of the arrangement of hyperplanes in the integrand is smaller than the generic one.[1] For real hyperplane arrangements, the signed Euler characteristic counts the number of bounded chambers. Here, we focus on the complement of hyperplanes $\mathcal{H}^G_{(X,Y)}$ associated to cosmological integrals. In the case in which the coefficients of the hyperplanes vary in a subspace Z of the coefficients space $\mathbb{C}^A$, as in Section 2.3, the singular locus is described by the Euler discriminant [12]. To give the definition, we first fix some notation. For each $z \in Z$, let

$$\mathcal{H}^G(z) := \{\alpha \in (\mathbb{C}^*)^n \,:\, L_i(z;\alpha) \neq 0,\ i = 1,\dots,k\},$$

and denote by χ_z the value of its signed Euler characteristic, $(-1)^n \cdot \chi(\mathcal{H}^G(z))$.

[1] This statement was communicated to us in conversations with S.-J. Matsubara-Heo. It is getting addressed by him in ongoing work.

The *Euler discriminant* is defined as the locus

$$\nabla_\chi(Z) = \{z \in Z : \chi_z < \chi^*\} \tag{19}$$

for which the Euler characteristic is strictly smaller than its maximal value $\chi^* = \max_{z\in Z}\{\chi_z\}$. Note that $\nabla_\chi(Z)$ is a closed subvariety of Z, see [12, Theorem 3.1].

Let us represent each hyperplane L_i as a scalar product between the vector $(\alpha, 1) = (\alpha_1, \dots, \alpha_n, 1)$ and the one describing the coefficients, denoted T_i. For instance, L_1 obtained from ℓ_1 in (16) has the coefficient vector $T_1 = (1, 0, X_1 + Y)$.

Consider the matrix $M'_G = [T_1 | \cdots | T_k]$ whose i-th column represents the coefficient vector of the hyperplane L_i. Let $M_G = [I_{n+1} | M'_G]$, where I_{n+1} is the identity matrix of size $n+1$. In [11], it was proven that the Euler discriminant is a hypersurface and its defining polynomial $E_\chi(Z)$ can be expressed in terms of maximal minors of the matrix M_G, namely

$$E_\chi(Z) = \prod_{\substack{I\subset[n+1+k],\, |I|=n+1 \\ \det(M_G^I)\neq 0}} \det\left(M_G^I\right),$$

where M_G^I denotes the submatrix of M_G whose columns are indexed by I.

In [3], the singularities of the cosmological integrals of trees were read off from their differential equations in matrix form, and they can be described and characterized in terms of complete tubings of another type of decorated graphs. However, these singularities constitute a subset of the factors of the polynomial $E_\chi(Z)$. In the physics literature, the linear factors occurring as entries of the dlog's in the connection matrix in ε-factorized form as in [3, (3.30)] are called "letters" of the differential equation. In our example, all letters occur as factors of the singular locus. Moreover, all of these factors describe configurations of the kinematic variables for which the solution can develop a singularity that is expected from physical principles. These configurations are called *physical singularities*, see e.g. [28].

Example 3.9. The singularities of the complements of hyperplanes $\mathcal{H}^G_{(X,Y)}$ associated to the 3-site chain can be read from [12, Example 5.4]. These in fact all correspond, up to a sign, to non-zero minors of the prescribed matrix M_G. However, the set of physical singularities are a subset of these, given by:

$$X_1+Y_{12},\ X_2+Y_{12}+Y_{23},\ X_3+Y_{23},\ X_1+X_2+X_3,\ X_1+X_2+Y_{23},\ X_2+X_3+Y_{12},$$
$$X_1-Y_{12},\ X_2+Y_{12}-Y_{23},\ X_2-Y_{12}+Y_{23},\ X_2-Y_{12}-Y_{23},$$
$$X_1+X_2-Y_{23},\ X_2+X_3-Y_{12},\ X_3-Y_{23}.$$

We observed a connection between the terms in the partial fraction decomposition of the integral induced by (17) and this subset of singularities.

In the remainder of this section, we will explain this relation in its full generality for arbitrary graphs.

Definition 3.10. Given an element $H_i \in \mathcal{F}$, we say that H_i is *totally time-ordered* if the associated rational function R_{H_i} is given by a single term.

The motivation for the naming comes from cosmology: the totally time-ordered diagrams specify the time-ordering of all the vertices, and not just of the nearest neighbors. A counter-example would be the gray diagram in Figure 4, for which the orientations are not sufficient to fix the ordering of the vertex X_1 with respect to X_3. We observed for $n = 2, 3, 4$ that all the physical singularities can be recovered by computing the singularities of the integrals in (18) that correspond to totally time-ordered subgraphs. The example below provides a more explicit explanation of this observation.

Example 3.11 (Example 3.9 continued)**.** Let $n = 3$. The partial fraction decomposition for the 3-site chain is exhibited in (17). We can construct a matrix $[I_4 \mid T_i T_j T_k]$ of size 4×7 for each fraction of the form $1/(L_i L_j L_k)$ induced by the terms in (17). One can check that the physical singularities, displayed in Example 3.9, coincide with the minors of the matrices arising from the integrals associated to totally time-ordered graphs.

4. Conclusion

Cosmological integrals have interesting mathematical properties, and, despite having been studied in various guises over the past twenty years, have only recently been looked at systematically, in the physics literature. It is even earlier days in their mathematical exploration. In this article, we took some first steps in uncovering their properties, using some algebraic tools.

Focusing on the integrals constructed in [3], we began to shed light on differential and difference equations satisfied by cosmological correlation functions, utilizing the Weyl algebra and shift algebras. For the two-site chain, we elaborated how the differential equations from [3] can be obtained in terms of a restricted GKZ system. It deserves further study whether this connection holds true in greater generality, and it would be interesting to see which results from the theory of GKZ systems can be imported to handle cosmological integrals.

We constructed a multivariate version of partial fraction decomposition for the flat space wavefunction. This decomposition is described via spanning subgraphs of the modeling graph, which we conjecture to hold true for arbitrary graphs. This beautiful connection to subgraphs has an interesting physical interpretation in terms of spacetime locality of interactions between particles. We are likely just scratching the surface of an interesting connection between the wavefunction and decorated graphs—a theme that has appeared in various forms in the past few years. We also hope to be able to shed light on the observed "dimension-drop" in the matrix differential system of cosmological integrals. The dimension of the full space of master integrals is that of the top twisted cohomology space of the underlying very affine variety. In [3], it was observed that certain subspaces of master integrals can be chosen to obtain a closed subsystem of physically relevant ones; for the three-site graph, e.g., 16 among originally 25. One should explore whether our method for partial fractioning of the flat space wavefunction can contribute to explaining this phenomenon.

There are several immediate directions for further exploration and development of our findings. It is clear that cosmological integrals still have many mathematical features yet to be fully understood, and the necessary tools to understand them will be very useful to physicists. Conversely, the examples from physics might suggest new mathematical structures, which are interesting and deserve of attention on their own right.

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