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CYCLIC POLYTOPES THROUGH THE LENS OF ITERATED INTEGRALS

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The volume of a cyclic polytope can be obtained by forming an iterated integral along a suitable piecewise linear path running through its edges. Different choices of such a path are related by the action of a subgroup of the combinatorial automorphisms of the polytope. Motivated by this observation, we look for other linear combinations of iterated integrals that are invariant under the subgroup action. This yields interesting polynomial attributes of the cyclic polytope. We prove that there are infinitely many of these invariants which are algebraically independent in the shuffle algebra.

1. Introduction

Iterated integrals and piecewise linear paths A *path*, for the purpose of this paper, is a continuous map $X : [0,1] \to \mathbb{R}^d$ such that the coordinate functions X_i are piecewise continuously differentiable. Given such a path X, its *(iterated integral) signature* is the linear form

$$S(X) : \mathbb{R}\langle \mathtt{1}, \dots, \mathtt{d} \rangle \to \mathbb{R}, \quad \mathtt{i}_1 \cdots \mathtt{i}_k \mapsto \int_{\Delta_k} dX_{i_1}(t_1) \dots dX_{i_k}(t_k) \tag{1}$$

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where k varies over all positive integers and Δ_k denotes the simplex $0 \le t_1 \le \cdots \le t_k \le 1$. Here, $\mathbb{R}\langle \mathtt{1}, \ldots, \mathtt{d} \rangle$ is the free associative algebra over the letters (that is, formal symbols) $\mathtt{1}, \ldots, \mathtt{d}$. The words $\mathtt{i}_1 \ldots \mathtt{i}_k$ form a basis of this space, such that (1) does indeed define a linear form. For example, $S(X)(\mathtt{11} + \mathtt{12}) = \int_0^1 \int_0^{t_2} X_1'(t_1) X_1'(t_2) + X_1'(t_1) X_2'(t_2) dt_1 dt_2$.

The signature of a path determines the path up to translation, reparametrization and tree-like equivalence [4, 6].

In this paper, we are interested in *piecewise linear paths*. Such a path is uniquely determined by its control points $x_1, \ldots, x_n \in \mathbb{R}^d$, that is, the (ordered) set of start and end points of all of its linear segments. The signature of such a piecewise linear path can be described explicitly in terms of the increments $a_i := x_i - x_{i-1}$. In fact, it defines a map

$$H_n^d : \mathbb{R}\langle \mathtt{1}, \ldots, \mathtt{d} \rangle \to \mathbb{R}[x_1, \ldots, x_n], \tag{2}$$

where $x_i = (x_{i1}, \ldots, x_{id})$, see Theorem and Definition 2.4 and Equation (6) for a recursive formula.

If the left-hand side is viewed as a commutative algebra $\mathbb{R}\langle \mathtt{1}, \ldots, \mathtt{d} \rangle_{\sqcup\!\sqcup}$ via the *shuffle product* $\sqcup\!\sqcup$ (see (7), and e.g. [7] for an introduction based on the recursive definition), this map becomes a homomorphism of graded algebras. Its image is a subalgebra of $\mathbb{R}[x_1, \ldots, x_n]$ which we will call *the ring of signature polynomials in $d \times n$ variables* in the following, denoted by $\mathcal{S}^d[x_1, \ldots, x_n]$.

The polynomials in $\mathcal{S}^d[x_1, \ldots, x_n]$ inherit some nice properties from their integral representation. For example, they are translation invariant,

$$p(x_1, \ldots, x_n) = p(x_1 + y, \ldots, x_n + y)$$

for $y \in \mathbb{R}^d$. Moreover, if $p \in \mathcal{S}^d[x_1, \ldots, x_n]$, then for $1 < i < n$ the polynomial

$$p(x_1, \ldots, x_{i-1}, \lambda x_{i-1} + (1 - \lambda) x_{i+1}, x_{i+1}, \ldots, x_n) \tag{3}$$

is independent of $\lambda \in [0, 1]$, due to reparametrization invariance of iterated integrals. In particular, (3) is a polynomial in variables $x_j, j \ne i$.

It follows that for an injective map $I \to J$ of finite totally ordered sets, there is a natural "restriction map" $S^d[x_J] \to \mathcal{S}^d[x_I]$ (where $x_I = \{x_i \mid i \in I\}$), given by the map $\mathcal{S}^d[x_J] \to \mathcal{S}^d[x_I]$ induced by replacing for $j \in J \backslash I$ the variable x_j by x_i where i is the largest element in I smaller than j or, by the above equivalently, the smallest element in I larger than j.

Now, for fixed n and a given subgroup G of S_n one can ask the following question: Which signature polynomials in $d \times n$ variables are invariant under the action of G on $x_1, \ldots, x_n$ by permutation? More precisely, we would like to determine the pullback $\mathsf{Inv}_n^d(G) \subseteq \mathbb{R}\langle \mathtt{1},\dots,\mathtt{d}\rangle$ in

$$\begin{array}{ccc} \mathsf{Inv}_n^d(G) & \longrightarrow & \mathbb{R}[x_1,\dots,x_n]^G \\ \downarrow & & \downarrow \\ \mathbb{R}\langle \mathtt{1},\dots,\mathtt{d}\rangle & \xrightarrow{H_n^d} & \mathbb{R}[x_1,\dots,x_n] \end{array}$$

where $\mathbb{R}[x_1,\dots,x_n]^G$ is the subring of G-invariants in $\mathbb{R}[x_1,\dots,x_n]$.

Towards positivity In this paper, we address this question for a specific choice of G. Namely, given d and $n \geq d+1$, S_n acts naturally by permutations of columns on the set of $(d+1)\times n$-matrices

$$\begin{pmatrix} 1 & \dots & 1 \\ x_1 & \dots & x_n \end{pmatrix} \tag{4}$$

and we want to choose G as the stabiliser C_n^d of the subset of matrices whose maximal minors are positive. Following the terminology of [3, Section 2], we call these matrices *positive*.

Our motivation is that for each such positive matrix, the volume of the polytope $\mathrm{conv}(x_1,\dots,x_n)$ can be obtained from the signature of the piecewise linear path X with control points $x_1 \to \dots \to x_n$ as the *signed volume*

$$\langle S(X), \tfrac{1}{d!}\mathsf{vol}_d\rangle = \frac{1}{d!}\int_{\Delta_d} \det\big(X'(t_1) \quad \dots \quad X'(t_d)\big)\, dt_1 \dots dt_d$$

where

$$\mathsf{vol}_d := \sum_{\sigma\in S_d} \mathrm{sgn}(\sigma)\sigma(\mathtt{1})\dots\sigma(\mathtt{d}) \in \mathbb{R}\langle \mathtt{1},\dots,\mathtt{d}\rangle, \tag{5}$$

see [12, Section 3.3]. In fact, one can deduce more generally that the volume of the convex hull of a so-called *convex* path agrees with its signed volume, cf. [2, Theorem 3.4]. Convex piecewise linear paths correspond precisely to matrices with nonnegative maximal minors [16, Section 2]. See the references in [2] for more context on convex paths.

As the set of $x_1,\dots,x_n$ with (4) positive is Zariski-dense in the set of all $x_1,\dots,x_n$, it follows that $\mathsf{vol}_d \in \mathsf{Inv}_n^d(C_n^d)$ for all $n \geq d+1$ (see Proposition 3.12). Thus, we expect $\mathsf{Inv}_n^d(C_n^d)$ to describe geometric features of polytopes of the form $\mathrm{conv}(x_1,\dots,x_n)$ for positive matrices (4). Such polytopes are known as cyclic d-polytopes admitting a *canonical labeling* $x_1,\dots,x_n$ (note however that this is not necessarily unique).

Of particular interest is the intersection

$$\mathsf{Inv}^d := \bigcap_{n\geq d+1} \mathsf{Inv}_n^d(C_n^d)$$

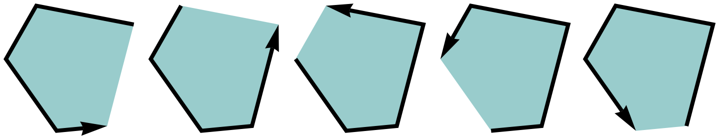

Figure 1: A cyclic 2-polytope with 5 vertices, spanned by 5 different piecewise linear paths, related by cyclic permutations of their control points. The volume of the polygon agrees with the signed volume of each of the paths.

which we call the *ring of volume invariants*. This terminology will be motivated in Proposition 3.12 where we show that if the signed volume of a piecewise linear path in general position is invariant under a permutation of the control points then its signature at any $w \in \mathsf{Inv}^d$ will be as well.

Note that Inv^d forms a subalgebra of $\mathbb{R}\langle \mathtt{1}, \dots, \mathtt{d} \rangle_{\sqcup\!\sqcup}$. One might view $H_n^d(\mathsf{Inv}^d)$ as functions on the set of canonically labeled cyclic d-polytopes with n vertices. In contrast to the discussion in [13, Section 6], these features are not necessarily SO-invariants. Note that by definition of Inv^d, $H_\bullet^d(\mathsf{Inv}^d)$ is compatible with the restriction maps of rings of signature polynomials. We can interpret them as restrictions to subpolytopes.

As noted above, we certainly have $\mathsf{vol}_d \in \mathsf{Inv}^d$. The main result of this paper is the following theorem in Section 4, showing that there is an abundance of volume invariants:

Theorem (4.9)**.** For any d, Inv^d contains infinitely many algebraically independent elements (with respect to the shuffle product) and is thus in particular infinitely generated as a (shuffle) subalgebra of $\mathbb{R}\langle \mathtt{1}, \dots, \mathtt{d} \rangle$.

Outline In Section 2, we precisely define and thoroughly discuss the ring homomorphism H_n^d. In particular, we describe how to obtain a useful recursive formula in (6).

Section 3 is devoted to introducing the subgroup $C_n^d \subset S_n$ as the stabilizer of positive matrices, i.e. matrices with positive maximal minors, under column permutation. We then show that C_n^d is exactly the subgroup of S_n under which the signed volume is invariant for piecewise linear paths in general position.

Finally, in Section 4, we prove our main results. In Propositions 4.5 and 4.8, we fully characterize the invariant rings $\mathsf{Inv}^d_{\geq d+3}$ for $d+3$ and more points in $\mathbb{R}^d$. In Theorem 4.9, we show that the rings of volume invariants Inv^d are ‘very large’, in the sense that they are infinitely generated, and even contain infinitely many algebraically independent elements.

2. Piecewise linear paths and signatures

Our goal in this section is to define and explain the homomorphism H_n^d.

Definition 2.1. A piecewise linear path with control points $x_1,\dots,x_n \in \mathbb{R}^d$ is a continuous map $X : [0,1] \to \mathbb{R}^d$ such that there are $0 = t_1 \le t_2 \le \cdots \le t_n = 1$ with the property that $X(t_i) = x_i$ for all i and X is an affine map on all intervals $[t_i,t_{i+1}]$. We write $\{x_1 \to \cdots \to x_n\}$ to denote such a path independent of the precise time parametrization, and we write PL_n^d for the set of all piecewise linear paths through $\mathbb{R}^d$ with n control points.

In particular, a piecewise linear path with two control points is a linear path. Up to reparametrization, any piecewise linear path can be viewed as a *concatenation* of linear paths.

Definition 2.2. Given paths $X : [0,1] \to \mathbb{R}^d$ and $Y : [0,1] \to \mathbb{R}^d$, their concatenation is the path $X \sqcup Y : [0,1] \to \mathbb{R}^d$ which is defined as $X(2t)$ for $t \in [0,\frac{1}{2}]$ and as $Y(2t-1)$ for $t \in [\frac{1}{2},1]$.

An important property of the signature is its compatibility with concatenation, in the following sense:

Proposition 2.3 (Chen's identity, Theorem 3.1 of [5])**.** Let X and Y be paths in $\mathbb{R}^d$. Then

$$S(X \sqcup Y) = S(X) \bullet S(Y).$$

Here $S(X) \bullet S(Y)$ is the composition

$$\mathbb{R}\langle \mathtt{1},\dots,\mathtt{d}\rangle \xrightarrow{\ \Delta\ } \mathbb{R}\langle \mathtt{1},\dots,\mathtt{d}\rangle \otimes \mathbb{R}\langle \mathtt{1},\dots,\mathtt{d}\rangle \xrightarrow{S(X)\otimes S(Y)} \mathbb{R}$$

where Δ denotes the coproduct of the Hopf algebra $\mathbb{R}\langle \mathtt{1},\dots,\mathtt{d}\rangle$. More explicitly, $S(X) \bullet S(Y)$ maps a word $\mathtt{i}_1 \cdots \mathtt{i}_k$ to

$$\sum_{j=0}^{k} \langle S(X), \mathtt{i}_1 \cdots \mathtt{i}_j\rangle \langle S(Y), \mathtt{i}_{j+1} \cdots \mathtt{i}_k\rangle.$$

We are now ready to define H_n^d.

Theorem and Definition 2.4. For any $w \in \mathbb{R}\langle \mathtt{1},\dots,\mathtt{d}\rangle$ the function

$$f_n(w) : \mathbb{R}^{d\times n} \to \mathbb{R}, \quad (x_1,\dots,x_n) \mapsto \langle S(\{x_1 \to \cdots \to x_n\}), w\rangle$$

is given by a polynomial in $x_1,\dots,x_n$. We define

$$H_n^d : \mathbb{R}\langle \mathtt{1},\dots,\mathtt{d}\rangle \to \mathbb{R}[x_1,\dots,x_n], \quad w \mapsto f_n(w)$$

Proof. First note that $f_n(w)$ is well-defined by reparametrization invariance of the signature S. We will now proceed by induction, starting with $n = 2$.

The signature of a linear path X with control points x_1, x_2 is easily calculated. Indeed, note that the integrals (1) only depend on the vector $a := x_2 - x_1$. The integrand of $\langle S(X), \mathtt{i}_1 \cdots \mathtt{i}_k \rangle$ is just the product $a_{i_1} \ldots a_{i_k}$ and thus

$$f_2(\mathtt{i}_1 \cdots \mathtt{i}_k) = \frac{1}{k!} a_{i_1} \ldots a_{i_k}$$

as the simplex Δ_k has volume $\frac{1}{k!}$, proving the base case. Now by Proposition 2.3 we have

$$\begin{aligned} &f_n(\mathtt{i}_1 \cdots \mathtt{i}_k)(x_1, \ldots, x_n) = \\ &\sum_{j=0}^{k} f_l(\mathtt{i}_1 \cdots \mathtt{i}_j)(x_1, \ldots, x_l) \cdot f_{n-l+1}(\mathtt{i}_{j+1} \cdots \mathtt{i}_k)(x_l, \ldots, x_n) \end{aligned}$$

for any n and all l. Choosing $l = 2$, we conclude by induction. □

Note that the proof yields the recursive formula

$$\begin{aligned} &H_n^d(\mathtt{i}_1 \cdots \mathtt{i}_k)(x_1, \ldots, x_n) = \\ &\sum_{j=0}^{k} \frac{1}{j!} H_{n-1}^d(\mathtt{i}_{j+1} \cdots \mathtt{i}_k)(x_2, \ldots, x_n) \prod_{m=1}^{j} (x_{2,i_m} - x_{1,i_m}). \end{aligned} \tag{6}$$

for $H_n^d(\mathtt{i}_1 \cdots \mathtt{i}_k)(x_1, \ldots, x_n)$.

Example 2.5. We have

$$\begin{aligned} &H_3^3(\mathtt{123})(x_1, x_2, x_3) = \\ &H_2^3(\mathtt{123})(x_2, x_3) H_2^3(\mathtt{e})(x_1, x_2) + H_2^3(\mathtt{23})(x_2, x_3) H_2^3(\mathtt{1})(x_1, x_2) \\ &\quad + H_2^3(\mathtt{3})(x_2, x_3) H_2^3(\mathtt{12})(x_1, x_2) + H_2^3(\mathtt{e})(x_2, x_3) H_2^3(\mathtt{123})(x_1, x_2) \\ &= \frac{1}{3!} a_{2,1} a_{2,2} a_{2,3} + \frac{1}{2!} a_{2,2} a_{2,3} \cdot a_{1,1} + a_{2,3} \cdot \frac{1}{2!} a_{1,1} a_{1,2} + \frac{1}{3!} a_{1,1} a_{1,2} a_{1,3} \end{aligned}$$

where $\mathtt{e}$ is the unit of $\mathbb{R}\langle \mathtt{1}, \ldots, \mathtt{d} \rangle$, the so-called empty word, which is mapped by all H_n^d to the unit constant polynomial.

Remark 2.6. In general, H_n^d can be shown to factor as

$$\begin{array}{ccc} \mathbb{R}\langle \mathtt{1}, \ldots, \mathtt{d} \rangle & & \\ \downarrow & \searrow^{H_n^d} & \\ \mathrm{Qsym}(\mathbb{R}[a_1, \ldots, a_{n-1}]) & \longrightarrow & \mathbb{R}[x_1, \ldots, x_n] \end{array}$$

where $\mathrm{Qsym}(\mathbb{R}[a_1,\ldots,a_{n-1}])$ denotes the ring of *quasi-symmetric functions of level d* in the vectors $a_1,\ldots,a_{n-1}$, cf. [9], and the bottom map sends a_i to $x_{i+1}-x_i$. We refer to [1] for further details about the signature of a piecewise linear path.

Let us now explain how to turn H_n^d into an algebra homomorphism. We do this by equipping the $\mathbb{R}$-vector space $\mathbb{R}\langle \mathtt{1},\ldots,\mathtt{d}\rangle$ with the commutative *shuffle product* $\sqcup\!\sqcup$. This can be defined on words in the following way:

$$\mathtt{i}_1\cdots\mathtt{i}_l \sqcup\!\sqcup \mathtt{i}_{l+1}\cdots\mathtt{i}_k := \sum_{\sigma\in G} \mathtt{i}_{\sigma^{-1}(1)}\cdots\mathtt{i}_{\sigma^{-1}(k)} \tag{7}$$

where G is the set of $\sigma \in S_k$ with $\sigma^{-1}(1) < \cdots < \sigma^{-1}(l)$ and $\sigma^{-1}(l+1) < \cdots < \sigma^{-1}(k)$. In other words, $\mathtt{i}_1\cdots\mathtt{i}_l \sqcup\!\sqcup \mathtt{i}_{l+1}\cdots\mathtt{i}_k$ is the sum of all ways of interleaving the two words $\mathtt{i}_1\cdots\mathtt{i}_l$ and $\mathtt{i}_{l+1}\cdots\mathtt{i}_k$.

The commutative algebra $(\mathbb{R}\langle \mathtt{1},\ldots,\mathtt{d}\rangle, \sqcup\!\sqcup)$ is well-understood. As an algebra, it is (infinitely) freely generated by the *Lyndon words*. For details, we refer to e.g. [21]. The connection to iterated integrals is the following:

Proposition 2.7 (Ree's shuffle identity [20])**.** Let X be a path in $\mathbb{R}^d$ and $p,q \in \mathbb{R}\langle \mathtt{1},\ldots,\mathtt{d}\rangle$. Then

$$\langle S(X), p \sqcup\!\sqcup q\rangle = \langle S(X), p\rangle\langle S(X), q\rangle$$

Corollary 2.8. The maps H_n^d are algebra homomorphisms

$$\mathbb{R}\langle \mathtt{1},\ldots,\mathtt{d}\rangle_{\sqcup\!\sqcup} := (\mathbb{R}\langle \mathtt{1},\ldots,\mathtt{d}\rangle, \sqcup\!\sqcup) \to (\mathbb{R}[x_1,\ldots,x_n],\cdot).$$

In particular, the kernel of H_n^d, which we denote by $\mathcal{I}(\mathsf{PL}_n^d)$, is an ideal in $\mathbb{R}\langle \mathtt{1},\ldots,\mathtt{d}\rangle_{\sqcup\!\sqcup}$. Geometrically, it can be viewed as the vanishing ideal of $S(\mathsf{PL}_n^d) \subseteq \mathrm{Spec}\,\mathbb{R}\langle \mathtt{1},\ldots,\mathtt{d}\rangle$, which is the image of PL_n^d under the signature (in [18], this is just called the vanishing ideal of PL_n^d). It follows that the ring of signature polynomials $\mathcal{S}^d[x_1,\ldots,x_n]$ is isomorphic to $\mathbb{R}\langle \mathtt{1},\ldots,\mathtt{d}\rangle_{\sqcup\!\sqcup}/\mathcal{I}(\mathsf{PL}_n^d)$.

3. The stabiliser of positive matrices

In this section, our goal is to determine the group C_n^d from the introduction. Recall that C_n^d is defined as the subgroup of S_n stabilising the set of positive $d\times n$-matrices under the action on columns.

Definition 3.1 (cf. [3, Section 2])**.** An $d\times n$ matrix is called positive if all its maximal minors are positive.

Remark 3.2. The *totally positive Grassmanian* is the quotient of positive matrices by the left GL^+-action, see [17, Definition 3.1]. However, not every positive matrix is *totally positive* in the sense of loc. cit.

The group S_n acts on the columns of an $(d+1)\times n$-matrix by permutation. C_n^d is defined as the stabiliser of positive matrices under this action. In other words, C_n^d is the subgroup of permutations of the columns of a $(d+1)\times n$ matrix with positive maximal minors such that the resulting matrix has again positive maximal minors.

It turns out that the parity of d has a large impact on the structure of C_n^d. We give a full description of this structure in Proposition 3.8 and Proposition 3.10 below. In fact, the group C_n^d is a subgroup of the automorphisms of a cyclic d-polytope with n vertices. To see this, we recall some elementary theory of (cyclic) polytopes.

Lemma 3.3. Let P be a d-dimensional polytope with vertex set $V := \{x_1,\dots,x_n\}$. Then $F = \{x_{i_1},\dots,x_{i_d}\}$ is the vertex set of a facet if and only if

$$\det\begin{pmatrix} 1 & \dots & 1 & 1 \\ x_{i_1} & \dots & x_{i_d} & y \end{pmatrix} \tag{8}$$

has a fixed sign for all $y \in V - F$.

Proof. Note that

$$\det\begin{pmatrix} 1 & \dots & 1 & 1 \\ x_{i_1} & \dots & x_{i_d} & y \end{pmatrix} = \det\begin{pmatrix} x_{i_2}-x_{i_1} & \dots & x_{i_d}-x_{i_1} & y-x_{i_1} \end{pmatrix}$$

As a function in y, this determinant vanishes exactly on the hyperplane spanned by $x_{i_1},\dots,x_{i_d}$ and has constant sign on the two associated open half-spaces. □

Corollary 3.4 (Gale's evenness criterion, [14])**.** Let P be a polytope with vertices $x_1,\dots,x_n$ such that all maximal minors of (4) have the same sign. Then the facets of P are exactly the sets $F = x_I := \{x_{i_1},\dots,x_{i_d}\}$ such that $\#\{i\in I \mid i>j\}$ has the same parity for all $j \in [n]-I$. In other words, P is a cyclic polytope.

Proof. This follows immediately from Lemma 3.3 as the sign of the determinant (8) is exactly $(-1)^{\#\{i\in I \mid i>j\}}$ times the sign of a maximal minor of (4) for $y = x_j \notin F$. □

Corollary 3.5. Let $x_1,\dots,x_n$ be such that the matrix

$$\begin{pmatrix} 1 & \dots & 1 \\ x_1 & \dots & x_n \end{pmatrix}$$

has positive maximal minors. Then for every $\pi \in S_n$ such that the matrix

$$\begin{pmatrix} 1 & \dots & 1 \\ x_{\pi(1)} & \dots & x_{\pi(n)} \end{pmatrix}$$

has positive maximal minors, $x_i \mapsto x_{\pi(i)}$ is a combinatorial automorphism of the cyclic polytope $P = \mathrm{conv}(x_1, \dots, x_n)$. That is, it defines an automorphism of its face lattice.

Proof. This is clear from Corollary 3.4 since the face condition there is only a condition on indices: it does not depend on the x_i themselves. □

We will see in Corollary 3.11 that for even d the converse is true up to sign, that is, combinatorial automorphisms preserve the property that all minors have the same sign. This is not true in odd dimensions:

Example 3.6. Let $x_1, \dots, x_6 \in \mathbb{R}^3$ be such that

$$\begin{pmatrix} 1 & \dots & 1 \\ x_1 & \dots & x_6 \end{pmatrix}$$

has positive maximal minors. Then $(x_1, x_2, x_3, x_4, x_5, x_6) \mapsto (x_6, x_2, x_3, x_4, x_5, x_1)$ is an automorphism of cyclic polytopes, but

$$\det \begin{pmatrix} 1 & 1 & 1 & 1 \\ x_6 & x_2 & x_3 & x_4 \end{pmatrix} < 0$$

while

$$\det \begin{pmatrix} 1 & 1 & 1 & 1 \\ x_2 & x_3 & x_4 & x_5 \end{pmatrix} > 0$$

Let us now give a full description of C_n^d. We will use the following characterization of the combinatorial automorphisms of a cyclic polytope:

Theorem 3.7 ([15, Theorem 8.3])**.** The combinatorial automorphism group of a cyclic d-polytope with n vertices is isomorphic to

	$n = d + 1$	$n = d + 2$	$n \geq d + 3$
d even	$\mathbb{S}_n$	$\mathbb{S}_{\frac{n}{2}}$ wr $\mathbb{Z}_2$	$\mathbb{D}_n$
d odd	$\mathbb{S}_n$	$\mathbb{S}_{\lceil \frac{n}{2} \rceil} \times \mathbb{S}_{\lfloor \frac{n}{2} \rfloor}$	$\mathbb{Z}_2 \times \mathbb{Z}_2$

We start with the case of odd dimension.

Proposition 3.8. Assume d is odd. Then the group C_n^d is

i) A_n if $n = d+1$,

ii) $A_n \cap (S_{\frac{n-1}{2}} \times S_{\frac{n+1}{2}})$ if $n = d+2$,

iii) $\mathbb{Z}/2$ if $n \geq d+3$ and $\frac{d+1}{2}$ is even and

iv) 1 if $n \geq d+3$ and $\frac{d+1}{2}$ is odd.

For the proof we need the following small lemma:

Lemma 3.9. Let

$$X = \begin{pmatrix} x_1 & \dots & x_{n+1} \end{pmatrix}$$

be a $n \times (n+1)$-matrix such that all minors have the same sign. Then switching two even or two odd columns switches the sign of all minors of X.

Proof. Let i, j be the indices of the columns that are switched and write X' for the matrix obtained from X by switching columns i and j. For $I = \{i_1, \dots, i_n\}$ with $i_1 < \dots < i_n$ we write X_I for the $n \times n$ matrix $\begin{pmatrix} x_{i_1} & \dots & x_{i_n} \end{pmatrix}$.
If I contains both i and j, then X'_I is obtained from X_I by switching two columns and thus inverts the sign of the determinant. Now assume that I does not contain j, so it contains i. X'_I is obtained from X_I by replacing x_i with x_j. In particular, if $J := I \cup \{j\} - \{i\}$, then X'_I has the same set of columns as X_J. Now note that every integer between i and j is contained in I and since $i - j$ is even the number of such integers is odd. Thus, an odd number of transpositions is required to obtain X_J from X'_I, and thus the determinant of X'_I and the determinant of X_J have inverse signs. □

Proof of Proposition 3.8. Let $x_1, \dots, x_n$ be such that (4) is positive. Set $P = \operatorname{conv}(x_1, \dots, x_n)$. We need to determine the subgroup of $\operatorname{Aut}(P)$ consisting of automorphisms that preserve positivity of all minors. Given $\pi \in \operatorname{Aut}(P)$ we write $\pi(X)$ for the matrix whose i-th column is the $\pi(i)$-th column of X.

- *i*) is clear from Theorem 3.7 as the determinant is an alternating map (here P is a simplex).
- For *ii*) we use that by Theorem 3.7 we have $\operatorname{Aut}(P) = S_{\frac{n-1}{2}} \times S_{\frac{n+1}{2}}$ if $n = d+2$, where the first factor acts on the even and the second on the odd vertices of P. Thus, the statement follows from Lemma 3.9.

- For *iii*) and *iv*) we use that, again by Theorem 3.7, $\mathrm{Aut}(P) = \mathbb{Z}/2 \times \mathbb{Z}/2$ where the first factor acts by switching the first and the last vertex and the second factor acts by inverting the order on the inner vertices. Let π be the generator of the first factor and s the generator of the second factor. If $\frac{d+1}{2}$ is even, then $\tau = \pi \circ s$ preserves the sign of minors, otherwise it flips the sign. This is because any minor of $\pi \circ s(X)$ is turned into a submatrix of X by $\frac{d+1}{2}$ transpositions. Next, aiming for a contradiction, assume s preserves positive minors. Then $\pi \circ s \circ s = \pi$ either preserves or flips the sign of all minors. But this is not the case: Consider the matrix of the first $d+1$ columns of $\pi(X)$. This matrix is turned into a submatrix of X by d transpositions of columns, so it has negative determinant. But the matrix of columns $2, \ldots, d+2$ of $\pi(X)$ is still a submatrix of X and has, in particular, positive determinant (see Example 3.6 for an example in the case $d = 3$).

□

Proposition 3.10. Assume d is even. Then the group C_n^d is

i) A_n if $n = d+1$,

ii) $(A_n \cap (S_{\frac{n}{2}} \times S_{\frac{n}{2}})) \rtimes \mathbb{Z}/2$ if $n = d+2$ and $\frac{d}{2}$ is even,

iii) $\ker \phi$ for the map $\phi : S_{\frac{n}{2}} \times S_{\frac{n}{2}} \rtimes \mathbb{Z}/2 \to \{-1, 1\}$ that maps (ω, π, τ) to $\mathrm{sgn}(\omega)\,\mathrm{sgn}(\pi)\gamma(\tau)$ (where γ maps τ to -1), if $n = d+2$ and $\frac{d}{2}$ is odd,

iv) $\mathbb{D}_n$ if $n \geq d+3$ and $\frac{d}{2}$ is even and

v) $\mathbb{Z}/n$ if $n \geq d+3$ and $\frac{d}{2}$ is odd.

Proof. Let again $x_1, \ldots, x_n$ be such that (4) is positive. We set $P = \mathrm{conv}(x_1, \ldots, x_n)$.

- *i*) is still clear from Theorem 3.7 as the determinant is an alternating map (again, P is a simplex).
- For *ii*) we need to adapt the proof of Theorem 3.7. Let $s \in S_n$ denote the order-reversing permutation. Since $\frac{d}{2}$ transpositions of columns turn any $d+1 \times d+1$ submatrix of $s(X)$ into a submatrix of X, s preserves the sign of minors if $\frac{d}{2}$ is even and flips it if $\frac{d}{2}$ is odd. In the first case, going through the proof of Theorem 3.7 and using Lemma 3.9, we obtain the semi-direct product $(A_n \cap S_{\frac{n}{2}} \times S_{\frac{n}{2}}) \rtimes \mathbb{Z}/2$. In the second case it is more difficult to describe the subgroup; we can define it as the kernel of the map $\phi : (S_{\frac{n}{2}} \times S_{\frac{n}{2}}) \rtimes \mathbb{Z}/2 \to \{-1, 1\}$ that maps (ω, π, τ) to $\mathrm{sgn}(\omega)\,\mathrm{sgn}(\pi)\gamma(\tau)$ where γ maps τ to -1.

- For *iv*) we use that, again by Theorem 3.7, $\mathrm{Aut}(P) = \mathbb{D}_n$. Let r be rotation by 1 and s the order-reversing permutation. They generate $\mathbb{D}_n$ and since $\frac{d}{2}$ is even, both of them preserve positive minors.
- Similarly, if $\frac{d}{2}$ is odd, then s will flip the signs of all minors. Thus, since $srs = r^{-1}$, we just obtain the subgroup generated by r in this case.

□

For case v), compare [17, Remark 3.3].

Corollary 3.11. For even d, every automorphism of a cyclic d-polytope with n vertices preserves the property that the maximal minors of (4) have constant sign.

Proof. Going through the proof of Proposition 3.10 again, we see that any combinatorial automorphism either preserves or flips the sign of all maximal minors. □

The following result adds further motivation to our interest in C_n^d and explains why we call Inv^d the ring of volume invariants.

Proposition 3.12. For all $n \geq d+1$ there is a non-empty Zariski open subset O of $\mathbb{R}^{d\times n}$ such that for all $X \in O$, $\sigma \in S_n$:

$$\langle S(X), \mathsf{vol}_d \rangle = \langle S(\sigma.X), \mathsf{vol}_d \rangle \text{ if and only if } \sigma \in C_n^d.$$

Here, we identify a piecewise linear path X with its ordered set of control points, i.e., a $d \times n$ matrix X.

Proof. The implication $\Leftarrow$ holds for all X as it holds for the Zariski dense subset of X with (4) positive, as discussed in the introduction. Thus it suffices to show the converse.

Let us denote the vanishing locus of the polynomial

$$\langle S(X), \mathsf{vol}_d \rangle - \langle S(\sigma.X), \mathsf{vol}_d \rangle = H_n^d(\mathsf{vol}_d) - \sigma.H_n^d(\mathsf{vol}_d)$$

in $\mathbb{R}^{d\times n}$ by Z_σ for $\sigma \in S_n$ and set $Z := \bigcup_{\sigma\in S_n\setminus C_n^d} Z_\sigma$. This is the locus of X violating the implication $\Rightarrow$. As it is closed, we only need to show that Z is not the whole space. Then we can choose $O := \mathbb{R}^{d\times n} - Z$.

Since $\mathbb{R}^{d\times n}$ is irreducible, it suffices to show that $Z_\sigma \neq \mathbb{R}^{d\times n}$ for all $\sigma \notin C_n^d$. Thus, let $\sigma \in S_n \setminus C_n^d$. Then we can choose X such that (4) is positive for X, but not for $\sigma.X$. We will now argue by contraposition that the impliciation $\Rightarrow$ is true for X.

Let $x_1, \dots, x_n$ denote the columns of X. Set $P = \operatorname{conv}(x_1, \dots, x_n)$. Then the matrix

$$\begin{pmatrix} 1 & \dots & 1 \\ x_{\sigma^{-1}(1)} & \dots & x_{\sigma^{-1}(n)} \end{pmatrix}$$

has a negative maximal minor. There is some triangulation $\mathcal{S}$ of P containing the simplex corresponding to the index set of this minor. To $\Delta \in \mathcal{S}$ we associate the indices of its vertices $\{i_1^\Delta, \dots, i_{d+1}^\Delta\}$, $i_1 < \dots < i_{d+1}$. Then we consider

$$\mathsf{vol}'_d(X) := \sum_{\Delta \in \mathcal{S}} \det \begin{pmatrix} 1 & \dots & 1 \\ x_{i_1^\Delta} & \dots & x_{i_{d+1}^\Delta} \end{pmatrix},$$

Since Δ is a triangulation and since all determinants appearing in the sum are positive if (4) is positive, vol'_d agrees with $\langle S(X), \mathsf{vol}_d \rangle$ on the Zariski dense subset of X with (4) positive and thus vol'_d and $\langle S(X), \mathsf{vol}_d \rangle$ are identical. But by construction $\mathsf{vol}'_d(\sigma.X)$ is strictly bounded by the volume of P (since at least one minor appearing in the sum will be negative) and so

$$\langle S(X), \mathsf{vol}_d \rangle - \langle S(\sigma.X), \mathsf{vol}_d \rangle > 0. \qquad \square$$

In particular, away from some exceptional closed set, we have the implication

$$\langle S(X), \mathsf{vol}_d \rangle = \langle S(\sigma.X), \mathsf{vol}_d \rangle \Rightarrow \forall\, w \in \mathsf{Inv}^d : \; \langle S(X), w \rangle = \langle S(\sigma.X), w \rangle$$

for all X.

4. Investigating the ring of volume invariants

In the following we write $\mathsf{Inv}_n^d := \mathsf{Inv}_n^d(C_n^d)$ for simplicity. Given the case distinction in Proposition 3.10, we will treat the cases $n \geq d+3$ simultanuously and put

$$\mathsf{Inv}_{\geq d+3}^d := \bigcap_{n \geq d+3} \mathsf{Inv}_n^d$$

so that we have

$$\mathsf{Inv}^d = \mathsf{Inv}_{d+1}^d \cap \mathsf{Inv}_{d+2}^d \cap \mathsf{Inv}_{\geq d+3}^d.$$

Note that the kernel $\mathcal{I}(\mathsf{PL}_{d+2}^d)$ of H_{d+2}^d is contained in $\mathsf{Inv}_{d+1}^d \cap \mathsf{Inv}_{d+2}^d$. We can write

$$\mathsf{Inv}^d = \frac{\mathsf{Inv}^d}{\mathsf{Inv}^d \cap \mathcal{I}(\mathsf{PL}_{d+2}^d)} \oplus \left(\mathsf{Inv}_{\geq d+3}^d \cap \mathcal{I}(\mathsf{PL}_{d+2}^d) \right)$$

While we can give a description of $\mathsf{Inv}_{\geq d+3}^d$, the problem for Inv_{d+1}^d and Inv_{d+2}^d is more difficult. We give examples and a conjecture instead, based on computations for low values of d:

Conjecture 4.1.

$$\mathsf{Inv}^d_{d+2}/\mathcal{I}(\mathsf{PL}^d_{d+2}) \cong \mathbb{R}[H^d_{d+2}(\mathsf{vol}_d)] \subseteq \mathcal{S}^d[x_1,\dots,x_{d+2}]$$

i.e. the subalgebra of $\mathcal{S}^d[x_1,\dots,x_{d+2}]$ generated by $H^d_{d+2}(\mathsf{vol}_d)$. Equivalently,

$$\mathsf{Inv}^d_{d+2} = \operatorname{span}\{(\mathsf{vol}_d)^{\sqcup\!\sqcup k}, k \geq 0\} \oplus \mathcal{I}(\mathsf{PL}^d_{d+2}).$$

Example 4.2. A computation using MACAULAY 2 shows that the vector space of invariants of degree ≤ 6 in Inv^3_4 is spanned by 18 elements, one in degree 3 (the signed volume), 6 in degree 5 and 11 in degree 6 (including the shuffle square of the signed volume). Here are two examples:

- $w_1 := \mathtt{12333} + \mathtt{13233} - \frac{2}{3}\cdot\mathtt{13323} - \frac{4}{3}\cdot\mathtt{13332} - \mathtt{21333} - \mathtt{23133} + \frac{2}{3}\cdot \mathtt{23313} + \frac{4}{3}\cdot\mathtt{23331} + \frac{5}{3}\cdot\mathtt{31323} - \frac{2}{3}\cdot\mathtt{31332} - \frac{5}{3}\cdot\mathtt{32313} + \frac{2}{3}\cdot\mathtt{32331} + \mathtt{33132} - \mathtt{33231} + \mathtt{33312} - \mathtt{33321}$

- $w_2 := -\mathtt{123333} - 3\cdot\mathtt{132333} + 4\cdot\mathtt{133233} + \mathtt{213333} + 3\cdot\mathtt{231333} + -4\cdot\mathtt{233133} + \mathtt{312333} - 2\cdot\mathtt{313233} - \mathtt{321333} + 2\cdot\mathtt{323133} + 2\cdot\mathtt{331323} - 4\cdot\mathtt{331332} - 2\cdot\mathtt{332313} + 4\cdot\mathtt{332331} - \mathtt{333123} + 3\cdot\mathtt{333132} + \mathtt{333213} + -3\cdot\mathtt{333231} + \mathtt{333312} - \mathtt{333321}$

Writing $x_i = (x_{i1},\dots,x_{id})$ and $a_i := x_{i+1} - x_i$, their images under (2) in the polynomial ring $\mathbb{R}[x_1,\dots,x_d]$ are given by

$$\begin{aligned}
&2\cdot(-3a_{13}^3a_{22}a_{31} + 3a_{12}a_{13}^2a_{23}a_{31} - 4a_{13}^2a_{22}a_{23}a_{31} + 4a_{12}a_{13}a_{23}^2a_{31}\\
&-4a_{13}a_{22}a_{23}^2a_{31} + 4a_{12}a_{23}^3a_{31} + 3a_{13}^3a_{21}a_{32} - 3a_{11}a_{13}^2a_{23}a_{32}\\
&+4a_{13}^2a_{21}a_{23}a_{32} - 4a_{11}a_{13}a_{23}^2a_{32} + 4a_{13}a_{21}a_{23}^2a_{32} - 4a_{11}a_{23}^3a_{32}\\
&-3a_{12}a_{13}^2a_{21}a_{33} + 3a_{11}a_{13}^2a_{22}a_{33} - 4a_{12}a_{13}a_{21}a_{23}a_{33} + 4a_{11}a_{13}a_{22}a_{23}a_{33}\\
&-4a_{12}a_{21}a_{23}^2a_{33} + 4a_{11}a_{22}a_{23}^2a_{33} - 2a_{13}^2a_{22}a_{31}a_{33} + 2a_{12}a_{13}a_{23}a_{31}a_{33}\\
&-4a_{13}a_{22}a_{23}a_{31}a_{33} + 4a_{12}a_{23}^2a_{31}a_{33} + 2a_{13}^2a_{21}a_{32}a_{33} - 2a_{11}a_{13}a_{23}a_{32}a_{33}\\
&+4a_{13}a_{21}a_{23}a_{32}a_{33} - 4a_{11}a_{23}^2a_{32}a_{33} - 2a_{12}a_{13}a_{21}a_{33}^2 + 2a_{11}a_{13}a_{22}a_{33}^2\\
&-4a_{12}a_{21}a_{23}a_{33}^2 + 4a_{11}a_{22}a_{23}a_{33}^2 - 3a_{13}a_{22}a_{31}a_{33}^2 + 3a_{12}a_{23}a_{31}a_{33}^2\\
&+3a_{13}a_{21}a_{32}a_{33}^2 - 3a_{11}a_{23}a_{32}a_{33}^2 - 3a_{12}a_{21}a_{33}^3 + 3a_{11}a_{22}a_{33}^3)
\end{aligned}$$

and

$$
\begin{aligned}
&24\cdot(-a_{13}^4a_{22}a_{31}+a_{12}a_{13}^3a_{23}a_{31}-2a_{13}^3a_{22}a_{23}a_{31}+2a_{12}a_{13}^2a_{23}^2a_{31}\\
&+a_{13}^4a_{21}a_{32}-a_{11}a_{13}^3a_{23}a_{32}+2a_{13}^3a_{21}a_{23}a_{32}-2a_{11}a_{13}^2a_{23}^2a_{32}\\
&-a_{12}a_{13}^3a_{21}a_{33}+a_{11}a_{13}^3a_{22}a_{33}-2a_{12}a_{13}^2a_{21}a_{23}a_{33}+2a_{11}a_{13}^2a_{22}a_{23}a_{33}\\
&-a_{13}^3a_{22}a_{31}a_{33}+a_{12}a_{13}^2a_{23}a_{31}a_{33}+a_{13}^3a_{21}a_{32}a_{33}-a_{11}a_{13}^2a_{23}a_{32}a_{33}\\
&-a_{12}a_{13}^2a_{21}a_{33}^2+a_{11}a_{13}^2a_{22}a_{33}^2+a_{13}^2a_{22}a_{31}a_{33}^2-a_{12}a_{13}a_{23}a_{31}a_{33}^2\\
&+2a_{13}a_{22}a_{23}a_{31}a_{33}^2-2a_{12}a_{23}^2a_{31}a_{33}^2-a_{13}^2a_{21}a_{32}a_{33}^2+a_{11}a_{13}a_{23}a_{32}a_{33}^2\\
&-2a_{13}a_{21}a_{23}a_{32}a_{33}^2+2a_{11}a_{23}^2a_{32}a_{33}^2+a_{12}a_{13}a_{21}a_{33}^3-a_{11}a_{13}a_{22}a_{33}^3\\
&+2a_{12}a_{21}a_{23}a_{33}^3-2a_{11}a_{22}a_{23}a_{33}^3+a_{13}a_{22}a_{31}a_{33}^3-a_{12}a_{23}a_{31}a_{33}^3\\
&-a_{13}a_{21}a_{32}a_{33}^3+a_{11}a_{23}a_{32}a_{33}^3+a_{12}a_{21}a_{33}^4-a_{11}a_{22}a_{33}^4)
\end{aligned}
$$

respectively.

Let us now give a description of $\mathsf{Inv}^d_{\geq d+3}$. Let $\mathcal{A}$ denote the antipode of the Hopf algebra $\mathbb{R}\langle \mathtt{1},\dots,\mathtt{d}\rangle$, that is, the map sending a word w to $(-1)^{d+1}w'$ where w' is obtained from w by reversing the order of its letters.

Definition 4.3. We define $\mathsf{TimeRevInv}^d := \{w \in \mathbb{R}\langle \mathtt{1},\dots,\mathtt{d}\rangle \mid w = \mathcal{A}w\}$. This is the subring of $w \in \mathbb{R}\langle \mathtt{1},\dots,\mathtt{d}\rangle$ such that $\langle S(X), w\rangle = \langle S(X^{-1}), w\rangle$ for all paths $X : [0,1] \to \mathbb{R}^d$, where $X^{-1} : [0,1] \to \mathbb{R}^d$, $t \mapsto X(1-t)$. Indeed, taking the antipode is the adjoint operation to time reversal (see e.g. [18]).

In the following, we will make crucial use of the following theorem (immediately equivalent to [11, Lemma 5.2]), which can be seen as a corollary of the Chen-Chow theorem for piecewise linear paths, or a weaker version thereof:

Theorem 4.4 (weak Chen-Chow)**.** Let $w, v \in \mathbb{R}\langle \mathtt{1},\dots,\mathtt{d}\rangle$. If $\langle S(X), w\rangle = \langle S(X), v\rangle$ for all piecewise linear paths X, then $w = v$.

Proof. By the proof of [12, Lemma 8], the image of piecewise linear paths under S spans the dual $(\mathbb{R}\langle \mathtt{1},\dots,d\rangle_{\leq k})^*$ of the vector space of length $\leq k$ words. □

Proposition 4.5. Assume d is odd. Then

$$\mathsf{Inv}^d_{\geq d+3} = \mathsf{TimeRevInv}^d$$

if $\frac{d+1}{2}$ is even, and $\mathsf{Inv}^d_{\geq d+3} = \mathbb{R}\langle \mathtt{1},\dots,\mathtt{d}\rangle$ if $\frac{d+1}{2}$ is odd.

Proof. Let X be a piecewise linear path in $\mathbb{R}^d$ with $n \geq d+3$ control points. If $\frac{d+1}{2}$ is odd then C_n^d is trivial by Proposition 3.8. If $\frac{d+1}{2}$ is even, then the proof of Proposition 3.8 shows that C_n^d is generated by the reflection τ which inverts the order of the vertices. On piecewise linear paths, this corresponds to $X \mapsto X^{-1}$. Thus, if $w \in \mathsf{Inv}_{\geq d+3}$, then $\langle S(X), \mathcal{A}w\rangle = \langle S(X^{-1}), w\rangle = \langle S(X), w\rangle$, concluding the proof by Theorem 4.4. □

Note that $\mathsf{TimeRevInv}$ can be identified as the image of

$$\mathbb{R}\langle \mathtt{1},\ldots,\mathtt{d}\rangle \to \mathbb{R}\langle \mathtt{1},\ldots,\mathtt{d}\rangle,\ w \mapsto w + \mathcal{A}w.$$

Example 4.6. In $d=3$, consider the concatenation square of signed 3-volume

$$\mathsf{vol}_3^{\bullet 2} = (\mathtt{123}+\mathtt{231}+\mathtt{312}-\mathtt{213}-\mathtt{132}-\mathtt{321})^{\bullet 2},$$

where the concatenation (of words) $\bullet$ is the bilinear non-commutative product on $\mathbb{R}\langle \mathtt{1},\ldots,\mathtt{d}\rangle$ given by

$$\mathtt{i}_1\ldots\mathtt{i}_k \bullet \mathtt{i}_{k+1}\ldots\mathtt{i}_m = \mathtt{i}_1\ldots\mathtt{i}_m$$

We have $\mathcal{A}\mathsf{vol}_3^{\bullet 2} = \mathsf{vol}_3^{\bullet 2}$, so $\mathsf{vol}_3^{\bullet 2} \in \mathsf{TimeRevInv}^3 = \mathsf{Inv}^3_{\geq 6}$.

Furthermore, $\langle S(X), \mathsf{vol}_3^{\bullet 2}\rangle$ is the integral

$$\int_{0\leq t_1\leq\cdots\leq t_6\leq 1} \det(X'(t_1),X'(t_2),X'(t_3))\det(X'(t_4),X'(t_5),X'(t_6))dt_1\ldots dt_6$$

so if X has only 4 segments, there is no choice of $0\leq t_1\leq\cdots\leq t_6\leq 1$ such that both determinants are non-zero. Thus, $\mathsf{vol}_3^{\bullet 2} \in \mathsf{Inv}^3_{\geq 6}\cap\mathcal{I}(\mathsf{PL}^3_5)\subset\mathsf{Inv}^3$.

Definition 4.7. We define $\mathsf{LoopClosureInv}^d$ as the subring of $u\in\mathbb{R}\langle \mathtt{1},\ldots,\mathtt{d}\rangle$ such that

$$\langle S(X),u\rangle = \langle S(X\sqcup\{X(1)\to X(0)\}),u\rangle = \langle S(\{X(1)\to X(0)\}\sqcup X),u\rangle,$$

where $\sqcup$ is concatenation of paths and $\{X(1)\to X(0)\}$ is the linear segment from $X(1)$ to $X(0)$. That is, the elements of $\mathsf{LoopClosureInv}^d$ correspond to signature values that are stable both under closing a path to a loop by concatenating a linear segment to the right (the *right loop closure*), as well as under closing a path to a loop by concatenating a linear segment to the left (the *left loop closure*). We refer to [19] for details.

For example, the signatures of the five paths from Figure 1 agree on all loop closure invariants (since the right closure of the first path is the left closure of the second path depicted, and so on).

Proposition 4.8. Assume d is even. Then

$$\mathsf{Inv}^d_{\geq d+3} = \mathsf{LoopClosureInv}^d\cap\mathsf{TimeRevInv}^d$$

if $\frac{d}{2}$ is even, and $\mathsf{Inv}^d_{\geq d+3} = \mathsf{LoopClosureInv}^d$ if $\frac{d}{2}$ is odd.

Proof. If $\frac{d}{2}$ is even then C_n^d is the group $\mathbb{D}_n$ by Proposition 3.10. It is spanned by the order-reversing permutation s_n and the rotation r_n. As observed in (the proof of) Proposition 4.5, invariants under s_n for all n simultaneously are precisely the elements of $\mathsf{TimeRevInv}$. On the other hand, invariants under all r_n for all n simultaneously are the elements of $\mathsf{LoopClosureInv}$, see [19]. Indeed, for $w \in \mathsf{Inv}^d_{\geq d+3}$ and the right closure $\bar{X}^R$ of the path X (which is a piecewise linear path with $n+1$ control points) we must have $\langle S(r_{n+1}(\bar{X}^R)), w\rangle = \langle S(\bar{X}), w\rangle$ for the rotation $r_{n+1} \in \mathbb{Z}/(n+1)$. But $\langle S(r_{n+1}(\bar{X}^R)), w\rangle = \langle S(X), w\rangle$ by reparametrisation invariance of iterated integrals. Similarly, for the left closure $\bar{X}^L$ of the path X we must have $\langle S(r_{n+1}^{-1}(\bar{X}^L)), w\rangle = \langle S(\bar{X}), w\rangle$. But again, $\langle S(r_{n+1}^{-1}(\bar{X}^L)), w\rangle = \langle S(X), w\rangle$. Using Theorem 4.4 and that both left- and right-closure admit an adjoint [19, Lemma 4.7] we see that $w \in \mathsf{LoopClosureInv}$. The reverse inclusion is immediate from [19, Proposition 4.3].

In the case that $\frac{d}{2}$ is odd we have that $C_n^d = \mathbb{Z}/n$ is just generated by r_n as shown in Proposition 3.10. Thus, both statements follow. □

Theorem 4.9. For any d, $\mathsf{Inv}^d_{\geq d+3} \cap \mathcal{I}(\mathsf{PL}^d_{d+2})$ (and in particular Inv^d) contains infinitely many algebraically independent elements (with respect to the shuffle product) and is thus in particular infinitely generated as a (shuffle) subalgebra of $\mathbb{R}\langle \mathtt{1}, \ldots, \mathtt{d}\rangle$.

Proof. If d is odd then we have $\mathsf{Inv}^d_{\geq d+3} = \mathsf{TimeRevInv}^d$ or $\mathsf{Inv}^d = \mathbb{R}\langle \mathtt{1}, \ldots, d\rangle$. If d is even then $\mathsf{Inv}^d_{\geq d+3} = \mathsf{LoopClosureInv}^d \cap \mathsf{TimeRevInv}^d$ or $\mathsf{Inv}^d_{\geq d+3} = \mathsf{LoopClosureInv}^d$. We claim that all of these algebras contain infinitely many algebraically independent elements. This is true for $\mathbb{R}\langle \mathtt{1}, \ldots, d\rangle$ as it is freely generated by the Lyndon words, see [21, Theorem 6.1]. Then it is also true for $\mathsf{TimeRevInv}^d$: It is the kernel of the map (of vector spaces) $\psi : \mathbb{R}\langle \mathtt{1}, \ldots, \mathtt{d}\rangle \to \mathbb{R}\langle \mathtt{1}, \ldots, \mathtt{d}\rangle, w \mapsto w - \mathcal{A}w$ and if it does not contain infinitely many algebraically independent elements, then there is a finite set S of Lyndon words such that each element is already algebraic over $\mathbb{R}[S]$, thus contained in it. It follows that the image of ψ would have to contain infinitely many algebraically independent $a_1, a_2, \ldots$ but then the elements $a_1^{\sqcup\!\sqcup 2}, a_2^{\sqcup\!\sqcup 2}, \cdots \in \mathsf{TimeRevInv}$ are still algebraically independent, yielding a contradiction.

In [19] it is shown that $\mathsf{LoopClosureInv}^d$ contains an infinite algebraically independent subset and by a similar argument as above it follows that the intersection $\mathsf{LoopClosureInv}^d \cap \mathsf{TimeRevInv}^d$ also does, using that $w - \mathcal{A}w$ is an element of $\mathsf{LoopClosureInv}^d$ for every $w \in \mathsf{LoopClosureInv}^d$ since $\mathcal{A}w$ is a loop closure invariant if w is (which follows from the definition).

So we have shown that $\mathsf{Inv}^d_{\geq d+3}$ contains infinitely many algebraically independent elements for any d. In particular, we can consider a composition

$$\mathbb{R}[s_1, s_2, \ldots] \hookrightarrow \mathsf{Inv}^d_{\geq d+3} \longrightarrow \mathbb{R}\langle \mathtt{1}, \ldots, \mathtt{d}\rangle \longrightarrow \mathbb{R}\langle \mathtt{1}, \ldots, \mathtt{d}\rangle / \mathcal{I}(\mathsf{PL}^d_{d+2})$$

where the first map is injective. The kernel of this composition must contain infinitely many algebraically independent elements as the quotient on the right is isomorphic to a subring of $\mathbb{R}[x_1,\dots,x_{d+2}]$ via (2). Indeed, otherwise there is again some N such that any element of the kernel is already algebraic over $\mathbb{R}[s_1,\dots,s_N]$ and we get an injection $\mathbb{R}[s_{N+1},s_{N+2},\dots] \to \mathbb{R}[x_1,\dots,x_{d+2}]$ which is absurd. The infinitely many algebraically independent elements are still algebraically independent in the larger ring $\mathsf{Inv}^d_{\geq d+3}$ and by construction contained in $\mathcal{I}(\mathsf{PL}^d_{d+2})$, proving the claim.

$\square$

5. Outlook

Computing volume invariants for even d In dimension 2, we simply have $\mathsf{Inv}^2 = \mathsf{LoopClosureInv}^2$. A lowest-degree example of a loop closure invariant for two-dimensional paths that is independent of signed area is the following:

$$
\begin{aligned}
&3\cdot\mathtt{111222}+5\cdot\mathtt{112122}+3\cdot\mathtt{112212}-3\cdot\mathtt{112221}+3\cdot\mathtt{121122}+\mathtt{121212}\\
&-5\cdot\mathtt{121221}+\mathtt{122112}-5\cdot\mathtt{122121}-3\cdot\mathtt{122211}-3\cdot\mathtt{211122}-5\cdot\mathtt{211212}\\
&+\mathtt{211221}-5\cdot\mathtt{212112}+\mathtt{212121}+3\cdot\mathtt{212211}-3\cdot\mathtt{221112}+3\cdot\mathtt{221121}\\
&+5\cdot\mathtt{221211}+3\cdot\mathtt{222111}
\end{aligned}
$$

However, starting from four dimensions the even case gets vastly more involved. The lowest-degree generators of $\mathcal{I}(\mathsf{PL}^4_6)$ are the following 8 on level 7,

$$
\begin{aligned}
&\mathsf{vol}_4\bullet\mathsf{vol}_3(1,2,3),\quad \mathsf{vol}_4\bullet\mathsf{vol}_3(1,2,4),\quad \mathsf{vol}_4\bullet\mathsf{vol}_3(1,3,4),\quad \mathsf{vol}_4\bullet\mathsf{vol}_3(2,3,4),\\
&\mathsf{vol}_3(1,2,3)\bullet\mathsf{vol}_4,\quad \mathsf{vol}_3(1,2,4)\bullet\mathsf{vol}_4,\quad \mathsf{vol}_3(1,3,4)\bullet\mathsf{vol}_4,\quad \mathsf{vol}_3(2,3,4)\bullet\mathsf{vol}_4
\end{aligned}
$$

where

$$
\begin{aligned}
\mathsf{vol}_4 &:= \mathsf{vol}_4(1,2,3,4)\\
&:= \mathsf{1234}-\mathsf{1243}-\mathsf{1324}+\mathsf{1342}+\mathsf{1423}-\mathsf{1432}-\mathsf{2134}+\mathsf{2143}\\
&\quad+\mathsf{2314}-\mathsf{2341}-\mathsf{2413}+\mathsf{2431}+\mathsf{3124}-\mathsf{3142}-\mathsf{3214}+\mathsf{3241}\\
&\quad+\mathsf{3412}-\mathsf{3421}-\mathsf{4123}+\mathsf{4132}+\mathsf{4213}-\mathsf{4231}-\mathsf{4312}+\mathsf{4321}
\end{aligned}
$$

is four dimensional signed volume and

$$\mathsf{vol}_3(\mathtt{i},\mathtt{j},\mathtt{k}) = \mathtt{ijk}+\mathtt{jki}+\mathtt{kij}-\mathtt{jik}-\mathtt{ikj}-\mathtt{kji}.$$

No linear combination of these eight is a loop-closure invariant.

Even though we know that $\mathsf{Inv}^4_{\geq d+3}\cap\mathcal{I}(\mathsf{PL}^4_6)$ is an infinitely generated subring, we need to compute very far to find the first generator.

The induced equivalence relation Recall that we can view Inv^d as features on cyclic polytopes with $n \geq d+1$ vertices. If we consider the equivalence relation

$$(x_1,\ldots,x_n) \sim (y_1,\ldots,y_m) :\Leftrightarrow \forall\, f \in \mathsf{Inv}^d : f(x_1,\ldots,x_n) = f(y_1,\ldots,y_m)$$

then we have $(x_1,...,x_n) \sim (x_{\sigma(1)},\ldots,x_{\sigma(n)})$ for any $\sigma \in C_d^n$. The properties of iterated integrals imply that $(x_1,\ldots,x_n) \sim (x_1+c,\ldots,x_n+c)$ for any $c \in \mathbb{R}^d$ and that $(x_1,\ldots,x_n) \sim (x_1,\ldots,\hat{x}_i,\ldots,x_n)$ (x_i is omitted in the second tuple) whenever x_i is a convex combination of x_{i-1} and x_{i+1}.

However, we do not expect these three types of relations to generate $\sim$: For example, if Conjecture 4.1 holds true, then any two d-polytopes with $d+2$ vertices and the same volume are equivalent under $\sim$.

Problem 5.1. How can the equivalence relation $\sim$ be described geometrically or combinatorial? Cf. [19, Section 5], [12, Conjecture 7.2] and [10].

Specializing to O and SL-invariants For O_d, the orthogonal group, we may reduce to invariants on demand (cf. [10]), for example through a projection

$$R : p \mapsto \int_{O_d} p(Ax_1,\ldots,Ax_n)d\mu(A),$$

where μ is the Haar measure of O_d with $\mu(O_d) = 1$. R preserves signature polynomials, and is an example of a so-called Reynold's operator. Now for SL_d, there is no finite Haar measure as it is a non-compact group. However, we may still compute the intersection of the subrings of Inv^d and the SL_d invariants. This yields functions on the positive Grassmannian, as the latter can be represented by positive matrices modulo SL_d action from the left. See for example [8] for the notions of Reynold's operator and Haar measure.

Invariants for other groups Instead of considering $G_n = C_n^d$, there are of course other interesting possibilities.

If we were to consider the maximal choice $G_n = S_n$, then $\mathsf{Inv}^d(G)$ would be the zero subring. Indeed, take any piecewise linear path P, with n vertices. Then one can double each vertex except the last to obtain a path with $2n-1$ vertices but the same signature. Permuting the order of the vertices allows then to obtain a tree-like path (i.e. a path with vanishing signature). Thus, any (simultaneous) invariant for the S_n-action must evaluate to 0 under the signature. By (weak) Chen-Chow, Theorem 4.4, this implies that the invariant itself must be 0.

For $G_n = \mathbb{Z}/n$, we exactly get $\mathsf{Inv}^d(G) = \mathsf{LoopClosureInv}^d$, and for $G_n = \mathbb{D}_n$, we have $\mathsf{Inv}^d(G) = \mathsf{LoopClosureInv}^d \cap \mathsf{TimeRevInv}^d$.

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