EXPERIMENTS WITH GORENSTEIN LIAISON

ROBIN HARTSHORNE

Dedicated to Silvio Greco in occasion of his 60-th birthday.

We give some experimental data of Gorenstein liaison, working with points in $\mathbb{P}^3$ and curves in $\mathbb{P}^4$, to see how far the familiar situation of liaison, biliaison, and Rao modules of curves in $\mathbb{P}^3$ will extend to subvarieties of codimension 3 in higher $\mathbb{P}^4$.

1. The Problem.

For curves in projective three-space $\mathbb{P}^3_k$, the usual theory of liaison and biliaison is well understood [9]. We will recall some of the basic facts, and then explore to what extent these results may generalize to liaison classes of varieties of higher codimension, such as curves in $\mathbb{P}^4$. Our method is to run experiments in various special cases, and look for examples which may indicate how the general situation will be.

First we recall the situation in $\mathbb{P}^3_k$. A *curve* will be a pure one-dimensional locally Cohen-Macaulay closed subscheme of $\mathbb{P}^3$. Two curves C_1 and C_2 are *linked* if there exists a complete intersection curve D such that $D = C_1 \cup C_2$ set-theoretically, and

$$\mathcal{I}_{C_1,D} \cong \mathcal{H}om(\mathcal{O}_{C_2}, \mathcal{O}_D)$$

$$\mathcal{I}_{C_2,D} \cong \mathcal{H}om(\mathcal{O}_{C_1}, \mathcal{O}_D).$$

AMS Classification: 14H50, 14M07.

The equivalence relation generated by linkage is called *liaison*. An even number of linkages generates the equivalence relation of *even liaison* or *biliaison*.

We say that C_2 is obtained from C_1 by an *elementary biliaison of height h* if there is a surface S in $\mathbb{P}^3$ containing C_1, and $C_2 \sim C_1 + hH$ on S, where $\sim$ denotes linear equivalence, and H is the hyperplane section. Here we use the theory of generalized divisors [3], so that any curve on any surface in $\mathbb{P}^3$ can be regarded as a divisor.

Then it is known that an elementary biliaison is an even liaison, and the equivalence relation generated by elementary biliaisons is the same as even liaison [3, 4.4].

Some of the main results of liaison theory for curves in $\mathbb{P}^3$ are contained in the following theorem.

Theorem 1.1. *For curves in $\mathbb{P}^3_k$, we have*

  a) *Two curves C_1, C_2 are in the same liaison equivalence class if and only if their Rao modules $M(C_i) = H^1_*(\mathcal{I}_{C_i}(n))$ are isomorphic, up to dualizing and shifting degrees. They are in the same biliaison equivalence class if and only if $M(C_1)$ and $M(C_2)$ are isomorphic up to shift of degrees.*
  b) *For each finite-length graded module M_0 over the homogeneous coordinate ring $R = k[x_0, x_1, x_2, x_3]$, there exists a smooth irreducible curve C in $\mathbb{P}^3$ and an integer h, such that $M(C) \cong M_0(h)$.*
  c) *For any finite length $M_0 \neq 0$, there is a minimum h for which there exist curves C_0 with $M(C_0) = M_0(-h)$. These are called **minimal** curves, and the family $\mathcal{L}_0(M_0)$ of minimal curves for M_0 is an irreducible subset of the Hilbert scheme .*
  d) *(The Lazarsfeld-Rao property): Any other curve C in the biliaison class associated to the module M_0 can be obtained by a finite number of ascending (i.e. $h \geq 0$) elementary biliaisons, plus a deformation, from a minimal curve C_0 in the biliaison class.*
  e) *For any module M and any postulation character γ, the subset $H_{\gamma, M}$ of the Hilbert scheme of curves with postulation character γ and Rao module M is irreducible (provided it is non-empty). (For a curve C with homogeneous coordinate ring $R(C) = R/I_C$, we define the postulation character γ_C to be the third difference function of the negative of the Hilbert function $\phi(\ell) = \dim_k R(C)_\ell$ of C.)*

  For proofs of these results, see [12] for a), b), and [9] for c), d), and e).

A curve C is *arithmetically Cohen-Macaulay* (ACM) if its homogeneous coordinate ring $R(C)$ is a Cohen-Macaulay ring. The ACM curves form a special case of the above theorem that requires slightly modified statements.

Theorem 1.2.

a) *A curve C is ACM if and only if its Rao module is 0. The ACM curves form one biliaison equivalence class.*

b) *Any ACM curve can be obtained from a line by a finite number of ascending elementary biliaisons, plus a deformation.*

c) *The postulation character γ of an ACM curve is* **positive** *in the following sense: $\gamma(0) = -1$; if s_0 is the least integer ≥ 1 for which $\gamma(s_0) \geq 0$, then $\gamma(n) \geq 0$ for all $n \geq s_0$. Conversely, for every positive postulation character, there exists an ACM curve with that character.*

d) *If the ACM curve C is integral, then its postulation character is* **connected**, *meaning that $\{n \in \mathbb{Z} \mid \gamma(n) > 0\}$ is an interval in $\mathbb{Z}$. Conversely, for every connected positive character, there is a smooth irreducible ACM curve with that character.*

e) *For any positive γ, the set of ACM curves with postulation character γ is an irredicible subset of the Hilbert scheme.*

For proofs, see [1], [2], and [9].

Now our problem is to what extent do these results extend to curves in $\mathbb{P}^4$, or more generally to subschemes of codimension ≥ 3 in any projective space ?

First of all, it is clear that the definition of liaison given above using complete intersections (which we denote by CI-liaison) is too restrictive. This has been made abundantly clear in the work of [7] — see the report of R. Miró-Roig in this volume [11]: there are other invariants besides the Rao module for CI-liaison in codimension 3, and using these, one can construct many examples of curves in $\mathbb{P}^4$ having the same Rao module, but not in the same CI-liaison class.

Therefore we will take Gorenstein liaison to be the natural generalization of CI-liaison to higher codimension. We state the definitions for curves in $\mathbb{P}^4$, though the generalization to subschemes of any dimension in any $\mathbb{P}^4$ is obvious [10].

A curve D in $\mathbb{P}^4$ is *arithmetically Gorenstein* (AG) if its homogeneous coordinate ring $R(D) = R/I_D$ is a Gorenstein ring, where now $R = k[x_0, x_1, x_2, x_3, x_4]$ is the coordinate ring of $\mathbb{P}^4$. Two curves C_1, C_2 in $\mathbb{P}^4$ are *G-linked* if there exists an AG curve D satisfying the same conditions as in the definition of liaison for curves in $\mathbb{P}^3$ above. The equivalence relation generated by G-linkage is *G-liaison*. The equivalence relation generated by even numbers of G-linkages is *even G-liaison*.

A curve C_2 is obtained from C_1 by an *elementary G-biliaison of height h* if there exists an ACM surface X in $\mathbb{P}^4$ satisfying also G_1 (Gorenstein in

codimension one), containing C_1, such that $C_2 \sim C_1 + hH$ on X, where again H is the hypersurface section of X.

It is easy to see that a G-biliaison is an even G-liaison [10, sec. 5.4]. The authors of [7] are fond of speaking of G-liaison "as a theory of divisors on arithmetically Cohen-Macaulay schemes," and indeed, most of their examples of G-liaison can also be accomplished by elementary G-biliaisons. However, the relation between these two notions is not yet clear, so we pose it as a question.

Question 1.3. Is the equivalence relation generated by elementary G-biliaisons equivalent to even G-liaison ?

This is true for CI-liaison in any codimension [3, 4.4], hence for G-liaison in codimension 2, but is already unknown for curves in $\mathbb{P}^4$.

It is easy to see that evenly G-linked curves have the same Rao module, up to twist [10, 5.3.3], but the converse is unknown:

Question 1.4. If two curves C_1, C_2 in $\mathbb{P}^4$ have isomorphic Rao modules, up to shift in degrees, are they in the same G-liaison class ? In particular, are any two ACM curves in the same biliaison class ?

(This is equivalent to asking if every ACM curve is *glicci*, an acronym for "Gorenstein liaison class of a complete intersection.")

For a given finite-length graded module $M \neq 0$, it is easy to see there is a minimum twist $M(h_0)$ for which there are curves with Rao module $M(h_0)$ [10, 1.2.8]. These are called *minimal curves*. Migliore has observed [10, 5.4.8] that the set of minimal curves for a given M may not be irreducible, so we state

Problem 1.5. For a given module $M \neq 0$, describe the set of minimal curves for the module M. Are they all in the same even G-liaison class ?

As an analogue of the Lazarsfeld-Rao property, we ask

Question 1.6. If C is a curve with Rao module $M \neq 0$, can C be obtained by a finite number of ascending elementary G-biliaisons from a minimal curve for the module M ? For ACM curves we ask, can any ACM curve be obtained by a finite number of ascending elementary G-biliaisons from a line ?

And lastly,

Question 1.7. Does the set of curves with given Rao module M and postulation character γ form an irreducible subset of the Hilbert scheme ?

In spite of the optimism of some of the researchers mentioned in the references, my expectation is that many of these questions will have negative answers. The purpose of this talk is to give negative answers to a couple of these questions, and to propose potential counterexamples to some others. We refer to the paper [4] for more details of results only stated here, and further references.

2. Points in $\mathbb{P}^3$.

Closed subschemes of dimension zero of $\mathbb{P}^3$ form the first non-trivial case of codimension 3 schemes in a $\mathbb{P}^4$. Any such scheme is ACM, so the questions to consider are:

a) is every such scheme glicci ?
  and

b) can any such scheme be obtained from a single point by a sequence of ascending G-biliaisons (on ACM curves in $\mathbb{P}^3$) ?

Since the structure of arbitrary zero-dimensional subschemes can be quite complicated (unlike the case of zero-schemes in $\mathbb{P}^2$, the Hilbert scheme of zero-schemes of degree d in $\mathbb{P}^3$ for fixed d may not even be irreducible! [6]), we decided to consider only reduced zero-schemes, i. e., finite sets of points, in general position. Here *general position* will always mean for a suitable Zariski-open subset of the Hilbert scheme, possibly subject to the condition of lying in a given curve or a given surface. We begin by studying points on low degree surfaces. It is easy to show

Proposition 2.1. *Any set of n general points in $\mathbb{P}^2$ can be obtained by a finite set of ascending biliaisons (in this case CI-biliaison is equivalent to G-biliaison) from a point.* [4, 2.1]

Similarly, using the ACM curves on a nonsingular quadric surface, one can show

Proposition 2.2. *Any set of n general points on a (fixed) nonsingular quadric surface $Q \subseteq \mathbb{P}^3$ can be obtained from a single point by a finite number of ascending biliaisons (by ACM curves on Q).* [4, 2.2]

On a nonsingular cubic surface the situation is more complicated.

Proposition 2.3. *A set of n general points on a (fixed) nonsingular cubic surface $X \subseteq \mathbb{P}^3$ can be connected by G-liaisons through sets of general points of other degrees on X to a single point. In particular a set of n general points on X is glicci* [4, 2.4]

However, in the proof, we were not able to accomplish this using ascending biliaisons only. We had to use ascending and descending liaisons and biliaisons. For example, to treat 18 general points, one has to link up to 20, then 28 points, before linking down in many steps to a single point.

Corollary 2.4. *Any set of $n \leq 19$ general points in $\mathbb{P}^3$ is glicci.*

Proof. Indeed, $n \leq 19$ general points lie on a nonsingular cubic surface $\mathbb{P}^3$.

Our experience in these results is that points lying on surfaces of low degree 1, 2, or 3, are manageable, but these methods fail for sets of points on higher degree surfaces. This is consistent with the examples of ACM curves in $\mathbb{P}^4$, proved to be glicci by [7, sec. 8]: they lie on rational ACM surfaces which are all contained in hypersurfaces of degree 1, 2, or 3. So we propose a problem for the first case not falling under the above results.

Problem 2.5. If Z is a set of 20 points in general position in $\mathbb{P}^3$, is Z glicci ? Can Z be obtained by ascending G-biliaisons from a point ?

3. ACM curves in $\mathbb{P}^4$.

Following the principle of the previous section, we focus our attention on general curves, usually integral or nonsingular, and sufficiently general in their component of the Hilbert scheme. Using elementary geometry of curves on the cubic scroll, the Del Pezzo surface of degree 4, and the Castelnuovo surface of degree 5, we find:

Proposition 3.1. *For each possible degree d and genus g of a nondegenerate integral ACM curve in $\mathbb{P}^4$ of degree $d \leq 9$, the Hilbert scheme $H_{d,g}$ of such curves is irreducible, and a general such curve can be obtained by ascending G-biliaisons from a line. In particular, these curves are glicci.* [4, 3.4]

A similar argument, using curves on the Bordiga surface of degree 6, gives the same result for ACM curves with $(d, g) = (10, 6)$.

Example 3.2. For $(d, g) = (10, 9)$ the Hilbert scheme of smooth ACM curves in $\mathbb{P}^4$ has two irreducible components. To see this, first consider a nondegenerate smooth (10,9) curve C in $\mathbb{P}^4$. Since $h^0(\mathcal{O}_C(2)) = 12$, we find $h^0(\mathcal{I}_C(2)) \geq 3$. It follows that C is contained in an irreducible surface of degree 3, which must be either a cubic scroll or the cone over a twisted cubic curve in $\mathbb{P}^3$.

We represent the cubic scroll S as $\mathbb{P}^2$ with one point blown up. If ℓ is the total transform of a line in $\mathbb{P}^2$, and e is the class of the exceptional line, we denote a divisor $D = a\ell - be$ by $(a; b)$. Then S is embedded in $\mathbb{P}^4$

by $H = (2; 1)$. In this notation there are two types of smooth (10,9) curves, $C_1 = (6; 2)$ and $C_2 = (7; 4)$. Note that each of these is obtained by G-biliaison from a line on S: $C_1 \sim L_1 + 3H$ where $L_1 = (0; -1)$ and $C_2 \sim L_2 + 3H$ where $L_2 = (1; 1)$. Hence both types are ACM.

The two types are distinguished by the following properties:

a) their self-intersection on S: $C_1^2 = 32$ while $C_2^2 = 33$.
b) their trisecants: since S is an intersection of quadric hypersurfaces, any trisecant to C_i must lie in S. The lines in S are of types L_1, L_2 above. So we see that C_1 has no trisecants, while C_2 has infinitely many trisecants of type L_2.
c) their gonality: C_2 is trigonal (a g_3^1 is cut out by the trisecants) while C_1 is not trigonal.
d) their multisecant planes. Let π be a plane containing the conic Γ of type $(1;0)$ on S. Then $C_1 \cdot \pi = 6$ and $C_2 \cdot \pi = 7$. The pencil of hyperplanes through π cuts out a g_4^1 on C_1 and a g_3^1 on C_2, computing the gonality of each curve .

Because each of these curves is contained in a unique cubic scroll, if C_t is a family of smooth (10,9) curves, then it is contained in a family S_t of cubic surfaces. Hence the self-intersection of C_t on S_t is constant in a family, and we conclude that neither type can specialize to the other. Hence the Hilbert scheme of smooth curves $H_{10,9}$ has two irreducible components, represented by the two types C_1 and C_2.

In contrast to the situation in $\mathbb{P}^3$ (where for example, the Hilbert scheme of smooth curves of $(d, g) = (9, 10)$ has two disconnected components), our two components of $H_{10,9}$ in $\mathbb{P}^4$ have a common intersection, formed by smooth (10,9) curves lying on the singular cubic surface S_0, the cone over a twisted cubic curve in $\mathbb{P}^3$. In a family S_t of smooth cubic scrolls, with limit S_0, both classes of lines L_1, L_2 have as limit a ruling L_0 of the cone S_0. So the two divisor classes C_1, C_2 both tend to the singular divisor class $L_0 + 3H$ on S_0. It is easy to see this divisor class on S_0 contains smooth curves. Then, imitating the proof of [5, 2.1], cf. [5, 1.6], one can show that every smooth (10,9) curve on S_0 is a limit of flat families of curves of either type C_1 or type C_2 on cubic scrolls.

Note finally, since $\mathcal{O}_C(2)$ is already nonspecial, it is easy to see that the postulation of all ACM (10,9) curves is the same, so we have an example where the Hilbert scheme of ACM curves with a fixed postulation is not irreducible, answering Question 1.7 above.

To show that this example is not an isolated phenomenon, we prove the following.

Theorem 3.3. *Let X be a smooth ACM surface in $\mathbb{P}^4$, let $C_0 \subseteq X$ be a curve, and assume either:*

a) *X is rational,*
   or
b) *$C_0 \sim aH + bK$ for $a, b \in \mathbb{Z}$, where H is the hyperplane class, and K the canonical divisor on X. Then for $m >> 0$, the set of curves $C \sim C_0 + mH$ on X, together with their deformations $C_t \subseteq X_t$ as X moves in the family of ACM surfaces X_t, forms an open subset of an irreducible component of the Hilbert scheme of curves in $\mathbb{P}^4$.*

Proof. For $m >> 0$, each such curve C will lie on a unique such X_t, so the dimension of the family of these curves will be equal to the dimension of the linear system $|C|$ on X, which is equal to $h^0(\mathcal{N}_{C/X})$, where $\mathcal{N}$ denotes normal bundle, plus the dimension of the family of ACM surfaces X, which is equal to $h^0(\mathcal{N}_{X/\mathbb{P}^4})$ by [1]. (Here the hypothesis a) or b) of the statement guarantees that when we deform X, the divisor class C extends to the deformed surface.) On the other hand, we know that the dimension of the family of these curves is $\leq h^0(\mathcal{N}_{C/\mathbb{P}^4})$ by the differential study of the Hilbert scheme.

Now from the exact sequence

$$0 \to \mathcal{N}_{C/X} \to \mathcal{N}_{C/\mathbb{P}^4} \to \mathcal{N}_{X/\mathbb{P}^4} \otimes \mathcal{O}_C \to 0$$

we find

$$h^0(\mathcal{N}_{C/\mathbb{P}^4}) \leq h^0(\mathcal{N}_{C/X}) + h^0(\mathcal{N}_{X/\mathbb{P}^4} \otimes \mathcal{O}_C).$$

On the other hand, consider the exact sequence

$$0 \to \mathcal{N}_{X/\mathbb{P}^4}(-C) \to \mathcal{N}_{X/\mathbb{P}^4} \to \mathcal{N}_{X/\mathbb{P}^4} \otimes \mathcal{O}_C \to 0 \, .$$

Since $C \sim C_0 + mH$, it follows from duality and Serre vanishing that $h^i(\mathcal{N}_{X/\mathbb{P}^4}(-C)) = 0$ for $i = 0, 1$ and for $m >> 0$. Hence $h^0(\mathcal{N}_{X/\mathbb{P}^4}) = h^0\mathcal{N}_{X/\mathbb{P}^4} \otimes \mathcal{O}_C)$ for $m >> 0$.

Putting these inequalities together, we find that $h^0(\mathcal{N}_{X/\mathbb{P}^4})$ is equal to the dimension of the family of curves in question. We conclude from this that they form an open subset of a (generically reduced) irreducible component of the Hilbert scheme.

Example 3.4. We can use this theorem to make more examples of non-irreducible Hilbert schemes of curves with given postulation and Rao module.

A first example is furnished by the families $C_1 + mH$ and $C_2 + mH$ on the cubic scroll, where C_1, C_2 are the curves of Example 3.2. For given m, they

will have the same degree, genus, and postulation; each forms an open set of an irreducible component of the Hilbert scheme, but the two families are different because the curves have a different self-intersection on S.

For another example , let X be a Bordiga surface, represented as $\mathbb{P}^2$ with 10 points $P_1, \ldots, P_{10}$ blown up, where the notation $(a; b_1, \ldots, b_{10})$ represents the divisor $a\ell - \sum b_i e_i$, and the embedding is given by $H = (4; 1^{10})$. Consider the divisors

$$
\begin{aligned}
L_1 &= (0; 0^9, -1) \\
L_2 &= (1; 1^3, 0^7) \\
L_3 &= (2; 1^7, 0^3).
\end{aligned}
$$

On a general Bordiga surface, L_1 is a line, and L_2, L_3 are not effective. But if P_1, P_2, P_3 are collinear, we get a special smooth Bordiga surface on which L_2 is represented by a line. If $P_1, \ldots, P_7$ lie on a conic, we get another special Bordiga surface on which L_2 is represented by a line. It follows that $C_i \sim L_i + mH$ are ACM curves with the same postulation on a general Bordiga surface, for $i = 1, 2, 3$, and $m >> 0$.

By the theorem, each of these C_i forms an (open set of) an irreducible component of the Hilbert scheme. Since $L_i^2 = -1$, $L_2^2 = -2$, $L_3^2 = -3$, the C_i have different self-intersection, so the components are distinct.

Problem 3.5. Find a way to distinguish the irreducible components of the Hilbert scheme of ACM curves in $\mathbb{P}^4$ with given degree, genus, and postulation. For example, would any of the properties suggested in a),b),c),d) of Example 3.2 force the family to be irreducible ?

Example 3.6. Our last experiment with ACM curves is the first case of an ACM curve not contained in a cubic hypersurface, namely smooth ACM curves with $(d, g) = (20, 26)$.

There are such curves defined by the 4×4 minors of a 4×6 matrix of general linear forms. These determinantal curves are glicci, by a theorem of [7] and form an irreducible family of dimension ≤ 69 [7, 10.3].

Allowing these determinantal curves to move in linear systems on smooth ACM surfaces X of degree 10 and sectional genus 11, we get a family of curves of dimension ≤ 74, whose general member can be obtained by ascending G-biliaisons from a line, and hence is glicci [4, 3.9].

On the other hand, the differential study of the Hilbert scheme shows that every irreducible component of smooth curves of $(d, g) = (20, 26)$ must have dimension $\geq 5d + 1 - g = 75$.

By a subtle study of the dimensions of linear systems of curves on the ACM surface of degree 10 mentioned above, we show that a general curve in the

Hilbert scheme of (20,26) curves cannot be obtained by ascending G-biliaisons from a line [4, 3.9]. This gives a negative answer to the second half of Question 1.6. What remains is a problem.

Problem 3.7. Is an ACM curve with $(d, g) = (20, 26)$ in $\mathbb{P}^4$ glicci ?

4. Curves on $\mathbb{P}^4$ with Rao module $M \neq 0$.

Here the questions to investigate are whether all curves with Rao module M belong to the same G-liaison class; what do the minimal curves look like; and can an arbitrary curve with Rao module M be obtained from a minimal curve by ascending Gorenstein biliaisons. As yet, there is very little experimental evidence for these questions, but what little there is shows that the situation is quite complicated.

We first consider the case $M = k$, of dimension one in one degree only. We can describe completely the minimal curves in this case, which have $M = k$ in degree 0.

Proposition 4.1. *For every $d \geq 2$ there are minimal curves in $\mathbb{P}^4$ with Rao module $M = k$ in degree 0. For each d these curves form an irreducible family. The general member of the family is a disjoint union of a line and a plane curve of degree $d - 1$ in general position in $\mathbb{P}^4$. Furthermore, all of these minimal curves are in the same G-liaison class.* [4, 4.1]

To begin the study of other curves with Rao module $M = k$, we look at smooth curves of low degree and genus. They exhibit many different behaviors.

Example 4.2. Every smooth nondegenerate $(d, g) = (5, 0)$ curve in $\mathbb{P}^4$ lies on a cubic scroll, has $M = k$ in degree 1, and is obtained by G-biliaison from a minimal curve of degree 2, namely two skew lines [4, 4.3].

Example 4.3. Smooth nondegenerate (6,1) curves in $\mathbb{P}^4$ form an irreducible family. They have $M = k$ in degree 1. They fall into two types. The general curve C_1 lies on a Del Pezzo surface, and is obtained by a G-biliaison from two skew lines. This curve has two trisecants. The special curve C_2 lies on a cubic scroll, and is obtained by G-biliaison from a minimal curve of degree 3. It has infinitely many trisecants. So in this case the two types are distinguished by which minimal curve they come from under G-biliaison [4, 4.4].

Example 4.4. Smooth nondegenerate (7,2) curves form an irreducible family, whose general member has $M = k$ in degree 1. In this case the general member of the family can be obtained by two different routes from minimal curves: one route is G-biliaison on the Del Pezzo surface from a minimal curve of degree 3; the other is a G-biliaison on the Castelnuovo surface from a minimal curve of degree 2 [4, 4.5].

Example 4.5. Next we consider smooth nondegenerate (11,7) curves in $\mathbb{P}^4$. They form an irreducible family, whose general member has $M = k$ in degree 2. There are such curves on a Bordiga surface, obtained by G-biliaison in two steps: from two skew lines to a smooth (5,0) curves on a cubic scroll, then to the (11,7) curve on the Bordiga surface. However, we can show by counting dimensions that the general (11,7) curve does not lie on a Bordiga surface, and cannot be obtained by ascending G-biliaisons from a minimal curve. This provides a negative answer to the first part of Question 1.6 above [4, 4.7]. There remains a problem.

Problem 4.6. Is a general (11,7) curve in $\mathbb{P}^4$ in the G-liaison class of two skew lines ?

Minimal curves in $\mathbb{P}^4$ with Rao module $M_a = R/(x_0, x_1, x_2, x_3, x_4^a)$ for $a \geq 2$ have been studied by Lesperance [8]. He shows

Proposition 4.7. *For $a \geq 2$, there are minimal curves with Rao module M_a of every degree $d \geq a+1$. A reduced minimal curve is one of the following (where we denote by P the point $(0, 0, 0, 0, 1)$.)*
  a) *A disjoint union of a line and a plane curve of degree a in $\mathbb{P}^3$, where P is the point of intersection of the line and the plane.*
  b) *A disjoint union of plane curves of degrees a, b, with $a \leq b$, where P is the point of intersection of the two planes, and P does not lie on either curve.*
  c) *A disjoint union of plane curves of degrees a, b, with $b \geq 1$, where P is the point of intersection of the two planes, but this time P lies on the curve of degree b. (For $b = 1$, we recover type a) above.)*
  d) *A disjoint union of a line and an ACM curve in $\mathbb{P}^3$, where P is the point of intersection of the line and the $\mathbb{P}^3$, and a is the least degree of a surface in $\mathbb{P}^3$ containing the ACM curve, but not containing P.*

Example 4.8. In particular, the set of minimal curves of a given degree may not be irreducible. The first example is $a = 2$, degree 4, where there are minimal curves of type b), a union of two conics, and type d), a union of a line and a twisted cubic curve, which form two irreducible families [8, 4.5].

A more serious problem arises with the question of G-liaison. Lesperance is able to show that most of the minimal curves described in Proposition 4.7 are in the same G-liaison class as the first (type a). However there remains an open question, of which we state the first case.

Problem 4.9. Let C_1 be a disjoint union of two conics of type b) above, and let C_2 be a disjoint union of a line and a twisted cubic curve, of type d) above. Then both have Rao module M_2. Are they in the same G-liaison class ?

Example 4.10. Applying G-biliaison on a Del Pezzo surface, we can rephrase Problem 4.9 in terms of smooth curves with $(d, g) = (8, 3)$.

On the Del Pezzo surface X, note that the divisor class $(1; 1, 0^4)$ is a conic, and two such are disjoint. So we can take $C_1 = (2; 2, 0^4)$ on X, and let $D_1 = C_1 + H = (5; 3, 1^4)$. This is a smooth (8,3) curve.

On the other hand, $(1; 0^5)$ is a twisted cubic, and $(0; 0^4, -1)$ is a line not meeting it, so we can take $C_2 = (1; 0^4, -1)$, and $D_2 = C_2 + H = (4; 1^4, 0)$. This is another smooth (8,3) curve.

The curves of type D_1, D_2 both have Rao module M_2, but neither type can specialize to the other, because each lies in a unique Del Pezzo surface, and on that surface, their set of intersection numbers with the sixteen lines are $(1^8, 3^8)$ for D_1 and $(0, 1^4, 2^6, 3^4, 4)$ for D_2.

Note also that the Hilbert scheme of smooth (8,3) curves in $\mathbb{P}^4$ is irreducible, but the general curve has Rao module $M = k$ in degree 1, and does not lie on a Del Pezzo surface. Thus our two families of curves are locally closed irreducible subsets of $H_{8,3}$.

Both types of curves D_1, D_2 have self-intersection 12. However, the two types can be distinguished by

a) their intersections with the 16 lines on X (mentioned above)
b) their multisecants: C_1 has trisecant lines, but no quadrisecant, while C_2 has a quadrisecant line
c) their multisecant planes: let π be the plane containing the conic $(2; 0, 1^4)$ in X. Then $C_1 \cdot \pi = 6$ while $C_2 \cdot \pi = 5$.
d) their gonality: C_1 is hyperelliptic, with a g_2^1 cut out by the hyperplanes through π, while C_2 has gonality 3, and a g_3^1 is cut out by the hyperplanes through π
e) the point P (determined by the Rao module) lies on the surface X for type D_2, but does not lie on X for type D_1.

Now we can rephrase Problem 4.9 as

Problem 4.11. Do the two types of smooth (8,3) curves with Rao module M_2 (described above) belong to the same G-liaison class ?

5. Conclusion.

For ACM curves in $\mathbb{P}^4$, we have shown that the family of ACM curves with given degree, genus, and postulation may not be irreducible (3.2); we have given examples of ACM curves that cannot be obtained by ascending Gorenstein biliaison from a line (3.6); and we have proposed examples of ACM curves that may not be glicci (3.7).

For curves with Rao module $M \neq 0$, we have described the minimal curves in two cases, illustrating their complexity (4.1),(4.7); we have given examples of curves that cannot be obtained from a minimal curve by ascending G-biliaisons (4.5); and we have proposed examples of curves with the same Rao modules that may not be in the same G-liaison class (4.6), (4.9).

We have seen by example that certain families of curves with the same Rao module can be distinguished by the least degree of an ACM surface containing the curve, or their self-intersection on an ACM surface of least degree containing the curve, or their multisecant lines, or their multisecant planes, or their gonality. What is lacking at this point is a better understanding of how these geometrical properties of the curve in its embedding behave under the operation of Gorenstein liaison.

REFERENCES

[1] G. Ellingsrud, *Sur le schéma de Hilbert des variétés de codimension 2 dans $\mathbb{P}^e$ à cône de Cohen-Macaulay,* Ann. Sc. ENS, 8 (1975), pp. 423–432.

[2] L. Gruson - C. Peskine, *Genre des courbes de l'espace projectif,* Springer LNM, 687 (1977), pp. 31–59.

[3] R. Hartshorne, *Generalized divisors on Gorenstein schemes,* K-Theory, 8 (1994), pp. 287–339.

[4] R. Hartshorne, *Some examples of Gorenstein liaison in codimension three,* Collect. Math., 53 (2002), pp.21–48.

[5] R. Hartshorne, *Familes of curves in $\mathbb{P}^3$ and Zeuthen's problem,* Memoirs AMS no. 617, vol. 130 (1997).

[6] A. Iarrobino, *Reducibility of the families of 0-dimensional schemes on a variety,* Invent. Math., 15 (1972), pp. 72–77.

[7]   J. Kleppe - J. Migliore - R. Miró-Roig - U. Nagel - C. Peterson, *Gorenstein liaison, complete intersection liaison invariants, and unobstructedness,* Memoirs AMS n. 732, vol. 154,(2001).

[8]   J. Lesperance,   *Gorenstein liaison of some curves in* $\mathbb{P}^4$, Collect. Math., 52 (2001), pp. 219–230.

[9]   M. Martin-Deschamps - D. Perrin, *Sur la classification des courbes gauches,* Astérisque, 184-185 (1990).

[10] J. Migliore, *Introduction to Liaison Theory and Deficiency Modules,* Birkhäuser, Boston, 1998.

[11] R.M. Miró-Roig, *Complete intersection liaison and Gorenstein liaison: new results and open problems,* This volume.

[12] A.P. Rao, *Liaison equivalence classes,* Math. Ann., 258 (1981), pp. 169–173.

Department of Mathematics,
University of California,
Berkeley, California 94720–3840 (USA)