

MODULI SPACES IN POSITIVE GEOMETRY

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These are lecture notes for five lectures given at MPI Leipzig in May 2024. We study the moduli space $\mathcal{M}_{0,n}$ of n distinct points on $\mathbb{P}^1$ as a positive geometry and a binary geometry. We develop mathematical formalism to study Cachazo-He-Yuan’s scattering equations and the associated scalar and Yang-Mills amplitudes. We discuss open superstring amplitudes and relations to tropical geometry.

Introduction

The new field of positive geometry is developing at the interface of combinatorial algebraic geometry in mathematics and scattering amplitudes in physics. The purpose of these notes is to give an introduction to some of the ideas in this area. Our main target audience is a mathematics graduate student with some experience in geometric combinatorics or algebraic geometry. At the urging of the organizers of the workshop “Combinatorial algebraic geometry from physics”, I have leant towards an algebro-geometric approach, though I have kept the language as concrete as possible. The text should be understandable to a motivated student who has taken one graduate class in algebraic geometry. No physics background is assumed.

When I started writing these notes, it became clear very quickly that I could not give a comprehensive survey of the area, such is the breadth of the ideas involved. I narrowed the scope as follows.

On the mathematical side, we focus on the space $\mathcal{M}_{0,n}$. This is the configuration space of n distinct points on the projective line $\mathbb{P}^1$. The space $\mathcal{M}_{0,n}$ is a central example in combinatorial geometry, as it is simultaneously:

1. a quotient of an open subset of the Grassmannian, or of a matrix space,
2. a hyperplane arrangement complement,
3. (naturally) an open subset of a toric variety,
4. a very affine variety,
5. a positive geometry,
6. a binary geometry,
7. a moduli space,
8. a cluster configuration space.

On the physics side, we consider three classes of amplitudes, all with tight connections to combinatorial algebraic geometry:

1. the planar scalar ϕ^3 -amplitude,
2. the planar gluon-amplitude for Yang-Mills theory, and
3. the open superstring-amplitude.

In all cases, we focus on the kinematic aspects of the amplitudes, and only consider *tree-level* amplitudes for massless scattering. Our aim is to give *precise* definitions of various functions on kinematic space, with *heuristic* explanations of the relations to quantum field theory and string theory. Since $\mathcal{M}_{0,n}$ is the main mathematical object we consider, our approach to amplitudes leans heavily on the Cachazo-He-Yuan *scattering equations* [CHY14a, CHY14c, CHY14b]. For gluon amplitudes, we borrow also from twistor string theory [Wit04, RSV04]. We regret the absence of loop amplitudes, of supersymmetry, and of momentum-twistors, all of which are the bread and butter of modern amplitudes theory.

A recurring theme in the notes is to ask whether amplitudes can be generalized beyond $\mathcal{M}_{0,n}$ to more general settings, such as that of binary geometries, positive geometries, and very affine varieties. In [Lam24], we develop these amplitudes for hyperplane arrangement complements.

Overview. We now give a more detailed survey of the contents of this work. In Section 1, we define and study the space $\mathcal{M}_{0,n}$. We use crucially a collection of cross-ratios u_{ij} called *dihedral coordinates* [KN69, Bro09]. These cross-ratios

are labeled by the diagonals, but not sides, of a n -gon. They form a basis of the character lattice of the intrinsic torus of $\mathcal{M}_{0,n}$ (Proposition 1.9), and define an affine variety $\widetilde{\mathcal{M}}_{0,n}$ (Section 1.4) that is a partial compactification of $\mathcal{M}_{0,n}$. The strata of $\widetilde{\mathcal{M}}_{0,n}$ are labeled by subdivisions of the n -gon, or equivalently, by faces of the associahedron polytope. Thus, $\widetilde{\mathcal{M}}_{0,n}$ is an affine patch in which we can see the combinatorics and geometry of the boundary stratification of $\overline{\mathcal{M}}_{0,n}$.

The dihedral coordinates u_{ij} depend on a choice of a connected component $(\mathcal{M}_{0,n})_{>0}$ of the real points $\mathcal{M}_{0,n}(\mathbb{R})$. We call $(\mathcal{M}_{0,n})_{>0}$ the positive part of $\mathcal{M}_{0,n}$. The analytic closure $(\mathcal{M}_{0,n})_{\geq 0}$ is a positive geometry (Definition 1.20) inside $\overline{\mathcal{M}}_{0,n}$, and we discuss properties of the canonical form $\Omega_{0,n} = \Omega((\mathcal{M}_{0,n})_{\geq 0})$, a holomorphic top-form on $\mathcal{M}_{0,n}$.

In Section 2, we formalize ideas from [AHHLT23] and define *binary geometries*. The dihedral coordinates u_{ij} of $\mathcal{M}_{0,n}$ satisfy *binary u -equations*. For example, with $n = 5$, we have five dihedral coordinates $u_{13}, u_{14}, u_{24}, u_{25}, u_{35}$, satisfying the five relations

$$\begin{aligned} u_{13} + u_{24}u_{25} &= 1 \\ u_{24} + u_{13}u_{35} &= 1 \\ u_{35} + u_{14}u_{24} &= 1 \\ u_{14} + u_{25}u_{35} &= 1 \\ u_{25} + u_{13}u_{14} &= 1 \end{aligned} \tag{0.1}$$

A key feature of dihedral coordinates is that if $u_{ij} = 0$ then we have $u_{kl} = 1$ for all diagonals (k, l) crossing (i, j) . This is the “binary” nature of the u -equations. In Definition 2.1 we define binary geometries, a class of affine varieties with distinguished coordinates u_i satisfying equations of a form similar to (0.1). We classify (Theorem 2.9) one-dimensional binary geometries and show (Theorem 2.10) that the face poset of a binary geometry is a simplicial complex that is a pseudomanifold. We suggest a number of potential directions for further exploration of binary geometries.

Section 3 is the core of the lecture series. We start with an informal discussion of scattering amplitudes in quantum field theory, leading to a precise definition of the *kinematic space* K_n for n -particle scattering (Definition 3.1). The link between kinematics and the geometry of $\mathcal{M}_{0,n}$ is made possible by a natural isomorphism between the integer points $K_n(\mathbb{Z})$ and the character lattice $\Lambda(\mathcal{M}_{0,n})$ (Proposition 3.3). The celebrated *scattering equations* of Cachazo-He-Yuan were a revolution in the construction of tree-level amplitudes, and we interpret them as a *scattering correspondence* (see (3.8))

$$\mathcal{I} \subset K_n \times \mathcal{M}_{0,n}. \tag{0.2}$$

This correspondence is cut out by the critical points of the scattering potential, also called a likelihood function in algebraic statistics, or master function in the theory of hyperplane arrangements.

At the heart of our philosophy is that the correspondence (0.2) is *canonically* associated to the very affine variety $\mathcal{M}_{0,n}$. The CHY scalar amplitudes are obtained by summing a rational function over the pre-images of a point $\underline{s} \in K_n$ under the map $\mathcal{I} \rightarrow K_n$. We make CHY's definition more conceptual in two ways. First, we define in Definition 3.22 an amplitude $A(\Omega|\Omega')$ that depends on two rational top-forms Ω, Ω' , which we expect to be taken to be canonical forms of positive geometries. Second, we define a scattering form on K_n as a push-pull of the canonical form of $\mathcal{M}_{0,n}$. This gives a conceptual approach to the scattering form of [AHBHY18].

In Section 4, we specialize to four-dimensional space-time and consider amplitudes for gluons (massless spin one particles). We introduce spinor coordinates on kinematic space. Mathematically, these coordinates come from the exceptional isomorphism $\text{Spin}(4) \simeq \text{SL}(2) \times \text{SL}(2)$ involving the spin group $\text{Spin}(4) \rightarrow \text{SO}(4)$ covering the complexification of the four-dimensional Lorentz group. Spinor coordinates capture in a beautiful way the special features of four-dimensional space-time and also encode the polarization vectors that appear in gluon scattering.

Restricting (0.2) to the subvariety of four-dimensional kinematics $K_n^4 \subset K_n$, the correspondence breaks up into $(n-3)$ irreducible components $J_1, \dots, J_{n-3}$, called “sectors” (Proposition 4.6). These sectors were discovered in Witten's twistor string theory [Wit04] and further studied by Roiban-Spradlin-Volovich [RSV04] and Cachazo-He-Yuan [CHY13]. The sector decomposition arises from the interpretation of (0.2) as an incidence subvariety $\tilde{\mathcal{Z}} \subset \mathcal{M}_{0,n}(\mathbb{P}^3, n-2)$, where $\mathcal{M}_{0,n}(\mathbb{P}^3, n-2)$ denotes the moduli space of rational n -pointed maps to $\mathbb{P}^3$ of degree $n-2$, and $\tilde{\mathcal{Z}}$ is the subvariety of maps with image in a quadric. We explain a construction of [AHBCT11], which reinterprets the scattering correspondence of the d -th sector as a subvariety of the Grassmannian $\text{Gr}(d+1, n)$. This is one of the starting points of the Grassmannian formulae for scattering amplitudes.

In Section 5, we study the tree-level n -point open superstring amplitude as an integral over $(\mathcal{M}_{0,n})_{>0}$, explaining how field theory amplitudes can be obtained as limits of string amplitudes as strings turn into particles. We consider a general class of stringy integrals [AHHL21b], also called Euler integrals in [MHMT23], which take the form

$$I(\tau, c) = \int_{U_{>0}} \phi_{\tau, c} \Omega = \int_{\mathbb{R}_{>0}^d} \frac{y_1^{\tau_1} \cdots y_n^{\tau_n}}{p_1^{c_1} \cdots p_r^{c_r}} \frac{dy_1}{y_1} \wedge \cdots \wedge \frac{dy_n}{y_n},$$

where $p_i = p_i(\mathbf{y})$ are polynomials in the variables $y_1, \dots, g_n$. Associated to such an integral function is a (very affine) subvariety U of a toric variety, and the integral is taken over a positive part $U_{>0} \subset U(\mathbb{R})$. The integrand $\phi_{\tau,c}$ is the natural potential function of U , and the condition that the integral $I(\tau, c)$ converges cuts out a class of bounded characters of the character lattice $\Lambda(U)$. In the case of the string amplitude itself, we have $U = \mathcal{M}_{0,n}$ and we recover the dihedral coordinates of Section 1 in this way.

We explain that the field theory limit of a string integral is naturally related to tropicalization. Indeed, the (positive) tropicalization $\text{trop}(\phi_{\tau,c})$ of the potential function controls the field theory limit of $I(\tau, c)$. In this way, the Feynman diagrams of ϕ^3 -theory (which are cubic trees) naturally appear, and we have the following satisfying “commutative diagram”:

$$\begin{array}{ccc}
 \mathcal{M}_{0,n} & \longleftrightarrow & \text{string amplitude } \int_{(\mathcal{M}_{0,n})_{>0}} \phi_X \Omega \\
 \text{tropicalization} \downarrow & & \downarrow \text{field theory limit} \\
 \text{cubic trees} & \longleftrightarrow & \text{Feynman amplitude } \sum_{\text{trees } T} \prod_{e \in E(T)} \frac{1}{X_e}
 \end{array}$$

In Section 6, we review basic definitions for very affine varieties, and we speculate on possible definitions of amplitudes in that setting.

Exercises and Problems. There are exercises at the end of each lecture, which we hope will be useful for a student reader. “Problems” are more open-ended: some of these are conjectures or open problems, and some are statements from the physics literature that I do not know proofs of in the mathematics literature.

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1. Positive geometry of $\mathcal{M}_{0,n}$

Let $[n] := \{1, 2, \dots, n\}$.

1.1. Matrices, Moduli space, and the Grassmannian

We represent points $\sigma \in \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ by non-zero vectors $[x_0 : x_1]$, with $[x_0 : x_1] \sim [\alpha x_0 : \alpha x_1]$ for a nonzero scalar α . The general linear group $\mathrm{GL}(2)$ of invertible 2×2 matrices acts on $\mathbb{C}^2$ and thus on $\mathbb{P}^1$. This action factors through the projective linear group $\mathrm{PGL}(2)$ which is the quotient group of $\mathrm{GL}(2)$ by the scalar matrices.

Definition 1.1. For $n \geq 3$, the space $\mathcal{M}_{0,n}$ is the moduli space of n distinct points $\sigma_1, \sigma_2, \dots, \sigma_n$ on $\mathbb{P}^1$ considered up to the simultaneous action of $\mathrm{PGL}(2)$.

We may represent a point $\underline{\sigma} \in \mathcal{M}_{0,n}$ as a $2 \times n$ matrix (with no zero columns)

$$\begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \end{bmatrix}$$

where the i -th column represents the point $\sigma_i \in \mathbb{P}^1$.

Let $\mathrm{Mat}_{2,n}$ denote the space of $2 \times n$ matrices. Let $\mathrm{Mat}_{2,n}^\circ \subset \mathrm{Mat}_{2,n}$ denote the subset where all 2×2 minors are nonzero. The group $\mathrm{GL}(2)$ acts on $\mathrm{Mat}_{2,n}^\circ$ on the left as row operations. The group $T \cong (\mathbb{C}^\times)^n$ acts on $\mathrm{Mat}_{2,n}^\circ$ (on the

right) by scaling the columns. Since one group acts on the left and the other on the right, the action of the two groups commute, and we obtain an action of the product group $\mathrm{GL}(2) \times T$. Let $Z \subset \mathrm{GL}(2)$ denote the subgroup of scalar matrices, and let $D = \{(t, t, \dots, t)\} \subset T$ denote the subgroup where all entries are equal. Then the action of Z and D on $\mathrm{Mat}_{2,n}^\circ$ are the same, so the action of $\mathrm{GL}(2) \times T$ factors through the quotient group where Z and D are identified. Equivalently, the action factors through the action of the group

$$\mathrm{PGL}(2) \times T \cong \mathrm{GL}(2) \times T'$$

where $T' = T/D \cong (\mathbb{C}^\times)^{n-1}$. We leave the proof of the following result as an exercise for the reader.

Lemma 1.2. The action of $\mathrm{PGL}(2) \times T \cong \mathrm{GL}(2) \times T'$ on $\mathrm{Mat}_{2,n}^\circ$ is free.

We have the following commutative diagram of quotient maps:

$$\begin{array}{ccc}
 & \mathrm{Mat}_{2,n}^\circ & \\
 \mathrm{GL}(2) \swarrow & & \searrow T \\
 \mathrm{Gr}^\circ(2,n) & & ((\mathbb{P}^1)^n)^\circ \\
 & \searrow T' & \swarrow \mathrm{PGL}(2) \\
 & \mathcal{M}_{0,n} &
 \end{array} \tag{1.1}$$

Here, $\mathrm{Gr}^\circ(2,n) \subset \mathrm{Gr}(2,n)$ is the locus inside the Grassmannian of 2-planes where all Plücker coordinates are non-vanishing, and $((\mathbb{P}^1)^n)^\circ$ denotes the space of n distinct points in $\mathbb{P}^1$, obtained by removing the “diagonals” from $(\mathbb{P}^1)^n$. Since $\mathcal{M}_{0,n}$ is the quotient of the smooth variety $\mathrm{Mat}_{2,n}^\circ$ by the free action of the algebraic group $\mathrm{PGL}(2) \times T$, we obtain the following.

Proposition 1.3. $\mathcal{M}_{0,n}$ is a smooth complex algebraic variety of dimension $n-3$.

The group $\mathrm{PGL}(2)$ acts simply-transitively on the space of 3 distinct ordered points in $\mathbb{P}^1$ (Exercise 1.22). Thus, the action of $\mathrm{PGL}(2)$ allows us to place $(\sigma_1, \sigma_2, \sigma_n)$ at $(0, 1, \infty)$ and $\sigma_3, \dots, \sigma_{n-1}$ are coordinates on $\mathcal{M}_{0,n}$. In particular, $\mathcal{M}_{0,3}$ is a single point and $\mathcal{M}_{0,4} \cong \mathbb{P}^1 \setminus \{0, 1, \infty\}$.

In general, $\mathcal{M}_{0,n}$ can be identified with the hyperplane arrangement complement in $\mathbb{C}^{n-3}$ for the $\binom{n}{2} - n$ hyperplanes

$$\begin{aligned}
 & \{\sigma_i = 0 \mid i = 3, 4, \dots, n-1\} \cup \{\sigma_i = 1 \mid i = 3, 4, \dots, n-1\} \\
 & \cup \{\sigma_i - \sigma_j = 0 \mid 3 \leq i < j \leq n-1\}.
 \end{aligned}$$

1.2. Cross ratios

For $i, j \in [n]$, we write (ij) to denote the 2×2 minor of a $2 \times n$ matrix, using the columns indexed i and j . For two distinct points $\sigma, \sigma' \in \mathbb{P}^1$, the expression $\sigma - \sigma'$ is simply the difference of two complex numbers; if one of σ or σ' is ∞ then it is taken to be $\pm\infty$. We also use the notation $\sigma_{ab} := (\sigma_a - \sigma_b)$.

Using the action of $\mathrm{PGL}(2) \times T$ we can put a $2 \times n$ matrix representing a point $\underline{\sigma} \in \mathcal{M}_{0,n}$ into the form

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \sigma_1 & \sigma_2 & \cdots & \sigma_n \end{bmatrix}.$$

In this form, we have $(ij) = \sigma_j - \sigma_i$.

Definition 1.4. For distinct $i, j, k, l \in [n]$, define the *cross ratio*

$$[ij|kl] := \frac{(ik)(jl)}{(il)(jk)} = \frac{(\sigma_i - \sigma_k)(\sigma_j - \sigma_l)}{(\sigma_i - \sigma_l)(\sigma_j - \sigma_k)}$$

Lemma 1.5. The cross ratio $[ij|kl]$ is a regular function on $\mathcal{M}_{0,n}$, satisfying the identities

$$\begin{aligned} [ij|kl] &= 1 - [ik|jl], & [ij|lk] &= [ij|kl]^{-1} = [ji|kl], \\ [ij|kl] &= [kl|ij] = [ji|lk] = [lk|ji]. \end{aligned}$$

When $n = 4$, any of the cross-ratios $[ij|kl]$ with (i, j, k, l) a permutation of $1, 2, 3, 4$, defines an isomorphism $\mathcal{M}_{0,4} \cong \mathbb{P}^1 \setminus \{0, 1, \infty\}$. More generally, for any n , we have the following.

Proposition 1.6. The cross-ratios define an embedding

$$\iota : \mathcal{M}_{0,n} \hookrightarrow (\mathbb{P}^1 \setminus \{0, 1, \infty\})^{\binom{[n]}{4}}. \quad (1.2)$$

Proof. Using the $\mathrm{PGL}(2)$ action, we place $(\sigma_1, \sigma_2, \sigma_n)$ at $(0, 1, \infty)$. The cross ratio $[1n|j2] = \sigma_j$ then determines σ_j . This shows that (1.2) is injective. Let $W \subset (\mathbb{P}^1 \setminus \{0, 1, \infty\})^{\binom{[n]}{4}}$ be the closed subvariety cut out by the ideal generated by the relations in Lemma 1.5. We define a morphism $\pi : W \rightarrow \mathcal{M}_{0,n}$ by sending $(\sigma_1, \sigma_2, \sigma_n)$ to $(0, 1, \infty)$ and σ_j to $[1n|j2]$. The relations of Lemma 1.5 can then be used to show that π is inverse to ι . $\square$

1.3. Dihedral coordinates

Define the *dihedral coordinates*

$$u_{ij} := [i, i+1 | j+1, j] = \frac{(\sigma_i - \sigma_{j+1})(\sigma_{i+1} - \sigma_j)}{(\sigma_i - \sigma_j)(\sigma_{i+1} - \sigma_{j+1})} \quad (1.3)$$

for $i, j \in [n]$ such that $i - j \notin \{-1, 0, 1\} \pmod n$, that is i, j are not equal or adjacent modulo n . We have that $u_{ij} = u_{ji}$, and thus the cross-ratios u_{ij} are labeled by the diagonals, or chords, (but not the sides) of a n -gon with vertices $1, 2, \dots, n$. Let diag_n denote the set of diagonals of an n -gon. There are $|\text{diag}_n| = \binom{n}{2} - n$ dihedral coordinates.

It is easy to check (Exercise 1.23) that the u_{ij} satisfy the following relations:

$$R_{ij} := u_{ij} + \prod_{(k,l) \text{ crosses } (i,j)} u_{kl} - 1 = 0. \quad (1.4)$$

A key feature of dihedral coordinates is that if $u_{ij} = 0$ then we have $u_{kl} = 1$ for all (k, l) crossing (i, j) . This follows from the relations $R_{kl} = 0$.

For example, for $n = 4$, we have two dihedral coordinates u_{13}, u_{24} satisfying the relation

$$u_{13} + u_{24} = 1.$$

For $n = 5$, the relations R_{ij} are given in (0.1).

Let $T_U = \text{Spec}(\mathbb{C}[u_{ij}, u_{ij}^{-1}])$ be the $(\binom{n}{2} - n)$ -dimensional torus with coordinates u_{ij} . Define the closed subvariety $U \subset T_U$ cut out by the relations (1.4).

Theorem 1.7 ([Bro09, AHHL21a]). The dihedral coordinates give a closed embedding $\mathcal{M}_{0,n} \hookrightarrow T_U$ with image equal to U . The ideal of U in T_U is equal to the ideal $I = (R_{ij})$ generated by the relations (1.4).

In particular, $\mathcal{M}_{0,n}$ is isomorphic (as a scheme) to $\text{Spec}(\mathbb{C}[u_{ij}, u_{ij}^{-1}]/I)$. In contrast to Proposition 1.6, Theorem 1.7 identifies $\mathcal{M}_{0,n}$ naturally with a closed subvariety of a *torus*.

For two subsets $A, C \subset [n]$, let $u_{A,C} = \prod_{a \in A, c \in C} u_{ac}$. Let (A, B, C, D) be a decomposition of $[n]$ into four non-empty cyclic intervals, in cyclic order, and define

$$R_{A,B,C,D} := u_{A,C} + u_{B,D} - 1.$$

For example, for $(A, B, C, D) = (\{2, 3\}, \{4\}, \{5, 6, 7\}, \{1\})$, we have $R_{A,B,C,D} = u_{25}u_{26}u_{27}u_{35}u_{36}u_{37} + u_{14} - 1$. Brown [Bro09] includes the generators $R_{A,B,C,D}$ in his definition of the ideal. It follows from [AHHL21a] that these additional generators are not necessary; we give a direct proof.

Proposition 1.8. The elements $R_{A,B,C,D}$ belong to the ideal $I = (R_{ij})$ of $\mathbb{C}[u_{ij}]$.

Proof. We prove the result using the following identity. Suppose $|A| \geq 2$ and a is the maximal element of A in cyclic order, and set $A' := A \setminus a$. Then we have (Exercise 1.23)

$$R_{A,B,C,D} - R_{\{a\},B,C,A' \cup D} - u_{a,C} R_{A',B \cup a,C,D} + u_{B,D} R_{A',\{a\},B,C \cup D} = 0. \quad (1.5)$$

We prove that $R_{A,B,C,D} \in I$ by reverse induction on $d = \max(|A|, |B|, |C|, |D|)$. The base case is $d = n - 3$, in which case after cyclic rotation we have $(|A|, |B|, |C|, |D|) = (1, 1, 1, n - 3)$ and $R_{A,B,C,D} = R_{a,c}$. Suppose the claim has been proven for all $d' > d$, and suppose that $|D| = d$. Suppose that $|A| \geq 2$. Then by (1.5), we have

$$R_{A,B,C,D} = R_{\{a\},B,C,A' \cup D} + u_{a,C} R_{A',B \cup a,C,D} - u_{B,D} R_{A',a,B,C \cup D}.$$

By induction, $-u_{B,D} R_{A',a,B,C \cup D}$ and $R_{\{a\},B,C,A' \cup D}$ belong to I . Repeatedly applying this relation, we reduce to proving that $R_{A,B,C,D} \in I$ for $|A| = 1$. Now if $|A| = 1$ and $|C| \geq 2$, we use the mirror reflection of (1.5) to reduce to the case $|A| = |C| = 1$, which belongs to I by definition. This completes the inductive step of the argument, and the proof. $\square$

The embedding $\mathcal{M}_{0,n} \hookrightarrow T_U$ is in fact intrinsic to $\mathcal{M}_{0,n}$. The group of units $\mathbb{C}[\mathcal{M}_{0,n}]^\times / \mathbb{C}^\times$ of the coordinate ring of $\mathcal{M}_{0,n}$, modulo scalars, is a free abelian group $\Lambda = \Lambda(\mathcal{M}_{0,n}) := \mathbb{C}[\mathcal{M}_{0,n}]^\times / \mathbb{C}^\times$ (see Section 6.1). The *intrinsic torus* of $\mathcal{M}_{0,n}$ is the torus T with character lattice Λ .

Proposition 1.9. The intrinsic torus of $\mathcal{M}_{0,n}$ is canonically isomorphic to the torus T_U . Equivalently, the lattice $\Lambda = \mathbb{C}[\mathcal{M}_{0,n}]^\times / \mathbb{C}^\times$ can be identified with the set of Laurent monomials $\{\prod u_{ij}^{X_{ij}} \mid X_{ij} \in \mathbb{Z}\}$ in the dihedral coordinates.

Proof. One way to obtain this result is to view $\mathcal{M}_{0,n}$ as the complement of $\binom{n}{2} - n$ hyperplanes in $\mathbb{C}^{n-3}$, as explained in Section 1.1. Let $f_1(x) = 0, f_2(x) = 0, \dots$ be the linear equations cutting out these hyperplanes, and note that the normal vectors to these hyperplanes span $\mathbb{C}^{n-3}$. Then the coordinate ring $\mathbb{C}[\mathcal{M}_{0,n}]$ is obtained from the polynomial ring $\mathbb{C}[x_1, \dots, x_{n-3}]$ by adjoining the inverses $f_1^{-1}, f_2^{-1}, \dots$. It follows that $\mathbb{C}[\mathcal{M}_{0,n}]^\times / \mathbb{C}^\times$ is isomorphic to the lattice of Laurent monomials in the f_i . Finally, we have an invertible monomial transformation between the $\binom{n}{2} - n$ dihedral coordinates u_{ij} and the $\binom{n}{2} - n$ linear functions $f_1, f_2, \dots$. This can be obtained directly, or from the equality (3.6). $\square$

We shall call

$$\Lambda = \left\{ \prod u_{ij}^{X_{ij}} \mid X_{ij} \in \mathbb{Z} \right\} = \mathbb{C}[\mathcal{M}_{0,n}]^\times / \mathbb{C}^\times \quad (1.6)$$

the *character lattice* of $\mathcal{M}_{0,n}$. The dual lattice $\Lambda^\vee$ is the *cocharacter lattice* of $\mathcal{M}_{0,n}$. While the torus T_U is intrinsic to $\mathcal{M}_{0,n}$, the basis of characters u_{ij} is not, but depends on a choice of connected component of $\mathcal{M}_{0,n}(\mathbb{R})$ (Exercise 1.24).

Example 1.10. Suppose $n = 4$. Then $\mathcal{M}_{0,4}$ is the subvariety of $(\mathbb{C}^\times)^2$ cut out by the single relation $u_{13} + u_{24} = 1$. The projection of $\mathcal{M}_{0,4}$ to the first factor gives an isomorphism $\mathcal{M}_{0,4} \cong \mathbb{C} \setminus \{0, 1\}$, and we have $\mathbb{C}[\mathcal{M}_{0,4}] \cong \mathbb{C}[u, u^{-1}, (1-u)^{-1}]$, where $u = u_{13}$. The character lattice is $\Lambda = \{u_{13}^{X_{13}} u_{24}^{X_{24}}\} = \{u^a (1-u)^b\} \cong \mathbb{Z}^2$.

1.4. Affine closure and compactification

The torus T_U is an open subset of the affine space $\mathbb{A}_U = \mathbb{C}^{\binom{n}{2}-n} = \text{Spec}(\mathbb{C}[u_{ij}])$. We define a partial compactification $\widetilde{\mathcal{M}}_{0,n}$ of $\mathcal{M}_{0,n}$ as the closure of $\mathcal{M}_{0,n} \hookrightarrow T_U \hookrightarrow \mathbb{A}_U$ in affine space. The variety $\widetilde{\mathcal{M}}_{0,n}$ has coordinate ring $\mathbb{C}[u_{ij}]/I$, where $I = (R_{ij})$.

The partial compactification $\widetilde{\mathcal{M}}_{0,n}$ has a natural stratification with strata indexed by subdivisions of the n -gon, or in other words, by the faces of the associahedron. For a point $\underline{\sigma} \in \widetilde{\mathcal{M}}_{0,n}$, define

$$M(\underline{\sigma}) := \{(i, j) \in \text{diag}_n \mid u_{ij}(\underline{\sigma}) \neq 0\}.$$

We may think of $M(\underline{\sigma})$ as analogous to the matroid of a matrix. It follows from the relations R_{ij} that if the diagonals (i, j) and (k, l) cross then at least one of (i, j) and (k, l) belong to $M(\underline{\sigma})$. Thus the diagonals not belonging to $M(\underline{\sigma})$ form a subdivision of the n -gon by non-intersecting diagonals.

Let $\Delta = \Delta_{0,n}$ be the simplicial complex (Section 2) with vertex set equal to the set of diagonals (ij) of an n -gon, and faces equal to subdivisions $\mathcal{D}$ of the n -gon, considered as a collection of diagonals. We have a disjoint union

$$\widetilde{\mathcal{M}}_{0,n} = \bigcup_{\mathcal{D} \in \Delta} \mathcal{M}_{\mathcal{D}}, \quad \mathcal{M}_{\mathcal{D}} = \{\underline{\sigma} \in \widetilde{\mathcal{M}}_{0,n} \mid \text{diag}_n \setminus M(\underline{\sigma}) = \mathcal{D}\}.$$

For example, $\mathcal{M}_{\emptyset} = \mathcal{M}_{0,n}$. Let $\widetilde{\mathcal{M}}_{\mathcal{D}}$ denote the closure of $\mathcal{M}_{\mathcal{D}}$ in $\widetilde{\mathcal{M}}_{0,n}$.

Proposition 1.11. Let $\mathcal{D}$ be a subdivision of the n -gon into polygons of sizes $n_1, n_2, \dots, n_s$. Then the stratum $\mathcal{M}_{\mathcal{D}}$ is isomorphic to $\mathcal{M}_{0,n_1} \times \dots \times \mathcal{M}_{0,n_s}$ and $\widetilde{\mathcal{M}}_{\mathcal{D}}$ is isomorphic to $\widetilde{\mathcal{M}}_{0,n_1} \times \dots \times \widetilde{\mathcal{M}}_{0,n_s}$. It is irreducible and codimension $|\mathcal{D}|$ in $\widetilde{\mathcal{M}}_{0,n}$.

Proof. We prove the isomorphism $\widetilde{\mathcal{M}}_{\mathcal{D}} \cong \widetilde{\mathcal{M}}_{0,n_1} \times \dots \times \widetilde{\mathcal{M}}_{0,n_s}$ for the case $s = 2$. The general case follows by induction and the other isomorphism is similar. The diagonal (ij) divides the n -gon into a polygon P_1 with vertices $i, i+1, \dots, j$ and a polygon P_2 with vertices $j, j+1, \dots, i$, where all vertex labels are taken cyclically modulo n . When $u_{ij} = 0$, we have $u_{kl} = 1$ for all (kl) crossing (ij) . The relation $R_{kl} = 0$ becomes $1 + 0 - 1 = 0$ while the relation $R_{ij} = 0$ becomes $0 + 1 - 1 = 0$, and are automatically satisfied.

Now let (ab) be a diagonal that does not cross (ij) . We assume that (ab) is a diagonal of P_1 . The diagonals (cd) that cross (ab) are of two types: diagonals

of P_1 that cross (ab) , or diagonals that cross both (ab) and (ij) . It follows that with $u_{ij} = 0$ and $u_{kl} = 1$ for all (kl) crossing (ij) , the relation $R_{ab} = 0$ becomes

$$R_{ab} = u_{ab} + \prod_{(cd)} u_{cd} - 1 = 0$$

where now (cd) varies (only) over diagonals of P_1 that cross (ab) . These are the u -equations for the polygon P_1 . Similarly, we obtain the u -equations for the polygon P_2 . More formally, let $I' = I + (u_{ij})$. Then we have proven that

$$\mathbb{C}[u_{ab}]/I' \cong (\mathbb{C}[u_{a_1b_1}]/I_1) \times (\mathbb{C}[u_{a_2b_2}]/I_2)$$

where I_1 and I_2 are the u -relations for the polygons P_1 and P_2 respectively. The last statement follows immediately from dimension counting $\dim(\mathcal{M}_{0,n_i}) = n_i - 3$. $\square$

The affine variety $\widetilde{\mathcal{M}}_{0,n}$ only partially compactifies $\mathcal{M}_{0,n}$. We can use it to study the compactification $\overline{\mathcal{M}}_{0,n}$, the moduli space of stable rational marked curves, as follows.

Definition 1.12. The moduli space $\overline{\mathcal{M}}_{0,n}$ of stable rational curves with n marked points is the (Zariski-)closure of $\mathcal{M}_{0,n}$ under the cross-ratio embedding ι of (1.2)

$$\overline{\mathcal{M}}_{0,n} = \overline{\mathcal{M}_{0,n}} \subset (\mathbb{P}^1)^{\binom{[n]}{4}}.$$

By definition, the cross-ratios $[ij|kl]$ are still well-defined on $\overline{\mathcal{M}}_{0,n}$, but now they take values in $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ rather than $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. Since the dihedral coordinates are some of the cross-ratios, it is not hard to see that $\widetilde{\mathcal{M}}_{0,n}$ is an open subset of $\overline{\mathcal{M}}_{0,n}$. Indeed, $\widetilde{\mathcal{M}}_{0,n}$ is the locus in $\overline{\mathcal{M}}_{0,n}$ where $u_{ij} \neq \infty$ for all diagonals (i, j) . The action of S_n on $\mathcal{M}_{0,n}$ by permuting points extends to an action as automorphisms on $\overline{\mathcal{M}}_{0,n}$. However, the action of $\alpha \in S_n$ sends $\widetilde{\mathcal{M}}_{0,n}$ to another subvariety $\widetilde{\mathcal{M}}_{0,n}^\alpha \subset \overline{\mathcal{M}}_{0,n}$. The subvariety $\widetilde{\mathcal{M}}_{0,n}^\alpha$ depends only on the dihedral ordering on $1, 2, \dots, n$ induced by α , and often we will view α as a dihedral ordering. A dihedral ordering on $1, 2, \dots, n$ is a permutation of the n points, where two permutations are considered equivalent if they are related by either cyclic rotation or the reversal $w_1 w_2 \cdots w_n \sim w_n w_{n-1} \cdots w_1$. Note that the cyclic rotation $(12 \cdots n) \in S_n$ sends $\widetilde{\mathcal{M}}_{0,n}$ to itself by sending $u_{ij} \mapsto u_{i+1, j+1}$, and similarly $(n, n-1, \dots, 1) \in S_n$ maps $u_{ij} \mapsto u_{n+1-j, n+1-i}$.

By Exercise 1.25, every point $\underline{\sigma} \in \overline{\mathcal{M}}_{0,n}$ belongs to $\widetilde{\mathcal{M}}_{0,n}^\alpha$ for some α . We view

$$\{\widetilde{\mathcal{M}}_{0,n}^\alpha \mid \alpha \text{ a dihedral ordering}\}$$

as open charts covering $\overline{\mathcal{M}}_{0,n}$.

Theorem 1.13. The moduli space $\overline{\mathcal{M}}_{0,n}$ is a smooth, projective algebraic variety and the boundary $\overline{\mathcal{M}}_{0,n} \setminus \mathcal{M}_{0,n}$ is a simple normal crossings divisor.

Sketch. Since the $\widetilde{\mathcal{M}}_{0,n}^\alpha$ cover $\overline{\mathcal{M}}_{0,n}$, smoothness of $\overline{\mathcal{M}}_{0,n}$ follows from smoothness of $\widetilde{\mathcal{M}}_{0,n}$. This would in principle follow from the Jacobian criterion and the relations R_{ij} (1.4). However, I do not know a direct proof along these lines.

Brown [Bro09] proves this result by induction, considering inclusion maps of the form $\widetilde{\mathcal{M}}_{0,n} \rightarrow \widetilde{\mathcal{M}}_{0,n_1} \times \widetilde{\mathcal{M}}_{0,n_2}$ where $n_1 + n_2 = n + 4$.

The statement also follows from the result in Section 5, where $\widetilde{\mathcal{M}}_{0,n}$ is identified with an open subset of a smooth toric variety. This toric variety is the toric variety associated to a particular associahedron, and the smoothness follows from results in the theory of cluster algebras; see [AHHL21a] for further details. $\square$

The moduli space $\overline{\mathcal{M}}_{0,n}$ has a natural stratification whose strata correspond to knowing which cross-ratios are equal to 0, 1, or ∞ . These strata are indexed by trees with n leaves labeled $1, 2, \dots, n$ (Exercise 1.26).

1.5. Positive coordinates

Let us consider the real points $\mathcal{M}_{0,n}(\mathbb{R})$. This is an open manifold of dimension $(n-3)$. There is an action of the symmetric group S_n on $\mathcal{M}_{0,n}$, acting by permuting the n points $\sigma_1, \dots, \sigma_n$. Viewing points of $\mathcal{M}_{0,n}$ as $2 \times n$ matrices, the symmetric group S_n acts by permuting the n columns.

Proposition 1.14. The symmetric group S_n acts transitively on the connected components of $\mathcal{M}_{0,n}(\mathbb{R})$. The manifold $\mathcal{M}_{0,n}(\mathbb{R})$ has $(n-1)!/2$ connected components, in bijection with dihedral orderings of $1, 2, \dots, n$.

Proof. Using the $\mathrm{PGL}(2)$ action, we place $(\sigma_1, \sigma_2, \sigma_n)$ at $(0, 1, \infty)$. Define the positive part $(\mathcal{M}_{0,n})_{>0} \subset \mathcal{M}_{0,n}(\mathbb{R})$ by

$$(\mathcal{M}_{0,n})_{>0} = \{(\sigma_3, \dots, \sigma_{n-1}) \mid 0 < \sigma_3 < \sigma_4 < \dots < \sigma_{n-1} < \infty\}, \quad (1.7)$$

the space of n points in $\mathbb{P}^1(\mathbb{R})$ arranged in order. This is clearly a connected component of $\mathcal{M}_{0,n}(\mathbb{R})$. By a direct calculation, $(\mathcal{M}_{0,n})_{>0}$ is the locus in $\mathcal{M}_{0,n}(\mathbb{R})$ where all dihedral coordinates u_{ij} take positive values. The symmetric group S_n sends cross-ratios to cross-ratios. The subgroup of S_n that sends dihedral coordinates to dihedral coordinates is exactly the dihedral subgroup $D_{2n} \subset S_n$ (generated by the cyclic rotation $(12 \cdots n)$ and the reflection $(n(n-1) \cdots 21)$). It follows that the connected components of $\mathcal{M}_{0,n}(\mathbb{R})$ are in bijection with the cosets S_n/D_{2n} , and the second statement follows. $\square$

We denote the connected components of $\mathcal{M}_{0,n}(\mathbb{R})$ by $\mathcal{M}_{0,n}(\alpha)$, where α is a dihedral ordering of $1, 2, \dots, n$.

The positive part $(\mathcal{M}_{0,n})_{>0}$ (1.7) of $\mathcal{M}_{0,n}$ is diffeomorphic to an open ball. We realize this using explicit parametrizations:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & \cdots & 0 \\ 0 & 1 & 1+x_1 & 1+x_1+x_2 & 1+x_1+x_2+x_3 & \cdots & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & \cdots & 0 \\ 0 & 1 & 1+y_1 & 1+y_1+y_1y_2 & 1+y_1+y_1y_2+y_1y_2y_3 & \cdots & 1 \end{bmatrix}$$

where $(x_1, \dots, x_{n-3}) \in \mathbb{R}_{>0}^{n-3}$ or $(y_1, \dots, y_{n-3}) \in \mathbb{R}_{>0}^{n-3}$. The rational coordinates $(x_1, \dots, x_{n-3})$ and $(y_1, \dots, y_{n-3})$ are called *positive coordinates*, and are related by the monomial transformation

$$x_i = y_1 y_2 \cdots y_i.$$

Each coordinate system consists of regular functions on $\mathcal{M}_{0,n}$ that generate the field $\mathbb{C}(\mathcal{M}_{0,n})$ of rational functions. Furthermore, the maps

$$(x_1, \dots, x_{n-3}) : \mathcal{M}_{0,n} \rightarrow (\mathbb{C}^\times)^{n-3}, \quad (y_1, \dots, y_{n-3}) : \mathcal{M}_{0,n} \rightarrow (\mathbb{C}^\times)^{n-3}$$

map $(\mathcal{M}_{0,n})_{>0}$ diffeomorphically onto $\mathbb{R}_{>0}^{n-3}$. In these coordinate systems, we have the following explicit formulae. For $1 \leq i < j < n$, the 2×2 minor (ij) is given by

$$(ij) = x_{i-1} + x_i + \cdots + x_{j-2} = y_1 y_2 \cdots y_{i-1} (1 + y_i + y_i y_{i+1} + \cdots + y_i y_{i+1} \cdots y_{j-2}),$$

where by convention $x_0 = 1$, and for $j = n$, we have $(in) = 1$. The dihedral coordinates are related to the positive coordinates by

$$u_{ij} = \begin{cases} \frac{y_i(1+y_{i+1}+y_{i+1}y_{i+2}+\cdots+y_{i+1}\cdots y_{n-3})}{(1+y_i+y_i y_{i+1}+\cdots+y_i y_{i+1}\cdots y_{n-3})} & \text{if } j = n-1 \\ \frac{(1+y_1+y_1 y_2+\cdots+y_1 y_2 \cdots y_{i-2})}{(1+y_1+y_1 y_2+\cdots+y_1 y_2 \cdots y_{i-1})} & \text{if } j = n \\ \frac{(1+y_i+y_i y_{i+1}+\cdots+y_i y_{i+1}\cdots y_{j-1})(1+y_{i+1}+y_{i+1}y_{i+2}+\cdots+y_{i+1}\cdots y_{j-2})}{(1+y_i+y_i y_{i+1}+\cdots+y_i y_{i+1}\cdots y_{j-2})(1+y_{i+1}+y_{i+1}y_{i+2}+\cdots+y_{i+1}\cdots y_{j-1})} & \text{else.} \end{cases}$$

We obtain, by inspection,

Proposition 1.15. The dihedral coordinates take values in $(0, 1)$ on $(\mathcal{M}_{0,n})_{>0}$.

For example, for $n = 4$ we have

$$u_{13} = \frac{y_1}{1+y_1}, \quad u_{24} = \frac{1}{1+y_1},$$

and for $n = 5$ we have

$$\begin{aligned} u_{13} &= \frac{1+y_1+y_1y_2}{(1+y_1)(1+y_2)}, & u_{14} &= \frac{y_1(1+y_2)}{1+y_1+y_1y_2}, & u_{24} &= \frac{y_2}{1+y_2}, \\ u_{25} &= \frac{1}{1+y_1}, & u_{35} &= \frac{1+y_1}{1+y_1+y_1y_2}. \end{aligned} \quad (1.8)$$

The next result follows immediately from these formulae.

Proposition 1.16. The rational function field $\mathbb{C}(\mathcal{M}_{0,n})$ of $\mathcal{M}_{0,n}$ is $\mathbb{C}(y_1, \dots, y_{n-3})$. The character lattice $\Lambda(\mathcal{M}_{0,n})$ is the lattice of Laurent monomials in the $n-3$ coordinates $y_1, y_2, \dots, y_{n-3}$ and the $\binom{n-2}{2}$ polynomials

$$p_{ij} = (1 + y_i + y_i y_{i+1} + \dots + y_i y_{i+1} \dots y_{j-2})$$

where $1 \leq i < j-1 < j \leq n-1$.

Note that $\binom{n-2}{2} + (n-3) = \binom{n}{2} - n$ is the number of diagonals of a n -gon.

1.6. Canonical form on $\mathcal{M}_{0,n}$

Let $(\mathcal{M}_{0,n})_{\geq 0} \subset \widetilde{\mathcal{M}}_{0,n} \subset \overline{\mathcal{M}}_{0,n}$ be the analytic closure of the positive component $(\mathcal{M}_{0,n})_{>0}$ in $\widetilde{\mathcal{M}}_{0,n}$. This is a compact semialgebraic set inside $\widetilde{\mathcal{M}}_{0,n}(\mathbb{R})$. It is cut out by the inequalities $u_{ij} \geq 0$. In fact, the coordinates u_{ij} take values in $[0, 1]$ on $(\mathcal{M}_{0,n})_{\geq 0}$, and identify $(\mathcal{M}_{0,n})_{\geq 0}$ with a closed subspace of the unit cube $[0, 1]^{\binom{n}{2}-n}$. The face structure of $(\mathcal{M}_{0,n})_{\geq 0}$ is the same as that of the *associahedron*: for each subdivision $\mathcal{D}$ of the n -gon, the (open) stratum $(\mathcal{M}_{0,n})_{\geq 0} \cap \mathcal{M}_{\mathcal{D}}$ is an open ball of codimension equal to the number of diagonals $|\mathcal{D}|$.

Example 1.17. Let $n = 5$. The two-dimensional space $(\mathcal{M}_{0,5})_{\geq 0}$ has five facets, given by $u_{ij} = 0$ for (ij) a diagonal of the pentagon. Consider one of the facets, which by cyclic symmetry we can take to be $u_{13} = 0$. The five u -equations then reduce to

$$u_{24} = 1, \quad u_{25} = 1, \quad u_{35} + u_{14} = 1.$$

So on this facet u_{13}, u_{24}, u_{25} are constant, and we have a one-dimensional space sitting inside two-dimensional u_{14}, u_{35} space, which is isomorphic to $(\mathcal{M}_{0,4})_{\geq 0}$, an interval. It is easy to see that $(\mathcal{M}_{0,5})_{\geq 0}$ is combinatorially a pentagon.

Definition 1.18. The *canonical form* of $(\mathcal{M}_{0,n})_{\geq 0}$, or the *Parke-Taylor form* is the top-degree differential form

$$\Omega = \Omega_{0,n} := \bigwedge_{i=1}^{n-3} d\log x_i = \bigwedge_{i=1}^{n-3} \frac{dx_i}{x_i} = \bigwedge_{i=1}^{n-3} d\log y_i = \bigwedge_{i=1}^{n-3} \frac{dy_i}{y_i}$$

Since y_i is a regular, non-vanishing function on $\mathcal{M}_{0,n}$, this is a regular top-form on $\mathcal{M}_{0,n}$. It has poles in the partial compactification $\widetilde{\mathcal{M}}_{0,n}$, and in general it has both poles and zeroes in $\widetilde{\mathcal{M}}_{0,n}$. By Problem 1.28, this form can also be written as

$$\Omega = \bigwedge_{(ij)} d\log \frac{u_{ij}}{1 - u_{i,j}} \quad (1.9)$$

where (ij) ranges over a triangulation $\mathcal{D}$ of the n -gon with no interior triangles (that is, triangles none of whose sides are sides of the n -gon). The “no interior triangle” condition is necessary. A recent result of Silversmith [Sil23] states that the field extension $\mathbb{C}(u_{ij} \mid (ij) \in \mathcal{D}) \subset \mathbb{C}(\mathcal{M}_{0,n})$ has degree 2^a where a is the number of interior triangles in the triangulation $\mathcal{D}$.

The Parke-Taylor form Ω encodes a remarkable amount of physics. By taking different kinds of integrals of this form, one obtains scalar ϕ^3 -amplitudes, string theory amplitudes, gauge theory amplitudes, and so on.

For each diagonal $(ij) \in \text{diag}_n$, let $\widetilde{\mathcal{M}}_{ij} \subset \widetilde{\mathcal{M}}_{0,n}$ be the divisor cut out by the equation $u_{ij} = 0$. Then by Proposition 1.11, $\widetilde{\mathcal{M}}_{ij}$ can be identified with $\widetilde{\mathcal{M}}_{0,n_1} \times \widetilde{\mathcal{M}}_{0,n_2}$, where $n_1 = j - i + 1$ and $n_2 = n + i - j + 1$. For the notion of residue in the following result, we refer the reader to [Lam22, Section 2].

Lemma 1.19. The canonical form Ω has a simple pole along $\widetilde{\mathcal{M}}_{ij}$ and we have

$$\text{Res}_{\widetilde{\mathcal{M}}_{ij}} \Omega = \Omega_{0,n_1} \wedge \Omega_{0,n_2}$$

under the isomorphism $\widetilde{\mathcal{M}}_{ij} \cong \widetilde{\mathcal{M}}_{0,n_1} \times \widetilde{\mathcal{M}}_{0,n_2}$.

Proof. Pick a triangulation $\mathcal{D}$ with no interior triangles that uses the diagonal (ij) . For all other diagonals $(kl) \in \mathcal{D}$ the function u_{kl} is a regular function on $\widetilde{\mathcal{M}}_{ij}$ that is neither identically 0 nor identically equal to 1. Thus $d\log \frac{u_{kl}}{1 - u_{kl}}$ has neither poles nor zeroes on $\widetilde{\mathcal{M}}_{ij}$. However, $d\log \frac{u_{ij}}{1 - u_{i,j}} = \frac{1}{u_{ij}(1 - u_{ij})} du_{ij}$ has a simple pole along $u_{ij} = 0$. We have

$$\text{Res}_{z=0} \frac{1}{z(1-z)} dz = 1$$

and it follows that

$$\text{Res}_{u_{ij}=0} \Omega = \bigwedge_{(kl) \in \mathcal{D} \setminus (ij)} d\log \frac{u_{kl}}{1 - u_{kl}} = \Omega_{0,n_1} \wedge \Omega_{0,n_2}. \quad \square$$

For example, the canonical form of $\mathcal{M}_{0,4}$ is $\frac{du_{13}}{u_{13}(1-u_{13})}$ which has simple poles with residues 1 and -1 at the two boundaries $u_{13} = 0$ and $u_{24} = 0$ ($u_{13} = 1$).

1.7. Positive geometry

Lemma 1.19 is close to saying that $(\mathcal{M}_{0,n})_{\geq 0}$ is a positive geometry with canonical form Ω . Informally, a positive geometry $X_{\geq 0}$ is a semialgebraic space equipped with a top-degree rational form $\Omega(X_{\geq 0})$ whose polar structure reflects the face structure of $X_{\geq 0}$.

Definition 1.20. A *positive geometry* is a d -dimensional (oriented) semialgebraic space $X_{\geq 0}$ in the real points of a projective d -dimensional complex algebraic variety X such that there exists a unique top-degree rational form $\Omega(X_{\geq 0})$ on X with the following properties:

1. either $d = 0$, and $X = X_{\geq 0}$ is a point and $\Omega(X_{\geq 0}) = \pm 1$ (depending on orientation), or
2. every facet $(F, F_{\geq 0})$ of $X_{\geq 0}$ is a $(d-1)$ -dimensional positive geometry, and $\Omega(X_{\geq 0})$ has simple pole along F satisfying

$$\text{Res}_F \Omega(X_{\geq 0}) = \Omega(F_{\geq 0})$$

and these are the only poles of $\Omega(X_{\geq 0})$.

For a more careful discussion of the definition of a positive geometry we refer the reader to [AHBL17, Lam22].

Note that even though $(\mathcal{M}_{0,n})_{\geq 0}$ is combinatorially a polytope (the associahedron), it is not actually a polytope. Instead, it can be identified with the nonnegative part of a toric variety (see Section 5) and it is also a *wonderpolytope* in the sense of Brauner, Eur, Pratt, Vlad [BEPV24].

It follows from induction on n and Lemma 1.19 that $(\overline{\mathcal{M}}_{0,n}, (\mathcal{M}_{0,n})_{\geq 0})$ is a positive geometry with canonical form Ω , modulo the statement that Ω has no poles on $\overline{\mathcal{M}}_{0,n}$ except the simple poles along the divisors $\widetilde{\mathcal{M}}_{ij}$. We leave this as Exercise 1.30.

Theorem 1.21 ([AHHL21a]). $(\overline{\mathcal{M}}_{0,n}, (\mathcal{M}_{0,n})_{\geq 0})$ is a positive geometry with canonical form Ω .

Another proof of Theorem 1.21 is given in [BEPV24].

1.8. Exercises and Problems

Exercise 1.22.

1. Show that $\text{PGL}(k)$ acts simply-transitively on the set of $k+1$ distinct ordered points in $\mathbb{P}^{k-1}$.

2. Verify the dimension of $\mathcal{M}_{0,n}$ stated in Proposition 1.3.

Exercise 1.23.

- (a) Prove Lemma 1.5.
- (b) Prove the relations (1.4).
- (c) Let (A, B, C, D) be a decomposition of $[n]$ into four non-empty cyclic intervals, in cyclic order. Show that the identity

$$u_{A,C} + u_{B,D} = 1$$

holds by writing both terms as cross-ratios.

- (d) Show that

$$\frac{1 - u_{ij}}{u_{ij}} \frac{1 - u_{i+1,j+1}}{u_{i+1,j+1}} = (1 - u_{i,j+1})(1 - u_{i+1,j}).$$

- (e) Prove (1.5).

Exercise 1.24. Call a character $\gamma \in \Lambda$ *bounded* if it takes bounded values on $(\mathcal{M}_{0,n})_{>0}$. Show that γ is bounded if and only if it is a monomial $\gamma = \prod_{ij} u_{ij}^{a_{ij}}$ in the u_{ij} , where $a_{ij} \geq 0$.

Exercise 1.25.

- (a) If you know another definition of $\overline{\mathcal{M}}_{0,n}$, show that it agrees with Definition 1.12.
- (b) Let $\underline{\sigma} \in \overline{\mathcal{M}}_{0,n}$. Show that there exists a dihedral ordering $\tau \in S_n$ such that $u_{\tau(i),\tau(j)}(\underline{\sigma}) \neq \infty$ for all diagonals (i, j) .

Exercise 1.26.

- (a) Prove that the closure $\widetilde{\mathcal{M}}_{\mathcal{D}}$ of the stratum $\mathcal{M}_{\mathcal{D}}$ in $\widetilde{\mathcal{M}}_{0,n}$ is the union $\bigcup_{\mathcal{D}'} \mathcal{M}_{\mathcal{D}'}$ over all subdivisions $\mathcal{D}'$ that refine $\mathcal{D}$.
- (b) Find a bijection between subdivisions of the n -gon and planar n -leaf trees. These are trees, embedded in the plane, with n leaves labeled $1, 2, \dots, n$ in order when traversing the outside of the tree. Thus the strata of $\widetilde{\mathcal{M}}_{0,n}$ are labeled by planar n -leaf trees.
- (c) Show that the strata of $\overline{\mathcal{M}}_{0,n}$ are labeled by all n -leaf trees with leaves labeled $1, 2, \dots, n$. Here, the trees are treated as abstract graphs with labeled leaves.

Exercise 1.27. The functions u_{ij} take nonzero real values on $\mathcal{M}_{0,n}(\mathbb{R})$. Prove that the signs of u_{ij} determine the connected component that a point lies in. What are the possible signs for $\{u_{ij}\}$? This can be considered a moduli space analogue of a matroid.

Problem 1.28. Prove (1.9). Is there a formula for Ω using $\{u_{ij} \mid (ij) \in \mathcal{D}\}$ where $\mathcal{D}$ is a triangulation that does have interior triangles?

1.8.1. Canonical form of $\text{Gr}(2, n)_{\geq 0}$

The *positive Grassmannian* $\text{Gr}(2, n)_{>0}$ is the locus in $\text{Gr}(2, n)$ represented by $2 \times n$ matrices all of whose 2×2 minors satisfy $(ij) > 0$. Its analytic closure in $\text{Gr}(2, n)$ is the (totally) nonnegative Grassmannian $\text{Gr}(2, n)_{\geq 0}$. Even though $(\mathcal{M}_{0,n})_{>0}$ is a rather straightforward quotient of $\text{Gr}(2, n)_{>0}$ by a positive torus $T_{>0} \cong \mathbb{R}_{>0}^{n-1}$, the combinatorics of $(\mathcal{M}_{0,n})_{\geq 0}$ and $\text{Gr}(2, n)_{\geq 0}$ is remarkably quite different [Lam16b, Pos06]! Like $(\mathcal{M}_{0,n})_{\geq 0}$, the totally nonnegative Grassmannian $\text{Gr}(2, n)_{\geq 0}$ is a regular CW-complex homeomorphic to a closed ball, but it is not a polytopal complex; see [GKL22b, GKL22a].

We investigate the canonical form $\Omega(\text{Gr}(2, n)_{\geq 0})$ of the positive Grassmannian. The map $\text{Gr}^\circ(2, n) \rightarrow \mathcal{M}_{0,n}$ from (1.1) is a locally-trivial fibration with fiber the $(n-1)$ -dimensional torus T' . In fact, this family can be trivialized and we have an isomorphism $\text{Gr}^\circ(2, n) = \mathcal{M}_{0,n} \times T'$. The torus T' has a canonical form $\Omega_{T'} = \bigwedge_{i=1}^{n-1} \text{dlog} t_i$, where t_i are a basis of characters of T' . Define a top-form on $\text{Gr}(2, n)$ by

$$\Omega(\text{Gr}(2, n)_{\geq 0}) := \Omega_{0,n} \wedge \Omega_{T'}$$

using the isomorphism $\text{Gr}^\circ(2, n) \cong \mathcal{M}_{0,n} \times T'$, and noting that $(n-3) + (n-1) = 2n-4 = \dim \text{Gr}(2, n)$.

Exercise 1.29. (a) Show that $\Omega(\text{Gr}(2, n)_{\geq 0})$ has no zeroes on $\text{Gr}(2, n)$, and simple poles only, exactly along the n positroid divisors $(i, i+1) = 0$ (see [Lam16b, KLS13] for more on positroid varieties).

(b) Now let $\pi : \text{Mat}_{2,n} \dashrightarrow \text{Gr}(2, n)$ be the natural quotient (rational) map. Then $\Theta := \pi^* \Omega(\text{Gr}(2, n)_{\geq 0})$ is a $2n-4$ form on the $2n$ -dimensional affine space $\text{Mat}_{2,n}$. Show that (up to a constant) we have

$$\Theta = \frac{1}{(12)(23)\cdots(n1)} \det(C_1 C_2 d^{n-2} C_1) \wedge \det(C_1 C_2 d^{n-2} C_2),$$

where C_1, C_2 denote the two rows of a $2 \times n$ matrix, the notation $d^{n-2} C_1$ denotes a $(n-2) \times n$ matrix of 1-forms, all of whose rows are dC_1 . For example, with

$n = 4$, we would have $\det(C_1 C_2 d^{n-2} C_1) =$

$$\det \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ C_{21} & C_{22} & C_{23} & C_{24} \\ dC_{11} & dC_{12} & dC_{13} & dC_{14} \\ dC_{11} & dC_{12} & dC_{13} & dC_{14} \end{bmatrix} = C_{11} C_{22} dC_{13} \wedge dC_{14} - C_{11} C_{22} dC_{14} \wedge dC_{13} \pm \dots$$

1.8.2. Canonical class of $\overline{\mathcal{M}}_{0,n}$ and zeroes of $\Omega_{0,n}$

We recall some statements concerning the intersection theory of $\overline{\mathcal{M}}_{0,n}$ [Kee92]. The dimension of the first Chow group $A^1(\overline{\mathcal{M}}_{0,n})$ (or the cohomology group $H^2(\overline{\mathcal{M}}_{0,n})$) is equal to $2^{n-1} - \binom{n}{2} - 1$. For example, for $n = 4$, we get $\dim H^2(\mathbb{P}^1) = 1$. A spanning set is indexed by the classes of the 2^{n-1} boundary divisors $D_S = D_{\bar{S}}$ where $S \subset 2^{[n]}$ is a subset of size $|S| \in [2, n-2]$, and $\bar{S} := [n] \setminus S$. In the notation of Exercise 1.26, D_S is the boundary divisor corresponding to a tree with two interior vertices, one connected to the leaves in S , and the other to the leaves in $\bar{S}$. For any four distinct $i, j, k, l \in [n]$, we have the relation

$$\sum_{\substack{i,j \in S \\ k,l \notin S}} [D_S] = \sum_{\substack{i,k \in S \\ j,l \notin S}} [D_S] = \sum_{\substack{i,l \in S \\ j,k \notin S}} [D_S] \quad \text{in } A^1(\overline{\mathcal{M}}_{0,n}).$$

The canonical class of $\overline{\mathcal{M}}_{0,n}$ is given by [Pan97]

$$K_{\overline{\mathcal{M}}_{0,n}} = \sum_{s=2}^{\lfloor n/2 \rfloor} \left(\frac{s(n-s)}{n-1} - 2 \right) \sum_{|S|=s} [D_S]. \quad (1.10)$$

For example, for $n = 4$, we get $K_{\overline{\mathcal{M}}_{0,4}} = -\frac{2}{3}([0] + [1] + [\infty]) = -2[\text{pt}]$, as expected. For $n = 5$, we get

$$K_{\overline{\mathcal{M}}_{0,5}} = -\frac{1}{2} \sum_{i,j} [D_{ij}].$$

Exercise 1.30. Compute the order of vanishing of $\Omega_{0,n}$ on every divisor $D_S \subset \overline{\mathcal{M}}_{0,n}$. In particular, show that the only poles of $\Omega_{0,n}$ are the divisors D_S such that $D_S \cap \widetilde{\mathcal{M}}_{0,n}$ is a divisor in $\widetilde{\mathcal{M}}_{0,n}$. Hint: use the action of the symmetric group S_n permuting the n points. For $w \in S_n$, write $w^*(\Omega_{0,n}) = f_w \Omega_{0,n}$, where f_w is a rational function on $\mathcal{M}_{0,n}$ depending on w . Now compute the poles of f_w on $\widetilde{\mathcal{M}}_{0,n}$.

Exercise 1.31. By definition, the class

$$[\Omega_{0,n}] = [\text{divisor of zeroes}] - [\text{divisor of poles}]$$

of the rational top-form $\Omega_{0,n}$ in $\overline{\mathcal{M}}_{0,n}$ is equal to $K_{\overline{\mathcal{M}}_{0,n}}$. Verify the formula (1.10) by computing $[\Omega_{0,n}]$. Hint: $w \in S_n$ acts on $A^1(\overline{\mathcal{M}}_{0,n})$ sending D_S to $D_{w(S)}$. The canonical class is invariant under the automorphisms, and thus invariant under the action of S_n . Thus we can compute $K_{\overline{\mathcal{M}}_{0,n}}$ by averaging $w \cdot [\Omega_{0,n}]$ over all permutations $w \in S_n$.

2. Binary geometry

We formalize some of the properties of the relations (1.4) in terms of binary geometries [AHHLT23, AHHL21a].

2.1. Binary geometries and u -equations

A simplicial complex Δ on $[n]$ is a collection of subsets $\Delta \subseteq 2^{[n]}$ satisfying:

1. If $F \in \Delta$ and $F' \subset F$ then $F' \in \Delta$.
2. Δ is non-empty.

The elements $F \in \Delta$ of a simplicial complex are called the *faces* of Δ . The dimension $\dim \Delta$ is the size of the largest face in Δ , minus one. Faces of Δ that are maximal under inclusion are called *facets*. We say that Δ is *pure* if all facets have the same dimension.

We say that Δ is a *flag complex* if each $\{i\}$ is a face of Δ for each i , and for $r > 2$, we have that $F = \{a_1, \dots, a_r\} \in \Delta$ whenever $\{a_i, a_j\} \in \Delta$ for all $1 \leq i < j \leq r$. We write $i \sim j$ if $\{i, j\}$ is a face of a flag complex Δ , and say that i and j are *compatible*. Otherwise, i and j are incompatible.

For $S \subset [n]$, let $H_S \subset \mathbb{C}^{[n]}$ denote the subspace where $x_s = 0$ for $s \in S$.

Definition 2.1. Let Δ be a flag simplicial complex on $[n]$. A *binary geometry* $\tilde{U}$ for Δ is an affine algebraic variety $\tilde{U} \subset \mathbb{C}^n$ of dimension $\dim \tilde{U} = \dim \Delta + 1$ cut out by n equations of the form

$$R_i := u_i + \prod_{j \not\sim i} u_j^{a_{ij}} - 1 = 0, \quad i = 1, 2, \dots, n \quad (2.1)$$

for integers $a_{ij} > 0$, satisfying the following property: for a subset $S \subset [n]$, the subvariety $\tilde{U}_S := \tilde{U} \cap H_S$ is non-empty exactly when $S \in \Delta$, and in this case $\tilde{U}_S$ is irreducible, and pure of codimension $|S|$ in $\tilde{U}$.

Note that when $u_i = 0$, the equation $R_j = 0$ gives $u_j = 1$ for any $j \not\sim i$. Suppose that $\tilde{U}$ is a binary geometry for Δ , and $x \in \tilde{U}$. Let

$$F_x := \{i \in [n] \mid u_i(x) = 0\}.$$

By the assumptions of Definition 2.1, we have $F_x \in \Delta$. We have a stratification of $\tilde{U}$ by the closed subvarieties $\tilde{U}_F$, for $F \in \Delta$. Let

$$U_F = \{x \in \tilde{U} \mid F_x = F\}$$

denote the locally-closed pieces in the stratification. Thus $\tilde{U} = \bigsqcup_{F \in \Delta} U_F$, and each U_F is itself non-empty, irreducible, and codimension $|F|$ in $\tilde{U}$.

We denote $U := U_\emptyset$. Thus U is the Zariski-open subset of $\tilde{U}$ obtained by intersecting $\tilde{U}$ with the torus $T = (\mathbb{C}^\times)^n \subset \mathbb{C}^n$. It is an irreducible, closed subvariety of the torus T , and thus by definition a very affine variety (see Section 6). Indeed, each U_F is a very affine variety.

Problem 2.2. Which flag simplicial complexes have binary geometries? When is a binary geometry smooth, and when are $\tilde{U}_F$ and U_F smooth? When do the equations (2.1) cut $\tilde{U}$ out scheme-theoretically?

Example 2.3. Let $\Delta = \{\emptyset, 1, 2, 3, 12, 23, 13\}$ be dual to the face complex of a triangle. (This simplicial complex is not flag, but it still makes sense to consider the u -equations.) The equations for a binary geometry $\tilde{U}$ for Δ take the form

$$u_1 + 1 = 1, \quad u_2 + 1 = 1, \quad u_3 + 1 = 1,$$

which has a unique solution $(u_1, u_2, u_3) = (0, 0, 0)$. Since this has the wrong dimension, there is no binary geometry for Δ .

Example 2.4. Let $\Delta = \{\emptyset, 1, 2, 3, 4, 12, 23, 14, 23\}$ be dual to the face complex of a square. The equations for a binary geometry $\tilde{U}$ for Δ take the form

$$u_1 + u_3^a = 1, \quad u_2 + u_4^b = 1, \quad u_3 + u_1^c = 1, \quad u_4 + u_2^d = 1,$$

for positive integers a, b, c, d . If we choose $a = b = c = d = 1$, we obtain a binary geometry $U \cong \mathbb{C}^2$. For all other choices of a, b, c, d , we have $\dim U < 2$ and we do not have a binary geometry.

Example 2.5. Let $\Delta = \Delta_{0,n}$ be the simplicial complex with vertex set equal to the set of diagonals (ij) of an n -gon, and faces equal to subdivisions $\mathcal{D}$ of the n -gon, considered as a collection of diagonals. Then $\widetilde{\mathcal{M}}_{0,n}$ is a binary geometry for $\Delta_{0,n}$.

Other examples of binary geometries are given in [AHHL21a] and [HLRZ20].

Proposition 2.6. Let $\tilde{U}$ be a binary geometry for Δ and $F \in \Delta$. The variety $\tilde{U}_F$ is a binary geometry for the link

$$\mathrm{lk}_\Delta(F) = \{G \in \Delta \mid F \cap G = \emptyset \text{ and } F \cup G \in \Delta\}.$$

In particular, $\dim \mathrm{lk}_\Delta(F) + \dim F = \dim \Delta - 1$.

Proof. The link $\text{lk}_\Delta(F)$ is a simplicial complex on the set $S = \{j \notin F \mid F \cup \{j\} \in \Delta\}$. On $\tilde{U}_F$ we have $u_i = 0$ for $i \in F$ and by (2.1) we have $u_k = 1$ for $k \notin S$. Thus $\tilde{U}_F$ is naturally embedded in the affine space $\mathbb{C}^S \subset \mathbb{C}^{[n]}$ given by $\{u_i = 0 \mid i \in F\} \cap \{u_k = 1 \mid k \notin S \cup F\}$. The equations R_i for $i \in F$ and R_k for $k \notin S$ are automatically satisfied. The equations R_j for $j \in S$ become

$$R_j|_{\mathbb{C}^S} = u_j + \prod_{l \in S \setminus \{j\}} u_l^{a_{jl}} - 1.$$

We have that $(\tilde{U}_F)_G = \tilde{U}_{F \cup G}$. It follows that $\tilde{U}_F$ is a binary geometry for $\text{lk}_\Delta(F)$. $\square$

Corollary 2.7. Suppose that a binary geometry for Δ exists. Then Δ is pure.

Proof. If $F, F' \in \Delta$ are maximal then $\text{lk}_\Delta(F) = \{\emptyset\} = \text{lk}_\Delta(F')$. It follows from Proposition 2.6 that $|F| = |F'|$. $\square$

Remark 2.8. The u -equations (2.1) that we consider are sometimes called *perfect u -equations*. For examples of some more general u -equations see [AHHL21b, Section 11] and [HLRZ20]. More recent appearances of u -equations include [AHCD⁺24, AHFS⁺23].

2.2. One-dimensional binary geometries

Theorem 2.9. Suppose that $\tilde{U}$ is a one-dimensional binary geometry for Δ . Then Δ consists of two isolated vertices and $U \cong \mathbb{C}$.

Proof. By Corollary 2.7, $\Delta = \{\emptyset, 1, 2, \dots, n\}$ must be 0-dimensional, consisting of n isolated vertices. If $n = 1$, we see directly that no binary geometry for Δ exists. If $n = 2$, we have the two relations

$$u_1 + u_2^a = 1, \quad u_2 + u_1^b = 1$$

and we have binary geometry if and only if $a = b = 1$, giving $U \cong \mathbb{C}$.

Now suppose that $n \geq 3$. We will show that no binary geometry exists. The relations are $R_i = u_i + \prod_{j \neq i} u_j^{a_{ij}} - 1 = 0$, where $a_{ij} > 0$. One solution to these equations is the point $p = (u_1, u_2, u_3, \dots, u_n) = (0, 1, 1, \dots, 1)$. We shall show that this is an isolated point of $\tilde{U}$, contradicting the requirement that $\tilde{U}$ is one-dimensional and irreducible.

Consider the Jacobian matrix $(\frac{\partial R_i}{\partial u_j})_{i,j=1}^n$ at the point p . It has the form

$$J = \begin{bmatrix} 1 & a_{12} & a_{13} & a_{14} & \cdots & a_{1n} \\ \beta_2 & 1 & 0 & 0 & \cdots & 0 \\ \beta_3 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & & \end{bmatrix}$$

where $\beta_i = \delta_{a_{i1},1}$ is the Kronecker delta. If the Jacobian is nonsingular at p then p must be an isolated point of $\tilde{U}$. We have $\det J = 1 - a_{12}\beta_2 - a_{13}\beta_3 - \cdots - a_{1n}\beta_n$. Since each a_{ij} is a positive integer and $\beta_i \in \{0, 1\}$, we have $\det J = 0$ exactly when one of the terms $a_{1j}\beta_j$ is equal to 1, and the rest of these terms vanish.

We now rule out the case $\det J = 0$. By relabeling, we may assume that $j = 2$. Thus $a_{12} = a_{21} = 1$ and $a_{i1} \geq 2$ for $i = 3, \dots, n$. We search for a formal power series solution $(u_1(t), \dots, u_n(t)) \in \mathbb{C}[[t]]^n$ to the equations $R_1, \dots, R_n$, where $(u_1(0), \dots, u_n(0)) = p$. (Assuming $\tilde{U}$ is a curve, the parameter t may be taken to be a local analytic parameter on the normalization of $\tilde{U}$, in the neighborhood of a preimage of $p \in U$.) We may assume that not all the linear terms of $u_1(t), \dots, u_n(t)$ vanish, and considering the linear coefficients $[t]R_1, [t]R_2, \dots$, we deduce that

$$\begin{aligned} u_1(t) &= \alpha t + \beta t^2 + \cdots, & u_2(t) &= 1 - \alpha t + \beta' t^2 + \cdots, \\ u_3(t), u_4(t), \dots, u_n(t) &\in 1 + O(t^2), \end{aligned}$$

where $\alpha \neq 0$. (This also follows from the Jacobian calculation.) For $i \geq 3$, we have, writing $[t^e]f(t)$ for the coefficient of t^e in $f(t) \in \mathbb{C}[[t]]$,

$$[t^e]R_i = \begin{cases} [t^e]u_i(t) & \text{if } 0 < e < a_{i1}, \\ [t^e]u_i(t) + \alpha^{a_{i1}} & \text{if } e = a_{i1}. \end{cases}$$

and thus the equation $R_i = 0$ gives

$$u_i(t) = 1 - \alpha^{a_{i1}} t^{a_{i1}} + O(t^{a_{i1}+1}), \quad i = 3, \dots, n.$$

Now let $a = \min_{i \geq 3} (a_{i1}) \geq 2$. Then

$$\begin{aligned} [t^a]R_1 &= [t^a]u_1(t) + [t^a]u_2(t) + \sum_{i \geq 3, a_{i1}=a} a_{1i}(-\alpha^a) = 0 \\ [t^a]R_2 &= [t^a]u_1(t) + [t^a]u_2(t) = 0, \end{aligned}$$

and since $a_{1i} > 0$ for all i , it follows that $\alpha^a = 0$, contradicting our choice that $\alpha \neq 0$. Thus there are no non-trivial power series solutions near p to the equations $R_1 = 0, \dots, R_n = 0$, and it follows that p is an isolated point. $\square$

A pure d -dimensional simplicial complex Δ is a *pseudomanifold* if any $(d-1)$ -dimensional face $F \in \Delta$ is contained in exactly two facets of Δ .

Theorem 2.10. Suppose that $\tilde{U}$ is a binary geometry for Δ . Then Δ is a pseudomanifold.

Proof. Apply Proposition 2.6 to a codimension-one face F of Δ , to deduce that $\tilde{U}_F$ is a one-dimensional binary geometry for $\text{lk}_\Delta(\mathbb{F})$. Then it follows from Theorem 2.9 that $\text{lk}_\Delta(F)$ consists of two vertices, or equivalently, F is contained in two facets of Δ . $\square$

2.3. Two-dimensional binary geometries

Proposition 2.11. Suppose $\tilde{U}$ and $\tilde{U}'$ are binary geometries for flag complexes Δ and Δ' on disjoint vertex sets. Then $\tilde{U} \times \tilde{U}'$ is a binary geometry for $\Delta \times \Delta' = \{F \sqcup F' \mid F \in \Delta, F' \in \Delta'\}$.

In the product binary geometry, any vertex of Δ is compatible with any vertex of Δ' . By Proposition 2.11 and Theorem 2.9, there is a unique two-dimensional binary geometry that is a product of two one-dimensional binary geometries. This is Example 2.4.

Now let Δ be a 1-dimensional flag simplicial complex with vertex set $[n]$ that is a pseudomanifold. Then Δ can be viewed as a graph, and as a graph it is a disjoint union of cycles. If we further assume that this graph is connected, then Δ is the face complex of the n -gon. If $n = 4$, we have Example 2.4. For $n = 5$, we have $\widetilde{\mathcal{M}}_{0,5}$ with u -equations (0.1). We now give examples of binary geometries for the hexagon and the octagon.

Proposition 2.12 ([AHHLT23, AHHL21a]). The following u -equations give a binary geometry for Δ the face complex of the hexagon, with vertices labeled $u_1, v_1, u_2, v_2, u_3, v_3$ in cyclic order:

$$u_i + u_{i+1}v_{i+1}u_{i+2} = 1, \quad v_i + v_{i+1}u_{i+2}^2v_{i+2} = 1,$$

where the indexing of u_i and v_i is taken modulo 3.

Proposition 2.13 ([AHHLT23, AHHL21a]). The following u -equations give a binary geometry for Δ the face complex of the octagon, with vertices labeled $u_1, v_1, u_2, v_2, u_3, v_3, u_4, v_4$ in cyclic order:

$$u_i + u_{i+1}v_{i+1}u_{i+2}^2v_{i+2}u_{i+3} = 1, \quad v_i + v_{i+1}u_{i+2}^3v_{i+2}^2u_{i+3}^3v_{i+3} = 1,$$

where the indexing of u_i and v_i is taken modulo 4.

Conjecture 2.14. Let Δ be the face complex of a n -gon. Then a binary geometry for Δ exists if and only if $n \in \{4, 5, 6, 8\}$.

2.4. Positive binary geometry

Let $\tilde{U}$ be a binary geometry. The *nonnegative part* $U_{\geq 0}$ (resp. *positive part* $U_{> 0}$) is the locus inside $\tilde{U}$ where $u_i \geq 0$ (resp. $u_i > 0$) for all $i \in [n]$. It follows from (2.1) that $u_i(x) \in [0, 1]$ for $x \in U_{\geq 0}$. Thus

$$(u_1, \dots, u_n) : U_{\geq 0} \hookrightarrow [0, 1]^n$$

embeds $U_{\geq 0}$ into a hypercube. We define $U_{F, \geq 0} := U_F \cap U_{\geq 0}$ and $U_{F, > 0} := U_F \cap U_{> 0}$.

We call the binary geometry $\tilde{U}$ *polytopal* if $U_{F,>0}$ is an open ball, $U_{F,\geq 0}$ is a closed ball, and $U_{F,\geq 0} = \bigcup_{G \supseteq F} U_{G,>0}$. In this case $U_{\geq 0}$, together with the stratification $U_{F,\geq 0}$ is a regular CW complex. Thus Δ is the dual of the face poset of this regular CW complex.

Problem 2.15. Find natural conditions for a binary geometry to be polytopal.

Viewing $\tilde{U} \subset \mathbb{C}^n$ as contained in the closure $\bar{U} \subset \mathbb{P}^n$, a projective variety, we may apply the Definition 1.20 of a positive geometry.

Definition 2.16. Suppose that $\tilde{U}$ is d -dimensional and thus $\dim \Delta = d - 1$. Let Ω_U be a rational form d -form on $\tilde{U}$. We say that Ω_U is a canonical form for $\tilde{U}$ if either

1. $d = 0$ and $U = \text{pt}$ and $\Omega_U = \pm 1$, or
2. $d > 0$, and Ω_U has only poles along $U_{\{i\}}$, each of these poles is simple, and for all $i \in [n]$ we have $\text{Res}_{u_i=0} \Omega_U = \Omega_{U_{\{i\}}}$, where $\Omega_{U_{\{i\}}}$ is a canonical form for $U_{\{i\}}$ (see Proposition 2.6). Furthermore, we require that Ω_U has no poles at the hyperplane at infinity $\mathbb{P}^n \setminus \mathbb{C}^n$.

In this case we call $\tilde{U}$ a positive binary geometry.

One way to construct a potential canonical form Ω_U is by positively parametrizing U . Namely, let $y_1, \dots, y_d$ be regular functions on U which restrict to an isomorphism:

$$(y_1, \dots, y_d) : U_{>0} \rightarrow \mathbb{R}_{>0}^d.$$

Then $\Omega_U := \bigwedge \frac{dy_i}{y_i}$ is a candidate canonical form. Indeed, in many cases we have the following desirable situation: the functions $y_1, \dots, y_d$ generate the rational function field $\mathbb{C}(U)$, and furthermore, each function u_i is a rational function

$$u_i = \frac{a_i(y)}{b_i(y)}$$

where $a_i(y)$ and $b_i(y)$ are polynomials with positive coefficients. This is the case for the positive parametrization of Section 1.5, and we will revisit this situation in Section 5.

Remark 2.17. According to the “pushforward heuristic” of [AHBL17, Heuristic 4.1], we more generally expect to be able to construct a canonical form if we are given a dominant rational map $(\mathbb{C}^\times)^d \dashrightarrow U$, which restricts to a diffeomorphism $\mathbb{R}_{>0}^d \rightarrow U_{>0}$, even if it is not necessarily a birational map.

2.5. Exercises and Problems

Exercise 2.18.

- (a) Prove Proposition 2.12 and Proposition 2.13.
- (b) The binary geometries of Proposition 2.12 and Proposition 2.13 are positive binary geometries. Find canonical forms for them.

Problem 2.19.

- (a) Show that there are no monomial relations among the generators u_i of a binary geometry. That is, we have $\prod_i u_i^{a_i} = 1$ if and only if $a_i = 0$ for all i .
- (b) Is it true that the intrinsic torus of U (Section 6.1) has character lattice equal to the lattice $\Lambda = \{\prod_i u_i^{a_i} \mid a_i \in \mathbb{Z}\}$ of Laurent monomials in the u_i ?

We remark that binary geometries are defined over the integers since the equations (2.1) are integral.

Problem 2.20. Find the Euler characteristic of U for binary geometries $\tilde{U}$. When does U have polynomial point count? See Theorem 3.10 and Theorem 6.3 for context.

Problem 2.21. Study the u -equations

$$\begin{aligned} u_1 + u_6 u_8 &= 1, & u_4 + u_3 u_5 &= 1, & u_8 + u_1 u_2 &= 1, & u_3 + u_4 u_7 &= 1 \\ u_5 + u_2 u_4 u_6 &= 1, & u_2 + u_5 u_7 u_8 &= 1, & u_6 + u_1 u_5 u_7 &= 1, & u_7 + u_2 u_3 u_6 &= 1 \end{aligned}$$

for the simplicial complex Δ with maximal facets

$$\{123, 124, 135, 147, 157, 236, 246, 358, 368, 468, 478, 578\}.$$

Can you prove that this is a binary geometry? What can you say about $\tilde{U}$ and U ? How is this space related to $\mathcal{M}_{0,6}$? This problem is revisited in Section 5.6.2.

3. Scattering equations

3.1. Scattering amplitudes

In particle physics, we are interested in predicting the outcomes of elementary particle scattering experiments. Let $p_1, p_2, \dots, p_n \in \mathbb{R}^D$ be the space-time momenta of n -particles in D -dimensional space-time. The scattering amplitude is a function $A_n(p_1, p_2, \dots, p_n)$ of the momentum vectors, and for the amplitude

to be non-vanishing, the momenta $p_1, \dots, p_n$ must satisfy momentum conservation:

$$p_1 + \dots + p_n = 0. \quad (3.1)$$

The function $|A_n(p_1, p_2, \dots, p_n)|^2$ is the probability density of measuring an event with the specified momenta.

To determine the amplitude, we choose a quantum field theory (QFT), which is a choice of particles and a choice of possible interactions between these particles. The scattering amplitude $A_n(p_1, p_2, \dots, p_n)$ is then determined by specifying the type of each of the particles $1, 2, \dots, n$, together with information about the internal symmetries (“polarization vectors”, “quantum numbers”) of each particle. Roughly speaking, the type of a particle is a representation of some Lie group, and the additional internal information is a vector in this representation.

In perturbative QFT, the scattering amplitude has an expansion

$$A_n = \sum_{L=0}^{\infty} A_n^{(L)}, \quad A_n^{(L)} = \sum_{L\text{-loop Feynman diagrams } \Gamma} \int_{\mathbb{R}^{L \times D}} \omega_{\Gamma} \quad (3.2)$$

as a sum over the loop order L , where $A_n^{(L)}$ is a sum over graphs with n leaves labeled $1, 2, \dots, n$, called L -loop Feynman diagrams. An L -loop Feynman diagram is a graph with first Betti number equal to L . The form ω_{Γ} is a rational differential form that depends on the momenta $p_1, \dots, p_n$.

In these lectures we will focus exclusively on the tree amplitude $A_n^{(0)}$, and often omit the exponent (0) in our notation. The function $A_n^{(0)}$ is a rational function obtained as the sum over Feynman diagrams that are trees.

3.2. Kinematic space

Space-time $\mathbb{R}^D$ is equipped with a Lorentzian metric, a nondegenerate symmetric bilinear form with signature $(1, D-1)$. We assume that physics, and therefore the scattering amplitude, is invariant under the orthogonal group $O_{1,D-1}(\mathbb{R})$ preserving the Lorentz metric. For example, this group includes usual rotations of space, and other linear transformations of space-time studied by Lorentz.

We will mathematically simplify the situation by complexifying the set up, simultaneously obscuring the physics. Namely, we will consider space-time momenta as vectors $p_1, \dots, p_n$ in a complex vector space $\mathbb{C}^D$ equipped with a symmetric bilinear form, and the amplitude $A_n(p_1, \dots, p_n)$ is then a complex-valued function invariant under the (complex) orthogonal group $O(D)$, the complexification of $O_{1,D-1}(\mathbb{R})$. Let P denote the $n \times D$ matrix with row vectors equal to $p_1, \dots, p_n$. By the first fundamental theorem of invariant theory, the ring of

invariant (polynomial) functions $\mathbb{C}[p_1, \dots, p_n]^{O(D)}$ is generated by the Mandelstam variables

$$s_{ij} = (p_i + p_j)^2 = 2p_i \cdot p_j = s_{ji}$$

where $u \cdot v$ denotes the symmetric bilinear form on $\mathbb{C}^D$, and we use the standard physics convention that $u^2 := u \cdot u$. The Mandelstam variables are also (up to a multiple of 2) the matrix entries of the $n \times n$ symmetric matrix $M := PP^T$. The $n \times n$ matrix M has rank at most D . The second fundamental theorem of invariant theory states that the ideal of relations between the s_{ij} is generated by the $(D+1) \times (D+1)$ minors of M . We refer the reader to [Pro07] for more on the fundamental theorems of invariant theory.

The tree amplitude $A_n^{(0)} = A_n^{\text{tree}}$ is thus a rational function of the s_{ij} . Taking the dot product of (3.1) with p_i , we obtain the relations

$$\sum_{j \neq i} s_{ij} = 0, \quad \text{for each } i = 1, 2, \dots, n. \quad (3.3)$$

In fact, (3.3) is equivalent to (3.1) in the following sense: any symmetric $n \times n$ matrix M with rank $\leq D$ and row (and thus column) sums equal to 0 is of the form $M = PP^T$ where the rows of P satisfy (3.1). To see this, note that while M may not be diagonalizable, it has an Autonne-Yakagi factorization $M = UDU^T$, where U is unitary and D is diagonal. Since $M[1, 1, \dots, 1]^T = 0$, we have $DU^T[1, 1, \dots, 1]^T = 0$. We have $D = QQ^T$ for a $n \times D$ matrix Q , and thus $M = PP^T$ for $P = UQ$. It is easy to see that Q can be chosen so that $P^T[1, 1, \dots, 1]^T = Q^T U^T[1, 1, \dots, 1]^T = 0$, that is, P satisfies momentum conservation.

We will furthermore restrict ourselves to scattering of massless particles, that is, the momentum vectors satisfy $p_i^2 = 0$, or equivalently, $s_{ii} = 0$.

Definition 3.1. The (massless) kinematic space K_n is the $\binom{n}{2} - n$ dimensional complex vector space of symmetric $n \times n$ matrices with diagonal entries equal to 0, and row and column sums equal to 0. Equivalently, the dual K_n^* of kinematic space is the vector space spanned by matrix entry functionals $s_{ij} = s_{ji}$ satisfying $s_{ii} = 0$ and the n equations (3.3).

We will often view a point in K_n as a collection $\underline{s} = (s_{ij})$ of complex numbers satisfying $s_{ii} = 0$ and (3.3).

Note that in our definition of kinematic space we have not fixed a dimension D of space-time. We will view scattering amplitudes in general space-time dimension as complex analytic functions on K_n . Tree amplitudes A_n^{tree} are then rational functions on K_n . In Section 4, we return to discuss the special kinematics when we work in $(D = 4)$ -dimensional space-time of our real world.

3.3. Scattering potential on $\mathcal{M}_{0,n}$

Define the *planar kinematic functions* $X_{ij} \in K_n^*$ by the formula

$$X_{ij} = \sum_{i \leq a < b \leq j-1} s_{ab}, \quad (3.4)$$

where $[i, j-1] = \{i, i+1, \dots, j-1\}$ is a cyclic interval and the index of summation is considered cyclically. For example, $X_{25} = s_{23} + s_{24} + s_{34}$. We have that X_{ij} vanishes when (ij) is not a diagonal of the n -gon. The s_{ij} can be expressed in terms of X_{ij} via the equation

$$s_{ij} = X_{i,j+1} + X_{i+1,j} - X_{i,j} - X_{i+1,j+1}. \quad (3.5)$$

Since K_n^* has dimension $\binom{n}{2} - n$, we have the following.

Proposition 3.2. The planar kinematic functions $\{X_{ij} \mid (ij) \in \text{diag}_n\}$ form a basis of K_n^* .

Furthermore, we may define dual lattices $K_n(\mathbb{Z}) \subset K_n$ and $K_n^*(\mathbb{Z}) \subset K_n^*$ as follows:

$$\begin{aligned} K_n^*(\mathbb{Z}) &:= \text{span}_{\mathbb{Z}}\{s_{ij}\} = \text{span}_{\mathbb{Z}}\{X_{ij}\} \\ K_n(\mathbb{Z}) &:= \{(\underline{s}) \in K_n \mid f(\underline{s}) \in \mathbb{Z} \text{ for all } f \in K_n^*(\mathbb{Z})\} = \{(\underline{s}) \in K_n \mid s_{ij} \in \mathbb{Z}\}. \end{aligned}$$

The relation between X_{ij} and s_{ij} can be viewed in terms of the cross-ratios u_{ij} and minors $(\sigma_i - \sigma_j)$ on $\mathcal{M}_{0,n}$. Namely, for any point in kinematic space $K_n(\mathbb{Z})$, we have a regular function on $\mathcal{M}_{0,n}$:

$$\phi_X := \prod_{(ij)} u_{ij}^{X_{ij}} = \prod_{i < j} (\sigma_i - \sigma_j)^{s_{ij}}, \quad (3.6)$$

the equality following from (3.5) and (1.3). Thus, we have:

Proposition 3.3. The integral kinematic space $K_n(\mathbb{Z})$ is naturally isomorphic to the character lattice $\Lambda(\mathcal{M}_{0,n})$ of (1.6). Integral dual kinematic space $K_n^*(\mathbb{Z})$ is naturally isomorphic to the cocharacter lattice $\Lambda^\vee(\mathcal{M}_{0,n})$.

We also have natural vector space isomorphisms $K_n \cong \Lambda \otimes_{\mathbb{Z}} \mathbb{C}$ and $K_n^* \cong \Lambda^\vee \otimes_{\mathbb{Z}} \mathbb{C}$.

Corollary 3.4. Any cross-ratio, or more generally any Laurent monomial in cross-ratios, can be uniquely written as a Laurent monomial in the dihedral coordinates u_{ij} , times a sign.

Proof. Any Laurent monomial in cross-ratios is of the form $\pm \prod_{i < j} (\sigma_i - \sigma_j)^{s_{ij}}$ for integers s_{ij} that define a point in $K_n(\mathbb{Z})$. The sign $\pm$ arises since $(\sigma_i - \sigma_j) = -(\sigma_j - \sigma_i)$. $\square$

Definition 3.5. For arbitrary $X_{ij} \in \mathbb{C}$, or $X \in K_n$, the *Koba-Nielsen potential*, or *scattering potential* is the (multi-valued) function (3.6).

This function is called a likelihood function [ST21], or master function [Var95] in other contexts. While ϕ_X is multi-valued, its (log-)critical point set in $\mathcal{M}_{0,n}$, the solutions to the equation

$$\mathrm{dlog}\phi_X = \sum_{(ij) \in \mathrm{diag}_n} X_{ij} \mathrm{dlog} u_{ij} = \sum_{(ij) \in \mathrm{diag}_n} \frac{X_{ij}}{u_{ij}} du_{ij} = 0,$$

is a well-defined algebraic set.

3.4. Cachazo-He-Yuan scattering equations

The traditional approach to (perturbative) scattering amplitudes is via the Feynman diagrams of (3.2). Around ten years ago, Cachazo, He, and Yuan [CHY14b] introduced a new approach to the computation of tree-level scattering amplitudes by solving *scattering equations*. This approach is inspired by Witten's twistor string theory [Wit04] which leads to explicit formulae for tree-level Yang-Mills amplitudes [RSV04].

The scattering equations are the following n equations

$$Q_i := \sum_{j \neq i} \frac{s_{ij}}{\sigma_i - \sigma_j} = 0, \quad \text{for } i = 1, 2, \dots, n. \quad (3.7)$$

Here, s_{ij} denote the Mandelstam coordinates of a point in kinematic space K_n , and $\sigma_1, \dots, \sigma_n$ are n distinct points on $\mathbb{P}^1$. By convention, if $\sigma_i = \infty$ then the term $\frac{s_{ij}}{\sigma_i - \sigma_j}$ is declared to be 0.

Proposition 3.6. The n scattering equations (3.7) are equivalent to the critical point equation $\mathrm{dlog}\phi_X = 0$.

Proof. Follows from (3.6) and that Q_i is up to sign equal to $\frac{\partial}{\partial \sigma_i} \log \phi_X$. $\square$

Proposition 3.7. The scattering equations are invariant under the action of $\mathrm{PGL}(2)$ on $\{\sigma_1, \dots, \sigma_n\}$.

Proof. Exercise 3.38. $\square$

Thus we may view the σ_i in (3.7) as defining a point $\underline{\sigma} \in \mathcal{M}_{0,n}$. We view the scattering equations in the following three ways:

1. If a point $\underline{s} = (s_{ij}) \in K_n$ has been fixed, we may solve these equations to find solutions $\underline{\sigma} \in \mathcal{M}_{0,n}$. This is the original perspective of [CHY14b].
2. If a point $\underline{\sigma} \in \mathcal{M}_{0,n}$ has been fixed, we may solve these equations to find solutions $\underline{s} \in K_n$. There are usually infinitely many such solutions, but restricting to $(n-3)$ -dimensional subspaces of K_n we will find that there is generically a unique solution, defining a *scattering (rational) map* from $\mathcal{M}_{0,n}$ to a subspace of K_n . This is the perspective of [AHBHY18, AHHL21b].
3. The scattering equations define a scattering correspondence variety

$$\mathcal{I} = \{(\underline{s}, \underline{\sigma}) \mid (3.7) \text{ holds} \} \subset K_n \times \mathcal{M}_{0,n}. \quad (3.8)$$

This will be useful in Section 4. Note that the equations (3.7) are homogeneous in the variables s_{ij} , so the incidence variety is also naturally defined in $\mathbb{P}(K_n) \times \mathcal{M}_{0,n}$. This correspondence has appeared in likelihood geometry [HS14] but I have not seen it explicitly in the amplitudes literature.

If a point $\underline{s} \in K_n$ has been fixed, then CHY postulate that the (tree-)amplitude rational function $A_n \in \mathbb{C}(K_n)$ is obtained as a sum of some function over the solutions of the scattering equations. The choice of function is determined by the choice of QFT. While the CHY scattering amplitudes can be considered for a variety of theories, we focus on scalar particles in this section. Before we define the CHY amplitude, we first discuss the solutions to the scattering equations.

3.5. Solving the scattering equations

Let us suppose that $\underline{s} \in K_n$ has been fixed, or equivalently, $p_1, \dots, p_n$ have been fixed. We consider the solutions to (3.7) in $\mathcal{M}_{0,n}$.

Example 3.8. Suppose $n = 4$. Then $\dim K_n = 4 - 2 = 2$, and using (3.3), we have

$$s_{12} = s, \quad s_{13} = -s - t, \quad s_{14} = t, \quad s_{23} = t, \quad s_{24} = -s - t, \quad s_{34} = s.$$

where $s = X_{13}$ and $t = X_{24}$. We use the $\text{PGL}(2)$ action to fix $\sigma_1 = 0, \sigma_2 = 1, \sigma_4 = \infty$. The scattering equation for $i = 4$ is automatically satisfied, and the other equations become

$$\frac{s}{-1} + \frac{-s-t}{-\sigma} = 0, \quad \frac{s}{1} + \frac{t}{1-\sigma} = 0, \quad \frac{-s-t}{\sigma} + \frac{t}{\sigma-1} = 0,$$

where $\sigma = \sigma_3$. These three equations are equivalent, with solution $\sigma = \frac{s+t}{s}$.

Theorem 3.9. For generic $\underline{s} \in K_n$, the scattering equations have $(n-3)!$ solutions.

Theorem 3.9 follows from a theorem of Orlik and Terao [OT95] (in the setting of hyperplane arrangement complements), or its generalization by Huh (Theorem 6.3) to very affine varieties. These results state that the number of solutions to the scattering equations is (generically) equal to the absolute value of the Euler characteristic of $\mathcal{M}_{0,n}$ ¹. The Euler characteristic of $\mathcal{M}_{0,n}$ is easy to compute by counting points over $\mathbb{F}_q$. After gauge-fixing $\sigma_1 = 0, \sigma_2 = 1, \sigma_n = \infty$, we find that there are $(q-2)$ choices for σ_3 , and then $(q-3)$ choices for σ_4 , and so on. This gives

$$\#\mathcal{M}_{0,n}(\mathbb{F}_q) = (q-2)(q-3)\cdots(q-n+2).$$

According to the next result, a consequence of the Weil conjectures, we have $\chi(\mathcal{M}_{0,n}(\mathbb{C})) = (-1)^{n-3}(n-3)!$.

Theorem 3.10. Suppose that we have an algebraic variety X defined over $\mathbb{Z}$. That is, X is cut out by polynomial equations with integer coefficients. Further suppose that $X_{\mathbb{F}_q}$ is smooth and has polynomial point count $f(q)$ for all prime powers $q = p^n$ of some prime p . Then the Euler characteristic of the complex algebraic variety $X(\mathbb{C})$ is given by $\chi(X(\mathbb{C})) = f(1)$.

Let us also sketch the original argument of [CHY14a] for Theorem 3.9, which is based on *soft-limits*; see also [ABFK⁺23]. We proceed by induction on n . We gauge-fix $\sigma_1 = \infty, \sigma_2 = 0, \sigma_3 = 1$, and set $s_{nb}(\varepsilon) = \varepsilon s_{nb}$. Substituting this into (3.7), we obtain a collection of ε -dependent equations. Setting $\varepsilon = 0$, the last equation disappears, and we are left with $n-1$ equations that we can view as a set of CHY scattering equations for $n-1$ particles. (Note that we have to adjust s_{1j} to assure that momentum conservation is satisfied.). We have $(n-1)$ equations, not depending on σ_n , and by induction there are $(n-4)!$ solutions to this system.

Now, let $|\varepsilon|$ be nonzero but small. Then the last equation is of the form

$$\varepsilon \sum_{j=2}^{n-1} \frac{s_{nj}}{\sigma_n - \sigma_j} = 0.$$

If $\sigma_1, \dots, \sigma_{n-1}$ have been determined, then clearing denominators, this is a polynomial equation in σ_n of degree $n-3$, and has $n-3$ solutions. We now argue that in a small neighborhood of each of the $(n-4)!$ solutions $\underline{\sigma}' = (\sigma_1, \dots, \sigma_{n-1})$ there are $(n-3)$ solutions for $\underline{\sigma} = (\sigma_1, \dots, \sigma_n)$ that satisfy all the scattering equations.

¹For a discussion of scattering equations when $\underline{s}$ become non-generic, see [KKT24].

3.6. ϕ^3 -amplitude from scattering equations

Define the $n \times n$ matrix Φ by

$$\Phi_{ab} := \partial_{\sigma_a} Q_b,$$

with entries that are functions on $K_n \times \mathcal{M}_{0,n}$. Since $Q_b = \partial_{\sigma_b} \log \phi_X$, the matrix Φ is a Hessian matrix. It follows from Exercise 3.41 that the matrix Φ has rank at most $(n-3)$. Let Φ_{pqr}^{abc} be the $(n-3) \times (n-3)$ matrix obtained by deleting rows a, b, c and columns p, q, r from Φ .

Lemma 3.11. The *reduced determinant*

$$\det' \Phi := \frac{\det \Phi_{pqr}^{abc}}{\sigma_{ab} \sigma_{bc} \sigma_{ca} \sigma_{pq} \sigma_{qr} \sigma_{rp}}$$

does not depend on the choice of a, b, c, p, q, r .

Proof. Exercise 3.41. □

Definition 3.12. The CHY scalar amplitude is the function on kinematic space K_n given by

$$A_n := \sum_{\text{sols}} \frac{1}{\det' \Phi} \frac{1}{(\sigma_{12} \sigma_{23} \cdots \sigma_{n1})^2} \quad (3.9)$$

summed over the solutions to the scattering equations.

Example 3.13. Choose $\{a, b, c\} = \{1, 2, 4\} = \{p, q, r\}$ and use $\text{PGL}(2)$ to place $(\sigma_1, \sigma_2, \sigma_4)$ at $(0, 1, \infty)$. Then Φ_{124}^{124} is a 1×1 matrix with entry

$$\frac{s+t}{\sigma^2} - \frac{t}{(\sigma-1)^2}.$$

There are as many factors involving $\sigma_4 = \infty$ in the numerator as in the denominator of (3.9). Substituting $\sigma = \frac{s+t}{s}$ from Example 3.8, we obtain

$$A_4 = \frac{1}{\frac{s^2}{s+t} - \frac{s^2}{t}} \times \frac{1}{\frac{t}{s}} = -\frac{s+t}{st} = -\left(\frac{1}{s} + \frac{1}{t}\right) = -\left(\frac{1}{X_{13}} + \frac{1}{X_{24}}\right).$$

Note that Example 3.13 is deceptive: for $n > 4$, the solutions to the scattering equations are not rational functions in X_{ij} . For $n = 5$, it requires solving a quadratic equation.

The scalar amplitude defined by Cachazo-He-Yuan scattering equations recovers the scalar ϕ^3 -amplitude defined as a sum over Feynman diagrams.

Theorem 3.14 ([DG14]). We have, up to a sign,

$$A_n = A_n^{\phi^3} = \sum_T \prod_{e \in E(T)} \frac{1}{X_{ij}} \quad (3.10)$$

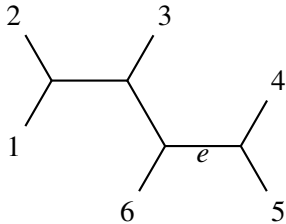
where the summation is over planar trees T (Exercise 1.26) with n leaves, interior vertices of degree three, and the product is over interior edges of e which separate leaves $i, i+1, \dots, j-1$ from $j, j+1, \dots, i-1$.

The proof of Theorem 3.14 in [DG14] is by induction, verifying that both sides of satisfy a recursion.

The cubic planar trees appearing in Theorem 3.14 are the “Feynman diagrams” (3.2) of a QFT called scalar ϕ^3 -theory, and $A_n = A_n^{(0)}$ is called the *planar ϕ^3 -amplitude* (at tree level). The choice of ϕ^3 -theory corresponds to the choice of degree 3 interior vertices. If we imagine inflowing momenta p_i at leaf i , and assume that momentum is conserved at every interior vertex, then

$$\text{momentum along edge } e = \pm(p_i + p_{i+1} + \dots + p_{j-1}) = \mp(p_j + p_{j+1} + \dots + p_{i-1}),$$

and X_{ij} is the dot product of this momentum vector with itself. The appearance of squared momenta, called “propagators”, in the denominator is a general feature of the rational functions associated to Feynman diagrams.



The momentum traveling along the edge e is equal to $\pm(p_4 + p_5) = \mp(p_1 + p_2 + p_3 + p_6)$, with square given by $X_e = X_{46} = s_{45}$.

Remark 3.15. More generally, for a permutation $\alpha \in S_n$, we define a Parke-Taylor factor

$$\text{PT}(\alpha) := \frac{1}{\sigma_{\alpha(1)\alpha(2)} \cdots \sigma_{\alpha(n)\alpha(1)}}. \quad (3.11)$$

If we replace the factor $\frac{1}{(\sigma_{12}\sigma_{23}\cdots\sigma_{n1})^2}$ in (3.9) by $\text{PT}(\alpha)\text{PT}(\beta)$ we obtain *partial amplitudes*, which are sums over another (usually smaller) collection of trees; see Exercise 3.44.

Remark 3.16. Let Y be a smooth complex algebraic variety and $f : Y \rightarrow \mathbb{C}$ be a function on Y . Suppose that a Lie group G acts freely on Y and f is a G -invariant

function. That is, for $g \in G$, we have the equality $f = f \circ g^*$, where $g^* : Y \rightarrow Y$ is the automorphism induced by the action of g . Taking differentials, we get

$$df = df \circ dg^*. \quad (3.12)$$

For a vector $v \in \mathfrak{g}$ in the Lie algebra $\mathfrak{g} = \text{Lie } G$, we have a corresponding vector field X_v on Y . Then (3.12) says that the 1-form df is annihilated by the vector field X_v . Indeed, df is annihilated by the $d = \dim G$ dimensional space of vector fields

$$\{X_v \mid v \in \text{Lie } G\}. \quad (3.13)$$

The quotient map $Y \rightarrow Y/G$ induces a map of tangent bundles $TY \rightarrow T(Y/G)$, with kernel equal to the subbundle with sections given by (3.13). The 1-form df naturally descends to a section of the cotangent bundle $T^*(Y/G)$.

Taking $Y = (\mathbb{P}^1)^n \setminus \{\text{diagonals}\}$, and $f = \phi_X$, and $G = \text{PGL}(2)$, this explains the three-dimensional kernel of Φ . The reduced Jacobian $\det' \Phi$ of Lemma 3.11 is a Jacobian associated to f , viewed as a function on $\mathcal{M}_{0,n} = Y/G$.

3.7. Delta functions

As we have explained in Proposition 3.3, the scattering potential, and thus the scattering correspondence (3.8) is intrinsically associated to $\mathcal{M}_{0,n}$. On the other hand, the reduced determinant of Lemma 3.11 is coordinate dependent: it depends on choosing some of the σ_i as coordinates. We develop formalism to reformulate the definition (3.9) in a more conceptual way.

We first introduce the formalism of “holomorphic delta functions” that are common in the physics literature. Recall that the delta function $\delta(x)$ is a distribution on the real line satisfying

$$\int_{-\infty}^{\infty} h(x) \delta(x) dx = h(0).$$

We will use this formula to motivate the definition of the *holomorphic delta function*, as follows. Let X be an n -dimensional complex algebraic variety, and suppose that $f_1, f_2, \dots, f_n$ are meromorphic functions defined on X . (If any of the f_i have singularities on X , by convention these singularities are removed from X .)

Definition 3.17. Let ω be a holomorphic n -form on X . Assume that the equations $f_1 = f_2 = \dots = f_n = 0$ have non-degenerate isolated solutions $\{p_1, \dots, p_r\}$ on X . The integral $\int \delta^n(f_1, f_2, \dots, f_n) \omega$ is defined as:

$$\int \delta^n(f_1, f_2, \dots, f_n) \omega = \sum_{p \in \{p_1, \dots, p_r\}} h(p) \det \left(\frac{\partial f_j}{\partial g_i} \right)^{-1} (p)$$

where for each solution p , we let $g_1, \dots, g_n$ be local coordinates such that in the neighborhood of p , we have

$$\omega(x) = h(x) dg_1 \wedge dg_2 \wedge \dots \wedge dg_n.$$

If $f'_1 = f'_2 = \dots = f'_n = 0$ defines the same solution set as $f_1 = f_2 = \dots = f_n = 0$, then

$$\int \delta^n(f'_1, \dots, f'_n) \omega = \int \delta^n(f_1, f_2, \dots, f_n) \det\left(\frac{\partial f'_i}{\partial f_j}\right)^{-1} \omega \quad (3.14)$$

So we have the formula $\delta^n(f'_1, \dots, f'_n) = \delta^n(f_1, f_2, \dots, f_n) \det\left(\frac{\partial f'_i}{\partial f_j}\right)^{-1}$. In particular, $\delta^2(f, g) = -\delta^2(g, f)$ and $\delta(\alpha f) = \delta(f)/\alpha$ for a scalar α . We also use the notation $\delta^n(f_1, f_2, \dots, f_n) = \delta(f_1)\delta(f_2)\dots\delta(f_n)$.

Now, define

$$\delta^{\text{CHY}} := \frac{\sigma_{ab}\sigma_{bc}\sigma_{ca}}{\sigma_{12}\sigma_{23}\dots\sigma_{n1}} \prod_{i \notin \{a,b,c\}} \delta(Q_i), \quad (3.15)$$

which (similar to Lemma 3.11) does not depend on the choice of $\{a, b, c\} \subset [n]$. The following reformulation of the CHY amplitude is essentially the one given in [BBBB⁺15].

Proposition 3.18. The definition (3.9) can also be written as

$$A_n = \int \delta^{\text{CHY}} \Omega_{0,n}$$

where the three points $\sigma_a, \sigma_b, \sigma_c$ have been gauge-fixed, for example to $\{0, 1, \infty\}$.

3.8. General formalism for scalar amplitudes

We suggest a formalism that produces scalar amplitudes just from the potential ϕ_X , which is canonically associated to the variety $\mathcal{M}_{0,n}$. There is some similarity with the approach of Mizera [Miz18] who worked in twisted (co)homology.

Let X be an n -dimensional complex algebraic variety. Let ϕ be a multi-valued function, which we call the potential function, and ω be a rational n -form on X . Let $g_1, \dots, g_n$ be local coordinates such that we have

$$\omega(x) = h(x) dg_1 \wedge dg_2 \wedge \dots \wedge dg_n.$$

We define the degree n delta function $\delta^n(\phi; \omega)$ by

$$\delta^n(\phi; \omega) := h(x) \prod_{i=1}^n \delta\left(\frac{\partial \log \phi}{\partial g_i}\right). \quad (3.16)$$

If $g'_1, \dots, g'_n$ is another set of local coordinates, then $\omega = h'(x)dg'_1 \wedge dg'_2 \wedge \dots \wedge dg'_n$ where

$$h'(x) = h(x) \det \left(\frac{\partial g_i}{\partial g'_j} \right).$$

It thus follows from (3.14) that $\delta^n(\phi; \omega)$ is well-defined.

Lemma 3.19. Let ϕ be a potential with isolated non-degenerate critical points, and let

$$\omega = h(x)dg_1 \wedge dg_2 \wedge \dots \wedge dg_n, \quad \eta = r(x)dq_1 \wedge dq_2 \wedge \dots \wedge dq_n$$

be two n -forms. We have

$$\int \delta^n(\phi; \omega) \eta = \sum_p h(p)r(p) \det \left(\frac{\partial^2 \log \phi}{\partial g_i \partial q_j} \right)^{-1} = \int \delta^n(\phi; \eta) \omega.$$

where the summation is over solutions p to $d \log \phi = 0$.

Proposition 3.20. We have $\delta^{n-3}(\phi_X; \Omega_{0,n}) = \delta^{\text{CHY}}$. Thus the CHY scalar amplitude is given by

$$A_n = \int \delta^{n-3}(\phi_X; \Omega_{0,n}) \Omega_{0,n}.$$

Proof. Fixing $\{\sigma_1, \sigma_2, \sigma_n\} = \{0, 1, \infty\}$ the potential $\phi = \phi_X$ and form $\Omega_{0,n}$ become

$$\phi = \prod_{j=3}^{n-1} (-\sigma_j)^{s_{1j}} (1 - \sigma_j)^{s_{2j}} \prod_{3 \leq i < j \leq n-1} \sigma_{ij}^{s_{ij}}, \quad \Omega = \frac{1}{\sigma_{23} \sigma_{34} \dots \sigma_{n-2, n-1}} d\sigma_3 \wedge \dots \wedge d\sigma_{n-1}.$$

For $i = 3, 4, \dots, n-1$, we have that $\partial_{\sigma_i} \log \phi = Q_i$, where the latter has been specialized to $\{\sigma_1, \sigma_2, \sigma_n\} = \{0, 1, \infty\}$. It follows that

$$\delta^{n-3}(\phi; \Omega) = \frac{1}{\sigma_{23} \sigma_{34} \dots \sigma_{n-2, n-1}} \delta(Q_3) \delta(Q_4) \dots \delta(Q_{n-1}).$$

This equals δ^{CHY} , after noting that in (3.15), two factors in both the numerator and denominator involve $\sigma_n = \infty$, and cancel out. $\square$

Example 3.21. Let $n = 4$ and write $\Omega_{0,4} = \frac{du}{u(1-u)}$ where $u = u_{13}$. We have

$$\phi = u^s (1-u)^t, \quad \frac{\partial \log \phi}{\partial u} = \frac{s}{u} - \frac{t}{1-u}, \quad \delta^1(\phi; \Omega) = \delta\left(\frac{s}{u} - \frac{t}{1-u}\right) \frac{1}{u(1-u)},$$

where $s = X_{13}$ and $t = X_{24}$. We calculate (noting that $\delta(af) = \delta(f)/a$)

$$\begin{aligned} \int \delta^1(\phi; \Omega) \Omega &= \int \delta\left(\frac{s}{u} - \frac{t}{1-u}\right) \frac{du}{u^2(1-u)^2} \\ &= -\frac{1}{s+t} \int \delta\left(u - \frac{s}{s+t}\right) \frac{du}{u(1-u)} \\ &= -\frac{1}{s+t} \frac{1}{(s/(s+t))(t/(s+t))} = -\left(\frac{1}{s} + \frac{1}{t}\right) = -A_4^{\phi^3}(s, t). \end{aligned}$$

The above formalism suggests the following very general definition.

Definition 3.22. Let U be an n -dimensional very affine variety with character lattice Λ , and let ϕ be the intrinsic potential on U . For two top-forms Ω, Ω' on U , we define the (partial) amplitude on $\Lambda_{\mathbb{R}}$,

$$A(\Omega|\Omega') := \int \delta^n(\phi; \Omega) \Omega' = \int \delta^n(\phi; \Omega') \Omega =: A(\Omega'|\Omega).$$

We expect to obtain top-forms Ω, Ω' from the general theory of positive geometries. See Section 6.1 for definitions of Λ and ϕ , and for further discussion.

3.9. From scattering correspondence to scattering form

Suppose we have a dominant rational map $f = (f_1, f_2, \dots, f_n) : X \dashrightarrow Y$ between complex algebraic varieties of the same dimension n , and suppose ω is a rational r -form on X . Then we define the *pushforward* $f_*\omega$ (also called the *trace*) as follows. Suppose that the map f has (generically) degree d . Let $W \subset Y$ be an (analytic) open set such that $f^{-1}(W) = V_1 \sqcup V_2 \sqcup \dots \sqcup V_d$ is a disjoint union of open sets $V_1, \dots, V_d \subset X$, where $f|_{V_i} : V_i \rightarrow W$ is biholomorphic. We then define

$$f_*\omega := \sum_{i=1}^d ((f|_{V_i})^{-1})^* \omega|_{V_i},$$

where the form on the right hand side, defined as a rational form on W , is extended to a rational form on Y by analytic continuation.

Let $y_1, y_2, \dots, y_n$ be local coordinates on Y . In these coordinates, we may write the rational map f as $f = (f_1, f_2, \dots, f_n)$. If f has degree d , the n -equations $f_1(x) - y_1 = f_2(x) - y_2 = \dots = f_n(x) - y_n = 0$ will generically have d isolated solutions. Suppose that ω has degree n , that is, ω is a top form. Then the integral $\int \delta^n(f_1(x) - y_1, f_2(x) - y_2, \dots, f_n(x) - y_n) \omega$ will be a well-defined rational function on Y .

Proposition 3.23. We have

$$f_*\omega = \left(\int \delta^n(f_1(x) - y_1, f_2(x) - y_2, \dots, f_n(x) - y_n) \omega \right) dy_1 \wedge \dots \wedge dy_n.$$

Example 3.24. Let $f: \mathbb{C} \rightarrow \mathbb{C}$ be given by $x \mapsto y = x^2$, and consider the 1-form $\omega = \frac{dx}{x-a}$. The pushforward is

$$f_*\omega = \omega|_{x=\sqrt{y}} + \omega|_{x=-\sqrt{y}} = \frac{d\sqrt{y}}{\sqrt{y}-a} + \frac{-d\sqrt{y}}{-\sqrt{y}-a} = \frac{dy}{y-a^2}.$$

On the other hand, using $\frac{\partial(x^2-y)}{\partial x} = 2x$, the delta function integral can be computed as

$$\int \delta(x^2-y) \frac{dx}{x-a} = \left(\frac{1}{2x(x-a)} \right)_{x \mapsto \sqrt{y}} + \left(\frac{1}{2x(x-a)} \right)_{x \mapsto -\sqrt{y}} = \frac{1}{y-a^2}.$$

Recall the scattering correspondence $\mathcal{I} \subset K_n \times \mathcal{M}_{0,n}$ (3.8). Consider the following diagram:

$$\begin{array}{ccc} & \mathcal{I} & \\ p \swarrow & & \searrow q \\ \mathcal{M}_{0,n} & & K_n \end{array}$$

Definition 3.25. The scattering form on K_n is the $(n-3)$ -form

$$\Psi_n := q_* p^* \Omega_{0,n}.$$

The pullback $p^* \Omega_{0,n}$ is a rational $(n-3)$ -form on $\mathcal{I}$. By Theorem 3.9, the map $q: \mathcal{I} \rightarrow K_n$ is a degree $(n-3)!$ map between two varieties of dimension $\binom{n}{2} - n$, and thus the pushforward $q_* p^* \Omega_{0,n}$ is well-defined.

We now give an explicit formula for Ψ_n .

Theorem 3.26. We have

$$\Psi_n = \sum_{\mathcal{D}} \text{sign}(\mathcal{D}) \bigwedge_{(ij) \in \mathcal{D}} d\log X_{ij} \quad (3.17)$$

summed over triangulations $\mathcal{D}$ of the n -gon, or equivalently over maximal simplices in $\Delta_{0,n}$. Here, the signs $\text{sign}(\mathcal{D}) \in \{+, -\}$ are chosen so that we obtain an orientation of the simplicial complex Δ .

There is an overall sign-ambiguity in Theorem 3.26, matching the sign-ambiguity of $\Omega_{0,n}$ which depends on an ordering of the positive coordinates, or equivalently, an orientation of the manifold $(\mathcal{M}_{0,n})_{>0}$.

We spell out details of $\text{sign}(\mathcal{D})$. The choice $\text{sign}(\mathcal{D})$ depends on the ordering of the elements $(ij) \in \mathcal{D}$ that appear in $\bigwedge_{(ij) \in \mathcal{D}} d\log X_{ij}$, so that the form $\text{sign}(\mathcal{D}) \bigwedge_{(ij) \in \mathcal{D}} d\log X_{ij}$ depends only on $\mathcal{D}$, and not the ordering. If we fix an ordering $(i_1 j_1), (i_2 j_2), \dots, (i_{n-3} j_{n-3})$ of the diagonals in $\mathcal{D}$, there is a unique

other triangulation $\mathcal{D}'$ obtained by flipping the diagonal (i_1, j_1) , swapping it for (i'_1, j'_1) . We ask that

$$\text{sign}(\mathcal{D}) \text{dlog} X_{i_1, j_1} \wedge \cdots \wedge \text{dlog} X_{i_{n-3}, j_{n-3}} = -\text{sign}(\mathcal{D}') \text{dlog} X_{i'_1, j'_1} \wedge \cdots \wedge \text{dlog} X_{i_{n-3}, j_{n-3}} \quad (3.18)$$

for any two $\mathcal{D}, \mathcal{D}'$ related in this way. This is equivalent to picking an orientation (that is, a generator for $H_{n-3}(|\Delta|, \mathbb{Z})$) on the geometric realization $|\Delta|$ of Δ , which is an orientable manifold (that is, $H_{n-3}(|\Delta|, \mathbb{Z}) \cong \mathbb{Z}$). Indeed, $|\Delta|$ is a sphere, the boundary sphere of the dual of the associahedron polytope, which is orientable.

Example 3.27. Let $n = 4$ and as usual we gauge-fix $(\sigma_1, \sigma_2, \sigma_4)$. The canonical form is $\frac{d\sigma}{\sigma-1} = \frac{d(\sigma-1)}{\sigma-1}$, and the scattering equations give $\sigma = \frac{X_{13}+X_{24}}{X_{13}}$. Substituting, we get

$$\Psi_4 = \text{dlog}(\sigma - 1) = \text{dlog}(X_{24}/X_{13}) = \text{dlog} X_{24} - \text{dlog} X_{13}.$$

The proof of Theorem 3.26 is given in Section 3.11.

Remark 3.28. The combinatorial formula in Theorem 3.26 is used as a definition of the scattering form in [AHBHY18]. Our definition of Ψ_n as a push-pull gives the scattering form a conceptual explanation which can be extended to the general setting, for example of a very affine variety as in Definition 6.7. See also [FM21] for a similar approach.

Let us consider the subspace $H(\underline{c}) \subset K_n$ given by the linear equations

$$X_{ij} + X_{i+1, j+1} - X_{i, j+1} - X_{i+1, j} = -s_{ij} = c_{ij} \quad (3.19)$$

where c_{ij} is a constant, where $1 \leq i < j-1 < j \leq n-1$. A quick calculation shows that these are $\binom{n-2}{2}$ linearly independent equations, so that $\dim H(\underline{c}) = n-3$.

Lemma 3.29. Let $\mathcal{D}$ be a triangulation of the n -gon. Then $\{X_{ij} \mid (ij) \in \mathcal{D}\}$ is a basis for $H(\underline{c})$.

Proposition 3.30. Let $\iota_{\underline{c}} : H(\underline{c}) \hookrightarrow K_n$ denote the inclusion. The pullback $d^{n-3}X := \iota_{\underline{c}}^* \text{sign}(\mathcal{D}) \wedge_{(ij) \in \mathcal{D}} \text{dlog} X_{ij}$ does not depend on $\mathcal{D}$. The pullback $\iota_{\underline{c}}^* \Psi_n$ is, up to sign, equal to the planar ϕ^3 -amplitude

$$\iota_{\underline{c}}^* \Psi_n = \pm A_n^{\phi^3} d^{n-3}X.$$

Proof. Exercise 3.43. □

The choice of the affine subspaces $H(\underline{c}) \subset K_n$ may appear mysterious. We explain them in Section 5.3. We remark that pullbacks of Ψ_n to other subspaces of K_n are also of physical interest, and are called *partial amplitudes* (Remark 3.15).

Example 3.31. Let $n = 4$. Then the scattering form is (up to sign) the 1-form

$$\Psi_n = \frac{dX_{13}}{X_{13}} - \frac{dX_{24}}{X_{24}}$$

on the two-dimensional space K_4 . Pulling back to the one-dimensional affine subspace $X_{13} + X_{24} = c$, we have $-dX_{13} = dX_{24}$, so

$$\iota_{\underline{c}}^* \Psi_n = \left(\frac{1}{X_{13}} + \frac{1}{X_{24}} \right) dX_{13},$$

so the coefficient of dX_{13} is exactly $A_4^{\phi^3}$.

Example 3.32. Let $n = 5$. We may use the two linear functions X_{13}, X_{14} as coordinates on $H(\underline{c})$, and then

$$X_{35} = -X_{14} + c_{14} + c_{24}, \quad X_{25} = -X_{13} + c_{13} + c_{14}, \quad X_{24} = X_{14} - X_{13} + c_{13}$$

on $H(\underline{c})$. One checks that the sign works out so that all $\iota_{\underline{c}}^* \text{sign}(\mathcal{D}) \wedge_{(ij) \in \mathcal{D}} d\log X_{ij}$ is always $\pm dX_{13} dX_{14}$ with the same sign.

Remark 3.33. Recall from Theorem 2.10 that if $\tilde{U}$ is binary geometry for Δ , then Δ is a pure pseudomanifold. In particular, it makes sense to ask that Δ is orientable, or oriented, using the same requirement as (3.18). Now suppose that $\tilde{U}$ is a binary geometry for a pure, orientable, pseudomanifold Δ . We then have a combinatorial scattering form Ψ_U on $\Lambda(U)_{\mathbb{R}}$, analogous to the formula in Theorem 3.26, given by

$$\Psi_U := \sum_{F \in \Delta} \text{sign}(F) \bigwedge_{i \in F} d\log X_i$$

where the sum is over the maximal simplices F of Δ , and the differential forms $\text{sign}(F) \wedge_{i \in F} d\log X_i$ are chosen to give an orientation of Δ .

3.10. Scattering map

Restricting the scattering correspondence projection $\mathcal{I} \rightarrow K_n$ to the subspace $H(\underline{c})$, we obtain a rational map [AHBHY18, AHHL21b]

$$\Phi = \Phi(\underline{c}) : \mathcal{M}_{0,n} \dashrightarrow H(\underline{c}), \quad (\underline{\sigma}) \mapsto (X_{ab}) \quad (3.20)$$

of degree $(n-3)!$, between two varieties of dimension $(n-3)$. The map $\Phi(\underline{c})$ can be described in coordinates explicitly. First, use $\text{PGL}(2)$ to take σ_n to ∞ , as

usual. Then the scattering equation $Q_n = 0$ disappears, and summing $Q_1, \dots, Q_k$, we obtain

$$s_{k,k+1} = - \sum_{\substack{1 \leq i \leq k \\ k+1 \leq j \leq n-1 \\ (i,j) \neq (k,k+1)}} \sigma_{k,k+1} \frac{s_{ij}}{\sigma_{ij}}.$$

Substituting this into (3.4), we can express each X_{ab} as a sum over s_{ij} where (ij) is a diagonal:

$$X_{ab} = - \sum_{\substack{1 \leq i < a \\ a < j < b}} \sigma_{a,j} \frac{s_{ij}}{\sigma_{ij}} - \sum_{\substack{a \leq i < b \\ b \leq j < n}} \sigma_{i,b-1} \frac{s_{ij}}{\sigma_{ij}} - \sum_{\substack{1 \leq i < a \\ b \leq j < n}} \sigma_{a,b-1} \frac{s_{ij}}{\sigma_{ij}}. \quad (3.21)$$

Note that this equality only holds if we assume the scattering equations $Q_i = 0$. For example, for $n = 5$, we may use the two linear functions X_{13}, X_{14} as coordinates on $H(\underline{c})$, and then gauge-fixing $(\sigma_1, \sigma_2, \sigma_n)$ to $(0, 1, \infty)$ as usual, the scattering map is

$$X_{13} = \frac{c_{13}}{\sigma_3} + \frac{c_{14}}{\sigma_4}, \quad X_{14} = \frac{\sigma_3 c_{14}}{\sigma_4} + \frac{(\sigma_3 - 1)c_{24}}{(\sigma_4 - 1)}. \quad (3.22)$$

Theorem 3.34. We have $\Phi(\underline{c})_* \Omega_{0,n} = \iota_{\underline{c}}^* \Psi_n$.

Proof. The map $\Phi(\underline{c})$ is the restriction of the map q to (the inverse image of) the subspace $H(\underline{c})$. The definition of the pushforward of a form commutes with such pullbacks. $\square$

Example 3.35. For $n = 4$, the scattering map is $X = \frac{c}{\sigma}$, where $X = X_{13}$, $\sigma = \sigma_3$ and $c = c_{13}$. The canonical form is

$$\Omega_{0,4} = d \log y_1 = \frac{d\sigma}{\sigma - 1},$$

where $y_1 = \sigma - 1$. Thus substituting $\sigma = \frac{c}{X}$, we get

$$\Phi(\underline{c})_* \Omega_{0,4} = -c \frac{dX}{X^2(c/X - 1)} = -c \frac{dX}{X(c - X)} = -\left(\frac{1}{X} + \frac{1}{c - X}\right) dX.$$

This agrees (up to sign) with $A_4^{\phi^3} dX$, after using $X_{13} + X_{24} = c$.

Remark 3.36. There are two other proofs of Theorem 3.34. For the first proof, combining with Proposition 3.30 and Theorem 3.14, the statement is equivalent to

$$A_n^{\phi^3} = \int \Omega((\mathcal{M}_{0,n})_{\geq 0}) \prod_{(ij) \in \mathcal{D}} \delta(X_{ij} - \Phi(\underline{c})_{ij})$$

where $\mathcal{D}$ is a triangulation of the n -gon and $\Phi(\underline{c})_{ij}$ denotes the X_{ij} -coordinate of the point $\Phi(\underline{c}) \in H(\underline{c})$. Using Proposition 3.18, this reduces to showing the identity of delta functions

$$\prod_{(ij) \in \mathcal{D}} \delta(X_{ij} - \Phi(\underline{c})_{ij}) = \frac{\sigma_{ab}\sigma_{bc}\sigma_{ca}}{\sigma_{12}\sigma_{23}\cdots\sigma_{n1}} \prod_{i \notin \{a,b,c\}} \delta(Q_i),$$

which is a direct Jacobian calculation.

For the second proof, as explained in [AHHL21b, Section 7], the map $\Phi : \mathcal{M}_{0,n} \rightarrow H(\underline{c})$ can be interpreted as the algebraic moment map of the toric variety of the associahedron Ass_{n-3} (see Section 5). The result then follows from the pushforward formula of [AHBL17], together with the identification of $\iota_{\underline{c}}^* \Psi_n$ as the canonical form $\Omega(\text{Ass}_{n-3})$ of the $(n-3)$ -dimensional associahedron Ass_{n-3} , which is the intersection of the affine subspace $H(\underline{c})$ with the positive orthant $\{X_{ij} \geq 0 \mid (ij) \in \text{diag}_n\} \subset K_n$.

Remark 3.37. Another geometric statement [AHBHY18, AHHL21b] is that when $c_{ij} > 0$ are positive then $(\mathcal{M}_{0,n})_{\geq 0}$ is mapped diffeomorphically onto an associahedron polytope, and $\Phi_* \Omega((\mathcal{M}_{0,n})_{\geq 0})$ is the canonical form of the associahedron. For example, with $n = 5$, as $\underline{c}$ vary within $(\mathcal{M}_{0,5})_{>0}$, we have (σ_3, σ_4) in the region $1 < \sigma_3 < \sigma_4 < \infty$. The image of this region under the map (3.22) in $H(\underline{c})$ is a pentagon, the two-dimensional associahedron.

3.11. Proof of Theorem 3.26

Our proof is a variant of the proof of [AHBL17, Theorem 7.12], which states that the pushforward of the canonical form of a projective toric variety X_P under the algebraic moment map is equal to the canonical form of the corresponding polytope P .

Let Ψ'_n denote the right hand side of (3.17). We first note that both Ψ_n and Ψ'_n are projective, meaning that they are pullbacks of forms from $\mathbb{P}(K_n)$. For Ψ_n this follows from the fact that the scattering equations, and thus the scattering correspondence as well, are invariant under the dilation action of $\mathbb{C}^\times$ on K_n . For Ψ'_n this follows from the definition of the signs $\text{sign}(\mathcal{D})$; see Exercise 3.43. Thus it suffices to show that Ψ_n and Ψ'_n have the same poles on K_n , and the same residues at these poles. The poles of Ψ'_n are along the hyperplanes $K_{ij} = \{X_{ij} = 0\}$. The poles of Ψ_n come from the image of the poles of $\Omega_{0,n}$. By Lemma 1.19 and Theorem 1.21, the poles of $\Omega_{0,n}$ are along the divisors $\widetilde{\mathcal{M}}_{ij} = \{u_{ij} = 0\}$ of $\widetilde{\mathcal{M}}_{0,n}$.

As explained in Section 5.2 and [AHHL21a], the affine variety $\widetilde{\mathcal{M}}_{0,n}$ is the complement of a hypersurface H in the toric variety X_P , where P is an associahedron. Under the identification $\widetilde{\mathcal{M}}_{0,n} \cong X_P \setminus H$, the canonical form $\Omega_{0,n}$ is

identified with the canonical form of the torus of X_P . Furthermore, the stratification of X_P into torus orbit closures induces the stratification of $\widetilde{\mathcal{M}}_{0,n}$ into $\widetilde{\mathcal{M}}_{\mathcal{D}}$. Let us consider the inclusions

$$K_n \times \mathcal{M}_{0,n} \subset K_n \times \widetilde{\mathcal{M}}_{0,n} \subset K_n \times X_P$$

and taking the closure of the scattering correspondence we obtain inclusions $\mathcal{I} \subset \mathcal{I}' \subset \mathcal{I}''$. The projection $\mathcal{I}'' \rightarrow K_n$ is a proper map of degree $(n-3)!$. There is an open subset $a: V \hookrightarrow K_n$ such that the pullbacks $\mathcal{I}'' \times_{K_n} V$ and $\mathcal{I} \times_{K_n} V$ are isomorphic. Thus $\Psi_n = q_* \Omega$, where Ω is the canonical form of X_P , pulled back to $\mathcal{I}''$.

By [AHHL21b, Section 7.1], the restriction of q to (the inverse image of) a subspace $H(\underline{c})$ can be identified with the algebraic moment map of X_P . The divisors $\widetilde{\mathcal{M}}_{ij} = \{u_{ij} = 0\}$ are sent by q to the facets of P , which are exactly the hyperplanes $K_{ij} = \{X_{ij} = 0\}$. By [KR01, Proposition 2.5], taking residues commutes with pushforward, so we have

$$\text{Res}_{K_{ij}} \Psi_n = \text{Res}_{K_{ij}} q_* \Omega = q_* \text{Res}_{\widetilde{\mathcal{M}}_{ij}} \Omega = q_* (\Omega_{0,n_1} \wedge \Omega_{0,n_2})$$

where the last equality is by Lemma 1.19. On $\widetilde{\mathcal{M}}_{ij}$, all characters u_{kl} where (kl) crosses (ij) , are equal to 1. So over K_{ij} , the potential restricts to

$$\phi_X|_{K_{ij} \times \widetilde{\mathcal{M}}_{ij}} = \prod_{(ab)} u_{ab}^{X_{ab}}$$

where the product is over all diagonals that do not cross (ij) . This is exactly the potential of $\widetilde{\mathcal{M}}_{0,n_1} \times \widetilde{\mathcal{M}}_{0,n_2}$ in the notation of Lemma 1.19. Thus the map $q: q^{-1}(K_{ij}) \rightarrow K_{ij}$ can be identified with the map

$$q_1 \times q_2 \times \text{id}: (K_{n_1} \times X_{P,n_1}) \times (K_{n_2} \times X_{P,n_2}) \times W \rightarrow K_{n_1} \times K_{n_2} \times W$$

where the direct sum decomposition $H_{ij} = K_{n_1} \times K_{n_2} \times W$ corresponds to diagonals contained in the polygon on one side of (ij) , the diagonals on the other side of (ij) , and the diagonals that cross (ij) . We deduce that $\text{Res}_{K_{ij}} \Psi_n = \Psi_{n_1} \times \Psi_{n_2}$. On the other hand, it is immediate from the definition of Ψ'_n that $\text{Res}_{K_{ij}} \Psi'_n = \Psi'_{n_1} \times \Psi'_{n_2}$. By induction on n , with the base case $n=4$ checked in Example 3.27, we conclude that $\text{Res}_{K_{ij}} \Psi_n = \text{Res}_{K_{ij}} \Psi'_n$ for any of the hyperplanes K_{ij} . It follows that $\Psi_n = \Psi'_n$.

3.12. Exercises and Problems

Exercise 3.38. Prove Proposition 3.7, as follows. The group $\text{PGL}(2)$ acts on $\mathbb{P}^1$ as Möbius transformations. For each $i = 1, 2, \dots, n$, let $\sigma'_i = \frac{\alpha\sigma_i + \beta}{\gamma\sigma_i + \delta}$. Show that $(\sigma_1, \dots, \sigma_n)$ is a solution to the scattering equations if and only if $(\sigma'_1, \dots, \sigma'_n)$ is.

Exercise 3.39.

- (a) Fill out the details in the proof of Proposition 4.5.
- (b) Write down conditions that guarantee that $r(z)$ in (4.4) is a regular morphism $r: \mathbb{P}^1 \rightarrow \mathbb{P}^{D-1}$ of degree $n-2$.

Exercise 3.40. Solve the scattering equations for $n=5$ by hand or by computer.

Exercise 3.41.

- (a) Show that $\sum_{i=1}^n \sigma_i^\ell Q_i = 0$ for $\ell = 0, 1, 2$. Thus a solution to $n-3$ of the scattering equations Q_i is automatically a solution to the remaining 3.
- (b) Prove Lemma 3.11 by using (a) and the following general statement. Let M be a $n \times n$ matrix and suppose that the nullspace K of M has dimension k . Let $m_{b_1, \dots, b_k}^{a_1, \dots, a_k}$ denote the minor obtained from M by removing rows $a_1, \dots, a_k$ and columns $b_1, \dots, b_k$. Then we have

$$m_{b_1, \dots, b_k}^{a_1, \dots, a_k} \Delta_{a'_1, \dots, a'_k}(K) = m_{b_1, \dots, b_k}^{a'_1, \dots, a'_k} \Delta_{a_1, \dots, a_k}(K)$$

where $\Delta_I(K)$ denote the Plücker coordinates of K viewed as a point in the Grassmannian $\text{Gr}(k, n)$. A similar statement holds for changing $b_1, \dots, b_k$.

Exercise 3.42. Compute the Euler characteristics of $\widetilde{\mathcal{M}}_{0,n}$ and $\overline{\mathcal{M}}_{0,n}$ using the fact that Euler characteristic is additive under open-closed decompositions into algebraic sets.

Exercise 3.43.

- (a) Show that the formula (3.17) for the scattering form Ψ_n is cyclic invariant up to sign.
- (b) Prove that Ψ_n can be written as a polynomial in the forms $\text{dlog}(X_{ij}/X_{kl})$. This says that Ψ_n is *projective* in the sense of [AHBHY18].
- (c) Prove Proposition 3.30.

Exercise 3.44. Let α be a cyclic ordering of $[n]$. A (usual, cubic, n -leaf) planar tree T is *compatible* with α if there exists an embedding of T into the plane so that the leaves are cyclically arranged in the order given by α . The partial amplitude $A_n(12 \cdots n | \alpha)$ is given by

$$A_n^{\phi^3}(12 \cdots n | \alpha) := \sum_T \prod_{e \in E(T)} \frac{1}{X_{ij}}$$

where the sum is over planar trees compatible with α .

1. Check for $n = 4$ and $n = 5$, that the partial amplitudes can be obtained by using the Parke-Taylor factor $\text{PT}(12 \cdots n)\text{PT}(\alpha)$ in (3.9) instead of $\frac{1}{(\sigma_{12}\sigma_{23}\cdots\sigma_{n1})^2}$.
2. Check that $A_4(1234|2134)$ can be obtained by pulling back the planar scattering form Ψ_4 to the subspace $X_{24} = c$, for a constant c . Which partial amplitude do you get by pulling back to $X_{13} = c$?
3. For each partial amplitude $A_5(12345|\alpha)$ find a subspace $\iota : H \hookrightarrow K_5$ such that the amplitude can be obtained by the pullback $\iota^*\Psi_5$.

Problem 3.45. Express the reduced determinant of Lemma 3.11 in terms of the dihedral coordinates u_{ij} .

Problem 3.46. Explore the scattering equations and (4.3) within the setting of the moduli space of stable, rational, n -pointed maps to $\mathbb{P}^1$.

3.12.1. Tropical u -equations

Recall the u -equations (1.4) satisfied by the dihedral coordinates u_{ij} . This equation has no minus signs (if we place the 1 on the other side of the equality), and we can formally take its “positive tropicalization”, given by

$$u_{ij} \mapsto U_{ij}, \quad + \mapsto \min, \quad \times \mapsto +, \quad \text{constant} \mapsto 0$$

obtaining the *tropical u -equation*

$$\text{trop}(R_{ij}) := \min(U_{ij}, \sum_{(kl) \text{ crosses } (ij)} U_{kl}) = 0.$$

We now take $\{U_{ij} \in K_n\}$ to be the basis dual to $\{X_{ij} \in K_n^*\}$. Thus the tropical u -equations define a subset of K_n^* .

Lemma 3.47. The intersection of the $\binom{n}{2} - n$ tropical u -equations is a pure polyhedral fan $\text{trop}_{\geq 0} \mathcal{M}_{0,n}$ in K_n^* of dimension $n - 3$, with maximal cones given by

$$C(\mathcal{D}) = \text{span}_{\geq 0}(X_{ij} \mid (ij) \in \mathcal{D}) \subset K_n^*$$

for $\mathcal{D}$ a triangulation of the n -gon.

Proof. Let $X \in K_n^*$ belong to the intersection of $\text{trop}(R_{ij}) = 0$. Clearly, $U_{ij}(X) \geq 0$ for all (ij) . Let

$$\mathcal{D}_X = \{(ij) \mid U_{ij}(X) > 0\}.$$

If (ij) and (kl) are diagonals that cross, then $\text{trop}(R_{ij}) = 0$ implies that they cannot both be elements of $\mathcal{D}$. It follows that the possible choices of $\mathcal{D}_X$ are

exactly the subdivisions of the n -gon, and conversely, if $\mathcal{D}_X$ is a subdivision of the n -gon then X satisfies the tropical u -equations. For a fixed subdivision $\mathcal{D}'$ of the n -gon we obtain a cone $C(\mathcal{D}')$ with dimension equal to the number of diagonals used, and the set we seek is the union of these cones. The maximal cones correspond to triangulations $\mathcal{D}$. $\square$

The fan $\text{trop}_{\geq 0}\mathcal{M}_{0,n}$ is the *positive tropical pre-variety* associated to the equations $R_{ij} = 0$. It turns out that $\text{trop}_{\geq 0}\mathcal{M}_{0,n}$ is equal to the positive tropicalization of $\mathcal{M}_{0,n}$ in the sense of [SW05]. Note that the rays (that is, the one-dimensional cones) of the fan $\text{trop}_{\geq 0}U$ are exactly $\{\mathbb{R}_{\geq 0} \cdot X_{ij}\}$. The monoid of bounded characters (Exercise 1.24) on $\mathcal{M}_{0,n}$ are exactly the characters taking nonnegative values on these rays.

Let $\Delta = \Delta_{0,n}$ be the simplicial complex of subdivisions of the n -gon, as in Section 1.4. Then the tropical u -equations cut out the *cone over* Δ . This suggests that for binary geometries the following is a reasonable definition of positive tropicalization.

Definition 3.48. Let U be a binary geometry for a flag simplicial complex Δ on $[n]$. The *positive tropical pre-variety* $\text{trop}_{\geq 0}^{\text{pre}}U$ of U is the intersection of the tropical u -equations $\text{trop}(R_i) = 0$ for $i = 1, 2, \dots, n$.

It follows from the same calculation as in Lemma 3.47 that $\text{trop}_{\geq 0}^{\text{pre}}U$ is a pure polyhedral fan whose cones are exactly

$$C(F) := \text{span}_{\geq 0}(X_i \mid i \in F).$$

We do not know (Problem 3.49) the relation between $\text{trop}_{\geq 0}^{\text{pre}}U$ and the usual notion of (positive) tropicalization (Section 6.7). The intersection of the tropical u -equations is analogous to the definition of the *(positive) Dressian* [AHLS21b, SW21].

Problem 3.49. How is the positive tropical pre-variety $\text{trop}_{\geq 0}^{\text{pre}}U$ in Definition 3.48 related to the positive tropicalization of the very affine variety U ?

The study of the tropicalization of binary geometries has recently been initiated by Cox and Makhlin [CM24].

4. Positive geometry of four dimensional space-time

In Section 3, we looked at the kinematics of n -particle scattering in D -dimensional space-time. Space-time of the real world is four-dimensional, so the case $D = 4$ is of particular physical importance, so we specialize to $D = 4$ in this section. For physics background to the material of this section, see [Dix96, Bri, EH15].

4.1. Spinors

In Definition 3.1, kinematic space K_n for n -particle scattering is defined to be a complex vector space of dimension $\binom{n}{2} - n$. This is viewed as the space of symmetric $n \times n$ matrices of the form $M = PP^T$, where the row vectors of the matrix P are the momentum vectors $p_1, \dots, p_n \in \mathbb{C}^D$. For low D , the rank of the matrix M is bounded above by D , and the “correct” kinematic space is the subvariety $K_n^D \subset K_n$ cut out by the conditions that all $(D+1) \times (D+1)$ minors vanish.

Setting space-time dimension to $D = 4$ leads to the remarkable algebra and geometry of spinors. We view $\mathbb{C}^4 \cong \text{Mat}_{2 \times 2}$ as the space of 2×2 matrices, with isomorphism chosen so that the symmetric bilinear form is given by $p^2 = p \cdot p = \det(p)$. Equivalently, the inner product is

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} = \frac{1}{2}(ad' + a'd - bc' - b'c).$$

The massless condition that $p^2 = \det(p) = 0$ implies that p has rank ≤ 1 and can be written as $p = \lambda \tilde{\lambda}$ where λ (resp. $\tilde{\lambda}$) is a column (resp. row) vector in $\mathbb{C}^2$. The vectors $\lambda, \tilde{\lambda} \in \mathbb{C}^2$ are called *spinors*.

There is a redundancy in this description: for a non-zero scalar $t \in \mathbb{C}^\times$, the spinors $(\lambda, \tilde{\lambda})$ and $(t\lambda, t^{-1}\tilde{\lambda})$ represent the same momentum vector p . This feature is (perhaps?) a slight annoyance when we consider scattering of scalar particles, but an advantage when we consider scattering of gluons. Spinor coordinates are also preferred when considering scattering of fermions (spin $1/2$ particles), though we will only consider gluon scattering. For more on *spinor-helicity formalism*, see the textbook [EH15].

Lemma 4.1. Suppose the spinors of p and q are $(\lambda, \tilde{\lambda})$ and $(\mu, \tilde{\mu})$ respectively. Then

$$2p \cdot q = \det(\lambda, \mu) \det(\tilde{\lambda}, \tilde{\mu})$$

where the determinants are taken by forming 2×2 matrices, using two column (resp. row) vectors.

Proof. Exercise 4.14. □

Now, let $\lambda_1, \dots, \lambda_n$ and $\tilde{\lambda}_1, \dots, \tilde{\lambda}_n$ be spinors for the n particles. We use the standard shorthand:

$$\langle ij \rangle := \det(\lambda_i, \lambda_j), \quad [ij] := \det(\tilde{\lambda}_i, \tilde{\lambda}_j)$$

satisfying the identities

$$\langle ij \rangle = -\langle ji \rangle, \quad [ij] = -[ji], \quad s_{ij} = \langle ij \rangle [ij].$$

Momentum conservation $p_1 + p_2 + \cdots + p_n = 0$ becomes the identity $\sum_i \lambda_i \tilde{\lambda}_i = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, which combining with Lemma 4.1 gives

$$\sum_i \langle ji \rangle [ik] = 0, \text{ for all } j, k. \quad (4.1)$$

Let $\text{SO}(D)$ denote the complex special orthogonal group in $D \geq 3$ dimensions. The group $\text{SO}(D)$ is not simply-connected and its simply-connected cover is the spin group $\text{Spin}(D)$, which is a double cover of $\text{SO}(D)$. When $D = 4$, we have an exceptional isomorphism $\text{Spin}(4) \cong \text{SL}(2) \times \text{SL}(2)$ arising from the Dynkin diagram isomorphism $D_2 \cong A_1 \times A_1$. The covering map $\text{SL}(2) \times \text{SL}(2) \rightarrow \text{SO}(4)$ has kernel given by the two element group $\{(I, I), (-I, -I)\}$. The $\text{SO}(4)$ action on $\mathbb{C}^4$ is replaced by the $\text{SL}(2) \times \text{SL}(2)$ action on the pair $(\lambda, \tilde{\lambda})$.

We now view $\lambda = (\lambda_1, \dots, \lambda_n)$ as a $2 \times n$ matrix and $\tilde{\lambda} = (\tilde{\lambda}_1, \dots, \tilde{\lambda}_n)$ as a $n \times 2$ matrix. By the first fundamental theorem of invariant theory, the ring of $\text{SL}(2)$ -invariant polynomial functions on $2 \times n$ (resp. $n \times 2$) matrices is generated by the 2×2 determinants $\langle ij \rangle$ (resp. $[ij]$). This ring, also called the *Plücker ring*, is the coordinate ring of the cone over the Grassmannian $\text{Gr}(2, n)$. Thus the kinematics of n massless particle scattering in $D = 4$ -dimensional space-time involves two 2-planes $\lambda, \tilde{\lambda}$ in n -dimensional space, and (4.1) says that $\lambda, \tilde{\lambda}$ are orthogonal 2-planes (Exercise 4.15).

Let $\widehat{\text{Gr}}(2, n)$ denote the cone over the Grassmannian $\text{Gr}(2, n)$. Thus the coordinate ring of $\widehat{\text{Gr}}(2, n)$ is the ring generated by $\langle ij \rangle$.

Definition 4.2. *Spinor kinematic space* is the space Λ_n^4 of pairs $(\lambda, \tilde{\lambda}) \in \widehat{\text{Gr}}(2, n) \times \widehat{\text{Gr}}(2, n)$ satisfying $\lambda \perp \tilde{\lambda}$.

For more on the algebraic geometry of spinor kinematic space, we refer the reader to the recent work [EPS24].

Remark 4.3. In $D = 3$ dimensions, we have the exceptional isomorphism $\text{Spin}(3) \cong \text{SL}(2)$, arising from the Dynkin diagram isomorphism $A_1 \cong B_1$. In $D = 6$ dimensions, we have the exceptional isomorphism $\text{Spin}(6) \cong \text{SL}(4)$, arising from the Dynkin diagram isomorphism $A_3 \cong D_3$. Though we will focus on $D = 4$, there is special physics appearing for all three choices of space-time dimension $D = 3, 4, 6$.

4.2. Gluon scattering

Elementary particles come with a discrete parameter, *spin*, which takes values in $0, \frac{1}{2}, 1, \frac{3}{2}, 2, \dots$. Spin 0 particles are also called scalar particles, and the amplitudes that we looked at in Section 3.6 are scattering amplitudes for scalars.

A particle with nonzero spin has intrinsic angular momentum – it is allowed to be rotating, and the axis of rotation gives additional degrees of freedom. For scattering of particles with nonzero spin, the amplitudes depend on not just the momentum vector of the particle, but also additional information. In the case of spin 1 particles, the additional information is called a polarization vector. Thus scattering amplitudes are functions of momentum vectors $p_1, \dots, p_n$ and polarization vectors $\xi_1, \dots, \xi_n$.

Gluons are elementary particles that mediate the strong force, that is, the exchange of gluons leads to an interaction between elementary particles called the strong force. Gluons appearing in the strong force are massless spin 1 particles with “color”, and we commonly call any such particle in a QFT a gluon. Color refers to the symmetries of a compact Lie group called a gauge group. In the case of the strong force, this gauge group is $SU(3)$. The dependence of the scattering amplitude on the gauge group can often be separated out in the sense that the amplitude can be factored as the product of a “group theory factor” and a “kinematic factor”. We ignore the gauge group in our presentation and focus on the kinematics. That is, we consider “color-stripped amplitudes”. However, let us remark that some of the beautiful combinatorics and geometry of amplitudes arises exactly because of the gauge group. For example, taking the rank of the gauge group to infinity (that is, $SU(N)$ with $N \rightarrow \infty$) leads to the consideration of “planar amplitudes”.

For gluon scattering, polarization vectors have two degrees of freedom. We may consider the polarization vector as a linear combination of a positive helicity polarization vector and a negative helicity one. So to compute amplitudes, it suffices to compute the 2^n helicity amplitudes, where each of the n particles is given either positive or negative helicity. These amplitudes are typically denoted $A_n(1^+, 2^-, 3^-, \dots, n^+)$ (particle 1 has helicity +, particle 2 has helicity –, ...) and so on.

It is a remarkable and beautiful fact that the additional information of the polarization vector, or equivalently of helicity, is stored in the choice of spinors $(\lambda, \tilde{\lambda})$ (which previously appeared redundant). Let $T \cong (\mathbb{C}^\times)^n$ act on Λ_n^4 by letting $t = (t_1, \dots, t_n) \in T$ send $(\lambda_i, \tilde{\lambda}_i)$ to $(t_i \lambda_i, t_i^{-1} \tilde{\lambda}_i)$. Let $h = (h_1, \dots, h_n) \in \{\pm 1\}^n$ denote a helicity vector. Then we have the following assertion:

Helicity h amplitudes are weight vectors for the T -action with weight $(2h_1, \dots, 2h_n)$.

(4.2)

In other words, helicity h amplitudes are rational functions on Λ_n^4 that scale by $\prod_i t_i^{2h_i}$ under the T -action on Λ_n^4 . We can also state this projectively, using the projective variety $\text{Proj}(\mathbb{C}[\Lambda_n^4]) \subset \text{Gr}(2, n) \times \text{Gr}(2, n)$.

Definition 4.4. For an appropriate line bundle L on $\text{Proj}(\mathbb{C}[\Lambda_n^4])$, helicity h

amplitudes are rational sections of L with T -weight equal to $(2h_1, \dots, 2h_n)$.

The coordinate ring $\mathbb{C}[\Lambda_n^4]$ of the variety Λ_n^4 is the multi-homogeneous coordinate ring of (or multi-cone over) the flag variety $\text{Fl}(2, n-2; n)$ of partial flags $\{(0 \subset \lambda \subset \tilde{\lambda}^\perp \subset \mathbb{C}^n)\}$ of dimensions $2, n-2$ in $\mathbb{C}^n$. (Here, we have used the isomorphism $\text{Gr}(2, n) \cong \text{Gr}(n-2, n)$ sending $\tilde{\lambda}$ to $\tilde{\lambda}^\perp$.) The action of T on $\text{Fl}(2, n-2; n)$ is the usual one, though to preserve the conventions on weights we should take only the subtorus $\{(t_1, \dots, t_n) \mid \prod_i t_i = 1\}$. We could also think of helicity amplitudes as rational sections of a line bundle on an appropriate GIT quotient $\text{Fl}(2, n-2; n)//T$, though I do not know how useful this point of view is.

4.3. Scattering equations from marked rational maps

We briefly review the origin of the scattering equations, and in this subsection we work in general space-time dimension D . Let us consider the following rational map from a n -punctured Riemann sphere $\mathbb{P}^1 \setminus \{\sigma_1, \dots, \sigma_n\}$ to $\mathbb{C}^D$:

$$q(z) := \sum_{i=1}^n \frac{p_i}{z - \sigma_i}, \quad z \in \mathbb{P}^1 \setminus \{\sigma_1, \dots, \sigma_n\}. \quad (4.3)$$

The map (4.3) depends on a choice of $\underline{\sigma}$ and on momentum vectors $p_1, \dots, p_n \in \mathbb{C}^D$, which we assume are massless and satisfy momentum conservation. Let

$$r(z) = q(z) \prod_i (z - \sigma_i) = \sum_{i=1}^n p_i \prod_{j \neq i} (z - \sigma_j), \quad (4.4)$$

which is now a polynomial in z . Due to the momentum conservation assumption, we have that $r(z)$ has degree $\leq n-2$. We may view $r(z)$ as a rational map $r: \mathbb{P}^1 \rightarrow \mathbb{P}^{D-1}$. The specific form (4.4) says that $r(\sigma_i) = p_i$, now viewed as a point in $\mathbb{P}^{D-1}$. Let $(K_n \times \mathcal{M}_{0,n})^\circ \subset K_n \times \mathcal{M}_{0,n}$ denote the dense open subset where $r(z)$ is a regular morphism of degree $n-2$. We may then consider the formula for $r(z)$ (or $q(z)$) as defining a morphism

$$(K_n \times \mathcal{M}_{0,n})^\circ \longrightarrow \mathcal{M}_{0,n}(\mathbb{P}^{D-1}, (n-2)),$$

where $\mathcal{M}_{0,n}(\mathbb{P}^{D-1}, (n-2))$ is the moduli space of n -pointed rational curves of degree $n-2$ in $\mathbb{P}^{D-1}$. It would be interesting to study the scattering equations within the context of the moduli space $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^{D-1}, (n-2))$ of stable rational curves.

The null-cone, or light-cone, of $\mathbb{C}^D$ is the subset of $\mathbb{C}^D$ consisting of vectors satisfying $p^2 = p \cdot p = 0$ (the massless condition). It is a quadric in $\mathbb{C}^D$, singular only at the origin.

Proposition 4.5. The scattering equations are equivalent to the condition that $r(z)$ has image in the null-cone of $\mathbb{C}^D$.

Sketch Proof. Differentiating $r(z)^2 = 0$, we obtain the condition $r'(z) \cdot r(z) = 0$. Evaluating this equation at $z = \sigma_i$ gives equations equivalent to the scattering equations. $\square$

4.4. Scattering equations in four dimensions

When $D = 4$, we identify $\mathbb{C}^4 \cong \text{Mat}_{2 \times 2}$ and ask that $r(z)$ has image in rank one matrices. Alternatively, we view $r(z)$ as a degree- $(n-2)$ map

$$r : \mathbb{P}^1 \setminus \{\sigma_1, \dots, \sigma_n\} \rightarrow \mathbb{P}^3$$

and ask that the image of r lies in the quadric $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$.

In this case, we can write the 2×2 matrix $r(z)$ as $r(z) = \tau(z)\tilde{\tau}(z)$ where $\tau(z)$ is a column vector of polynomials, and $\tilde{\tau}(z)$ is a row vector of polynomials. This representation is unique (up to rescaling $(\tau, \tilde{\tau}) \mapsto (t\tau, t^{-1}\tilde{\tau})$) when the four matrix entries of $r(z)$ do not have a common root, or equivalently, when $r(z) = 0$ has no solutions. Additionally imposing that $r(z)$ has degree equal to $n-2$, this is exactly the situation when $r(z)$ defines a genuine map $\mathbb{P}^1 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ of degree $n-2$. The polynomials $\tau(z), \tilde{\tau}(z)$ are rational maps $\tau, \tilde{\tau} : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ obtained by composing r with the projections to one of the factors.

Recall the scattering correspondence $\mathcal{I} \subset K_n \times \mathcal{M}_{0,n}$. We may restrict the family $\mathcal{I} \rightarrow K_n$ to the subvariety $K_n^4 \subset K_n$ of rank ≤ 4 matrices. Let P_n^4 denote the space of $n \times 4$ matrices P whose rows are massless momentum vectors $p_1, \dots, p_n$ satisfying momentum conservation. Then we have a map $P_n^4 \rightarrow K_n^4$ (sending $P \mapsto PP^T$) and we can pull the family $\mathcal{I}$ back to a family $\tilde{\mathcal{I}}' \rightarrow P_n^4$. We let $\tilde{\mathcal{I}} \subset P_n^4 \times \mathcal{M}_{0,n}$ denote those irreducible components of $\tilde{\mathcal{I}}'$ that map dominantly to P_n^4 . Thus the map $\tilde{\mathcal{I}} \rightarrow P_n^4$ is a morphism of degree $(n-3)!$; the preimages of a point in P_n^4 are the solutions to the scattering equations.

Proposition 4.6. The correspondence $\tilde{\mathcal{I}} \subset P_n^4 \times \mathcal{M}_{0,n}$ has $n-3$ irreducible components $\tilde{\mathcal{I}}_1, \tilde{\mathcal{I}}_2, \dots, \tilde{\mathcal{I}}_{n-3}$, all of the same dimension $3n-4 = \dim P_n^4$. For $d = 1, 2, \dots, n-3$, the d -th irreducible component $\tilde{\mathcal{I}}_d$ corresponds to factorizations $r(z) = \tau(z)\tilde{\tau}(z)$ where $\tau(z)$ has degree d and $\tilde{\tau}(z)$ has degree $\tilde{d} = n-2-d$.

Proof. Given a point in $\tilde{\mathcal{I}}$, sitting over a generic point $P \in P_n^4$, we have a corresponding map $r(z) = \tau(z)\tilde{\tau}(z)$ defined by (4.4), mapping $\mathbb{P}^1 \setminus \{\sigma_1, \dots, \sigma_n\}$ to the quadric $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$. Since $r(z)$ has degree $n-2$, if $\tau(z)$ has degree d then $\tilde{\tau}(z)$ has degree $\tilde{d} = n-2-d$. First note that for a generic point $p_1, \dots, p_n \in P_n^4$, neither $\tau(z)$ nor $\tilde{\tau}(z)$ can be constant. This is because by (4.4) we have $r(\sigma_i) = p_i$, and

this would force all the λ_i to be parallel, or all the $\tilde{\lambda}_i$ to be parallel, which is generically not the case. Thus $1 \leq d \leq n-3$.

The possible maps $\tau: \mathbb{P}^1 \rightarrow \mathbb{P}^1$ (resp. $\tilde{\tau}$) is determined by a pair of polynomials of degree $\leq d$ (resp. $\leq \tilde{d} = n-d-2$), belonging to a $2(d+1)$ (resp. $2(\tilde{d}+1)$) dimensional affine space. Thus $\tau(z), \tilde{\tau}(z)$ is represented by a point in this affine space, and an open subset of this affine space arises in this way. The possible maps $r(z)$ obtained in this way is thus an irreducible $(2n-1)$ -dimensional space, since $(\alpha\tau(z))(\alpha^{-1}\tilde{\tau}(z)) = \tau(z)\tilde{\tau}(z)$ for a scalar α . Also $\dim \mathcal{M}_{0,n} = n-3$ so we have described an irreducible subvariety of $\tilde{\mathcal{I}}$ of dimension $3n-4$; let us call this family $\tilde{\mathcal{J}}_d$.

Now, $\dim P_n^4 = 4n-4-n = 3n-4$, with 4 constraints from momentum conservation and n from the massless condition. Since $\tilde{\mathcal{I}} \rightarrow P_n^4$ is generically finite of degree $(n-3)!$, we have $\dim \tilde{\mathcal{I}} = \dim P_n^4 = 3n-4$. It follows that each of the subvarieties $\tilde{\mathcal{J}}_d$ described above is full-dimensional in $\tilde{\mathcal{I}}$. Over the locus $(P_n^4)^\circ$ where $\tilde{\mathcal{I}} \rightarrow (P_n^4)^\circ$ has exactly degree $(n-3)!$, every solution belongs to one of the $\tilde{\mathcal{J}}_d$. Thus $\tilde{\mathcal{J}}_1, \dots, \tilde{\mathcal{J}}_{n-3}$ are exactly the irreducible components of $\tilde{\mathcal{I}}$. $\square$

The irreducible components $\tilde{\mathcal{J}}_d$ of $\tilde{\mathcal{I}}$ separate the solutions of the scattering equations into $n-3$ sectors. Let $E_{n,k}$ denote the Eulerian number, counting the number of permutations in S_n with k ascents. Thus $\sum_k E_{n,k} = n!$. For example, $E_{3,1} = 1$, $E_{3,2} = 4$, and $E_{3,3} = 1$. In [CHY13], Cachazo–He–Yuan explain the following beautiful formula (earlier observed in a slightly different setting in [RSV04]).

CHY Formula.

$$\#\{\text{solutions in the } k = d+1\text{-th sector}\} = E_{n-3,k-2}. \quad (4.5)$$

Equivalently, the degree of the map $\tilde{\mathcal{J}}_d \rightarrow P_n^4$ is equal to the Eulerian number $E_{n-3,d-1}$.

I do not know of an explanation of this CHY formula similar to the proof of Theorem 3.9 that depends on the calculation of Euler characteristic.

4.5. Twistor strings

According to a proposal of Witten [Wit04], and expanded upon by Roiban–Spradlin–Volovich [RSV04], gluon scattering amplitudes in Yang–Mills theory can be obtained from *twistor string theory*. Indeed, this proposal was one of the key motivations for the CHY scattering equations.

Substituting $z = \sigma_i$ into (4.4), we have the equality of 2×2 matrices

$$\tau(\sigma_i)\tilde{\tau}(\sigma_i) = r(\sigma_i) = \prod_{j \neq i} (\sigma_j - \sigma_i) p_i = \prod_{j \neq i} (\sigma_j - \sigma_i) \lambda_i \tilde{\lambda}_i$$

and it follows that for some $t_i, \tilde{t}_i \neq 0$, we have the equations

$$t_i \tau(\sigma_i) = t_i \sum_{m=0}^d \tau_m \sigma_i^m = \lambda_i, \quad \tilde{t}_i \tilde{\tau}(\sigma_i) = \tilde{t}_i \sum_{m=0}^{\tilde{d}} \tilde{\tau}_m \sigma_i^m = \tilde{\lambda}_i \quad (4.6)$$

where $t_i \tilde{t}_i = \prod_{j \neq i} (\sigma_j - \sigma_i)^{-1}$, and τ_m (resp. $\tilde{\tau}_m$) are the coefficients of the 2-component polynomial $\tau(z)$ (resp. $\tilde{\tau}(z)$). When $\lambda_i, \tilde{\lambda}_i$ have been fixed, we view (4.6) as constraints on the $k, \tilde{k}$ -planes (with $k = d + 1$ and $\tilde{k} = \tilde{d} + 1$) $C \in \text{Gr}(k, n)$, $\tilde{C} \in \text{Gr}(\tilde{k}, n)$:

$$C = \begin{bmatrix} t_1 & t_2 & \cdots & t_n \\ t_1 \sigma_1 & t_2 \sigma_2 & \cdots & t_n \sigma_n \\ \vdots & \vdots & \ddots & \vdots \\ t_1 \sigma_1^d & t_2 \sigma_2^d & \cdots & t_n \sigma_n^d \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} \tilde{t}_1 & \tilde{t}_2 & \cdots & \tilde{t}_n \\ \tilde{t}_1 \sigma_1 & \tilde{t}_2 \sigma_2 & \cdots & \tilde{t}_n \sigma_n \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{t}_1 \sigma_1^{\tilde{d}} & \tilde{t}_2 \sigma_2^{\tilde{d}} & \cdots & \tilde{t}_n \sigma_n^{\tilde{d}} \end{bmatrix} \quad (4.7)$$

stating that $\lambda \subset C$ and $\tilde{\lambda} \subset \tilde{C}$. In other words, the existence of $\tau(z), \tilde{\tau}(z)$, gives the 2-vectors $\tau_0, \dots, \tau_d, \tilde{\tau}_0, \dots, \tilde{\tau}_{\tilde{d}}$, which witness $\lambda \subset C$ and $\tilde{\lambda} \subset \tilde{C}$.

Lemma 4.7. With $t_i \tilde{t}_i = \prod_{j \neq i} (\sigma_j - \sigma_i)^{-1}$, we automatically have $C \cdot \tilde{C}^T = 0$.

Since $k + \tilde{k} = n$, we have that C and $\tilde{C}$ are orthogonal complements of each other. In other words, $C = \tilde{C}^\perp$ as points in $\text{Gr}(k, n)$. The scattering equations are now the constraint that $\lambda \subset C \subset \tilde{\lambda}^\perp$.

Following [AHBCT11], we view (4.7) using the Veronese map $\theta : \mathbb{P}^1 \rightarrow \mathbb{P}^{k-1}$ (with $k = d + 1$) induced by the map

$$\theta : \mathbb{C}^2 \rightarrow \text{Sym}^{k-1} \mathbb{C}^2 \cong \mathbb{C}^k, \quad v \mapsto v \otimes v \otimes \cdots \otimes v.$$

The representation $\text{Sym}^{k-1} \mathbb{C}^2$ of $\text{GL}(2)$ gives a homomorphism $\theta : \text{GL}(2) \rightarrow \text{GL}(k)$, and the Veronese map is equivariant with respect to the actions of $\text{GL}(2)$:

$$\theta(g \cdot v) = \theta(g) \cdot \theta(v), \quad g \in \text{GL}(2), v \in \mathbb{C}^2.$$

The map $\theta \times \theta \cdots \theta : (\mathbb{C}^2)^n \rightarrow (\mathbb{C}^{k+1})^n$ descends to a rational *Veronese map* $\theta : \text{Gr}(2, n) \dashrightarrow \text{Gr}(k, n)$ given by the formula

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \end{bmatrix} \mapsto \begin{bmatrix} a_1^{k-1} & a_2^{k-1} & \cdots & a_n^{k-1} \\ a_1^{k-2} b_1 & a_2^{k-2} b_2 & \cdots & a_n^{k-2} b_n \\ \vdots & \vdots & \ddots & \vdots \\ b_1^{k-1} & b_2^{k-1} & \cdots & b_n^{k-1} \end{bmatrix}$$

Note that θ is only a rational map. It is not, for example, defined on the $\binom{n}{2}$ points $\text{span}(e_i, e_j) \in \text{Gr}(2, n)$.

The matrices $C, \tilde{C}$ of (4.7) are, generically, those of the form $\theta(V)$ for $V \in \text{Gr}(2, n)$. The scattering correspondence can now be viewed as a subvariety of $\text{Gr}(2, n) \times \Lambda_n^4$:

$$\mathcal{J}_d = \mathcal{J}_{k-1} := \{(V, \lambda, \tilde{\lambda}) \mid \lambda \in \theta(V) \subset \tilde{\lambda}^\perp\} \subset \text{Gr}(2, n) \times \Lambda_n^4. \quad (4.8)$$

The fiber of $\mathcal{J}_d$ over generic $(\lambda, \tilde{\lambda})$ can be identified with the fiber of $\tilde{\mathcal{J}}_d$ over the corresponding $P \in P_n^4$, where $p_i = \lambda_i \tilde{\lambda}_i$.

Definition 4.8. Define the *ABCT variety* $V(k, n) := \overline{\theta(\text{Gr}(2, n))} \subset \text{Gr}(k, n)$ as the closure of the image of θ , a $(2n-4)$ -dimensional irreducible subvariety of the Grassmannian.

Remark 4.9. Let us mention the following natural appearance of more positive geometry.

Conjecture 4.10. The closure $V(k, n)_{\geq 0} := \overline{\theta(\text{Gr}(2, n)_{>0})}$ is a positive geometry.

Note that the face structure of $V(k, n)_{\geq 0}$ and $\text{Gr}(2, n)_{\geq 0}$ differ, even though their interiors are naturally diffeomorphic. I expect the face structure of $V(k, n)_{\geq 0}$ to be closely related to that of the momentum amplituhedron [DFLP19] (see Section 4.7.1) in the same way that the face structure of $(\mathcal{M}_{0, n})_{\geq 0}$ matches that of the associahedron.

4.6. Extracting spinor-helicity gluon amplitudes

In Section 3.10, we saw that the scalar amplitude could be obtained by interpreting the scattering equations as a scattering map from $\mathcal{M}_{0, n}$ to kinematic space K_n . To obtain spinor-helicity amplitudes, we similarly take the scattering correspondence $\mathcal{J}_d$ (4.8) and view it as a map from $\text{Gr}(2, n)$ to a subvariety of Λ_n^4 . Since $\dim \text{Gr}(2, n) = 2n-4$, we need a $(2n-4)$ -dimensional subvariety of Λ_n^4 .

Let $[n] = N \sqcup P$, where the particles in N have negative helicity and the particles in P have positive helicity. Let $k = |N|$ and $n-k = |P|$. We define an “am-bitwistor” subvariety $\Lambda_{N, P}$ of Λ_n^4 as follows. Let $\pi_N : \mathbb{C}^n \rightarrow \mathbb{C}^N$ and $\pi_P : \mathbb{C}^n \rightarrow \mathbb{C}^P$ denote the orthogonal projections. Viewing $(\lambda_i, i \in N)$ as a $2 \times k$ matrix defining a two-plane $\lambda_N \subset \mathbb{C}^N$ and similarly $\tilde{\lambda}_P \subset \mathbb{C}^P$. Define

$$\Lambda_{N, P} := \{(\lambda, \tilde{\lambda}) \in \Lambda_n^4 \mid (\lambda, \tilde{\lambda}) \in \text{Gr}(2, \pi_N^{-1}(\lambda_N)) \times \text{Gr}(2, \pi_P^{-1}(\tilde{\lambda}_P))\} \subset \Lambda_n^4.$$

Since $\dim \pi_N^{-1}(\lambda_N) = n-k+2$ and $\dim \pi_P^{-1}(\tilde{\lambda}_P) = k+2$, the product of Grassmannians has dimension $2(n-k)+2k = 2n$, and the subvariety of the product satisfying $\lambda \perp \tilde{\lambda}$ has dimension $2n-4$. This is our desired subvariety, and the correspondence $\mathcal{J}_d$ defines a dominant pushforward map

$$\Phi(\lambda_N, \tilde{\lambda}_P) : \text{Gr}(2, n) \rightarrow \Lambda_{N, P}.$$

Note that in $\Lambda_{N,P}$, the spinors $\lambda_i, i \in N$ and $\tilde{\lambda}_i, i \in P$ are considered fixed, while the remaining spinors $\lambda_i, i \in P$ and $\tilde{\lambda}_i, i \in N$ are considered varying.

By [MS14, HZ18, DFLP19, HKZ22], the spinor-helicity amplitude $A_{N,P}$ can be obtained as the coefficient of

$$\prod_{i \in N} (d\tilde{\lambda}_i^1 \wedge d\tilde{\lambda}_i^2) \prod_{j \in P} (d\lambda_j^1 \wedge d\lambda_j^2)$$

in

$$\Phi(\lambda_N, \tilde{\lambda}_P) * \Omega(\text{Gr}(2, n)_{\geq 0}) \wedge d^4(\lambda \tilde{\lambda}).$$

To this computation, we pull the $(2n-4)$ -form $\Theta = \Phi(\lambda_N, \tilde{\lambda}_P) * \Omega(\text{Gr}(2, n)_{\geq 0})$ back to $\hat{\Lambda}_n^4$. Wedging with $d^4(\lambda \tilde{\lambda})$ gives us a well-defined $2n$ -form on $\text{Mat}_{2,n} \times \text{Mat}_{2,n}$. We explain in the following example.

Example 4.11. Let us consider the case $|N| = k = 2$ and $|P| = n - 2$, called “MHV amplitudes”. Then $\dim \pi_N^{-1}(\lambda_N) = n$ so $\pi_N^{-1}(\lambda_N) = \mathbb{C}^n$ while $\dim \pi_P^{-1}(\lambda_P) = 4$. If $(\lambda, \tilde{\lambda}) \in \Lambda_{N,P}$ then $\dim \lambda^\perp = n - 2$ so generically we must have $\tilde{\lambda} = \lambda^\perp \cap \pi_P^{-1}(\lambda_P)$. In other words, the projection

$$\Lambda_{N,P} \rightarrow \text{Gr}(2, \pi_N^{-1}(\lambda_N)) \cong \text{Gr}(2, n), \quad (\lambda, \tilde{\lambda}) \mapsto \lambda$$

is birational. The composition of $\Phi(\lambda_N, \tilde{\lambda}_P)$ with this projection is an isomorphism, and thus the pushforward form $\Omega = \Phi(\lambda_N, \tilde{\lambda}_P) * \Omega(\text{Gr}(2, n)_{\geq 0})$ is simply the canonical form on $\text{Gr}(2, n)$. As in Exercise 1.29, we pull the form back to a $(2n-4)$ -form Θ on $\text{Mat}_{2,n}$. We have, with $N = \{a, b\}$, up to a constant,

$$\Theta = \frac{\langle ab \rangle^2}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle} \prod_{j \in P} (d\lambda_j^1 \wedge d\lambda_j^2) + \text{other terms},$$

where the other terms involve $d\lambda_i^\alpha$ for $i \in N$, and vanish if we assume that λ_i is constant for $i \in N$. (Note that this form is invariant under scaling each of the spinors λ_i .) Now, the momentum conservation constraint $\lambda \tilde{\lambda} = 0$ consists of four equations, and $d^4(\lambda \tilde{\lambda})$ is the wedge of them. We have, up to a constant,

$$d^4(\lambda \tilde{\lambda}) = \langle ab \rangle^2 d\tilde{\lambda}_a^1 \wedge d\tilde{\lambda}_a^2 \wedge d\tilde{\lambda}_b^1 \wedge d\tilde{\lambda}_b^2 + \text{other terms}.$$

The other terms either involve $d\tilde{\lambda}_j$ for $j \in P$, or they vanish in the product $\Theta \wedge d^4(\lambda \tilde{\lambda})$. Thus,

$$\Theta \wedge d^4(\lambda \tilde{\lambda}) = \frac{\langle ab \rangle^4}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle} \prod_{i \in N} (d\tilde{\lambda}_i^1 \wedge d\tilde{\lambda}_i^2) \prod_{j \in P} (d\lambda_j^1 \wedge d\lambda_j^2)$$

and

$$A_{N,P} = A_{N,P}^{\text{gluon}} = \frac{\langle ab \rangle^4}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle}.$$

This is the famed *Parke-Taylor formula* for MHV amplitudes.

We can also use the scattering correspondence diagram

$$\begin{array}{ccc} & \mathcal{I}_d & \\ p \swarrow & & \searrow q \\ \text{Gr}(2, n) & & \Lambda_n^4 \end{array}$$

to define a scattering form on spinor kinematic space, analogous to Definition 3.25.

Definition 4.12. The Yang-Mills scattering form is the $(2n-4)$ -form

$$\Upsilon_n := q_* p^* \Omega(\text{Gr}(2, n)_{\geq 0})$$

on spinor kinematic space Λ_n^4 .

We expect that Υ_n is equal to the scattering form defined in [HZ18]. We formulate this as a conjecture.

Conjecture 4.13. The $(2n-4)$ -form Υ_n coincides with the $\mathcal{N} = 4$ SYM scattering form defined in [HZ18].

4.7. Exercises and Problems

Exercise 4.14.

- (a) Prove Lemma 4.1.
- (b) Let $n = 4$. Prove that

$$\frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 14 \rangle} = \frac{[kl]^4}{[12][23][34][14]}$$

where $\{i, j, k, l\} = \{1, 2, 3, 4\}$.

- (c) Prove Lemma 4.7.
- (d) Figure out the meaning of, and then prove, the following identity of four-point amplitudes:

$$A_4(1^-, 2^-, 3^+, 4^+) + A_4(2^-, 1^-, 3^+, 4^+) + A_4(2^-, 3^+, 1^-, 4^+) = 0.$$

- (e) Prove the identity

$$A_5(1, 2, 5, 3, 4) = A_5(1, 2, 4, 3, 5) + A_5(1, 4, 2, 3, 5) + A_5(1, 4, 3, 2, 5)$$

for any choice of helicities for 1, 2, 3, 4, 5.

Exercise 4.15. Let $X \subset \text{Gr}(k, n) \times \text{Gr}(\ell, n)$ denote the subvariety of pairs $(V, W) \in \text{Gr}(k, n) \times \text{Gr}(\ell, n)$ such that $V \perp W$. Figure out or look up the generators for the ideal that cuts out X . In the case $k = 2 = \ell$, check that one recovers the equations of (4.1).

Exercise 4.16. Calculate the helicity amplitude $A_{N,P}$ when $|N| = |P| = 3$ and $n = 6$.

Problem 4.17. Use the geometry and topology of $\tilde{\mathcal{I}}$ to prove the CHY formula (4.5).

Problem 4.18. ² Recall the subvariety $V(k, n) \subset \text{Gr}(k, n)$ from Definition 4.8.

- (a) The variety $V(3, 6)$ is a codimension one irreducible subvariety of $\text{Gr}(3, 6)$. It is cut out by the condition that 6 points on $\mathbb{P}^2$ lie on a conic. Find, in Plücker coordinates, the relation that cuts out this variety.
- (b) Calculate the ideal of $V(k, n)$.
- (c) What is the degree of the subvariety $V(k, n) \subset G(k, n)$? What is the cohomology class $[V(k, n)] \in H^*(\text{Gr}(k, n))$? (We expect this to be closely related to, or identical to, the sum of the cohomology classes of the $(2n - 4)$ -dimensional positroid varieties which appear in the formula for the amplitude in [AHBC⁺16].)

4.7.1. Spinor-helicity polytopes

As mentioned in Remark 3.37, the pullback $\iota_{\underline{c}}^* \Psi_n$ of the scattering form to $H(\underline{c})$ is the canonical form of the associahedron polytope, a positive geometry. Analogously, in momentum-twistor coordinates, the (super-)Yang-Mills amplitude can be obtained from the canonical form of the *amplituhedron* [AHT14], another positive geometry. In recent years, interest has turned to the momentum amplituhedron $\mathcal{M}_{n,k} \subset \Lambda_n^4$ [DFLP19, HZ18, HKZ22].

Let L be a full-rank $(n - k + 2) \times n$ matrix and $\tilde{L}$ be a full-rank $(k + 2) \times n$ matrix, viewed as linear maps. They induce (rational) maps on Grassmannians

$$\tilde{L} : \text{Gr}(k, n) \dashrightarrow \text{Gr}(k, k + 2), \quad L : \text{Gr}(k, n) \dashrightarrow \text{Gr}(n - k, n - k + 2)$$

where the first map is just $V \mapsto \tilde{L}(V)$ for $V \in \text{Gr}(k, n)$ and the second map is $V \mapsto L(V^\perp)$. These combine to a rational map

$$(\tilde{L}, L) : \text{Gr}(k, n) \dashrightarrow \text{Gr}(k, k + 2) \times \text{Gr}(n - k, n - k + 2).$$

²This problem is addressed by Agostini, Ramesh, and Shen [ARS24]. In particular, (b), (c) have been solved for $k = 3$.

Let $\text{span}(L), \text{span}(\tilde{L}) \subset \mathbb{C}^n$ denote the $(n-k+2)$ -dimensional (resp. $(k+2)$ -dimensional) subspaces spanned by the rows of L (resp. $\tilde{L}$). We have a map

$$\begin{aligned} \text{Gr}(k, k+2) \times \text{Gr}(n-k, n-k+2) &\rightarrow \text{Gr}(2, k+2) \times \text{Gr}(2, n-k+2) \\ &\cong \text{Gr}(2, \text{span}(\tilde{L})) \times \text{Gr}(2, \text{span}(L)) \quad (4.9) \\ &\hookrightarrow \text{Gr}(2, n) \times \text{Gr}(2, n), \end{aligned}$$

where the first map takes orthogonal complements. In the following, we abuse notation by letting Λ_n^4 refer to a subvariety of $\text{Gr}(2, n) \times \text{Gr}(2, n)$ (instead of the cones over such). Let $\Phi_{(\tilde{L}, L)}$ denote the composition of $(\tilde{L}, L)$ with (4.9).

Lemma 4.19 ([DFLP19]). The map $\Phi_{(\tilde{L}, L)}$ lands in Λ_n^4 .

Proof. Equip $\mathbb{C}^n$ and $\mathbb{C}^{n-k+2}$ with their natural inner products, and view $L : \mathbb{C}^n \rightarrow \mathbb{C}^{n-k+2}$ and $L^T : \mathbb{C}^{n-k+2} \rightarrow \mathbb{C}^n$ as linear maps so that $\langle a, L(b) \rangle_{\mathbb{C}^{n-k+2}} = \langle L^T(a), b \rangle_{\mathbb{C}^n}$. Let $Y = L(V^\perp)$. Let $v \in V^\perp$ and $u \in Y^\perp \subset \mathbb{C}^{n-k+2}$. Then we have $\langle v, L^T u \rangle_{\mathbb{C}^n} = \langle L(v), u \rangle_{\mathbb{C}^{n-k+2}} = 0$ since $L(v) \in Y$. Thus $V^\perp \subset (L^T(Y^\perp))^\perp$ or $L^T(Y^\perp) \subset V$. Here, $L^T(Y^\perp) \subset \mathbb{C}^n$ is a 2-dimensional subspace. Similarly, with $\tilde{Y} = \tilde{L}(V)$, we have $\tilde{L}^T(\tilde{Y}^\perp) \subset V^\perp$. It follows that $L^T(Y^\perp)$ and $\tilde{L}^T(\tilde{Y}^\perp)$ are orthogonal. $\square$

The image of $(\tilde{L}, L)$ in $\text{Gr}(k, k+2) \times \text{Gr}(n-k, n-k+2)$ has codimension 4, and thus dimension $2k+2(n-k)-4 = 2n-4$. The subvariety $\Phi_{(\tilde{L}, L)}(\text{Gr}(k, n)) \subset \Lambda_n^4$ also has dimension $2n-4$, and it is exactly these subvarieties that the scattering form Υ_n of Definition 4.12 should be pulled back to, to obtain Yang-Mills amplitudes.

We now specialize to the case where all Grassmannians and matrices are defined over $\mathbb{R}$. The *momentum amplituhedron* $\mathcal{M}_{n,k}$ [DFLP19] is defined to be the image of $\text{Gr}(k, n)_{\geq 0}$ under the map $(\tilde{L}, L)$, or under the map $\Phi_{(\tilde{L}, L)}$, when $(\tilde{L}, L)$ satisfies the appropriate positivity condition. Positivity conditions are discussed in [DFLP19, (2.16) and below (2.32)], which we do not review here. An important motivation of our presentation is the work of He-Kuo-Zhang [HKZ22], who conjecture that the scattering map of Section 4.5 sends $\text{Gr}(2, n)_{>0}$ diffeomorphically onto the interior of the momentum amplituhedron.

Remark 4.20. The amplituhedron, defined as a subspace of the Grassmannian $\text{Gr}(k, k+m)$ is a direct generalization of a polytope, with the Grassmannian replacing projective space. In contrast, the momentum amplituhedron lives in the rather more complicated Λ_n^4 which can be identified with a partial flag variety. There are nevertheless advantages to the latter setup. The analogue of the scattering equations, and the twistor string variety $V(k, n) \subset \text{Gr}(k, n)$ is not known in the setting of amplituhedron, nor is the analogue of the HKZ conjecture [HKZ22]. In momentum space, the pairs $(\lambda, \tilde{\lambda}) \in \Lambda_n^4$ exhibits the natural

parity duality $\lambda \leftrightarrow \tilde{\lambda}$, whereas for the amplituhedron parity duality is more involved [GL20].

We suggest the following definition, an analogue of the Grassmann polytopes of [Lam16b].

Definition 4.21. A *spinor-helicity polytope* is the image

$$P := \Phi_{(\tilde{L}, L)}(\text{Gr}(k, n)_{\geq 0}) \cong (\tilde{L}, L)(\text{Gr}(k, n)_{\geq 0}),$$

when the map is well-defined on the totally-nonnegative Grassmannian.

Problem 4.22. Find conditions on $(\tilde{L}, L)$ to guarantee that $\Phi_{(\tilde{L}, L)}(\text{Gr}(k, n)_{\geq 0})$ is well-defined.

In the case of Grassmann polytopes, this problem is solved by Karp [Kar17]. A sufficient condition is discussed earlier in [Lam16b] for “tame Grassmann polytopes”.

Let us also suggest the following Schubert calculus problem which is the spinor-helicity version of [Lam16a]. Recall the positroid varieties $\Pi_f \subset \text{Gr}(k, n)$ [KLS13].

Problem 4.23. What is the cohomology class of $[\Phi_{(\tilde{L}, L)}(\Pi_f)]$? When do we have $\dim \Phi_{(\tilde{L}, L)}(\Pi_f) = \dim \Pi_f$, and in that case, what is the degree of the map $\Pi_f \rightarrow \Phi_{(\tilde{L}, L)}(\Pi_f)$?

5. String amplitudes

In this section, we introduce string theory amplitudes, following the presentation of [AHHL21b]. For other perspectives, see for example [SS13, Miz20].

5.1. Open string amplitudes at tree level

Recall that we defined the Koba-Nielsen potential ϕ_X on $\mathcal{M}_{0,n}$ in Definition 3.5.

Definition 5.1. The *tree level n -point open superstring amplitude* is the function on kinematic space K_n defined as the following integral:

$$I_n(\underline{s}) := (\alpha')^{n-3} \int_{(\mathcal{M}_{0,n})_{>0}} \phi_{\alpha'X} \Omega((\mathcal{M}_{0,n})_{>0}) = (\alpha')^{n-3} \int_{(\mathcal{M}_{0,n})_{>0}} \prod_{a < b} (ab)^{\alpha' s_{ab}} \Omega_{0,n}. \quad (5.1)$$

This function should be thought of as the analytic continuation of the integral in the domain where the integral converges.

In string theory, particles are replaced by either circles S^1 (in closed string theory) or intervals $[0, 1]$ (in open string theory). The inverse string tension α' is a parameter that measures the size of the string. In the usual formula for vibrating strings, we have that string tension and string length are inversely related. As $\alpha' \rightarrow 0$, the strings shrink in size and one obtains point particle scattering in the limit. Whereas the field theory scattering amplitude is a sum over L -loop Feynman diagrams with n distinguished leaves, the string theory amplitude is an integral over $\mathcal{M}_{L,n}$, the moduli space of genus L curves with n marked points.

We consider only genus zero string amplitude. In closed string theory, the integral is over all complex points of $\mathcal{M}_{0,n}$. In open string theory, the integral is over the set of real points of $\mathcal{M}_{0,n}$. The choice of a component $(\mathcal{M}_{0,n})_{>0}$ in (5.1) is precisely analogous to the sum over *planar* trees in Theorem 3.14.

Let us consider (5.1) for $n = 4$. In this case, setting $s = s_{12}$ and $t = s_{23}$, we have

$$I_4(s, t) = \alpha' \int_0^1 \frac{du}{u(1-u)} u^{\alpha's} (1-u)^{\alpha't} = \alpha' B(\alpha's, \alpha't), \quad (5.2)$$

where $B(s, t)$ is the Euler Beta function

$$B(s, t) = \int_0^1 \frac{du}{u(1-u)} u^s (1-u)^t = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)}. \quad (5.3)$$

The integral converges when $\text{Re}(s), \text{Re}(t) > 0$ and $B(s, t)$ extends to a meromorphic function in the two complex variables s, t . Since the gamma function has no zeroes, the poles of $B(s, t)$ are the poles of $\Gamma(s)\Gamma(t)$ which occur when s or t is a nonpositive integer. Thus $B(s, t)$ has a pole at $s = 0$, at $t = 0$, and a zero at $s + t = 0$. As a function of α' , the integral $I_4(s, t)$ thus has a simple pole at $\alpha' = 0$; the prefactor α' in (5.2) ensures a well-defined limit as $\alpha' \rightarrow 0$. Indeed, field theory amplitudes are recovered from string theory amplitudes in the limit as inverse string tension goes to 0:

$$\lim_{\alpha' \rightarrow 0} I_4(s, t) = \frac{s+t}{st} = \frac{1}{s} + \frac{1}{t} = A_4^{\phi^3}(s, t), \quad (5.4)$$

where we have used the approximation $\alpha'\Gamma(\alpha's) \sim \frac{1}{s}$ when $\alpha' \rightarrow 0$. This limit is also called a “low-energy” limit of string theory.

The Beta function was studied by Euler and Legendre in the 18th and 19th centuries. In the 20th century, Veneziano [Ven68] noticed the interpretation of the Beta function as an amplitude, and this was one of the key ingredients that started string theory.

The connected components of $\mathcal{M}_{0,n}(\mathbb{R})$ are in bijection with dihedral orderings of $1, 2, \dots, n$. For a dihedral ordering β , we denote by $\mathcal{M}_{0,n}(\beta)$ the

corresponding connected component, and by $\Omega(\beta)$ its canonical form. We have the following larger family of partial string amplitudes

$$I_n(\beta|\gamma) := (\alpha')^{n-3} \int_{\mathcal{M}_{0,n}(\gamma)} \phi_{\alpha'X} \Omega(\beta),$$

where β, γ are two dihedral orderings. These are string theory versions of partial ϕ^3 -amplitudes; see Remark 3.15 and [Miz20] for further discussion. When β is fixed, say $\beta = +$ is the positive ordering, and γ varies we have a family of integral functions $I_n(+|\gamma)$. These integral functions are not linearly independent – they span a vector space of dimension $(n-3)!$.

5.2. Toric stringy integrals

The string integral (5.1) belongs to a class of integral functions called *stringy integrals* in [AHLS21b] and *Euler integrals* in [MHMT23], which have the following form:

$$\int \frac{y_1^{\tau_1} \cdots y_n^{\tau_n}}{p_1^{c_1} \cdots p_r^{c_r}} \frac{dy_1}{y_1} \wedge \cdots \wedge \frac{dy_n}{y_n},$$

where $p_1, p_2, \dots, p_r$ are polynomials in $y_1, \dots, y_n$, and $\tau_1, \dots, \tau_n, c_1, \dots, c_r$ are complex parameters. Besides string amplitudes, Euler integrals also include hypergeometric integrals and various generalizations, and also Feynman integrals. We refer the reader to the survey [MHMT23] for a discussion of the many beautiful properties of these integrals.

Here we focus on an algebro-geometric perspective. Let $p(y) \in \mathbb{C}[y]$ be a polynomial in variables $y_1, \dots, y_d$ with non-vanishing constant term, and let $p_1(y), p_2(y), \dots, p_r(y)$ be the irreducible factors of p . To fix conventions, we assume that the constant terms in $p_1(y), \dots, p_r(y)$ are all equal to 1. Let $T_y \cong (\mathbb{C}^\times)^d = \text{Spec}(\mathbb{C}[y_1^{\pm 1}, \dots, y_d^{\pm 1}])$ denote the torus with coordinates $y_1, \dots, y_d$. Let $U \subset T_y$ be the open subset where $p(y) \neq 0$. The variety U is a very affine variety.

Lemma 5.2. The very affine variety U has character lattice Λ equal to the lattice of Laurent monomials in the elements $y_1, y_2, \dots, y_d, p_1(y), \dots, p_r(y)$.

Proof. Suppose that $r(y) = f(y)/g(y) \in \mathbb{C}(y_1, \dots, y_d)$ is a rational function that is a unit in $\mathbb{C}[U]$, and $f(y), g(y)$ have no common factors. Any irreducible factor $h(y)$ of $f(y)$ or $p(y)$ which is not a monomial in $y_1, \dots, y_d$ defines a non-empty hypersurface $h(y) = 0$ in the torus T_y . It follows that $h(y)$ must equal one of the $p_1(y), \dots, p_r(y)$. Thus $r(y)$ is a Laurent monomial in the functions $y_1, y_2, \dots, y_d, p_1(y), \dots, p_r(y)$. Furthermore, irreducibility implies that there are no monomial relations between these Laurent monomials (as regular functions on U). The result follows. $\square$

Thus U has an intrinsic scattering potential (see Section 6.2)

$$\phi_{\tau,c} := \prod_{i=1}^d y_i^{\tau_i} \prod_{j=1}^r p_j(y)^{-c_j}.$$

We have chosen the signs of the exponents for reasons that will be clear later.

The variety U can naturally be identified with an open subset of a toric variety. Let $P \subset \mathbb{Z}^d$ be the Newton polytope of $p(y)$; this is the convex hull of the exponent vectors of the monomials in $p(y)$. We assume that P is full-dimensional and let $A = P \cap \mathbb{Z}^d = \{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_p\}$ be the set of lattice points in P . Consider the toric variety X_A that is the closure of the image of the map

$$T_y \ni (y_1, \dots, y_d) \mapsto (y^{\mathbf{a}_1} : \dots : y^{\mathbf{a}_p}) \in \mathbb{P}^{p-1}. \quad (5.5)$$

This map is an inclusion as long as the $\mathbb{Z}$ -affine span of $\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_p\}$ is equal to $\mathbb{Z}^d$. We will assume this by replacing $p(y)$ with a power of it, not changing the variety U .

The polynomial $p(y)$ is a linear combination of the monomials $y^{\mathbf{a}_i}$. Thus the hypersurface $\{p(y) = 0\} \subset T_y$ is the intersection of T_y with a hyperplane $H \subset \mathbb{P}^{p-1}$ cut out by the corresponding linear equation. Then U can be identified with $T_y \setminus H$.

If $p(y)$ has positive coefficients, then one of the connected components of U is simply the positive part $(T_y)_{>0} = \mathbb{R}_{>0}^d$ of the torus T_y . We denote this positive component by $U_{>0}$. Henceforth, we make the

Assumption 5.3. The polynomial $p(y)$ has positive coefficients.

Definition 5.4. The string integral of U is the integral function

$$I(\tau, c) := (\alpha')^d \int_{U_{>0}} \phi_{\alpha'\tau, \alpha'c} \Omega$$

where $\Omega = \frac{dy_1}{y_1} \wedge \dots \wedge \frac{dy_n}{y_n}$ is the canonical form of $U_{>0}$, and $(\tau, c) \in \Lambda_{\mathbb{C}}$.

We investigate the convergence of the string integral, assuming that $(\tau, c) \in \Lambda_{\mathbb{R}}$.

Example 5.5. We consider the Beta function (5.3). Setting $y = u/(1-u)$, we have

$$B(s, t) = \int_0^1 \frac{du}{u(1-u)} u^s (1-u)^t = \int_0^\infty y^s (1+y)^{-t-s} \frac{dy}{y}.$$

So $B(s, t)$ is a string integral where we have a single polynomial $p(y) = 1 + y$. We may break up the integral as $\int_0^\infty = \int_0^1 + \int_1^\infty$. When $y \rightarrow 0$, the integrand behaves like $y^{s-1} dy$ and $\int_0^1 y^{s-1} dy$ converges when $s > 0$. When $y \rightarrow \infty$, both the integrand behaves like $y^{s+(-t-s)} = y^{-t}$, and $\int_1^\infty y^{-t-1} dy$ converges when $t > 0$. Thus the integral converges (absolutely) if $s > 0$ and $t > 0$.

Let $\Lambda_y^\vee$ be the cocharacter lattice of T_y . The vector space $\Lambda_{y,\mathbb{R}}^\vee$ can be viewed as the Lie algebra of the real Lie group $T_y(\mathbb{R})$. The exponential map $\exp: \Lambda_{y,\mathbb{R}}^\vee \rightarrow U_{>0} = (T_y)_{>0}$ is an isomorphism, and we may rewrite

$$I(\tau, c) = (\alpha')^d \int_{\Lambda_{y,\mathbb{R}}^\vee} \prod_{i=1}^d \exp(\alpha' \tau_i Y_i) \prod_{j=1}^r p_j(\exp(Y))^{-\alpha' c_j} dY_1 \cdots dY_d. \quad (5.6)$$

where $Y_i = \log y_i$ are coordinates on $\Lambda_{y,\mathbb{R}}^\vee$. Define the (positive) tropicalization of rational functions in $y_1, \dots, y_d$ by

$$y_i \mapsto Y_i, \quad + \mapsto \min, \quad \times \mapsto +, \quad \div \mapsto -.$$

This assignment makes sense for the potential $\phi_{\alpha'\tau, \alpha'c}$ even when α', τ, c are not integers, and we obtain

$$\text{trop}(\phi) = \text{trop}(\phi_{\alpha'\tau, \alpha'c}) := \alpha' \left(\sum_{i=1}^d \tau_i Y_i - \sum_{j=1}^r c_j \text{trop}(p_j)(Y) \right).$$

We view this as a piecewise linear function on $\Lambda_{y,\mathbb{R}}^\vee$. We call a piecewise linear function *positive* if it is positive on $\Lambda_{y,\mathbb{R}}^\vee \setminus \{0\}$. The analysis of Example 5.5 can be generalized to give the following result. It is often formulated in terms of Minkowski sums of Newton polytopes [AHHL21b, NP13, MHMT23].

Theorem 5.6 ([AHHL21b, Claim 3]). Suppose $(\tau, c) \in \Lambda_{\mathbb{R}}$. The integral (5.6) converges if and only if the integral

$$\int_{\Lambda_{y,\mathbb{R}}^\vee} \exp(-\text{trop}(\phi_{\alpha'\tau, \alpha'c})(Y)) dY_1 \cdots dY_d \quad (5.7)$$

converges. This is the case if and only if the piecewise linear function $\text{trop}(\phi)(Y)$ is positive on $\Lambda_{y,\mathbb{R}}^\vee$.

Idea of Proof. Bound the integrand $\prod_{i=1}^d \exp(\tau_i Y_i) \prod_{j=1}^r p_j(\exp(Y))^{-c_j}$ above and below by multiples of $\exp(-\text{trop}(\phi_{\tau,c})(Y))$. $\square$

Example 5.7. We continue Example 5.5. We have

$$\text{trop}(y^s(1+y)^{-t-s}) = sY - (s+t) \min(0, Y) = \begin{cases} sY & \text{if } Y > 0, \\ -tY & \text{if } Y < 0. \end{cases}$$

This function is positive if and only if $s > 0$ and $t > 0$.

The domains of linearity of the piecewise linear function $\text{trop}(\phi)$ give a complete fan $\mathcal{F}$ in $\Lambda_{y,\mathbb{R}}^\vee$, where $\text{trop}(\phi)$ restricts to a linear function on each maximal cone of $\mathcal{F}$. The integral (5.7) becomes a sum of integrals over the maximal cones C of $\mathcal{F}$,

$$\int_C \exp(-(a_1 Y_1 + \cdots + a_d Y_d)) dY_1 \cdots dY_d,$$

where a_i are linear functions in α', τ, c . These integrals are Laplace transforms of cones, and are rational functions in the a_i .

Proposition 5.8. Let C be a full-dimensional cone and ℓ a linear function that is positive on $C \setminus \{0\}$. Then

$$\int_C \exp(-\ell(Y)) dY_1 \cdots dY_d = \text{Vol}(C \cap \{\ell(Y) \leq 1\})$$

where Vol denotes the normalized volume where the standard simplex has volume 1.

Example 5.9. We have

$$\int_{\mathbb{R}_{>0}^d} \exp(-(a_1 Y_1 + \cdots + a_d Y_d)) dY_1 \cdots dY_d = \prod_{i=1}^d \int_{\mathbb{R}_{>0}} \exp(-a_i Y_i) dY_i = \prod_{i=1}^d \frac{1}{a_i}$$

which is the normalized volume of a simplex with vertices $0, \frac{1}{a_1} e_1, \dots, \frac{1}{a_d} e_d$, where e_i are coordinate vectors dual to the functions Y_i .

Corollary 5.10 ([AHHL21b, Claim 3]). Suppose that (τ, c) are such that $\text{trop}(\phi)$ is positive. Then we have $\lim_{\alpha' \rightarrow 0} I(\tau, c) = \text{Vol}(\text{trop}(\phi_{\tau, c})) \leq 1$.

Proof. Summing over maximal cones of $\mathcal{F}$, we deduce from Proposition 5.8 that

$$\int_{\Lambda_{y,\mathbb{R}}^\vee} \exp(-\text{trop}(\phi_{\tau, c})(Y)) dY_1 \cdots dY_d = \text{Vol}(\text{trop}(\phi)) \leq 1.$$

Now,

$$\text{Vol}(\text{trop}(\phi_{\alpha' \tau, \alpha' c}) \leq 1) = \text{Vol}(\alpha' \text{trop}(\phi) \leq 1) = \frac{1}{(\alpha')^d} \text{Vol}(\text{trop}(\phi) \leq 1).$$

The factor $\frac{1}{(\alpha')^d}$ cancels out the prefactor of $(\alpha')^d$ in the definition of $I(\tau, c)$. Finally, by the proof of Theorem 5.6, the integral $I(\tau, c)$ is bounded above and below by $(\alpha')^d$ times the integral (5.7), up to a factor of $\gamma^{\alpha'}$ for positive constants γ . Since $\lim_{\alpha' \rightarrow 0} \gamma^{\alpha'} = 1$, the result follows. $\square$

Example 5.11. We continue Example 5.7. When $s, t > 0$, the inequality $\text{trop}(y^s(1+y)^{-t-s}) \leq 1$ cuts out the region $[-1/t, 1/s]$ which has volume $1/s + 1/t$. This agrees with (5.4).

We return to the string amplitude.

Theorem 5.12. The field theory limit of the open string amplitude is given by the planar ϕ^3 -amplitude:

$$\lim_{\alpha' \rightarrow 0} I_n(\underline{s}) = A_n^{\phi^3}(\underline{s}).$$

Proof. Let $y_1, \dots, y_d$, where $d = n - 3$ be the positive parameters of Section 1.5. By Proposition 1.16, the lattice Λ of Laurent monomials in u_{ij} is isomorphic to the lattice of Laurent monomials in $\{y_1, \dots, y_d\} \cup \{p_{ij}(y)\}$. Thus $\mathcal{M}_{0,n} \cong U$ where $U \subset T_y$ is defined by the non-vanishing of $p(y) = \prod p_{ij}(y)$.

Let $\Lambda^\vee$ be the intrinsic cocharacter lattice of $\mathcal{M}_{0,n}$. The functions U_{ij} and the functions Y_i are linear functions on $\Lambda_{\mathbb{R}}^\vee$. Define a linear map $\pi : \Lambda_{\mathbb{R}}^\vee \rightarrow \Lambda_{y,\mathbb{R}}^\vee$ by projection, that is, by the equality $Y_i = Y_i \circ \pi$. The key statement is that π sends maximal cones of $\text{trop}_{\geq 0} U$ to maximal cones of $\mathcal{F}$ (Exercise 5.21). The summation of Theorem 3.14 can be written as

$$A_n^{\phi^3} = \sum_{C \subset \text{trop}_{\geq 0} U} \prod_{(ij) \in \mathcal{D}} \frac{1}{X_{ij}}$$

summed over maximal cones of $\text{trop}_{\geq 0} U$ spanned by $X_{ij}, (ij) \in \mathcal{D}$. This in turn matches the summation

$$\int_{\Lambda_{y,\mathbb{R}}^\vee} \exp(-\text{trop}(\phi_{\tau,c})(Y)) dY_1 \cdots dY_d = \text{Vol}(C \cap (\text{trop}(\phi) \leq 1))$$

over maximal cones of $\mathcal{F}$. □

5.3. Scattering map revisited

We now interpret the scattering equations in terms of the integrals $I(\tau, c)$. The “method of steepest descent” or “saddle-point method” estimates the asymptotics of integrals

$$\int f(x) \exp(\alpha' g(x)) dx$$

when $\alpha' \rightarrow \infty$, by considering the behavior of the integral near the critical points of $g(x)$. In our setting, $g = \log \phi_{\tau,c}$, and the critical point equations are those for $\log \phi_{\tau,c}$, which are exactly the scattering equations in the case of the string integral. This is also the setting of the positive models of [ST21, Section 6].

Definition 5.13. The scattering equations for the string integral $I(\tau, c)$ are the critical point equations

$$d\log \phi_{\tau, c} = 0$$

on U .

In the string integral, positive coordinates $y_1, \dots, y_d$ on U have been distinguished. This choice gives a particularly elegant way to formulate the scattering map of Section 3.10 and an explanation of the subspaces $H(\underline{c}) \subset K_n$. Indeed, we may write

$$\prod_{(ij)} (ij)^{s_{ij}} = \prod_{(ij)} u_{ij}^{X_{ij}} = \prod_{i=1}^{n-3} y_i^{\tau_i} \prod_{ij} p_{ij}^{-c_{ij}}(y).$$

Under the positive parametrization of Section 1.5, when i, j satisfy $1 \leq i < j-1 < j \leq n-1$ we have

$$(ij) = p_{ij}(y) \times \text{monomial in } y_1, \dots, y_{n-3}.$$

It follows that for such i, j we have $c_{ij} = -s_{ij}$. Thus the subspace $H(\underline{c})$ is obtained by fixing the exponents c_{ij} and allowing τ_i to vary. More abstractly, the coordinates $(y_1, \dots, y_n)$ provide a sublattice $\Lambda' \subset \Lambda$. The orthogonal sublattice $(\Lambda')^\perp \subset \Lambda^\vee$ is a space of linear functions that are fixed on each subspace $H(\underline{c})$. The constants $\underline{c}$ are the values that elements of $(\Lambda')^\perp$ take.

We now generalize the scattering map $\Phi(\underline{c})$ of (3.20) to the general setting of a stringy integral. The scattering equations (Definition 5.13) can be written as a collection of d equations

$$\frac{1}{\phi} \frac{\partial \phi}{\partial y_i} = 0, \quad i = 1, 2, \dots, d$$

which we can write as

$$\frac{\tau_i}{y_i} = \sum_{j=1}^r \frac{c_j}{p_j(y)} \frac{\partial p_j(y)}{\partial y_i}, \quad i = 1, 2, \dots, d.$$

or

$$\tau_i = y_i \sum_{j=1}^r \frac{c_j}{p_j(y)} \frac{\partial p_j(y)}{\partial y_i}, \quad i = 1, 2, \dots, d.$$

Recall that (τ, c) are coordinates on $\Lambda_{\mathbb{R}}$. For fixed c , we view this as a (rational) map

$$\Phi(c) : U(\mathbb{R}) \rightarrow H(c) \cong \Lambda_{y, \mathbb{R}}$$

where $H(c) \subset \Lambda_{\mathbb{R}}$ is an affine subspace where the c -coordinates are held fixed and the τ -coordinates vary.

Theorem 5.14 ([AHHL21b]). Suppose that $I(\tau, c)$ converges, or equivalently by Theorem 5.6, that $\text{trop}(\phi_{\tau, c})$ is positive. Then we have

$$\Phi(c)_* \Omega_y = \lim_{\alpha' \rightarrow 0} I(\tau, c) d\tau_1 \wedge \cdots \wedge d\tau_d$$

as forms on $H(c)$. Furthermore, $\Phi(c)$ sends $\mathbb{R}_{>0}^d$ to the interior of the polytope $R^\vee$ polar dual to

$$R = \{\text{trop}(\phi) \leq 1\}$$

and $\Phi(c)_* \Omega_y$ is the canonical form of R .

Remark 5.15. The scattering map $\Phi(c) : U(\mathbb{R}) \rightarrow H(c) \cong \Lambda_{y, \mathbb{R}}$ can also be identified with the algebraic moment map of the toric variety X_A . See [AHBL17, Section 7.3] and [AHHL21b].

5.4. u -coordinates

Our setup distinguishes a subsemigroup inside the character lattice Λ . Call a character $\chi \in \Lambda$ *bounded* if it takes bounded values on $U_{>0}$. Write $\chi = \frac{a(y)}{b(y)}$ in reduced form, where $a(y)$ and $b(y)$ are polynomials in $y_1, \dots, y_d$.

Lemma 5.16. For a character $\chi = \frac{a(y)}{b(y)} \in \Lambda$, the following are equivalent:

1. χ is bounded,
2. the Newton polytope of $a(y)$ is contained inside the Newton polytope of $b(y)$,
3. the function $\text{trop}(\chi)$ is positive.

For example, all the five rational functions in (1.8) satisfy this criterion. Denote by $\Gamma \subset \Lambda$ the subsemigroup of bounded characters.

Lemma 5.17. With our assumption that the Newton polytope P of $p(y)$ is full-dimensional, we have that

1. Γ generates the rational function field $\mathbb{C}(y) = \mathbb{C}(U)$, and
2. $\Gamma \cup \Gamma^{-1}$ generate the coordinate ring $\mathbb{C}[U]$.

Proof. We may, and will, assume that P is sufficiently large by replacing $p(y)$ by a large power of it. By Lemma 5.16, the rational functions

$$\frac{y^{\mathbf{a}}}{p(y)}, \quad \mathbf{a} \in P \cap \mathbb{Z}^d$$

belong to Γ . Taking lattice points $\mathbf{a}'$ and $\mathbf{a}' + e_i$ shows that $y_i \in \mathbb{C}(\Gamma)$. It follows that $\mathbb{C}(\Gamma) = \mathbb{C}(y) = \mathbb{C}(U)$.

The condition that $\text{trop}(\chi)$ is positive from Lemma 5.16 is an open condition in $\Lambda_{\mathbb{R}}$, and defines an open polyhedral cone $B \subset \Lambda_{\mathbb{R}}$ so that $B \cap \Lambda = \Gamma$. Since $\Gamma \neq \emptyset$ we have that B is a non-empty full-dimensional cone. It follows that for B contains adjacent lattice points in all coordinate directions. In particular $y_1^{\pm 1}, \dots, y_d^{\pm 1}, p_1^{\pm 1}(y), \dots, p_r^{\pm 1}(y)$ all belong to $\Gamma\Gamma^{-1}$. Thus $\mathbb{C}[\Gamma \cup \Gamma^{-1}] = \mathbb{C}[U]$. $\square$

Definition 5.18. The u -coordinates on U are the generators of Γ . The affine closure of U is $\tilde{U} = \text{Spec}(\mathbb{C}[\Gamma])$.

Proposition 5.19. The partial compactification of $\tilde{U}$ is the affine open $X_A \setminus H$ in the projective toric variety X_A of (5.5). In particular, $\tilde{U}$ has a natural stratification indexed by the faces of the Newton polytope P .

In the case that $U = \mathcal{M}_{0,n}$, we have $\tilde{U} = \widetilde{\mathcal{M}}_{0,n}$. In this case the polytope P is an associahedron.

5.5. CEGM amplitudes and Grassmannian string integrals

An important generalization of $\mathcal{M}_{0,n}$ is the configuration space $X(k, n)$ of n points $\sigma_1, \dots, \sigma_n \in \mathbb{P}^{k-1}$ in *general linear position*, that is, no k of the points belong to the same hyperplane in $\mathbb{P}^{k-1}$. Let $\text{Gr}(k, n)^\circ \subset \text{Gr}(k, n)$ be the subset of the Grassmannian where all Plücker coordinates are non-vanishing. The space $X(k, n)$ is isomorphic to the torus quotient $\text{Gr}(k, n)^\circ / T'$, and is a very affine variety.

The space $X(k, n)$ has a positive part $X(k, n)_{>0}$, called *positive configuration space* [AHLS21b, AHLS21a], the image of the component $\text{Gr}(k, n)_{>0} \subset \text{Gr}(k, n)(\mathbb{R})$ where all Plücker coordinates are positive. Positive configuration space can be positively parametrized [Pos06, SW05, AHLS21b] with coordinates $y_1, y_2, \dots, y_d$, where $d = \dim X(k, n) = k(n - k) - (n - 1)$. For example, the following coordinates y_1, y_2, y_3, y_4 positively parametrize the four-dimensional space $X(3, 6)$:

$$\begin{bmatrix} 0 & 0 & -1 & -1 & & -1 & & & \\ 0 & 1 & 0 & -1 & -1 - y_1 & & -1 - y_1 - y_1 y_3 & & \\ 1 & 0 & 0 & 1 & 1 + y_1 + y_1 y_2 & 1 + y_1 + y_1 y_2 + y_1 y_2 y_3 + y_1 y_3 + y_1 y_2 y_3 y_4 & & & \end{bmatrix}.$$

The Plücker coordinates $\Delta_I = p_I(y)$ are nonnegative polynomials in $y_1, \dots, y_d$. We let $p_1(y), p_2(y), \dots$ be the distinct irreducible factors of these polynomials. This puts $X(k, n)$ in the setting of Section 5.2. The corresponding partial compactification $\tilde{U}$ from Proposition 5.19 is studied in [AHLS21b].

Cachazo-Early-Guevara-Mizera [CEGM19] generalized the scattering equations to $X(k, n)$ and defined corresponding scalar amplitudes, called *CEGM amplitudes* or *generalized biadjoint scalar amplitudes*; see [CE20, CEZ22] for some recent work in this direction. These amplitudes are the field theory limits of the *Grassmannian string integrals* [AHHL21b] which have the form

$$I(\underline{s}) = (\alpha')^d \int_{X(k, n)_{>0}} \prod_{I \in \binom{[n]}{k}} p_I^{\alpha' s_I} \Omega(X(k, n)_{>0}) = \int_{\mathbb{R}_{>0}^d} \prod_{I \in \binom{[n]}{k}} p_I^{\alpha' s_I}(y) \frac{dy_1}{y_1} \wedge \cdots \wedge \frac{dy_d}{y_d}$$

where the s_I satisfy $\sum_{a_2, \dots, a_k} s_{a_1, a_2, \dots, a_k} = 0$ for any a_1 , the analogue of momentum conservation, which also guarantees the integrand is torus-invariant and descends to $X(k, n)$.

5.6. Exercises and Problems

Exercise 5.20.

- (a) Verify that the integral (5.3) converges when $\operatorname{Re}(s) > 0$ and $\operatorname{Re}(t) > 0$.
- (b) Work out the details in the proof of Theorem 5.6. To do so, decompose the integration domain $\Lambda_{y, \mathbb{R}}^\vee \cong \mathbb{R}^d$ into the maximal cones of the domains of linearity of $\operatorname{trop}(\phi)$ and analyze the convergence on each cone separately.

Exercise 5.21. Show that the map $\pi : \Lambda_{\mathbb{R}}^\vee \rightarrow \Lambda_{y, \mathbb{R}}^\vee$ in the proof of Theorem 5.12 sends maximal cones of $\operatorname{trop}_{\geq 0} U$ to maximal common domains of linearity of $\operatorname{trop}(p_{ij})(Y)$. Then check that the contributions of these cones to $A_n^{\phi^3}$ and to $\lim_{\alpha' \rightarrow 0} I_n$ match.

Exercise 5.22. Prove Lemma 5.16.

Exercise 5.23. Prove Proposition 5.19. How is the stratification of Proposition 5.19 related to the vanishing and non-vanishing of the generators of Γ ?

Exercise 5.24. Recall the positive parametrization of $\mathcal{M}_{0,5}$ by parameters y_1, y_2 . The five variables $u_{13}, u_{14}, u_{24}, u_{25}, u_{35}$ are subtraction-free rational functions in y_1, y_2 .

- (a) Consider the (positive) tropicalization $\operatorname{trop}(\phi_X)$ as a piecewise-linear function on $\Lambda_{y, \mathbb{R}}^\vee$. Compute the domains of linearity of this piecewise-linear function.
- (b) The domains of linearity of $\operatorname{trop}(\phi_X)$ give the structure of a complete fan in $\Lambda_{y, \mathbb{R}}^\vee$. This complete fan has five one-dimensional cones. Compute the minimal lattice generators $r_1, r_2, \dots, r_5 \in \Lambda_y^\vee$ of these cones.
- (c) Verify that the generators r_i can be labeled as $r_{13}, r_{14}, r_{24}, r_{25}, r_{35}$ so that

$$\operatorname{trop}(u_{ij})(r_{ab}) = \delta_{(ij), (ab)}.$$

5.6.1. Simplicial Γ and u -equations

Consider the setting of Section 5.4. The nicest situation is when Γ is generated by $u_1, \dots, u_d$ and these generators also form a basis for the character lattice Λ . This is the case for $\mathcal{M}_{0,n}$, and the generators are exactly the dihedral coordinates u_{ij} . We say in this case that Γ is *simplicial*.

Problem 5.25. Find examples of simplicial Γ .

See [AHHL21b, Section 9] for related discussion. Even when Γ is simplicial, and we have natural coordinates $u_1, \dots, u_d$ on Λ , it is not clear when we obtain a binary geometry in the sense of Section 2. Note that the u -equation R_i for u_i exists exactly because the function $1 - u_i$ is a monomial in $u_1, \dots, u_d$.

Problem 5.26. Let u be a generator of Γ . When do we have $1 - u \in \Lambda$, and when do we have $1 - u \in \Gamma$? More generally, for which pairs $(c, u^X) \in \mathbb{C}^\times \times \Lambda$ do we have $c - u^X \in \Lambda$?

5.6.2. Pell's space

This problem comes from [HLRZ20, Section 4]. Consider the polynomials $p_i(y) = (1 + y_i)$ for $i = 1, 2, \dots, d$ and $p_{d+i}(y) = (1 + y_i + y_i y_{i+1})$ for $i = 1, 2, \dots, d-1$. The following is my interpretation of statements in [HLRZ20], which we may consider conjectures. It extends Problem 2.21 which corresponds to the case $d = 3$.

Problem 5.27. ³

- (a) Show that the Newton polytope P of $p(y) = \prod_i p_i(y)$ is simple, has $3d - 1$ facets, and P_d vertices, where P_d is Pell's number recursively defined by $P_1 = 1, P_2 = 2, P_d = 2P_{d-1} + P_{d-2}$.
- (b) We have $\dim \Lambda = (2d - 1) + d$. Show that the minimal generators of Γ are a basis of Λ . Call these generators $u_1, \dots, u_{3d-1}$.
- (c) Show that $\tilde{U}$, “Pell's space”, is a binary geometry for the simplicial complex Δ , defined to be the face complex of P . Find and prove the u -equations in terms of $u_1, \dots, u_{3d-1}$.
- (d) Study the geometry of $\tilde{U}$. Is there a moduli interpretation of this space?

³This problem has been addressed by Bossinger, Telek, and Tillmann-Morris [BTTM24].

6. Very affine amplitudes?

Definition 6.1. A *very affine variety* U is an irreducible closed subvariety of a torus.

Examples of very affine varieties include: $\mathcal{M}_{0,n}$, complements of essential hyperplane arrangements, configuration spaces of points in $\mathbb{P}^k$, and so on. In this section, we speculate on amplitudes for very affine varieties. In [Lam24], we study these amplitudes in more detail in the situation of hyperplane arrangement complements.

6.1. Intrinsic torus

The group $\Lambda = \Lambda(U) := \mathbb{C}[U]^\times / \mathbb{C}^\times$ of units of the coordinate ring of U , modulo scalars, is a free abelian group of finite rank by a theorem of Rosenlicht [Ros57] and Samuel [Sam66]. Let $T = \text{Hom}(\Lambda, \mathbb{C}^\times)$ be the torus with character group $\Lambda \cong \mathbb{Z}^n$, and we denote by $\Lambda^\vee$ the dual group of cocharacters of T . There is an inclusion

$$\iota : U \hookrightarrow T, \quad p \mapsto (f \mapsto f(p)),$$

and any closed embedding of U into a torus T' is a composition of ι with a homomorphism $T \rightarrow T'$.

6.2. Scattering potential

Let $U \subset T \cong (\mathbb{C}^\times)^n$ be a d -dimensional very affine variety, and let $u_1, u_2, \dots, u_n$ be the coordinates of T . Then U has a (multi-valued) potential function

$$\phi_X = \prod_{i=1}^n u_i^{X_i}$$

for $X = (X_1, \dots, X_n) \in \Lambda_{\mathbb{C}} \cong \mathbb{C}^n$. When T is the intrinsic torus, we call ϕ_X the intrinsic potential on U .

Definition 6.2. The *scattering equations* of U are the critical point equations of the log potential on U :

$$d \log \phi_X = \sum_{i=1}^n X_i d \log u_i = \sum_{i=1}^n \frac{X_i}{u_i} du_i = 0.$$

While ϕ_X is in general multi-valued, the equations $d \log \phi_X = 0$ give a well-defined subvariety of U .

The following result of Huh generalizes earlier work of Orlik and Terao [OT95] in the case of hyperplane arrangement complements.

Theorem 6.3 ([Huh13]). Suppose that U is smooth. For generic X , the critical point set consists of $|\chi(U)|$ reduced points, where $\chi(U)$ denotes the Euler characteristic.

6.3. Very affine positive geometries

Let U be a very affine variety. We consider positive geometries inside compactifications X of U . For example, one can take X to be a tropical compactification in the sense of Tevelev [Tev07].

Definition 6.4. A very affine positive geometry is a triple $(U, X, X_{\geq 0})$ where $(X, X_{\geq 0})$ is a positive geometry, X is a compactification of U , and $X_{\geq 0}$ is the analytic closure in X of a connected component of $U(\mathbb{R})$.

For a very affine positive geometry, there is a natural definition of string amplitude. Let ϕ_Z denote the intrinsic potential of U .

Definition 6.5. The string amplitude of a very affine positive geometry $(U, X, X_{\geq 0})$ is the function on $\Lambda_{\mathbb{C}}$ defined by analytic continuation of the integral function

$$I(Z) := (\alpha')^d \int_{X_{>0}} \phi_{\alpha'Z} \Omega(X_{\geq 0}),$$

where $Z \in \Lambda_{\mathbb{C}}$.

Definition 6.6. The scalar amplitude of a very affine positive geometry $(U, X, X_{\geq 0})$ is the rational function on $\Lambda_{\mathbb{C}}$ defined by

$$A(Z) := \int \delta^d(\phi_Z; \Omega(X_{\geq 0})) \Omega(X_{\geq 0}),$$

where $Z \in \Lambda_{\mathbb{C}}$, and the delta function $\delta^d(\phi_Z, \Omega(X_{\geq 0}))$ is defined in Section 3.7.

We expect that the scalar amplitude should be the field-theory limit of the string amplitude, that is, $A(Z) = \lim_{\alpha' \rightarrow 0} I(Z)$. More generally, if we are given two very affine positive geometries $(U, X, X_{\geq 0})$ and $(U, X, Y_{\geq 0})$ with the same very affine variety U , we can define the partial string amplitude

$$I_{X_{>0}, Y_{>0}}(Z) := (\alpha')^d \int_{Y_{>0}} \phi_{\alpha'Z} \Omega(X_{\geq 0}), \quad I_{Y_{>0}, X_{>0}}(Z) := (\alpha')^d \int_{X_{>0}} \phi_{\alpha'Z} \Omega(Y_{\geq 0})$$

and the partial scalar amplitude (cf. Definition 3.22)

$$A_{X_{>0}, Y_{>0}}(Z) = A_{Y_{>0}, X_{>0}}(Z) := A(\Omega(X_{\geq 0}) | \Omega(Y_{\geq 0})) = \int \delta^d(\phi_Z; \Omega(Y_{\geq 0})) \Omega(X_{\geq 0}).$$

We may also define a scattering form using the scattering correspondence:

$$\begin{array}{ccc} & \mathcal{I} & \\ p \swarrow & & \searrow q \\ U & & \Lambda_{\mathbb{C}} \end{array}$$

Definition 6.7. For a very affine positive geometry $(U, X, X_{\geq 0})$, the scattering form $\Upsilon(X_{\geq 0})$ on $\Lambda_{\mathbb{C}}$ is given by

$$\Upsilon := q_* p^* \Omega(X_{\geq 0}).$$

6.4. Positive parametrizations of very affine varieties

In some cases, we can positively parametrize the very affine U , putting us in the situation of Section 5, and Definition 6.5 and Definition 6.6 will hold.

Let $U \subset T$ be a d -dimensional very affine variety in the intrinsic torus T , and let u_i be the coordinates of the T . Let $\Lambda = \{u_1^{a_1} \cdots u_n^{a_n} \mid (a_1, \dots, a_n) \in \mathbb{Z}^n\}$ be the character lattice. Suppose that we have $y_1, \dots, y_d \in \Lambda$ such that $\mathbb{C}(y_1, \dots, y_d) = \mathbb{C}(U)$. In particular, each $u_i \in \mathbb{C}(y)$ is a rational function $u_i(y) \in \mathbb{C}(y)$ in $y_1, \dots, y_d$. Let $p_1(y), p_2(y), \dots, p_r(y) \in \mathbb{C}[y]$ be the list of all distinct irreducible factors (assumed to have constant term 1) appearing in the numerator or denominator of $u_i(y)$, excluding $y_1, \dots, y_d$ themselves.

Definition 6.8. In this situation, suppose that $p_i(y) \in \Lambda$ for $i = 1, 2, \dots, r$. Then we say that $y_1, \dots, y_d$ give a parametrization of U . We call this a *positive parametrization* if the coefficients of $p_i(y)$ are all positive.

Lemma 6.9. Suppose $y_1, \dots, y_d \in \Lambda$ give a parametrization of U . Then there is an invertible monomial transformation between $\{y_1, y_2, \dots, y_d, p_1(y), \dots, p_r(y)\}$ and $\{u_1, \dots, u_n\}$, and $\{y_1, y_2, \dots, y_d, p_1(y), \dots, p_r(y)\}$ also form a basis of Λ .

Proof. There are no monomial relations between $y_1, \dots, y_d, p_1(y), \dots, p_r(y)$, since we have chosen all the polynomials to be irreducible with constant term 1. It follows that they form a basis of Λ . $\square$

Thus in the case of a (positive) parametrization, we are in the situation Section 5.2.

6.5. Tropicalizations of very affine varieties

Now suppose that U is a very affine variety, but we do not have (or do not know) any natural top-form Ω on U (arising from a positive geometry), nor do we have a positive parametrization of U . In this case, we can still try to define a

scalar amplitude by viewing the formula Theorem 3.14 as a Laplace transform of $\text{trop}_{\geq 0} U$, cf. Proposition 5.8. We begin by defining the tropicalization of a very affine variety, and refer the reader to [MS15, Tev07] for more on tropicalization.

Let $\mathcal{C} = \bigcup_{n \geq 1} \mathbb{C}((t^{1/n}))$ denote the field of Puiseux series, equipped with a valuation

$$\text{val}: \mathcal{C} \rightarrow \mathbb{R} \cup \{\infty\}, \quad \text{val}(f(t)) = \begin{cases} \text{minimal degree appearing in } f(t) & \text{if } f \neq 0 \\ \infty & \text{if } f = 0. \end{cases}$$

Definition 6.10. The tropicalization $\text{trop}(U)$ is the subset of $\Lambda_{\mathbb{R}}^{\vee} := \Lambda^{\vee} \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^n$ obtained by taking the closure of the valuations of all $\mathcal{C}$ points of U :

$$\text{trop}(U) := \overline{\{\text{val}(p) \mid p \in U(\mathcal{C})\}} \subset \Lambda_{\mathbb{R}}^{\vee}.$$

The tropicalization $\text{trop}(U)$ can be given the structure of a polyhedral fan, pure of dimension $d = \dim(U)$. There are a number of different definitions of the fan structure on $\text{trop}(U)$. We assume that one such fan structure has been fixed.

Example 6.11. Suppose that $U = \mathcal{M}_{0,4}$. Then the intrinsic torus has coordinates (u_{13}, u_{24}) , and U is given by the single equation $u_{13} + u_{24} = 1$. The tropicalization $\text{trop}(U)$ consists of the three 1-dimensional cones (rays): $\mathbb{R}_{\geq 0} \cdot (1, 0)$, $\mathbb{R}_{\geq 0} \cdot (0, 1)$, $\mathbb{R}_{\geq 0} \cdot (-1, -1)$. The positive tropicalization $\text{trop}_{>0}(U)$ consists of the two 1-dimensional cones: $\mathbb{R}_{\geq 0} \cdot (1, 0)$, $\mathbb{R}_{\geq 0} \cdot (0, 1)$.

Let $C \subset \Lambda_{\mathbb{R}}^{\vee}$ be a maximal cone of $\text{trop}(U)$. The affine span of C is a d -dimensional subspace $W \subset \Lambda_{\mathbb{R}}^{\vee}$. We define a d -form on W that will serve as an integration measure. Since W is rationally defined, the intersection $\Lambda^{\vee} \cap W$ is a sublattice of $\Lambda^{\vee}$ that spans W . We define an integration measure $d^d W$ by declaring that a unit cube in $\Lambda^{\vee} \cap W$ has unit volume with respect to $d^d W$. The standard simplex in $\Lambda^{\vee} \cap W$ has normalized volume equal to one.

We propose the following general notion of (partial) amplitude for a very affine variety.

Definition 6.12. Let $\mathcal{T}$ denote a subset of maximal cones of $\text{trop}(U)$. Then the $\mathcal{T}$ -partial amplitude is the rational function on $\Lambda_{\mathbb{R}}$

$$A_{\mathcal{T}}(X) := \sum_{C \in \mathcal{T}} \int_C e^{-X} d^d W = \text{Vol}((\bigcup_{C \in \mathcal{T}} C) \cap \{X \leq 1\}) = \sum_{C \in \mathcal{T}} \text{Vol}(C \cap \{X \leq 1\}), \quad (6.1)$$

where $X \in \Lambda_{\mathbb{R}}$ is viewed as a linear function on $\Lambda_{\mathbb{R}}^{\vee}$. The second equality is by Proposition 5.8.

We view $A_{\mathcal{T}}(X)$ as the Laplace transform of $\mathcal{T} \subset \Lambda_{\mathbb{R}}^{\vee}$. In the case the cones C is simplicial with appropriately chosen generators $r_1, r_2, \dots, r_d$ that generate W , we have

$$\text{Vol}(C \cap \{X \leq 1\}) = \prod_{i=1}^d \frac{1}{(r_i, X)}$$

is a product of inverses of linear functions (r_i, X) on $\Lambda_{\mathbb{R}}^{\vee}$; see Example 5.9.

Remark 6.13. In the case of the configuration space $X(k, n)$ of Section 5.5, Cachazo-Early-Zhang [CEZ22] have studied subfans $\mathcal{T}$ called *chirotopal tropical Grassmannians*. See also the recent work of Antolini and Early [AE24].

6.6. Positive components

To proceed to the definition of *planar* amplitudes, we need to specify a set of cones $\mathcal{T}$. This roughly amounts to picking a connected component of $U(\mathbb{R})$. We henceforth assume that U is defined over $\mathbb{R}$.

In the case $U = \mathcal{M}_{0,n}$, the following are roughly equivalent:

1. the choice of a connected component of $\mathcal{M}_{0,n}(\mathbb{R})$;
2. the choice of a positive parametrization of $\mathcal{M}_{0,n}$;
3. the choice a canonical form $\Omega_{0,n}$ on $\mathcal{M}_{0,n}$;
4. the choice of “planar trees” among all n -leaf trees;
5. the choice of an affine compactification $\mathcal{M}_{0,n} \subset \widetilde{\mathcal{M}}_{0,n}$;
6. the choice of a subset of “planar” boundary divisors of $\overline{\mathcal{M}}_{0,n}$;
7. the choice of a subsemigroup $\Gamma \subset \Lambda$ of bounded characters;
8. the choice of the Parke-Taylor factor $\text{PT}(\alpha)$;
9. the choice of a subfan $\text{trop}_{\geq 0} \mathcal{M}_{0,n} \subset \text{trop} \mathcal{M}_{0,n}$.

Let $\kappa : \Lambda \rightarrow \mathbb{Z}/2\mathbb{Z}$ be a group homomorphism. If a basis $u_1, \dots, u_n$ of Λ is chosen, we write κ as a sequence in $\{+, -\}^n$ with a $+$ indicating $\kappa(u_i) = 0 \in \mathbb{Z}/2\mathbb{Z}$ and a $-$ indicating $\kappa(u_i) = 1 \in \mathbb{Z}/2\mathbb{Z}$. The torus T , as an abstract algebraic variety, has an alternative group structure denoted T_{κ} , given by sending the identity $(1, 1, \dots, 1)$ to element $((-1)^{\kappa_1}, \dots, (-1)^{\kappa_n})$, where $\kappa_i = \kappa(u_i)$. Equivalently, T_{κ} has characters given by $(-1)^{\kappa(u)} u$, for $u \in \Lambda$.

The set of real points $T(\mathbb{R})$ of the torus has 2^n components. We let $T_{>0}$ be the connected component containing the identity, and let $T_{\kappa}(\mathbb{R}) := (T_{\kappa})_{>0}$. Thus $T(\mathbb{R}) = \sqcup_{\kappa} T_{\kappa}(\mathbb{R})$.

Define a *component* of $U(\mathbb{R})$ to be a non-empty intersection

$$U_\kappa := U \cap T_\kappa(\mathbb{R})$$

of the very affine variety with $T_\kappa(\mathbb{R})$. If non-empty, this is a union of connected components of $U(\mathbb{R})$, and in many interesting cases U_κ is a single connected component of $U(\mathbb{R})$. This is the case for $\mathcal{M}_{0,n}$ (Exercise 1.27). While U_κ can consist of multiple connected components, we will use $\kappa \in \pi_0(T(\mathbb{R}))$ as an input to our definition of amplitudes.

6.7. Positive part of tropicalization

Given a choice of component $T_\kappa(\mathbb{R})$, we may define $\text{trop}_\kappa U$ in a number of ways. The first one is the positive tropicalization of [SW05]. Let $\mathcal{R}_{>0} \subset \mathcal{C}$ be the semifield consisting of nonzero Puiseux series such that the coefficient of the lowest degree term is real and positive. Let $U(\mathcal{C})$ denote the $\mathcal{C}$ -points of U . Let $U(\mathcal{R}_{>0}) \subset U(\mathcal{C})$ denote the subset of points $p \in U(\mathcal{C})$ where $u(p) \in \mathcal{R}_{>0}$ for every character $u \in \Lambda$ of T .

Definition 6.14. The positive tropicalization $\text{trop}_{>0}(U)$ is the subset of $\Lambda_{\mathbb{R}}^\vee$ obtained by taking the closure of the valuations of all $\mathcal{R}_{>0}$ points of U :

$$\text{trop}_{>0}(U) := \overline{\{\text{val}(p) \mid p \in U(\mathcal{R}_{>0})\}} \subset \Lambda_{\mathbb{R}}^\vee.$$

The positive tropicalization $\text{trop}_{>0}(U)$ is a subfan of $\text{trop}(U)$, also pure of dimension d . Let $U(\mathcal{R}_\kappa) \subset U(\mathcal{C})$ denote the subset of points $p \in U(\mathcal{C})$ where $(-1)^{\kappa(u)}u(p) \in \mathcal{R}_{>0}$ for every character $u \in \Lambda$ of T .

Definition 6.15. The κ -tropicalization $\text{trop}_\kappa(U)$ is the set

$$\text{trop}_\kappa(U) := \overline{\{\text{val}(p) \mid p \in U(\mathcal{R}_\kappa) \text{ for all } u \in L\}} \subset \Lambda_{\mathbb{R}}^\vee.$$

We view $\text{trop}_\kappa(U)$ as the tropicalization of the component U_κ .

Example 6.16. Continuing Example 6.11,

- $\text{trop}_{++}(U) = \text{trop}_{>0}(U)$ consists of the two 1-dimensional cones $\mathbb{R}_{\geq 0} \cdot (1, 0)$ and $\mathbb{R}_{\geq 0} \cdot (0, 1)$.
- $\text{trop}_{+-}(U)$ consists of the two 1-dimensional cones $\mathbb{R}_{\geq 0} \cdot (0, 1)$ and $\mathbb{R}_{\geq 0} \cdot (-1, -1)$.
- $\text{trop}_{-+}(U)$ consists of the two 1-dimensional cones $\mathbb{R}_{\geq 0} \cdot (1, 0)$ and $\mathbb{R}_{\geq 0} \cdot (-1, -1)$.
- $\text{trop}_{--}(U)$ is empty.

This agrees with $|\pi_0(\mathcal{M}_{0,4}(\mathcal{R}))| = (4-1)!/2 = 3$.

6.8. Bounded invertible functions

We suggest an alternative to positive tropicalization, using bounded invertible functions, as in Exercise 1.24.

Definition 6.17. Let U_κ be a non-empty component. Define the submonoid $B_\kappa \subset \Lambda$ by

$$B_\kappa = \{u \in \Lambda \mid u|_{U_\kappa} \text{ takes bounded values}\}.$$

Define the κ -positive part of $\text{trop}(U)$ by

$$\text{trop}^\kappa(U) := \{q \in \text{trop}(U) \mid u(q) \geq 0 \text{ for all } u \in B_\kappa\}.$$

Here, $u \in \Lambda$ is viewed as a linear function on $\Lambda_{\mathbb{R}}^\vee$.

We expect that under good conditions, u belongs to B_κ if and only if $u(\text{trop}_\kappa(U)) \subset \mathbb{R}_{\geq 0}$.

Example 6.18. For $U = \mathcal{M}_{0,4}$,

- B_{++} is generated by u_{13}, u_{24} .
- B_{+-} is generated by $u_{13}^{-1}, u_{24}u_{13}^{-1}$.
- B_{-+} is generated by $u_{24}^{-1}, u_{13}u_{24}^{-1}$.

Finally, we can take $\mathcal{T} = \text{trop}_\kappa(U)$ or $\mathcal{T} = \text{trop}^\kappa(U)$ in (6.1) as the definition of the planar scalar amplitude of U .

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