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BINARY GEOMETRIES FROM PELLYTOPES

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Binary geometries have recently been introduced in particle physics in connection with stringy integrals. In this work, we study a class of simple polytopes, called *pellytopes*, whose number of vertices are given by Pell's numbers. We provide a new family of binary geometries determined by pellytopes as conjectured by He–Li–Raman–Zhang. We relate this family to the moduli space of curves by comparing the pellytope to the ABHY associahedron.

1. Introduction

Binary geometries are affine varieties with stratifications determined by certain simplicial complexes. These curious geometric objects – like positive geometries – first arose from the study of canonical forms on polytopes and amplituhedra, as a novel method to compute scattering amplitudes [2, 5]. The stratification of the binary geometry leads to a factorization of the amplitude. Besides the systematically studied examples of binary geometries arising from simplicial complexes associated with finite type cluster algebras and generalized permutahedra [4, 9, 11], for instance the ABHY kinematic associahedron [3], other classes of binary geometries related to amplitudes for Grassmannians have been studied in [6–8]. In this paper we initiate the study of binary geometries more abstractly by focusing on an elementary example.

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More precisely, given a flag simplicial complex Δ on $[n]$ we write $i \nsim j$ for $i, j \in \Delta$ if $\{i, j\} \notin \Delta$. We associate to each $i \in [n]$ a polynomial in $\mathbb{C}[u_1, \dots, u_n]$ determining a *u-equation*

$$R_i = u_i + \prod_{j \nsim i} u_j^{a_{ij}} - 1 = 0 \tag{1}$$

for some integers $a_{ij} > 0$. If $a_{ij} \in \{0, 1\}$ for all i and j the binary geometry is called *perfect*. The affine variety in $\mathbb{C}^n$ defined by the n equations of this form is a binary geometry if it satisfies certain boundary conditions (see Definition 2.2). In particular, a binary geometry is stratified by binary geometries corresponding to links of Δ, see Lemma 2.5. Notice that requiring all coordinates to be real and non-negative forces them to be in the interval $[0, 1]$; hence the name *binary*.

In this paper we focus on the simplicial complex arising from a polytope not belonging to either of the aforementioned classes: the *pellytope* is defined as

$$\mathcal{P}_d := \mathrm{Newt}\left(\prod_{i=1}^{d} (1 + y_i) \prod_{j=1}^{d-1} (1 + y_j + y_j y_{j+1}) \right) \subset \mathbb{R}^d. \tag{2}$$

It is a d-dimensional simple polytope with $3d - 1$ facets and its number of vertices is given by *Pell's number* n_{d+1}, defined recursively by

$$n_1 = 1, \quad n_2 = 2 \quad \text{and} \quad n_d = 2n_{d-1} + n_{d-2} \tag{3}$$

(see Corollary 3.2). The (inner) normal fan of $\mathcal{P}_d$, denoted Σ_d, determines a flag simplicial complex. The pellytope has been studied by physicists in [9, §4] as an example of a simplicial fan with desirable combinatorial properties: stars of Σ_d factor as products of copies of Σ_i for $i < d$ (see (8) and Lemma 3.5). These lead the authors in [9] to conjecture that the pellytope determines a binary geometry. We verify their conjecture:

Theorem 1.1. *The pellytope $\mathcal{P}_d$ determines a binary geometry $\widetilde{\mathcal{U}}_d \subset \mathbb{C}^{3d-1}$ defined by $3d - 1$ u-equations.*

We call $\widetilde{\mathcal{U}}_d$ the *Pellspace*. It is closely related to the ABHY realisation of the associahedron. The ABHY associahedron $\mathcal{A}_d$ is realized as the Newton polytope of a polynomial divisible by the polynomial defining $\mathcal{P}_d$ in (2), so the normal fan of $\mathcal{A}_d$ is a refinement of Σ_d. The binary geometry associated to $\mathcal{A}_d$ is an affine chart of the moduli space of stable rational curves, denoted by $\widetilde{\mathcal{M}}_{0,n}$ where $n = d + 3$. This enables us to prove the following statement, which addresses an open question in [10, Problem 5.27 (d)].

Corollary 1.2. *$\widetilde{\mathcal{M}}_{0,n}$ is an affine subset of a blowup of the binary geometry $\widetilde{\mathcal{U}}_d$ of the pellytope $\mathcal{P}_d$.*

We therefore expect that there exists an alternative compactification of $\mathcal{M}_{0,n}$ in which $\widetilde{\mathcal{U}}_d$ is an affine chart. We explore the case of $d = 3$ in the final section. It would be interesting to understand the moduli interpretation and combinatorial structure of this space more generally.

Outline. In §2 we recall the definition of a binary geometry and the necessary concepts from polyhedral geometry. In §3 we study the pellytope and its combinatorics. In §4.1 we study the very affine variety determined by the pellytope, which is followed by the proof of Theorem 1.1 in §4.2 and the proof of Corollary 1.2 in §4.3.

2. Binary geometries

Let $[n] = \{1, \ldots, n\}$ and consider a simplicial complex Δ on $[n]$. More precisely, Δ is a non-empty collection of subsets of $[n]$ satisfying the following property:

(i) If $S \in \Delta$ and $S' \subset S$ then $S' \in \Delta$.

If, additionally, a simplicial complex Δ satisfies:

(ii) $\{k\} \in \Delta$ for each $k \in [n]$ and

(iii) if $\{k_1, \ldots, k_r\} \subset [n]$ with $\{k_i, k_j\} \in \Delta$ for all $1 \leq i < j \leq r$, then $\{k_1, \ldots, k_r\} \in \Delta$

we call Δ a *flag complex*. A simplicial complex Δ is *pure* if all the maximal sets in Δ (with respect to inclusion) have the same cardinality. We say that $i, j \in [n]$ are *incompatible* if $\{i, j\} \notin \Delta$, and write $i \nsim j$.

Before we proceed to recall the definition of a binary geometry, we discuss two examples of flag complexes that will serve as the running examples of this section.

Example 2.1. (a) On $[2] = \{1, 2\}$, the collection of subsets $\Delta = \{\{1\}, \{2\}\}$ form a pure flag complex where $1, 2$ are incompatible.

(b) On $[4] = \{1, 2, 3, 4\}$, the collection of subsets

$$\Delta = \{\{1\}, \{2\}, \{3\}, \{4\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}\}$$

is a pure flag complex. This flag complex is combinatorially isomorphic to the inner normal fan of a square (see Figure 1). In this paper, we will mostly consider flag complexes that correspond to the inner normal fans of certain polytopes.

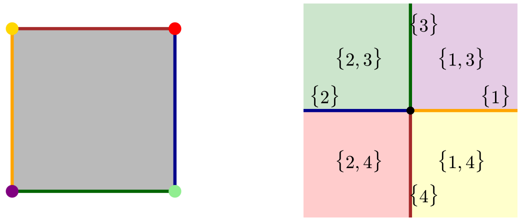


Figure 1: A square and its inner normal fan, which is isomorphic to the flag complex in Example 2.1(b).

Definition 2.2. [10, Definition 2.1] Let Δ be a flag simplicial complex on $[n]$. A *binary geometry* $\widetilde{U}$ for Δ is an affine algebraic variety $\widetilde{U} \subset \mathbb{C}^n$ cut out by n equations of the form (1), satisfying the following properties:

(i) $\dim \widetilde{U} = \max_{S \in \Delta} \#S$;

(ii) for $S \subset [n]$ the subvariety $\widetilde{U}_S = \widetilde{U} \cap \{u \in \mathbb{C}^n \mid u_i = 0\ \forall i \in S\}$ is non-empty if and only if $S \in \Delta$;

(iii) if $\widetilde{U}_S$ is non-empty, it is irreducible of codimension $\#S$ in $\widetilde{U}$.

Example 2.3. The flag complex from Example 2.1(a) gives rise to the equations

$$R_1 = u_1 + u_2 - 1 = 0, \qquad R_2 = u_2 + u_1 - 1 = 0.$$

The variety $\widetilde{U} = \{u \in \mathbb{C}^2 \mid R_1(u) = 0,\ R_2(u) = 0\}$ has dimension 1 – thus property (i) in Definition 2.2 is satisfied. For $S = \{1,2\}$, the subvariety

$$\widetilde{U}_S = \left\{u \in \mathbb{C}^2 \mid u_1 + u_2 - 1 = 0,\ u_1 = 0,\ u_2 = 0\right\}$$

is empty. For $S = \{1\}$ and $S = \{2\}$, we have

$$\widetilde{U}_{\{1\}} = \left\{u \in \mathbb{C}^2 \mid u_1 + u_2 - 1 = 0,\ u_1 = 0\right\} = \left\{(0,1)\right\}, \quad \widetilde{U}_{\{2\}} = \left\{(1,0)\right\},$$

which are irreducible subvarieties of codimension 1. Thus, $\widetilde{U}$ is a binary geometry. By [10, Theorem 2.9], $\widetilde{U}$ is the only one dimensional binary geometry.

If $\widetilde{U}$ is a binary geometry, then the non-empty subvarieties $\widetilde{U}_S$ for $S \in \Delta$ are also binary geometries. Their underlying flag complexes can be described as follows. Following the terminology used in [10, Proposition 2.6], we define the *link* $\mathrm{lk}_\Delta S$ of $S \in \Delta$ as the simplicial complex

$$\mathrm{lk}_\Delta S := \left\{T \in \Delta \mid S \cap T = \varnothing \text{ and } S \cup T \in \Delta\right\} \tag{4}$$

on the set $W_S = \{j \in [n] \setminus S \mid S \cup \{j\} \in \Delta\}$. In other words, the set W_S contains all j that are compatible with S but do not lie in S.

Example 2.4. Consider the flag complex from Example 2.1(b). The link of $S = \{3\}$ equals $\mathrm{lk}_\Delta(\{3\}) = \{\{1\},\{2\}\}$.

If there exists a binary geometry for a flag complex Δ, then Δ is pure by [10, Corollary 2.7]. Lemma 2.5 relates $\widetilde{U}_S$ to the link $\mathrm{lk}_\Delta(S)$. Part (b) will be used in the proof of Theorem 1.1.

Lemma 2.5. *Let Δ be a pure flag complex on $[n]$, and let $\widetilde{U}$ be an affine algebraic variety $\widetilde{U} \subset \mathbb{C}^n$ cut out by n equations of the form as in* (1)*. For $S \in \Delta$, the subvariety $\widetilde{U}_S = \widetilde{U} \cap \{u \in \mathbb{C}^n \mid u_k = 0 \ \forall k \in S\}$ is cut out by the equations:*

$$u_k = 0 \quad \textit{for } k \in S, \qquad u_j = 1 \quad \textit{for } j \notin W_S,$$
$$R_i = u_i + \prod_{\ell \in W_S\colon \ell \nsim i} u_\ell^{a_{i\ell}} - 1 = 0 \quad \textit{for } i \in W_S.$$

Moreover, the following hold:

(a) If $\widetilde{U}$ is a binary geometry and $S \in \Delta$, then $\widetilde{U}_S$ is a binary geometry with underlying simplicial complex $\mathrm{lk}_\Delta S$.

(b) If $\widetilde{U}$ is an irreducible variety of dimension $\max_{S\in\Delta} \#S$, and $\widetilde{U}_{\{k\}}$ is a binary geometry with underlying simplicial complex $\mathrm{lk}_\Delta\{k\}$ for each $k \in [n]$, then $\widetilde{U}$ is a binary geometry.

Proof. We prove (b); part (a) follows directly from the proof of [10, Proposition 2.6]. Let $S \in \Delta$ and choose $k \in S$. By assumption, $\widetilde{U}_{\{k\}}$ is a binary geometry with

$$\dim \widetilde{U}_{\{k\}} = \max_{T \in \mathrm{lk}_\Delta\{k\}} \#T = \left(\max_{V\in\Delta} \#V\right) - 1 = \dim \widetilde{U} - 1.$$

For the second equality above we use that Δ is pure, which implies that there exists $V \in \Delta$ such that $k \in V$ and $\#V = \max_{S\in\Delta} \#S$. Since $(S \setminus \{k\}) \in \mathrm{lk}_\Delta(\{k\})$ and $\widetilde{U}_{\{k\}}$ is a binary geometry, it follows that $\widetilde{U}_S = (\widetilde{U}_{\{k\}})_{S\setminus\{k\}}$ is a non-empty irreducible variety of codimension $\#S$ in $\widetilde{U}$. Thus, $\widetilde{U}$ is a binary geometry. □

The product of two binary geometries is also a binary geometry. To state this result, we recall the definition of the *product of two simplicial complexes*. Let Δ, Δ' be simplicial complexes defined on $[n]$, $\{n+1, \ldots, n+m\}$ respectively. The product simplicial complex is defined as $\Delta \times \Delta' := \{F \cup F' \mid F \in \Delta, F' \in \Delta'\}$. Note that every $\{k\} \in \Delta$ is compatible with every $\{k'\} \in \Delta'$ in $\Delta \times \Delta'$. The following simple facts about flag complexes will be used in Section 3.

Lemma 2.6. *(i) If Δ and Δ' are both flag complexes, then $\Delta \times \Delta'$ is a flag complex. (ii) If Δ is a flag complex and Δ' is the subdivision of Δ induced by bisecting a single 1-simplex, then Δ' is a flag complex.*

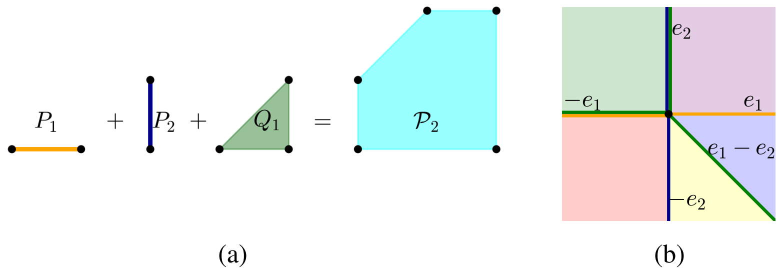


Figure 2: (a) The pellytope $\mathcal{P}_2$. (b) The inner normal fan of $\mathcal{P}_2$, which is the common refinement of the inner normal fans of P_1, P_2 and Q_1.

Proposition 2.7. *[10, Proposition 2.11] Let $\widetilde{U}$ and $\widetilde{U}'$ be binary geometries for flag complexes Δ and Δ' on disjoint sets. Then $\widetilde{U} \times \widetilde{U}'$ is a binary geometry for the product simplicial complex $\Delta \times \Delta'$.*

Example 2.8. For the binary geometry $\widetilde{U}$ from Example 2.3, we have

$$\widetilde{U} \times \widetilde{U} = \left\{ u \in \mathbb{C}^4 \,\middle|\, u_1 + u_2 - 1 = 0,\ u_3 + u_4 - 1 = 0 \right\}.$$

This variety equals the binary geometry given by the flag complex from Example 2.1(b). For $S = \{3\}$, the subvariety $(\widetilde{U} \times \widetilde{U})_{\{3\}}$ is cut out by the equations $u_3 = 0$, $u_4 = 1$ and $u_1 + u_2 - 1 = 0$. Thus $(\widetilde{U} \times \widetilde{U})_{\{3\}}$ is isomorphic to the one dimensional binary geometry $\widetilde{U}$ from Example 2.3.

3. Pellytopes

For each $d \in \mathbb{N}$, the pellytope $\mathcal{P}_d \subset \mathbb{R}^d$ defined in (2) is the Minkowski sum of

$$P_i = \mathrm{Conv}\{0, e_i\}, \quad Q_j = \mathrm{Conv}\{0, e_j, e_j + e_{j+1}\}, \tag{5}$$

where $i \in [d], j \in [d-1]$ and $e_1, \ldots, e_d$ denote standard basis vectors of $\mathbb{R}^d$. In Corollary 3.2, we show that the number of vertices of $\mathcal{P}_d$ is given by Pell's number (3). For $d = 1$, the pellytope is the segment $\mathrm{Conv}\{0, e_1\}$ and its inner normal fan is combinatorially isomorphic to the flag complex in Example 2.1(a). For $d = 2$, we depicted the pellytope and its inner normal fan in Figure 2. The main result of this section is:

Proposition 3.1. *For each $d \in \mathbb{N}$, the inner normal fan Σ_d of the pellytope $\mathcal{P}_d$ is a simplicial fan with n_{d+1} maximal cones and $3d - 1$ rays spanned by the vectors $e_1, \ldots, e_d, -e_1, \ldots, -e_d, e_1 - e_2, \ldots, e_{d-1} - e_d$.*

Before proving Proposition 3.1, we recall some basic notions of polyhedral geometry – for details see [14]. Given a polytope $P \subset \mathbb{R}^n$ we denote its *inner normal fan* by Σ_P. To a fan Σ one associates a simplicial complex as follows. For a fixed order $\rho_1,\ldots,\rho_k$ of the rays of Σ, we define

$$\Delta(\Sigma) := \left\{ S \subset [k] \;\middle|\; \mathrm{Cone}(\rho_i \mid i \in S) \in \Sigma \right\}.$$

For an example, we refer to Figure 1. If Σ is a *simplicial fan*, that is, each cone $C \in \Sigma$ is generated by linearly independent vectors, then $\Delta(\Sigma)$ is a simplical complex.

Denote the Minkowski sum of two polytopes $P, Q \subset \mathbb{R}^n$ by $P+Q$; its inner normal fan is the *common refinement* $\Sigma_P \wedge \Sigma_Q$ of Σ_P and Σ_Q, that is,

$$\Sigma_{P+Q} = \Sigma_P \wedge \Sigma_Q = \left\{ C \cap C' \mid C \in \Sigma_P, C' \in \Sigma_Q \right\}. \tag{6}$$

For $P \subset \mathbb{R}^n$ (resp. $Q \subset \mathbb{R}^m$) denote $\iota_n(P)$ (resp. $\iota_m(Q)$) the inclusion of P (resp. Q) into the span of the first n (resp. last m) basis vectors of $\mathbb{R}^{n+m}$. A simple computation shows that

$$\Sigma_{\iota_n(P)+\iota_m(Q)} = \Sigma_P \times \Sigma_Q = \left\{ C \times C' \;\middle|\; C \in \Sigma_P, C' \in \Sigma_Q \right\}. \tag{7}$$

For a set $S \subset \mathbb{R}^d$, we denote the linear span of S by $L(S)$.

Proof of Proposition 3.1. By definition, the pellytope $\mathcal{P}_d$ is the Minkowski sum $\mathcal{P}_{d-1} + P_d + Q_{d-1}$. The fan Σ_{P_d} has two rays, generated by $e_d, -e_d$. For brevity we write $L := L(e_1,\ldots,e_{d-2})$, and denote by $(\Sigma_{Q_{d-1}} + L)$ the inner normal fan of Q_d viewed as a polytope in $\mathbb{R}^d$. That is, this fan contains the following cones

(1) L, $\mathrm{Cone}(e_d)+L$, $\mathrm{Cone}(-e_{d-1})+L$, $\mathrm{Cone}(e_d,-e_{d-1})+L$,

(2) $\mathrm{Cone}(e_{d-1}-e_d)+L$, $\mathrm{Cone}(e_d, e_{d-1}-e_d)+L$, $\mathrm{Cone}(-e_{d-1}, e_{d-1}-e_d)+L$.

From (6) and (7) it follows that $\Sigma_d = \left(\Sigma_{d-1} \times \Sigma_{P_d}\right) \wedge (\Sigma_{Q_{d-1}} + L)$. Using (7), the fan $\Sigma_{d-1} \times \Sigma_{P_d}$ is straightforward to compute. One adds to every cone in Σ_{d-1} either a ray $\mathrm{Cone}(e_d)$, $\mathrm{Cone}(-e_d)$ or 0. The intersection of the cones in $\Sigma_{d-1} \times \Sigma_{P_d}$ with the cones in (1) is either $\{0\}$ or a cone in $\Sigma_{d-1} \times \Sigma_{P_d}$. Intersecting the cones in $\Sigma_{d-1} \times \Sigma_{P_d}$ with the cones in (2) gives the following new cones in the common refinement: $\mathrm{Cone}(e_{d-1}-e_d)$, $\mathrm{Cone}(e_{d-1}-e_d, -e_d)$, $\mathrm{Cone}(e_{d-1}-e_d, e_{d-1})$. Thus, Σ_d is the refinement of $\Sigma_{d-1} \times \Sigma_{P_d}$ by adding the ray generated by $e_{d-1}-e_d$.

Using the above observations, we complete the proof by induction on d. For $d = 1,2$, the normal fan Σ_1 (resp. Σ_2) is simplicial and has 2 (resp. 5) rays and maximal cones (see Figure 2(b) for $d = 2$). Assume that the statement of the proposition is true for $k \leq d-1$ for some $d \geq 3$. Since Σ_{d-1} is simplicial, the fan $\Sigma_{d-1} \times \Sigma_{P_d}$ is also simplicial and has twice as many maximal cones as Σ_{d-1}.

The ray $\mathrm{Cone}(e_{d-1} - e_d)$ is contained in $\mathrm{Cone}(e_{d-1}, -e_d) \in \Sigma_{d-1} \times \Sigma_{P_d}$, which implies that Σ_d is a simplicial fan. Since Σ_d is the refinement of $\Sigma_{d-1} \times \Sigma_{P_d}$ by adding the ray generated by $e_{d-1} - e_d$, it follows that Σ_d has $3(d-1)+2+1 = 3d-1$ rays. Every maximal cone of $\Sigma_{d-1} \times \Sigma_{P_d}$ containing $e_{d-1} - e_d$ contributes two maximal cones of Σ_d (so one extra). The number of maximal cones of $\Sigma_{d-1} \times \Sigma_{P_d}$ containing $e_{d-1} - e_d$ is the number of $(d-1)$-dimensional cones in Σ_{d-1} that contain e_{d-1} which is the number of maximal cones in Σ_{d-1}. Thus, Σ_d has $2n_d + n_{d-1}$ maximal cones. □

Corollary 3.2. *For each $d \in \mathbb{N}$, the pellytope $\mathcal{P}_d$ is a simple polytope with $3d-1$ facets and n_{d+1} vertices.*

Lemma 3.3. *The simplicial complex $\Delta(\Sigma_d)$ is a flag complex.*

Proof. The result follows from Lemma 2.6 and the recursive construction of Σ_d. We have $\Delta(\Sigma_{d-1} \times \Sigma_{P_d}) = \Delta(\Sigma_{d-1}) \times \Delta(\Sigma_1)$. Since Σ_d is the refinement of $\Sigma_{d-1} \times \Sigma_{P_d}$ by adding the ray $\mathrm{Cone}(e_{d-1} - e_d)$, the simplical complex $\Delta(\Sigma_d)$ is the subdivision of $\Delta(\Sigma_{d-1}) \times \Delta(\Sigma_1)$ induced by bisecting the 1-simplex given by the 2-cone $\mathrm{Cone}(e_{d-1}, -e_d) \in \Sigma_{d-1} \times \Sigma_{P_d}$. It is clear that $\Delta(\Sigma_1)$ is a flag complex, and so it follows that $\Delta(\Sigma_d)$ is a flag complex for all $d > 0$. □

The two simplices in $\Delta(\Sigma_d)$ given by the rays ρ and ρ' are compatible if and only if $\mathrm{Cone}(\rho, \rho') \in \Sigma_d$. In this case we say that the rays are compatible.

Lemma 3.4. *The incompatible pairs of rays in Σ_d are generated by the following $5d-5$ pairs of vectors:*

$$\begin{aligned} e_i \not\sim -e_i \quad & \forall i \in [d] \\ e_i \not\sim -e_{i+1}, \quad -e_i \not\sim e_i - e_{i+1}, \quad e_{i+1} \not\sim e_i - e_{i+1} \quad & \forall i \in [d-1] \\ e_i - e_{i+1} \not\sim e_{i+1} - e_{i+2} \quad & \forall i \in [d-2]. \end{aligned}$$

Proof. The rays generated by $\pm e_i, \pm e_{i+1}$ and $e_i - e_{i+1}$ are coplanar, so any pair of these five rays are compatible only if they are adjacent in the two-dimensional fan $\Sigma_d \cap (L(e_i) + L(e_{i+1}))$. This gives incompatibility between the rays of Σ_d generated by the first four types of pairs listed in the Lemma. The cone spanned by e_i and $-e_{i+2}$ lies in Σ_d and bisects the cone spanned by $e_i - e_{i+1}$ and $e_{i+1} - e_{i+2}$, so the two rays $\mathrm{Cone}(e_i - e_{i+1})$ and $\mathrm{Cone}(e_{i+1} - e_{i+2})$ are incompatible.

All other pairs of rays are compatible. This can be checked by induction, using the recursive construction of Σ_d from Σ_{d-1}. The d-dimensional fan $\Sigma_{d-1} \times \Sigma_1 e_d$ contains two rays which are not contained in Σ_{d-1}, namely $\mathrm{Cone}(e_d)$ and $\mathrm{Cone}(-e_d)$. These two rays are incompatible with each other, and compatible with every ray in Σ_{d-1}. Any two rays in Σ_{d-1} are compatible in $\Sigma_{d-1}\times\Sigma_1 e_d$ if and only if they are compatible in Σ_{d-1}. Now Σ_d is the refinement of $\Sigma_{d-1}\times\Sigma_{P_d}$ by the ray $\rho := \mathrm{Cone}(e_{d-1}-e_d)$. This ray bisects the two-cone spanned by e_{d-1} and $-e_d$, so the two rays $\mathrm{Cone}(e_{d-1})$ and $\mathrm{Cone}(-e_d)$ are the only compatible pair in $\Sigma_{d-1}\times\Sigma_{P_d}$ to become incompatible in Σ_d. Moreover, ρ is compatible with a ray ρ' in Σ_d if and only if ρ' is compatible with both $\mathrm{Cone}(e_{d-1})$ and $\mathrm{Cone}(-e_d)$ in $\Sigma_{d-1}\times\Sigma_{P_d}$. Thus the only rays of Σ_d which are incompatible with ρ are $\mathrm{Cone}(-e_{d-1})$, $\mathrm{Cone}(e_d)$ and $\mathrm{Cone}(e_{d-2}-e_{d-1})$. □

We conclude this section by investigating the links of simplices in $\Delta(\Sigma_d)$. Let τ be a cone in a fan $\Sigma\subset\mathbb{R}^n$. The *star of* τ is defined as the projection of the cones in Σ containing τ under the natural projection map $\pi\colon\mathbb{R}^n\to\mathbb{R}^n/L(\tau)$, i.e.,

$$\mathrm{star}_\Sigma(\tau) := \{\pi(\sigma)\mid \tau\subset\sigma\in\Sigma\}. \tag{8}$$

The simplicial complex $\Delta(\mathrm{star}_\Sigma(\tau))$ is combinatorially isomorphic to the link $\mathrm{lk}_{\Delta(\Sigma)}\,\tau$. Combined with Lemma 2.5, the following characterisation of the stars of rays in Σ_d allows us to prove that the pellytopes define binary geometries by induction on d.

Lemma 3.5. *Let ρ_v denote the ray in Σ_d generated by the vector $v\in\mathbb{R}^d$, then*

$$\mathrm{star}_{\Sigma_d}(\rho_v)\cong\begin{cases}\Sigma_{i-1}\times\Sigma_{d-i} & \text{if } v\in\{\pm e_i\mid i=1,\dots,d\},\\ \Sigma_{i-1}\times\Sigma_1\times\Sigma_{d-i-1} & \text{if } v\in\{e_i-e_{i+1}\mid i=1,\dots,d-1\},\end{cases}$$

where $\cong$ denotes linear isomorphism of fans, and Σ_0 is a single point.

Proof. Note that $\Delta(\mathrm{star}_{\Sigma_d}(\rho_v))$ is a flag complex, since any link of a flag complex is itself a flag complex. The fans on the right hand side of the equation also induce flag complexes, by Lemma 2.6(i). To show that the fans are isomorphic, it therefore suffices to check that their 2-skeletons are isomorphic. When $v=\pm e_i$, the linear isomorphism $\mathbb{R}^{i-1}\times\mathbb{R}^{d-i}\to\mathbb{R}^d/L(e_i)$ given by $(e_j,0)\mapsto e_j$ and $(0,e_k)\mapsto e_{k+i}$ clearly induces a bijection between the rays of $\Sigma_{i-1}\times\Sigma_{d-i}$ and the rays of $\mathrm{star}_{\Sigma_d}(\rho_v)$. Similarly, the linear isomorphism $\mathbb{R}^{i-1}\times\mathbb{R}\times\mathbb{R}^{d-i-1}\to\mathbb{R}^d/L(e_i-e_{i+1})$ given by $(e_j,0,0)\mapsto e_j$, $(0,e_1,0)\mapsto e_i$ and $(0,0,e_k)\mapsto e_{k+i+1}$ induces a bijection between the rays of $\Sigma_{i-1}\times\Sigma_1\times\Sigma_{d-i-1}$ and the rays of $\mathrm{star}_{\Sigma_d}(\rho_{e_i-e_{i+1}})$. Two rays in $\mathrm{star}_{\Sigma_d}(\rho_v)$ are compatible if and only if their preimages under π are compatible rays in Σ_d. To see that the linear isomorphisms above induce isomorphisms of the 2-skeletons, we may therefore use the characterisation of the compatibility between rays of Σ_k given above.

For example, suppose that $v=e_i$ for some $i\in\{2,\dots,d-1\}$. The only rays in Σ_d incompatible with ρ_v are the three rays generated by $-e_i$, $-e_{i+1}$ and $e_{i-1}-e_i$. The compatibility structure between rays of Σ_d contained in $L(e_1,\dots,e_{i-1})$ gives the compatibility structure of Σ_{i-1}, and every one of these rays is compatible with every ray in $\Sigma_d \cap L(e_{i+1},\dots,e_d)$ apart from $-e_{i+1}$. Since $-e_{i+1} \equiv e_i - e_{i+1}$ mod $L(e_i)$, the linear map $(0,e_k) \mapsto (0,e_{k+i})$ identifies $\mathrm{Cone}(-e_1) \in \Sigma_{d-i}$ with $\pi(\mathrm{Cone}(e_i - e_{i+1})) \in \mathrm{star}_{\Sigma_d}(\rho_v)$. The ray $\mathrm{Cone}(e_i - e_{i+1})$ is incompatible with the rays $\mathrm{Cone}(e_{i+1})$ and $\mathrm{Cone}(e_{i+1} - e_{i+2})$ in Σ_d, so the compatibility structure of $\mathrm{star}_{\Sigma_d}(\rho_v) \cap L(e_i,\dots,e_d)/L(e_i)$ matches the compatibility structure of Σ_{d-i} under the linear map. Thus $\mathrm{star}_{\Sigma_d}(\rho_{e_i}) \cong \Sigma_{i-1} \times \Sigma_{d-i}$ – the other cases are proved by a similar comparison of the 2-skeletons. □

4. Pellspace

In this section, we introduce the Pellspace, compute its character lattice, and show that it forms a binary geometry for the simplicial complex defined by the inner normal fan of the pellytope. Throughout this section, we closely follow the notation from [10]. However, we have slightly modified the name "Pell's space" [10, Section 5.6.2] to "Pellspace" for reasons related to pronunciation.

4.1. The character lattice and bounded characters

For $i \in [d]$, $j \in [d-1]$, consider the polynomials $p_i = 1 + y_i$ and $q_j = 1 + y_j + y_j y_{j+1}$ in $C[y_1,\dots,y_d]$.. The Newton polytope of p_i (resp. q_j) equals P_i (resp. Q_j) from (5). We define the *open Pellspace* as the very affine variety in $(\mathbb{C}^*)^d$ complement to the vanishing set of the p_i's and q_j's:

$$\mathcal{U}_d := \left\{ x \in (\mathbb{C}^*)^d \;\middle|\; p_i(x) \neq 0 \;\forall i \in [d] \text{ and } q_j(x) \neq 0 \;\forall j \in [d-1] \right\}$$

Since the p_i's and q_j's only have positive coefficients, the positive real orthant $\mathbb{R}^d_{>0}$ is a connected component of $\mathcal{U}_d$. The *character lattice* Λ of $\mathcal{U}_d$ is defined as the set of units in its coordinate ring modulo scalars. It is in fact a lattice due to [12, 13] and by [10, Lemma 5.2] it is the lattice of Laurent monomials in $y_1,\dots,y_n, p_1,\dots,p_d, q_1,\dots,q_{d-1}$. In particular, every element of Λ can be written uniquely as

$$y^a p^b q^c = \prod_{i=1}^{d} y_i^{a_i} \prod_{i=1}^{d} p_i^{b_i} \prod_{i=1}^{d-1} q_i^{c_i}, \quad (a,b,c) \in \mathbb{Z}^d \times \mathbb{Z}^d \times \mathbb{Z}^{d-1}. \tag{9}$$

A character $y^a p^b q^c \in \Lambda$ is *bounded* if it takes bounded values on $\mathbb{R}^d_{>0}$. We denote by $\Gamma \subset \Lambda$ the semigroup of bounded characters. In what follows, we show that the minimal generators of Γ form a basis of Λ.

To find the minimal generators of Γ, we recall the method from [3]. The *tropicalization* of rational functions in $y_1,\dots,y_d$ is defined by $y_i \mapsto Y_i$, $+ \mapsto \min$,

$\times \mapsto +$, $\div \mapsto -$. For example, the tropicalization of $y^a p^b q^c$ from (9) is

$$\operatorname{trop}(y^a p^b q^c)(Y) = \sum_{i=1}^{d} a_i Y_i + \sum_{i=1}^{d} b_i \min\{0, Y_i\} + \sum_{i=1}^{d-1} c_i \min\{0, Y_i, Y_i + Y_{i+1}\}.$$

By [10, Lemma 5.16] we have

$$\Gamma = \left\{ y^a p^b q^c \in \Lambda \,\middle|\, \operatorname{trop}(y^a p^b q^c)(Y) \geq 0 \quad \forall Y \in \mathbb{R}^d \right\}. \tag{10}$$

A simple computation shows that $\operatorname{trop}(y^a p^b q^c)(Y) = 0$ for all $Y \in \mathbb{R}^d$ if and only if $(a,b,c) = 0$. Thus, if $y^a p^b q^c \neq 1$, the inequality $\operatorname{trop}(y^a p^b q^c)(Y_*) \geq 0$ is strict for at least one $Y_* \in \mathbb{R}^d$. Since $\operatorname{trop}(y^a p^b q^c)(Y)$ is a piecewise linear function on $\mathbb{R}^d$ and linear on each cone of Σ_d, it is enough to check whether $\operatorname{trop}(y^a p^b q^c)(v) \geq 0$ for generators v of the rays in Σ_d.

Lemma 4.1. *Let V be a set of generators of the rays in Σ_d. The following are equivalent*

(i) $\operatorname{trop}(y^a p^b q^c)(Y) \geq 0$ *for all* $Y \in \mathbb{R}^d$ *and* $\operatorname{trop}(y^a p^b q^c)(Y_*) > 0$ *for at least one* $Y_* \in \mathbb{R}^d$,

(ii) $\operatorname{trop}(y^a p^b q^c)(v) \geq 0$ *for all* $v \in V$ *and* $\operatorname{trop}(y^a p^b q^c)(v_*) > 0$ *for at least one* $v_* \in V$.

Proof. *(i)* $\Leftarrow$ *(ii).* Let $Y \in \mathbb{R}^d$. Since Σ_d is a complete fan, there exists a cone $C \in \Sigma_d$ (not necessarily full dimensional) with $Y \in C$. Let $v_1, \ldots, v_s \in V$ be the generators of C and let $\lambda_1, \ldots, \lambda_s \geq 0$ such that $\sum_{i=1}^{s} \lambda_i v_i = Y$. Since $\operatorname{trop}(y^a p^b q^c)$ is linear on C it follows that

$$\operatorname{trop}(y^a p^b q^c)(Y) = \operatorname{trop}(y^a p^b q^c)\left(\sum_{i=1}^{s} \lambda_i v_i\right) = \sum_{i=1}^{s} \lambda_i \underbrace{\operatorname{trop}(y^a p^b q^c)(v_i)}_{\geq 0} \geq 0.$$

The second part of (i) follows by taking $Y_* = v_*$.

(i) $\Rightarrow$ *(ii).* Assumption (i) implies $\operatorname{trop}(y^a p^b q^c)(v) \geq 0$ for all $v \in V$. It therefore suffices to find one $v_* \in V$ such that $\operatorname{trop}(y^a p^b q^c)(v_*) > 0$. By assumption, there exists $Y_* \in \mathbb{R}^d$ such that $\operatorname{trop}(y^a p^b q^c)(Y_*) > 0$. Again, we write $Y_* = \sum_{i=1}^{s} \lambda_i v_i$ for some $v_1, \ldots, v_s \in V$, $\lambda_1, \ldots, \lambda_s \geq 0$. If $\operatorname{trop}(y^a p^b q^c)(v_i) = 0$ for all $i = 1, \ldots, s$, then $\operatorname{trop}(y^a p^b q^c)(Y_*) = 0$, which is a contradiction. Thus, there exists $v_* \in \{v_1, \ldots, v_s\}$ with $\operatorname{trop}(y^a p^b q^c)(v_*) > 0$. □

Now let $V = (e_1, \ldots, e_d, -e_1, \cdots - e_d, e_1 - e_2, \ldots, e_{d-1} - e_d)$ be the tuple of ray generators for Σ_d. Consider the real matrix $M_d = (m_{ij})_{i,j \in [3d-1]}$ defined by

$$m_{ij} = \operatorname{trop}(F_i)(V_j), \tag{11}$$

where $(F_1, \ldots, F_{3d-1}) = (y_1, \ldots, y_d, p_1, \ldots, p_d, q_1, \ldots, q_{d-1})$.

Example 4.2. Consider the case $d = 2$. We have

$$\operatorname{trop}(F_5)(e_1 - e_2) = \operatorname{trop}(1 + y_1 + y_1 y_2)(1,-1) = \min\{0,1,1-1\} = 0.$$

Thus, the entry in the bottom left corner of M_2 equals 0. We compute the remaining entries of M_2 using the same approach and obtain

$$M_2 = \begin{pmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & -1 & 0 \end{pmatrix} \in \mathbb{R}^{5\times 5}. \tag{12}$$

Combining (10) and Lemma 4.1, it follows that the bounded characters on $\mathcal{U}_d$ are given by

$$\Gamma = \left\{ y^a p^b q^c \in \Lambda \,\middle|\, (a,b,c) M_d \geq 0 \right\}. \tag{13}$$

Here we use the notation $v \geq 0$ for $v \in \mathbb{R}^n$ to indicate that each entry of the vector v is non-negative.

Lemma 4.3. *The matrix M_d from* (11) *is invertible over $\mathbb{Z}$. The rows of the inverse matrix M_d^{-1} are*

$$\beta_i = e_i + e_{d+i+1} - e_{2d+i}, \qquad \beta_d = e_d - e_{2d}, \qquad \beta_{d+1} = -e_{d+1},$$
$$\beta_{d+j} = e_{d+j-1} - e_{2d+j-1} \qquad \beta_{2d+i} = -e_{d+i} - e_{d+i+1} + e_{2d+i}.$$

for $i = 1,\dots,d-1$ and $j = 2,\dots,d$.

Proof. Let $A_1,\dots,A_d,B_1,\dots,B_d,C_1,\dots,C_{d-1}$ denote the columns of M_d. Explicitly we have $A_i = e_i$ for $i \in [d]$,

$$B_1 = -e_1 - e_{d+1} - e_{2d+1}, \quad B_i = -e_i - e_{d+i} - e_{2d+i-1} - e_{2d+i}, \quad B_d = -e_d - e_{2d} - e_{3d-1},$$

for $i = 2,\dots,d-1$, and

$$C_i = e_i - e_{i+1} - e_{d+i+1} - e_{2d+i+1}, \qquad C_{d-1} = e_{d-1} - e_d - e_{2d}.$$

for $i = [d-2]$. A direct computation shows that $M_d^{-1} M_d$ is the identity matrix. □

For example, when $d = 2$ we have

$$M_2^{-1} = \begin{pmatrix} 1 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & -1 & 1 \end{pmatrix}.$$

Proposition 4.4. *Let* $\beta_i = (a_i, b_i, c_i)$, $i = 1, \dots 3d-1$ *denote the rows of* M_d^{-1}. *The elements*

$$u_i = z^{\beta_i} = y^{a_i} p^{b_i} q^{c_i} = \prod_{j=1}^{d} y_j^{a_{ij}} \prod_{j=1}^{d} p_j^{b_{ij}} \prod_{j=1}^{d-1} q_j^{c_{ij}}$$

are the minimal generators of Γ. *Moreover,* $u_1, \dots, u_{3d-1}$ *is a basis of the character lattice* Λ.

Proof. Since $\beta_i M_d = e_i \geq 0$, it follows from (13) that $u_i = z^{\beta_i} \in \Gamma$. We show that each u_i is a minimal element of Γ by contradiction. Assume that there exists $w_1, w_2 \in \Gamma$ such that $u_i = w_1 w_2$. Let $\alpha_1, \alpha_2 \in \mathbb{Z}^{3d-1}$ such that $w_1 = z^{\alpha_1}, w_2 = z^{\alpha_2}$. By construction $\alpha_1 + \alpha_2 = \beta_i$. After multiplying these vectors with the matrix M_d from the right, we have

$$\alpha_1 M_d + \alpha_2 M_d = \beta_i M_d = e_i \tag{14}$$

Since $w_1, w_2 \in \Gamma$, we also have that $\alpha_1 M_d \geq 0$ and $\alpha_2 M_d \geq 0$. If there exists $k \neq \ell$ such that the k^{th} coordinate of $\alpha_1 M_d$ and ℓ^{th} coordinate of $\alpha_2 M_d$ are both nonzero, then $\alpha_1 M_d + \alpha_2 M_d = e_i$ has at least two non-zero coordinates, which is a contradiction. If there exists k such that the k^{th} coordinate is positive for both $\alpha_1 M_d$ and $\alpha_2 M_d$, then the k^{th} coordinate of $\alpha_1 M_d + \alpha_2 M_d = e_i$ is larger than one, which is again a contradiction. Thus u_i cannot be written as $w_1 w_2$ for $w_1, w_2 \in \Gamma$.

It remains to show that $u_1, \dots, u_{3d-1}$ generate Γ and that they form a basis of Λ. Let $w = z^{\alpha} \in \Lambda$ and choose $\lambda_1, \dots, \lambda_{3d-1} \in \mathbb{Z}$ such that

$$\alpha M_d = \sum_{i=1}^{3d-1} \lambda_i e_i = \sum_{i=1}^{3d-1} \lambda_i (\beta_i M_d) = \Big(\sum_{i=1}^{3d-1} \lambda_i \beta_i\Big) M_d.$$

Multiplying with $(M_d)^{-1}$ from the right, we have $\alpha = \sum_{i=1}^{3d-1} \lambda_i \beta_i$, which shows that $u_1, \dots, u_{3d-1}$ form a basis of Λ. If we assume additionally that $w \in \Gamma$, then $\alpha M_d \geq 0$ and we can choose $\lambda_1, \dots, \lambda_{3d-1} \in \mathbb{Z}_{\geq 0}$. From this it follows that $u_1, \dots, u_{3d-1}$ generate Γ. □

Corollary 4.5. *The minimal generators of* Γ *have the following form:*

$$u_i = \frac{y_i p_{i+1}}{q_i}, \quad u_d = \frac{y_d}{p_d}, \quad u_{d+1} = \frac{1}{p_1}, \quad u_{i+d+1} = \frac{p_i}{q_i}, \quad u_{i+2d} = \frac{q_i}{p_i p_{i+1}} \quad \textit{for } i = 1, \dots, d-1.$$

Example 4.6. For $d = 2$, the minimal generators of Γ are

$$u_1 = \frac{y_1 p_2}{q_1}, \quad u_2 = \frac{y_2}{p_2}, \quad u_3 = \frac{1}{p_1}, \quad u_4 = \frac{p_1}{q_1}, \quad u_5 = \frac{q_1}{p_1 p_2}.$$

4.2. The u-equations for the Pellspace

Given the expressions in Corollary 4.5, we observe that the minimal generators of Γ satisfy the following equations:

$$u_i = \begin{cases} 1 - u_{d+1}u_{d+2} & i = 1 \\ 1 - u_{i+d}u_{i+d+1}u_{i-1+2d} & i = 2, \dots, d-1 \\ 1 - u_{3d-1}u_{2d} & i = d \\ 1 - u_1 u_{1+2d} & i = d+1 \\ 1 - u_j u_{j+1} u_{j+1+2d} & i = j+d+1 \text{ where } 1 \le j \le d-2 \\ 1 - u_{d-1}u_d & i = 2d \\ 1 - u_{d+1}u_2 u_{2+2d} & i = 1+2d \\ 1 - u_{j-1+2d}u_{j+d}u_{j+1}u_{j+1+2d} & i = j+2d \text{ where } 2 \le j \le d-2 \\ 1 - u_d u_{3d-2} u_{2d-1} & i = 3d-1 \end{cases} \tag{15}$$

As expected these equations are u-equations for the pellytope:

Lemma 4.7. *The equations* (15) *satisfied by the minimal generators of* Γ *are* u*-equations for* $\Delta(\Sigma_d)$ *as defined in* (1).

Proof. The matrix M_d induces the following correspondence between rays of Σ_d and generators of Γ:

$$u_i \longleftrightarrow \begin{cases} \text{Cone}(e_i) & \text{if } i \in [d] \\ \text{Cone}(-e_j) & \text{if } i = d+j \text{ where } j \in [d] \\ \text{Cone}(e_j - e_{j+1}) & \text{if } i = 2d+j \text{ where } j \in [d-1] \end{cases} \tag{16}$$

It follows from the characterization of compatibility between the rays of Σ_d in Lemma 3.4 that the equations in (15) are of the desired form (1). □

We note that correspondence (16) can also be observed on the level of the Newton polytopes: after multiplying each variable u_i so that the denominator is of form $q := \prod_{j\in[d]} p_j \prod_{j\in[d-1]} q_j$, we see that the Newton polytope of the numerator is the convex hull of all lattice points in $\mathcal{P}_d$ that do not lie in the facet determined by the associated ray in Σ_d.

Example 4.8. When $d = 3$ the minimal generators of Γ are

$$u_1 = \frac{y_1 p_2}{q_1},\ u_2 = \frac{y_2 p_3}{q_2},\ u_3 = \frac{y_3}{p_3},\ u_4 = \frac{1}{p_1},\ u_5 = \frac{p_1}{q_1},\ u_6 = \frac{p_2}{q_2},\ u_7 = \frac{q_1}{p_1 p_2},\ u_8 = \frac{q_2}{p_2 p_3}.$$

As indicated above they correspond to the rays $e_1, e_2, e_3, -e_1, -e_2, -e_3, e_1 - e_2$ and $e_2 - e_3$ respectively. For example, we have

$$u_3 = \frac{y_3}{p_3} = \frac{y_3 p_1 p_2 q_1 q_2}{p_1 p_2 p_3 q_1 q_2}$$

As $y_3 = p_3 - 1$ all lattice points in $\mathrm{Newt}(y_3p_1p_2q_1q_2)$ have a positive y_3 coordinate. Hence, it is the complement of the facet with normal vector e_3.

Definition 4.9. The vanishing locus of the u-equations for $\Delta(\Sigma_d)$ in $\mathbb{C}^{3d-1}$ is called *Pellspace* and it is denoted by $\widetilde{\mathcal{U}}_d$.

Theorem 1.1 states that the Pellspace is a binary geometry with simplicial complex given by the inner normal fan of the pellytope. We proceed by showing that the Pellspace is irreducible by realizing its defining ideal as the kernel of a ring homomorphism. Define $S := \mathbb{C}[y_1^{\pm 1}, \dots, y_d^{\pm 1}, p_1^{\pm 1}, \dots, p_d^{\pm 1}, q_1^{\pm 1}, \dots, q_{d-1}^{\pm 1}]$ and consider the ideal $I \subset S$ generated by $p_i - (1 + y_i)$ and $q_i - (1 + y_i + y_iy_{i+1})$. Then Corollary 4.5 determines a map

$$\widetilde{f} : \mathbb{C}[u_1, \dots, u_{3d-1}] \to S/I \tag{17}$$

sending each u_i to its corresponding monomial in S. The kernel of $\widetilde{f}$ is the prime ideal we denote by $\widetilde{K}$.

Example 4.10. For $d = 3$ we compute $\widetilde{K}$ in *Macaulay2* and find that it is minimally generated by

$$u_1 + u_4u_5 - 1, \quad u_2 + u_5u_7u_6 - 1, \quad u_3 + u_6u_8 - 1, \quad u_4 + u_1u_7 - 1,$$
$$u_5 + u_2u_1u_8 - 1, \quad u_6 + u_3u_2 - 1, \quad u_7 + u_2u_4u_8 - 1, \quad u_8 + u_3u_5u_7 - 1.$$

Up to relabelling the variables these equations coincide with the equations (15), as well as the u-equations from [10, Problem 2.23].

In order to see that the Pellspace is irreducible we need to show that the u-equations in (15) are the generators of $\widetilde{K}$. The following corollary is a direct consequence of Corollary 4.5.

Corollary 4.11. *We have for* $i = 2, \dots, d-1$

$$y_1 = \frac{u_1u_{2d+1}}{u_{d+1}}, \quad y_i = \frac{u_iu_{2d+i}}{u_{2d+i-1}u_{d+i}}, \quad y_d = \frac{u_d}{u_{2d}u_{3d-1}}, \quad q_1 = \frac{1}{u_{d+2}u_{d+1}},$$
$$p_1 = \frac{1}{u_{d+1}}, \quad p_i = \frac{1}{u_{d+i}u_{2d+i-1}}, \quad p_d = \frac{1}{u_{2d}u_{3d-1}}, \quad q_i = \frac{1}{u_{d+i}u_{d+i+1}u_{2d+i-1}}.$$

Proposition 4.12. *The ideal* $\widetilde{K} \subset \mathbb{C}[u_1, \dots, u_{3d-1}]$*, the kernel of the map* $\widetilde{f}$ *defined in* (17)*, is generated by the u-equations for* $\Delta(\Sigma_d)$ *in* (15).

Proof. Consider the maps

$$\mathbb{C}[u_1^{\pm 1}, \dots, u_{3d-1}^{\pm 1}] \xrightarrow{f} S \xrightarrow{g} \mathbb{C}(y_1, \dots, y_d),$$

where f is an isomorphism given by the matrix M_d^{-1} and g is the map sending y_i to y_i, p_i to $1 + y_i$ and q_i to $1 + y_i + y_iy_{i+1}$. Let $K := \ker(g \circ f)$, so $K \cap$

$\mathbb{C}[u_1,\ldots,u_{3d-1}] = \widetilde{K}$. The kernel of g is generated by $p_i - (1+y_i)$ and $q_j - (1+y_j+y_jy_{j+1})$ (equivalently, $q_j - (1+y_jp_{j+1})$) for $1 \le i \le d$ and $1 \le j \le d-1$; since f is an isomorphism, the preimages generate $K = f^{-1}(\ker(g))$. Denote by $J \subset \mathbb{C}[u_1^{\pm 1},\ldots,u_{3d-1}^{\pm 1}]$ the ideal generated by the polynomials in (15). As the generators of J are polynomial, $K = J$ implies the claim.

$\underline{K \subset J}$: It is convenient to change the generating set of K to be $p_1 - 1 - y_1, p_d - 1 - y_d$ and $q_i - y_ip_{i+1} - 1$, $1 \le i \le d-1$, and $p_i - q_i + y_iy_{i+1}$, $2 \le i \le d-1$. For each of these generators we compute

$$\begin{aligned}
f^{-1}(p_1 - 1 - y_1) &= \frac{1 - u_{d+1} - u_1u_{2d+1}}{u_{d+1}},\\
f^{-1}(p_d - 1 - y_d) &= \frac{1 - u_d - u_{2d}u_{3d-1}}{u_{2d}u_{3d-1}},\\
f^{-1}(p_i - q_i + y_iy_{i+1}) &= -\frac{1 - u_{d+i+1} - u_iu_{i+1}u_{2d+i+1}}{u_{d+i}u_{2d+i-1}u_{d+i+1}}, \quad i = 2,\ldots,d-1\\
f^{-1}(q_1 - y_1p_2 - 1) &= \frac{1 - u_1 - u_{d+2}u_{d+1}}{u_{d+2}u_{d+1}},\\
f^{-1}(q_i - y_ip_{i+1} - 1) &= \frac{1 - u_i - u_{d+i}u_{d+i+1}u_{2d+i-1}}{u_{d+i}u_{d+i+1}u_{2d+i-1}}, \quad i = 2,\ldots,d-1.
\end{aligned}$$

$\underline{J \subset K}$: The first part of the proof shows that the u-equations for $i = 1\ldots,d+1$ are contained in K. The u-equations for $i = d+2,\ldots,3d-1$ require additional computations; for example, $1 - u_{2d+1} - u_2u_{d+1}u_{2d+2}$ is obtained from multiplying the following expression by u_{d+1}:

$$f^{-1}(p_1 - y_1 - 1) - u_{1+2d}u_{d+2}f^{-1}(q_1 - y_1p_2 - 1) - u_{d+3}u_{2d+1}f^{-1}(p_2 - y_2 - 1).$$

Similar expressions exist for the u-equations for u_{2d+i}, $i = 2,\ldots,d-1$. Finally, the expression $1 - u_{i+d+1} - u_{i+1}u_{i+1+2d}$ equals

$$u_{i-1+d+1}u_{i+d+1}u_{i-1+2d}f^{-1}\big((q_i - 1 - y_i - y_iy_{i+1}) - (p_i - y_i - 1)\big).$$

This recovers the u-equations for u_{i+d+1} for $1 \le i \le d-1$. □

We are now prepared to prove Theorem 1.1:

Proof of Theorem 1.1. The variety $\widetilde{\mathcal{U}}_d$ is by definition the affine closure of the variety defined by the vanishing of $K = \ker(g \circ f)$ in $(\mathbb{C}^*)^{3d-1}$ (or equivalently, $\widetilde{\mathcal{U}}_d$ is the vanishing set of $\widetilde{K} = \ker(\widetilde{f})$ in $\mathbb{C}^{3d-1}$ as defined above). By Proposition 4.12, K is generated by the equations (15), which are of the desired form (1) by Lemma 4.7. We proceed by verifying the items (i), (ii) and (iii) of Definition 2.2.

For (i) we need to show that $\widetilde{\mathcal{U}}_d$ is an irreducible variety of dimension d. Irreducibility follows from Proposition 4.12. Recall that a surjection of coordinate rings corresponds to an inclusion of the affine varieties. In particular, the existence of the map g shows that

$$\mathbb{C}(\widetilde{\mathcal{U}}_d) = \mathrm{Frac}(\mathbb{C}[u_1,\dots,u_{3d-1}]/K) \cong \mathbb{C}(y_1,\dots,y_d),$$

so $\mathcal{U}_d$ is of dimension d.

For (ii) and (iii), we use Lemma 2.5 and induction on d. For $d=1$, Example 2.3 shows that $\widetilde{\mathcal{U}}_1$ is a binary geometry. Assume that for all $d' < d$ we have that $\widetilde{U}_{d'}$ is a binary geometry. From Lemma 2.5 and Lemma 3.5 for each $k \in [3d-1]$ it follows that $(\widetilde{\mathcal{U}}_d)_{\{k\}}$ is a product of binary geometries. Hence $(\widetilde{\mathcal{U}}_d)_{\{k\}}$ is a binary geometry by Proposition 2.7. Using Lemma 2.5(b), we conclude that the Pellspace $\widetilde{\mathcal{U}}_d$ is a binary geometry for $\Delta(\Sigma_d)$. □

4.3. Relationship between the Pellspace and $\widetilde{\mathcal{M}}_{0,n}$

In this section we compare the binary geometries given by the pellytope and the associahedron. Let $n = d+3$ and consider an n-gon with cyclically labelled vertices. We label the arcs of the n-gon by ij where $1 \le i < j-1 \le n-2$, and say that two arcs are incompatible if they cross each other. The u-equations determined by the associahedron are given by

$$u_{ij} + \prod_{kl \nparallel ij} u_{kl} = 1. \tag{18}$$

The ABHY construction of the associahedron in kinematic space is of particular interest due to its connection to the positive geometry on $\mathcal{M}_{0,n}$ [1]. This realisation $\mathcal{A}_{n-3}$ of the asociahedron can be defined [3, (5.5)] as the Newton polytope of the polynomial

$$G_{n-3} := \prod_{ij} (1 + y_i + y_i y_{i+1} + \cdots + y_i y_{i+1} \cdots y_{j-2}) \in \mathbb{C}[y_1,\dots,y_{n-3}]. \tag{19}$$

Cones in the normal fan $\Sigma_{\mathcal{A}_{n-3}}$ of the associahedron correspond to subdivisions of the n-gon – rays correspond to arcs ij. The ABHY realisation is equivalent to setting the positive orthant in $\mathbb{R}^{n-3}$ to be the maximal cone corresponding to the triangulation of the n-gon given by every arc centred at a single point. Labelling this point $n-1$ gives the following dictionary between arcs on the n-gon and primitive generators of the rays of $\Sigma_{\mathcal{A}_{n-3}}$:

$$e_i \longleftrightarrow i(n-1), \quad -e_i \longleftrightarrow (i+1)n \quad \text{and} \quad e_i - e_k \longleftrightarrow i(k+1). \tag{20}$$

We note that $\Sigma_{\mathcal{A}_d}$ is a refinement of Σ_d – the polynomial defining $\mathcal{P}_d$ clearly divides G_d, so the pellytope $\mathcal{P}_d$ is a Minkowski summand of $\mathcal{A}_d$.

Example 4.13. When $d = 1$ or 2, the pellytope and the ABHY associahedron coincide. In the case $d = 3$, the normal fan to $\mathcal{A}_3$ is given by adding a single ray $\mathrm{Cone}(e_1 - e_3)$ to Σ_3.

The binary geometry defined by the associahedron $\mathcal{A}_{n-3}$ is $\widetilde{\mathcal{M}}_{0,n}$, an affine chart on the moduli space of stable curves $\overline{\mathcal{M}}_{0,n}$ (cf. [10, Lectures 1-2]). We can therefore use the relationship between $\mathcal{P}_d$ and $\mathcal{A}_d$ to give a moduli interpretation of $\widetilde{\mathcal{U}}_d$. Corollary 1.2 is a direct consequence of the following Lemma:

Lemma 4.14. *Suppose that f_1 and $f_2 \in \mathbb{C}[y_1, \dots, y_d]$ are polynomials with full-dimensional Newton polytopes, positive coefficients and non-vanishing constant term. Let $U_i \subset (\mathbb{C}^*)^d$ be the very affine variety given by the locus where $f_i \neq 0$. Let $\widetilde{U}_i = \mathrm{Spec} A_i$ be the affine closure, where A_i is the subring of $\mathbb{C}[U_i]$ generated by the bounded characters on U_i.*

If f_2 divides f_1, then there is a birational morphism $\pi : \widetilde{U}_1 \to \widetilde{U}_2$ which is the restriction of a toric blowup of projective toric varieties $X_1 \to X_2$.

Proof. Let X_i be the projective toric variety associated to the normal fan of the Newton polytope $P_i = \mathrm{Newt}\, f_i$. The polynomial f_i determines a section of the very ample line bundle on X_i associated to P_i – let $H_i \subset X_i$ be the zero locus of this section. The affine variety $\widetilde{U}_i$ may be identified with $X_i \setminus H_i$ (cf. [10, Proposition 5.19]). If f_2 divides f_1, then the normal fan of P_1 is a refinement of the normal fan of P_2, which induces a toric blowup $\pi : X_1 \to X_2$. Moreover, one can check that $\pi^{-1}(H_2) \subset H_1$, so the toric morphism π restricts to a map $\widetilde{U}_1 \to \widetilde{U}_2$. □

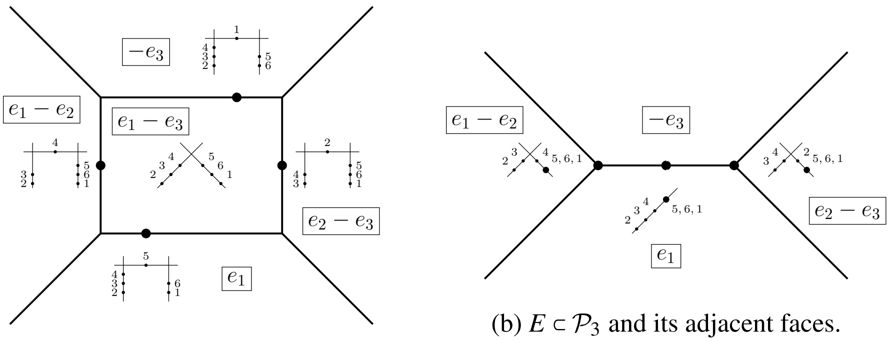
Moreover, Lemma 3.5 shows that the boundary of $\widetilde{\mathcal{U}}_d$ has a recursive structure similar to that of $\overline{\mathcal{M}}_{0,n}$ – the strata of $\widetilde{\mathcal{U}}_d$ are isomorphic to products of lower-dimensional Pellspaces $\widetilde{\mathcal{U}}_i$ for a collection of $i < d$.

Since the blowup $X_{\Sigma_{\mathcal{A}_d}} \to X_{\Sigma_d}$ is toric, the exceptional locus is contained in the complement of the very affine varieties $\mathcal{M}_{0,n}$ and $\mathcal{U}_d$. This suggests we can consider $\widetilde{\mathcal{U}}_d$ to be an affine chart on some smaller compactification of $\mathcal{M}_{0,n}$ than the space of n-pointed stable curves $\overline{\mathcal{M}}_{0,n}$. We give a detailed description of this compactification in the case $d = 3$ in the following example.

Example 4.15. When $d = 3$, the exceptional divisor of the toric blowup is the toric divisor $D_\rho \subset X_{\Sigma_{\mathcal{A}_3}}$ associated to the ray $\rho = \mathrm{Cone}(e_1 - e_3)$, the only ray of $\Sigma_{\mathcal{A}_3}$ not contained in Σ_3. By the dictionary (20), the intersection $D_\rho \cap \widetilde{\mathcal{M}}_{0,6}$ is given by the equation $u_{14} = 0$.

Which stable curves are contained in this stratum of $\widetilde{\mathcal{M}}_{0,6}$? The variables u_{ij} are *dihedral coordinates* on $\mathcal{M}_{0,n}$ – these are cross-ratios

$$u_{ij} = \frac{(x_i - x_{j+1})(x_{i+1} - x_j)}{(x_i - x_j)(x_{i+1} - x_{j+1})}, \tag{21}$$



(a) $F \subset \mathcal{A}_3$ and its adjacent faces.

(b) $E \subset \mathcal{P}_3$ and its adjacent faces.

Figure 3: Figures (a) and (b) above respectively show $F \subset \mathcal{A}_3$, the face of the associahedron corresponding to the diagonal 14 on the hexagon, and $E \subset \mathcal{P}_3$, the edge of the pellytope dual to $\mathrm{Cone}(e_1, -e_3) \in \Sigma_3$, as well as their adjacent faces. Each of these faces is labelled with its primitive inward normal vector. The face F and its edges, as well as the edge E and its endpoints, are also labelled with diagrams of the curves represented by points in the interior of the corresponding strata of $\widetilde{\mathcal{M}}_{0,6}$ and $\widetilde{\mathcal{U}}_3$ respectively.

where each point on $\mathcal{M}_{0,n}$ represents a choice of (a PGL(2)-orbit of) n distinct points $p_i := [x_i : 1] \in \mathbb{P}^1$ – that is, the isomorphism class of a smooth rational curve with n marked points. The cross-ratio (21) determines the image $[u_{ij} : 1]$ of the point p_{j+1} under the PGL(2)-transformation that sends (p_i, p_j, p_{i+1}) to $(0, 1, \infty)$. Thus the point p_{j+1} collides with p_i as $u_{ij} \to 0$, and p_{j+1} collides with p_j as $u_{ij} \to 1$.

One sees that the marked points p_5 and p_1 collide on our six-pointed stable curve as $u_{14} \to 0$. However, the relations (18) satisfied by the dihedral coordinates imply that $u_{25}, u_{26}, u_{35}, u_{36} \to 1$ as $u_{14} \to 0$, so the three points p_5, p_6 and p_1 collide as $u_{14} \to 0$. We consider a point in $\{u_{14} = 0\}$ to represent the *stabilisation* of the limit of a one-parameter family in $\mathcal{M}_{0,6}$ of smooth curves C_t in which the marked points p_5, p_6 and p_1 collide as $t \to 0$. In the stabilisation C_0, the three colliding points break off into a second component C', and are distributed on $C' \cong \mathbb{P}^1$ according to their ratios of approach to one another. Indeed, one can check that $\{u_{14} = 0\} \cap \widetilde{\mathcal{M}}_{0,6} \cong \widetilde{\mathcal{M}}_{0,4} \times \widetilde{\mathcal{M}}_{0,4}$ – each fibre $\widetilde{\mathcal{M}}_{0,4}$ parametrises one of the two irreducible components of C_0 (each considered as four-pointed curves in order to encode the position of the intersection point q).

The image of the stratum of $\widetilde{\mathcal{M}}_{0,n}$ corresponding to a cone $\sigma \in \Sigma_{\mathcal{A}_d}$ is an open subset of the stratum $(\widetilde{\mathcal{U}}_d)_\tau$, where τ is the smallest cone in Σ_d containing σ. In particular we have $\pi(\{u_{14} = 0\}) \subset \{u_1 = u_6 = 0\} \cap \widetilde{\mathcal{U}}_3$, where u_1 and u_6 are the variables associated to the rays $\mathrm{Cone}(e_1)$ and $\mathrm{Cone}(-e_3) \in \Sigma_d$ by the identification (16). By Lemma 3.5, this stratum of $\widetilde{\mathcal{U}}_3$ is isomorphic to $\widetilde{\mathcal{U}}_1 \cong \widetilde{\mathcal{M}}_{0,4}$, and one can see from analysis of the adjacent strata that π contracts the fibre of $\{u_{14} = 0\}$ which parametrises the distribution of p_5, p_6, p_1 and q on C'. We may therefore consider points on $\{u_1 = u_6 = 0\} \cap \widetilde{\mathcal{U}}_3$ to represent smooth (but unstable) curves on which the marked points p_5, p_6 and p_1 coincide and the other marked points are distinct from each other and p_1.

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