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COMPLETE INTERSECTION LIAISON AND GORENSTEIN LIAISON: NEW RESULTS AND OPEN PROBLEMS

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Dedicated to Silvio Greco in occasion of his 60-th birthday.

1. Introduction.

This expository paper is a slightly modified version of a talk I gave at the "Conference on Commutative Algebra and Algebraic Geometry", Catania, April 11-13, 2001. The purpose of the talk was to review some of the recent results on Gorenstein liaison (simply, G-liaison) confronting them with classical results in complete intersection liaison theory (simply, CI-liaison) and this note is only meant as a tiny introduction to what has recently become a very lively area of research. No complete proofs are given and I refer to the papers quoted in the bibliography for proofs, more material, and further references.

The notion of using complete intersections to link varieties has been used for a long time ago, going back at least to work of Noether, Macaulay and Severi. Since then liaison theory has been largely studied; in codimension 2, liaison theory has reached a very satisfying state but in higher codimension there are still many open questions/problems. Much of the theory has been built around linking with complete intersections schemes, which in codimension 2 coincide with Gorenstein schemes, and recently the attention has been focused on Gorenstein liaison. In my talk, I will try to convince you that Gorenstein

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liaison is a more natural approach if we want to carry out a program in higher codimension and I refer to the monograph [13] for a more detailed treatment.

It is a classical result originally proved by Gaeta [8], in 1948, and reproved in modern language by Peskine-Szpiro [20], that every arithmetically Cohen-Macaulay (briefly, ACM), codimension 2 subscheme $X \subset \mathbb{P}^n$ can be CI-linked in a finite number of steps to a complete intersection or, equivalently, all codimension 2, ACM subschemes $X \subset \mathbb{P}^n$ are licci. The first goal of this work is to see that in the CI-liaison context Gaeta's Theorem does not generalize well to subschemes $X \subset \mathbb{P}^n$ of higher codimension. More precisely, I will prove the existence of infinitely many different CI-liaison classes containing ACM curves $C \subset \mathbb{P}^4$. I will give two different kind of examples: (1) I will see that two ACM curves $C_t, C_{t'} \subset \mathbb{P}^4$ with a t-linear resolution:

$$0 \longrightarrow R(-t-2)^{\frac{t^2+t}{2}} \longrightarrow R(-t-1)^{t^2+2t} \longrightarrow R(-t)^{\frac{t^2+3t+2}{2}} \longrightarrow I(C_t) \longrightarrow 0$$

belong to different CI-liaison classes provided $t \neq t'$ (Corollary 3.9) and; (2) many ACM curves $C \subset \mathbb{P}^4$ on a Castelnuovo (resp. Bordiga) surface $S \subset \mathbb{P}^4$ give rise to an infinite number of CI-liaison classes containing ACM curves $C + tH$ by just adding different number of hyperplane sections (Example 3.11).

The second goal is to convince the reader that G-liaison is in many ways more natural than CI-liaison and among other results I will state that ACM curves $C \subset \mathbb{P}^4$ lying on a general, smooth, ACM surface $S \subset \mathbb{P}^4$ are glicci, i.e., they belong to the G-liaison class of a complete intersection (Theorems 4.6 and 4.10). Using the fact that, roughly speaking, Gorenstein liaison is a theory about generalized divisors on arithmetically Cohen-Macaulay schemes which collapses to the setting of CI-liaison theory as a theory of generalized divisors on a complete intersection and the fact that the Picard group of a rational normal scroll surface is well known, we get that any effective divisor X on a rational normal scroll surface $S \subset \mathbb{P}^n$ is glicci provided X is arithmetically Cohen-Macaulay as a subscheme of $\mathbb{P}^n$ (Theorem 4.11) and a surprising number of applications (Corollaries 4.13-15).

The last goal is to generalize Gaeta's Theorem and prove that standard determinantal schemes are glicci (Theorem 5.3). Since in codimension 2, ACM schemes are standard determinantal and since in codimension 2, arithmetically Gorenstein schemes and complete intersection schemes coincide, this result is indeed a full generalization of Gaeta's Theorem.

Next we outline the structure of the paper. In section 2, we provide the necessary background information about G-liaison needed later on. In section 3, we introduce some graded modules which are liaison invariants under CI-liaison but not under G-liaison (Theorem 3.3, Theorem 3.5 and Remark 6.2) and we use them to prove the existence of infinitely many different CI-liaison classes containing ACM curves $C \subset \mathbb{P}^4$ (Corollary 3.9 and Example 3.11). In section 4, we determine huge families of ACM curves $C \subset \mathbb{P}^n$ which are glicci (Theorem 4.6, 4.10 and 4.11; and Corollaries 4.13-15); in section 5, we generalize Gaeta's Theorem (Theorem 5.3); and we end the paper with some final comments and open questions.

In view of the already vast literature I have only included the references that are directly related to the topics discussed here. I apologize to the many whose beautiful and deep contributions could not even be mentioned without overly enlarging the perspective of this note and those whose work I may have failed to cite properly.

Acknowledgment. I am greatly in debted to my co-authors of [13] for the enjoyable collaboration which led to most of the material described in this paper: they are J. Kleppe, J. Migliore, U. Nagel and C. Peterson.

I am also very grateful to M. Casanellas part of section 4 grew out of a long time cooperation.

Notation. Throughout this paper we work over an algebraically closed field $\mathbf{k}$ of characteristic 0. By $\mathbb{P}^N$ we denote the N-dimensional projective space over $\mathbf{k}$, by R the polynomial ring $\mathbf{k}[X_0, \ldots, X_N]$ and $m = (X_0, \ldots, X_N)$. For any closed subscheme V of $\mathbb{P}^N$ we denote by I_V its ideal sheaf, $I(V)$ its saturated homogeneous ideal (note that $I(V) = H^0_*(I_V) := \bigoplus_{t \in \mathbb{Z}} H^0(\mathbb{P}^n, I_V(t))$), $A(V) = R/I(V)$ the homogeneous coordinate ring, $N_V = \mathrm{Hom}(I_V, \mathcal{O}_V)$ the normal sheaf of V and $M_i(V) = H^i_*(I_V) := \bigoplus_{t \in \mathbb{Z}} H^i(\mathbb{P}^n, I_V(t))$, $i = 1, \ldots, \dim(V)$, the i-th deficiency module of V.

Let $X \subset \mathbb{P}^N$ be a locally Cohen-Macaulay and equidimensional scheme of codimension c. X is said to be arithmetically Cohen-Macaulay (briefly, ACM) if and only if $M_i(X) = 0$ for $1 \leq i \leq N - c$ or, equivalently, $A(X)$ is a Cohen-Macaulay ring. X is said to be arithmetically Gorenstein (briefly, AG) if and only if $I(X)$ has a resolution

$$0 \longrightarrow R(-t) \longrightarrow \oplus_{i=1}^{\alpha_{c-1}} R(-n_i^{c-1}) \longrightarrow \ldots$$
$$\ldots \longrightarrow \oplus_{i=1}^{\alpha_1} R(-n_i^1) \longrightarrow I(X) \longrightarrow 0.$$

In particular, X is arithmetically Cohen-Macaulay. It is well known that in codimension two AG subschemes and complete intersection subschemes coincide. In higher codimension, any complete intersection subscheme is AG but not vice versa (indeed, a set of $N + 2$ points in $\mathbb{P}^N$ in linear general position is AG but not complete intersection).

2. Background material.

In this section, we collect the basic definitions about G-liaison needed in this paper as well as elementary examples.

Definition 2.1. (See also [13]; Definitions 2.3, 2.4 and 2.10). Let V_1 and $V_2 \subset \mathbb{P}^N$ be two equidimensional schemes without embedded components. We say that V_1 and V_2 are *directly CI-linked* (resp. *directly G-linked*) if there exists a complete intersection scheme (resp. an AG scheme) X such that $I_{V_1}/I_X \cong \mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^N}}(\mathcal{O}_{V_2}, \mathcal{O}_X)$ and $I_{V_2}/I_X \cong \mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^N}}(\mathcal{O}_{V_1}, \mathcal{O}_X)$. If V_1 and V_2 do not share any common component then this is equivalent to $X = V_1 \cup V_2$.

Example 2.2.

(i) A simple example of schemes directly CI-linked is the following one: Let C_1 be a twisted cubic in $\mathbb{P}^3$ and let C_2 be a secant line to C_1. The union of C_1 and C_2 is a degree 4 curve which is the complete intersection X of two quadrics Q_1 and Q_2. So C_1 and C_2 are directly CI-linked by the complete intersection X.

(ii) As a simple example of schemes directly G-linked we have: We consider a set $Y_1 \subset \mathbb{P}^3$ of four points in linear general position and a sufficiently general point Y_2. Since $X = Y_1 \cup Y_2$ is an AG scheme, Y_1 and Y_2 are directly G-linked.

Definition 2.3. Let V_1 and $V_2 \subset \mathbb{P}^N$ be two equidimensional schemes without embedded components. We say that V_1 and V_2 are in the same *CI-liaison class* (resp. *G-liaison class*) if and only if there exists a sequence of schemes $Y_1, \ldots, Y_r$ such that Y_i is *directly CI-linked* (resp. *directly G-linked*) to Y_{i+1} and such that $Y_1 = V_1$ and $Y_r = V_2$. If V_1 is linked to V_2 in two steps by complete intersection (resp. AG) schemes we say that they are *CI-bilinked* (resp. *G-bilinked*).

In other words *CI-liaison* (resp. *G-liaison*) is the equivalence relation generated by directly CI-linkage (resp. directly G-linkage) and roughly speaking liaison theory is the study of these equivalence relations and the corresponding equivalence classes.

In codimension two CI-liaison and G-liaison generate the same equivalence relation, since complete intersections and AG schemes coincide. In higher codimension it is no longer true. Indeed, a simple counterexample is the following: Consider a set X of four points in $\mathbb{P}^3$ in linear general position. By Example 2.2 (ii) we can G-link X to a single point. Therefore, X is glicci. On the other hand, it follows from [12]; Corollary 5.13 that X is not licci.

Definition 2.4. A scheme $X \subset \mathbb{P}^N$ is said to be *licci* if it is in the CI-liaison class of a complete intersection. Analogously, we say that a scheme $X \subset \mathbb{P}^N$ is *glicci* if it is in the G-liaison class of a complete intersection.

Although the goal of my talk was to show the merits of studying Gorenstein liaison, it is worth to mention some disadvantages:

(1) It is easy to check that both CI-links and G-links are preserved under hyperplane sections. Nevertheless, CI-links lift and G-links do not lift, in general (see [13]; Example 2.12).
(2) Given a scheme $V \subset \mathbb{P}^N$ it is, in general, very difficult to find "good" G-links, i.e., "good" Gorenstein ideals $I_X \subset I_V$ of the same high ("good" often means "small")

3. Liaison invariants and applications.

Definition 3.1. Let $X \subset \mathbb{P}^N$ be a locally Cohen-Macaulay equidimensional scheme. A graded R-module $C(X)$ which depends only on X is a *CI-liaison* (resp. *G-liaison*) *invariant* as an R-module (resp., **k**-module) if there exists a homogeneous R (resp. **k**)-module isomorphism $C(X) \cong C(X')$ for any X' in the CI-liaison (resp. G-liaison) class of X.

It is well known that for equidimensional, locally Cohen-Macaulay schemes $X \subset \mathbb{P}^N$, the i-th module of deficiency $M_i(X) := \bigoplus_{t\in\mathbb{Z}} H^i(\mathbb{P}^n, I_V(t))$, $1 \le i \le \dim(X)$, are CI-liaison invariants (up to shifts and duals). Even more they are G-liaison invariants. In this section, we describe other CI-liaison invariants which allow us to distinguish between many CI-liaison classes which cannot be distinguished by deficiency modules alone.

Let $X \subset \mathbb{P}^{n+c}$ be a closed subscheme, locally CM, equidimensional of $\dim n > 0$ [1]. If X is ACM all the CI-liaison invariants $M_i(X)$, $1 \le i \le \dim(X)$, vanish. Our first goal is to describe non-trivial CI-liaison invariants of ACM schemes. To this end, we consider a graded R-free resolution of $I = I(X)$:

$$\dots \oplus_i R(-n_i^2) \longrightarrow \oplus_i R(-n_i^1) \longrightarrow I \longrightarrow 0. \tag{1}$$

We apply $\mathrm{Hom}(-, \mathcal{O}_X)$ to the exact sequence (1) and we obtain

$$0 \longrightarrow N_X \longrightarrow \oplus_i \mathcal{O}_X(-n_i^1) \longrightarrow \oplus_i \mathcal{O}_X(-n_i^2).$$

[1] Throughout this paper we work with schemes of dimension $n > 0$. We want to point out that the results we give generalize to 0-dimensional schemes and we assume $n > 0$ to avoid technical complications.

We take cohomology ($H^n_*\mathcal{O}_X \cong H^{n+1}_m(R/I)$, A=R/I); and we get a natural map

$$\delta_X : H^n_* N_X \longrightarrow \operatorname{Hom}_R(I, H^{n+1}_m(A)) \cong \operatorname{Hom}_R(I, H^{n+2}_m(I)).$$

This map δ_X plays an important role; in particular, its kernel and cokernel are CI-liaison invariants (See Theorem 3.3).

Theorem 3.2. If I/I^2 is a free R/I-module, then δ_X is an isomorphism. In particular, if $X \subset \mathbb{P}^{n+c}$ is a global complete intersection, then δ_X is an isomorphism.

Theorem 3.3. *Let $X, X' \subset \mathbb{P}^{n+c}$ be ACM subschemes of dimension $n > 0$ algebraically linked by a complete intersection $Y \subset \mathbb{P}^{n+c}$. Then :*

(1) As graded R-modules:
$H^i_* N_X \cong H^i_* N_{X'}$ *for* $1 \leq i \leq n-1$,
$\ker(\delta_X) \cong \ker(\delta_{X'})$.
(2) As graded $\mathbf{k}$*-modules:*
$\operatorname{coker}(\delta_X) \cong \operatorname{coker}(\delta_{X'})$.
(3) Moreover, if $Y \subset \mathbb{P}^{n+c}$ is a complete intersection of type $f_1, \ldots, f_c$, we have $h^0 N_X = h^0_* N_{X'} + \sum_i h^0(I_{X'}(f_i)) - \sum_i h^0(I_X(f_i))$.

Proof. See [13]; Theorem 6.1 □

As application, we get the following useful criterion to check if an ACM scheme is licci.

Corollary 3.4. *Let $X \subset \mathbb{P}^{n+c}$ be a closed subscheme of dimension $n > 0$. If X is licci, then:*

(1) $H^i_* N_X = 0$ *for* $1 \leq i \leq n-1$,
(2) δ_X *is an isomorphism.*

Proof. It follows from Theorem 3.3 and the fact that for complete intersections $Y \subset \mathbb{P}^{n+c}$, $H^i_* N_Y = 0$ for $1 \leq i \leq n-1$ and δ_Y is an isomorphism (Remark 3.2). □

From now until the end of this section, we will restrict our attention to closed subschemes $X \subset \mathbb{P}^{n+3}$, $n > 0$, of codimension 3 and we will deduce from the previous results the CI-liaison invariance of the local cohomology groups

$$H^i_m(K_{R/I} \otimes_R I) \quad i = 0, \ldots, n$$

being $K_{R/I} = Ext^3_R(R/I, R)(-n-4)$ the canonical module of X.

Indeed, using basic facts on local cohomology, the spectral sequence relating local and global Ext:

$$E_2^{pq} := H^p(X, \mathcal{E}xt^q(\mathcal{F}, \mathcal{G})) \Rightarrow Ext^{p+q}(\mathcal{F}, \mathcal{G}),$$

and the spectral sequence:

$$E_2^{pq} :=_{\mu} Ext^p_R(M_1, H^q_m(M_2)) \Rightarrow_{\mu} Ext^{p+q}_m(M_1, M_2),$$

we obtain

Therorem 3.5. *Let $X \subset \mathbb{P}^{n+3}$ be an ACM subscheme of codimension 3 ($n > 0$) and $K := Ext^3_R(A, R)(-n-4)$ its canonical module. Then, we have*

(1) $H^{i+1}_* N_X \cong H^i_m(K \otimes_R I)(n+4)$, $0 \leq i \leq n-2$, *as graded R-modules.*
(2) There exists an exact sequence:

$$0 \to H^{n-1}_m(K \otimes_R I)(n+4) \to H^n_* N_X \xrightarrow{\delta_X}$$

$$\mathrm{Hom}(I, H^{n+1}_m(A)) \to H^n_m(K \otimes_R I)(n+4) \to 0.$$

In particular,

(3) $H^i_m(K \otimes_R I)$ *are CI-liaison invariant as graded R (resp. $\mathbf{k}$)-modules, $0 \leq i < n$ (resp. $0 \leq i \leq n$). Moreover, if X is locally complete intersection then*

$$H^i_m(K \otimes_R I)(n+4) \cong H^{n-i}_m(K \otimes_R I)^v \quad i = 0, \ldots, n$$

as R-modules.

Proof. See [13]; Proposition 6.8. □

As application we get another useful criterion to check if an ACM subscheme X of $\mathbb{P}^N$ is licci.

Corollary 3.6. *Let $X \subset \mathbb{P}^{n+3}$ be a closed subscheme of dimension $n > 0$. If X is licci then $H^i_m(K \otimes_R I) = 0$, $0 \leq i \leq n$.*

Proof. It follows from Theorem 3.5 and the fact that for complete intersections $Y \subset \mathbb{P}^{n+3}$, $H^i_m(K_{R/I(Y)} \otimes_R I(Y)) = 0$, $0 \leq i \leq n$. □

Now, we will illustrate with an example how to use Theorem 3.5.

Example 3.7. *Let $C \subset \mathbb{P}^4$ be a smooth, connected curve of degree d and genus g with an "almost linear" resolution:*

$$0 \to R(-s-3)^a \to R(-s-2)^b \to R(-s-1)^{c_1} \oplus R(-s)^{c_0} \to I(C) \to 0.$$

If $d+g-1-ac_0 \neq 0$ then C is not licci.

Idea of the Proof. We compute the dimension, $l(C)_\mu := \dim_{\mu+5} H^0_m(K_A \otimes_R I(C))$, of the CI-liaison invariants ${}_{\mu+5}H^0_m(K_A \otimes_R I)$. The exact sequence and the duality of Theorem 3.5 gives us (small letters mean dimension)

$$l(C)_\mu - l(C)_{-\mu-5} = h^1 N_C(\mu) -_\mu \operatorname{Hom}_R(I(C), H^2_m(A)).$$

Since ${}_{-2}\operatorname{Hom}_R(I, H^2_m(A)) = ac_0$ and $h^1 N_C(-2) = -\chi N_C(-2) = d+g-1$ (Riemann-Roch's Theorem), we obtain

$$l(C)_{-2} - l(C)_{-3} = h^1 N_C(-2) -_{-2} \operatorname{Hom}_R(I(C), H^2_m(A)) = d+g-1-ac_0.$$

Therefore, by Corollary 3.6, if $d+g-1-ac_0 \neq 0$ then C is not licci. □

Remark 3.8.
(1) The only smooth connected curve in $\mathbb{P}^4$ with a linear resolution ($c_0 = 0$) which is licci is a line.
(2) The smooth rational quartic is not licci. Indeed, $(a, b, c_1, c_0, s) = (3, 8, 6, 0, 1)$ and $d+g-1-ac_0 = 3 \neq 0$.

We will now deduce the existence of infinitely many different CI-liaison classes containing ACM curves $C \subset \mathbb{P}^4$.

Corollary 3.9. *Let $C_t \subset \mathbb{P}^4$ be an ACM curve with a linear resolution:*

$$0 \longrightarrow R(-t-2)^{\frac{t^2+t}{2}} \longrightarrow R(-t-1)^{t^2+2t} \longrightarrow R(-t)^{\frac{t^2+3t+2}{2}} \longrightarrow I(C_t) \longrightarrow 0.$$

For $t \neq q$, C_t and C_q belong to different CI-liaison classes.

Proof. We have $d(C_t) = \binom{t+3}{4} - \binom{t+2}{4}$, $p_a(C_t) = (t-1)d(C_t) + 1 - \binom{t+3}{4}$ and $d(C_t) + p_a(C_t) - 1 \neq d(C_q) + p_a(C_q) - 1$ for $t \neq q$. Therefore, by Example 3.7, C_t and C_q belong to different liaison classes provided $t \neq q$. □

Remark 3.10. Corollary 3.9 shows that in the context of CI-liaison Gaeta's Theorem does not generalize to ACM subschemes $X \subset \mathbb{P}^n$ of higher codimension. In next sections, I will try to convince the reader that G-liaison is a more natural approach if we want to carry out a program in higher codimension.

As another example about the existence of infinitely many different CI-liaison classes containing ACM curves $C \subset \mathbb{P}^4$ we have the following one

Example 3.11. Let $S \subset \mathbb{P}^4$ be a Castelnuovo (resp. Bordiga) surface and let $C \subset S$ be a rational, normal quartic. Consider an effective divisor $C_t \in |C+tH|$, where H is a hyperplane section of S and $0 \le t \in \mathbb{Z}$. It holds:

- C_t is not licci, $\forall t \ge 0$;
- C_t and $C_{t'}$ belong to different CI-liaison classes provided $0 \le t < t'$.

In next section, we will see that all these examples of ACM curves $C_t = C + tH \subset S \subset \mathbb{P}^4$ which belong to different CI-liaison classes, belong to the same G-liaison class. So the situation drastically changes when we link by means of arithmetically Gorenstein schemes instead of complete intersections.

4. Glicci curves in $\mathbb{P}^n$.

In this section, using the fact that the Picard group of a "general" ACM surface $X \subset \mathbb{P}^4$ and of a rational normal scroll surface $S \subset \mathbb{P}^n$ are well known together with the fact that roughly speaking G-liaison is a theory about divisors on ACM schemes, we will see that there is only one G-liaison class containing ACM curves $C \subset \mathbb{P}^4$ lying on a general, smooth, ACM surface $S \subset \mathbb{P}^4$ (Theorems 4.6 and 4.10) and that all ACM curves $C \subset \mathbb{P}^n$ lying on a rational, normal scroll surface $S \subset \mathbb{P}^n$ are glicci (Theorem 4.11).

We start with some preliminary results.

Definition 4.1. A noetherian ring A (resp. a noetherian scheme X) satisfies the condition G_1, "Gorenstein in codimension ≤ 1", if every localization A_p (resp. every local ring $\mathcal{O}_x$) of dimension ≤ 1 is a Gorenstein local ring.

Lemma 4.2. *Let $X \subset \mathbb{P}^n$ be an ACM subscheme satisfying the property G_1, and let C be a subcanonical divisor on X. Let $F \in I(C)$ be a homogeneous polynomial of degree d such that F does not vanish on any component of X. Let H_F be the divisor cut out on X by F. Then the effective divisor $H_F - C$ on X, viewed as a subscheme of $\mathbb{P}^n$, is AG. In fact, any effective divisor in the linear system $|H_F - C|$ is AG.*

Sketch of the Proof. We are assuming that C is the divisor associated to a regular section of $\omega_X(l)$ for some $l \in \mathbb{Z}$. Let Y be the residual divisor, $Y \in |H_F - C|$. We have $I_{Y|X}(d) \cong \mathcal{O}_X(dH - Y) \cong \mathcal{O}_X(C) \cong \omega_X(l)$ and the exact sequence

$$0 \longrightarrow I(X) \longrightarrow I(Y) \longrightarrow H^0_*(\omega_X)(l-d) \longrightarrow 0.$$

Using the minimal free resolutions of $I(X)$ and $H^0_*(\omega_X)(l-d)$ together with the Horseshoe Lemma ([21], 2.2.8, pg. 37) we deduce that Y is AG. □

In next Proposition we are going to prove that in contrast to the fact that adding hyperplane sections does not preserve the CI-liaison class (see Example 3.11), it preserves the G-liaison class.

Proposition 4.3. *Let $X \subset \mathbb{P}^n$ be a smooth ACM subscheme and let $C \subset X$ be an effective divisor. Take any divisor C_t in the linear system $|C + tH|$ being H a hyperplane section of X and $t \in \mathbb{Z}$. Then, C and C_t are G-bilinked. (Notice that if $t = 0$ then C and C_t are linearly equivalent.)*

Sketch of the proof. Let K be a subcanonical divisor of X. Take $A \in I(K)$ a form of degree $a >> 0$ not vanishing on any component of X. So $H_A - K$ is effective (We denote by H_A the codimension one subscheme of X cut out by A). Now we choose forms $F \in I(C)$ and $G \in I(C_t)$ with $\deg F + t = \deg G$ and a divisor D on X such that $H_F - C = D = H_G - C_t$. By lemma 4.2, $H_{AF} - K$ and $H_{AG} - K$ are arithmetically Gorenstein. Moreover, $H_{AF} - K - C = (H_A - K) + (H_F - C) = H_A - K + D$ and $H_{AG} - K - C_t = (H_A - K) + (H_G - C_t) = H_A - K + D$. So C and C_t are Gorenstein linked to $H_A - K + D$ as subschemes of $\mathbb{P}^n$ or, equivalently, C and C_t are G-bilinked. □

Proposition 4.3 motivates the following definition

Definition 4.4. Let $X \subset \mathbb{P}^n$ be a smooth scheme. We say that an *effective divisor* C on X is *minimal* if there is no effective divisor in the linear system $|C - H|$ being H a hyperplane section divisor of X.

Terminology 4.5. To say that a statement holds for a general point of a projective variety Y means that there exists a countable union Z of proper subvarieties of Y such that the statement holds for every $x \in Y \setminus Z$. In particular, we say that a statement holds for a *general* surface $X \subset \mathbb{P}^4$ with Hilbert polynomial $p(t)$ if the statement holds for a general point of an irreducible component of $Hilb^{\mathbb{P}^4}_{p(t)}$.

From now on, unless otherwise specified the word general, when referred to elements of projective varieties, will have this meaning. We have:

Theorem 4.6. *All ACM curves $C \subset \mathbb{P}^4$ lying on a general, smooth, rational, ACM surface $S \subset \mathbb{P}^4$ are glicci, i.e., they belong to the G-liaison class of a complete intersection.*

Sketch of the Proof. According to the classification of general, smooth, rational, ACM surfaces S is either

(1) A cubic scroll: $S = Bl_{\{p_1\}}(\mathbb{P}^2)$ embedded in $\mathbb{P}^4$ by means of the linear system $|2E_0 - E_1|$, $\deg(S) = 3$, and $\mathrm{Pic}(S) \cong \mathbb{Z}^2 =< E_0; E_1 >$, or
(2) A Del Pezzo surface: $S = Bl_{\{p_1,\dots,p_5\}}(\mathbb{P}^2)$ embedded in $\mathbb{P}^4$ by means of the linear system $|3E_0 - \sum_{i=1}^{5} E_i|$, $\deg(S) = 4$, and $\mathrm{Pic}(S) \cong \mathbb{Z}^6 =< E_0; E_1, \dots, E_5 >$, or
(3) A Castelnuovo surface: $S = Bl_{\{p_1,\dots,p_8\}}(\mathbb{P}^2)$ embedded in $\mathbb{P}^4$ by means of the linear system $|4E_0 - 2E_1 - \sum_{i=2}^{8} E_i|$, $\deg(S) = 5$, and $\mathrm{Pic}(S) \cong \mathbb{Z}^9 =< E_0; E_1, \dots, E_8 >$, or
(4) A Bordiga surface: $S = Bl_{\{p_1,\dots,p_{10}\}}(\mathbb{P}^2)$ embedded in $\mathbb{P}^4$ by means of the linear system $|4E_0 - \sum_{i=1}^{10} E_i|$, $\deg(S) = 6$, and $\mathrm{Pic}(S) \cong \mathbb{Z}^{11} =< E_0; E_1, \dots, E_{10} >$.

For each general, smooth, rational, ACM surface, we classify the minimal ACM curves C on S (see [13]; 8). Finally, we check that each minimal ACM curve C on S is glicci by direct examination. $\square$

To generalize Theorem 4.6 to other ACM surfaces $S \subset \mathbb{P}^4$, we need the following result of López.

Theorem 4.7. *Let $X \subset \mathbb{P}^4$ be a general, ACM surface not complete intersection with degree matrix $[u_{i,j}]$, $u_{i,j} > 0$ for all i, j. Then, three cases are possible for the Picard group of X:*

(i) $\mathrm{Pic}(X) \cong \mathbb{Z}^9$ *and X is a Castelnuovo surface, or*
(ii) $\mathrm{Pic}(X) \cong \mathbb{Z}^{11}$ *and X is a Bordiga surface, or*
(iii) $\mathrm{Pic}(X) \cong \mathbb{Z}^2$ *if X is none of the above.*

Proof. See [15]; Theorem III. 4.2. $\square$

Remark 4.8. In the last case, Theorem 4.7 (iii), $\mathrm{Pic}(X)$ is generated by $H = \mathcal{O}_X(1)$ and K, being K the canonical sheaf of X.

Remark 4.9. Let $X \subset \mathbb{P}^4$ be a smooth general ACM surface. Assume that either X is a complete intersection or X is rational. Then, any ACM curve C on X is glicci. Indeed, either X is rational and the result follows Theorem 4.6, or X is a complete intersection, $\deg(X) > 4$ and $\mathrm{Pic}(X) \cong \mathbb{Z} =< H >$. In this last case, the result follows from Proposition 4.3 and the fact that the hyperplane section H of X is an ACM curve C contained in $\mathbb{P}^3$, and according to Gaeta's Theorem [8], H is licci.

From now on, we restrict our attention to general ACM surfaces $X \subset \mathbb{P}^4$ which are neither rational, nor complete intersection. We will also assume that the degree matrix $[u_{i,j}]$ of X verifies $u_{i,j} > 0$ for all i, j. Hence, according to Theorem 4.7 and Remark 3.8, $\mathrm{Pic}(X) \cong \mathbb{Z}H \oplus \mathbb{Z}K$; and we are ready to prove one of the main result of [5].

Theorem 4.10. *Let $X \subset \mathbb{P}^4$ be a general, ACM surface with degree matrix $[u_{i,j}]$, $u_{i,j} > 0 \quad \forall i, j$. Then, every ACM effective divisor C on X is glicci.*

Sketch of the Proof. According to Lemma 4.2 and Proposition 4.3, we only have to study the effective divisors D of type $D = H$ (which is glicci because according to [13]; Theorem 3.6, S is glicci and by [16]; Theorem 5.2. 15 its hyperplane section H is also glicci) and of type $D = aK + bH$, with $a > 0$ (indeed, $aK + bH$ and $-(a+1)K + mH$ are G-linked because $aK + bH - (a+1)K + mH = -K + (b+m)H$ is AG by Lemma 4.2, so $aK + bH$ is glicci if and only if $-(a+1)K + mH$ is glicci).

By [5]; Theorem 3.12 $aK + bH$ is not ACM for $a \geq 2$. Now, we will check that any effective divisor in the linear system $\mid K + lH \mid$ is glicci:

Let L be the $(n+1) \times (n+2)$ matrix defining the surface X and let $A = [L, M]$ be the matrix obtained adding to L a column M. Thus, A defines a codimension 3 standard determinantal scheme $D \subset X \subset \mathbb{P}^4$. By Theorem 5.3 (see below), D is glicci. Moreover, $\mathcal{O}_X(D) \cong \omega_X(t)$ for some $t \in \mathbb{Z}$, i.e., $D \in |K + tH|$ (see [13]; Proposition 10.7). Hence, $K + lH$ and D are G-bilinked (Proposition 4.3). So $K + lH$ is glicci which proves what we want. $\square$

Using the fact that, roughly speaking, G-liaison is a theory about generalized divisors on ACM schemes which collapses to the setting of CI-liaison theory as a theory of generalized divisors on a complete intersection (for more details see [13] and [10]) and the fact that the Picard group of rational normal scrolls is well known, in [5], the authors study G-liaison classes of ACM divisors on rational normal scrolls surfaces and they proved:

Theorem 4.11. *Let $C \subset \mathbb{P}^n$ be an ACM curve lying on a rational normal scroll surface $S = S(a_0, a_1) \subset \mathbb{P}^n$, $a_0 + a_1 = n - 1$. Then C is glicci.*

Remark 4.12. In [3], M. Casanellas has extended the above results to ACM divisors on an arbitrary rational normal scroll $S = (Sa_0, \cdots, a_k) \subset \mathbb{P}^{c+k}$ being $c = \sum_{i=0}^{k} a_i$.

As a first application of Theorem 4.11 we will prove that extremal curves $C \subset \mathbb{P}^r$, i.e. curves of maximum arithmetic genus, are glicci. To this end, let us introduce some notation:

Given two integers d, r wit $d \geq r \geq 3$, we denote by $\pi(d, r)$ the upper bound (Castelnuovo bound) for the arithmetic genus of irreducible, non-degenerate curves of degree d in $\mathbb{P}^r$. More precisely,

Theorem/Definition 4.13. (Castelnuovo Theorem) *Let $C \subset \mathbb{P}^r$ be an integral, non-degenerate curve of degree d; set $m = \left[\frac{d-1}{r-1}\right]$ and write $d = m(r-1) + \varepsilon + 1$, $0 \leq \varepsilon \leq r-2$. Then the arithmetic genus of C satisfies*

$$p_a(C) \leq \pi(d, r) := \binom{m}{2}(r-1) + m\varepsilon.$$

Integral, non-degenerate curves $C \subset \mathbb{P}^r$ for which the bound is attained are called **Castelnuovo curves***; they are ACM curves and were classified by Castelnuovo.*

Proof. See, for instance [9] p. 42. □

As a consequence of Theorems 4.11 and 4.13 we obtain:

Corollary 4.14. *Let C be a Castelnuovo curve of degree d in $\mathbb{P}^r$, $d \geq 2r+1$, $r \geq 3$. Then, C is glicci.*

Proof. By [1]; Theorem III. 2.5, C lies either on the Veronese surface or on a rational normal scroll surface. In the first case, C is glicci because the Picard group of the Veronese surface is generated by a conic C_0 and the hyperplane section H is linearly equivalent to $2C_0$; since C_0 and H are glicci, applying Proposition 4.3 we get that every curve lying on the Veronese surface is ACM and glicci. In the second case, C is ACM and lies on a rational normal scroll surface, so by Theorem 4.11 we get that C is glicci. □

As another application of Theorem 4.11 we will see that ACM curves $C \subset \mathbb{P}^n$ with maximum Castelnuovo-Mumford regularity are glicci. Recall that if $C \subset \mathbb{P}^n$ is an integral ACM curve of degree d, then its Castelnuovo-Mumford regularity, $reg(V)$, is bounded above by $reg(V) \leq \lceil \frac{d-1}{c} \rceil + 1$ ([19]; Theorem 1.2), where $\lceil m \rceil$ is the smallest integer $\geq m$ for $m \in \mathbb{Q}$. We have:

Corollary 4.15. *Let $C \subset \mathbb{P}^n$ be an integral ACM curve of degree $d > (c+1)^2$. Assume that V has maximum Castelnuovo-Mumford regularity i.e.: $reg(V) = \lceil \frac{d-1}{c} \rceil + 1$. Then, V is glicci.*

Proof. By [19]; Theorem 1.2, C is a divisor on a variety of minimal degree. A variety of minimal degree is either a rational normal scroll, a cone over a quadric hypersurface or a cone over the Veronese surface in $\mathbb{P}^5$ (see, for instance, [9]; pg. 51). By Theorem 4.11, any ACM effective divisor on a rational normal scroll surface is glicci; and it is easy to check that any ACM effective divisor on a cone over a quadric hypersurface or a cone over the Veronese surface in $\mathbb{P}^5$ is glicci. $\square$

As a last application of Theorem 4.11 we will see that smooth ACM hyperelliptic curves $C \subset \mathbb{P}^r$ are also glicci. Recall that a smooth curve $C \subset \mathbb{P}^r$ is said to be **hyperelliptic** if it carries a g_2^1 linear series and by [1]; pg. 221, it holds: $\deg(C) \geq 2g + 1$ and $r \geq g + 1$.

Corollary 4.16. *All smooth, hyperelliptic ACM curves $C \subset \mathbb{P}^r$ are glicci.*

Proof. By [7]; Theorem 2, any smooth, ACM, hyperelliptic curve $C \subset \mathbb{P}^r$ of degree d lies on a rational normal scroll surface $S = S(a_0, a_1) \subset \mathbb{P}^r$, $a_0 + a_1 + 1 = r$, and $S \sim 2H + (d - 2r + 2)F$, as a divisor on S. By Theorem 4.11, any ACM, effective divisor on a rational normal scroll surface is glicci and we are done. $\square$

In [3], Casanellas studies the G-liaison class of smooth, hyperelliptic curves $C \subset \mathbb{P}^r$ not necessarily ACM.

5. Generalization of Gaeta's Theorem.

In this section, we generalize Gaeta's theorem and we prove that any standard determinantal subscheme $X \subset \mathbb{P}^n$ is in the G-liaison class of a complete intersection. We start fixing some notation.

Definition 5.1. A subscheme $X \subset \mathbb{P}^n$ of codimension $c + 1$ is said to be standard determinantal if $I(V)$ is defined by the maximal minors of a $t \times (t+c)$ homogeneous matrix A. To simplify, we will often write $I(X) = I(A)$.

If $X \subset \mathbb{P}^n$ is standard determinantal then X is ACM. Moreover, the Hilbert-Burch Theorem states that, in codimension 2, the converse is also true.

In section 3, we have pointed out that if $X \subset \mathbb{P}^n$ is licci then it is ACM, and hence, if we also have $\text{codim}(X) = 2$, then X is standard determinantal. The important contribution to liaison theory of Gaeta's theorem (See [20] for a rigorous, modern proof of Gaeta's theorem) is the converse:

Theorem 5.2. *Let $V \subset \mathbb{P}^n$ be a pure codimension 2 subscheme defined by the maximal minors of a $t \times (t + 1)$ homogeneous matrix A. Then, V is licci.*

Sketch of the Proof. We link V to a scheme V_1 by means of a complete intersection X defined by two minimal generators of V. V_1 is ACM and, hence, standard determinantal. Gaeta proved that the matrix A_1 defining $I(V_1)$ is obtained from A deleting two columns and transposing. Going on, in a finite number of steps, we reach a 1×2 matrix, i.e. a complete intersection. $\square$

In the context of G-liaison, a generalization of gaeta's Theorem does hold and we have

Theorem 5.3. *Let $V \subset \mathbb{P}^n$ be a pure codimension c subscheme defined by the maximal minors of a $t \times (t+c-1)$ homogeneous matrix A. Then, V is glicci.*

Idea of the Proof. The proof is rather technical and the main idea is the following one:

We denote by B the matrix obtained deleting a "suitable" column of A and we call X the subscheme defined by the maximal minors of B. ("Suitable" means that $\operatorname{codim}(X) = c - 1$. First take, if necessary, a general linear combination of the rows and columns of A.)

We denote by A' the matrix obtained deleting a "suitable" row of B and we call V' the subscheme defined by the maximal minors of A'. ("Suitable" means that $\operatorname{codim}(V') = c$. First take, if necessary, a general linear combination of the rows and columns of B.)

We consider V and V' as divisors on X, we show that V and V' are G-bilinked. Hence in $2t-2$ steps we reach a scheme defined by a 1×3 matrix, i.e., we arrive at a complete intersection. $\square$

Remark 5.4. Gaeta's original theorem says that all ACM subschemes of codimension 2 are licci. Since it is well known for subschemes of codimension two that ACM subschemes are standard determinantal and that AG subschemes and complete intersections coincide, Theorem 5.3 is a full generalization of Gaeta's Theorem.

Finally, we want to stress that this last result drastically differs from the one we obtain when we link by means of complete intersection schemes. Indeed, since any ACM curve D_p in $\mathbb{P}^4$ defined by the maximal minors of a $p \times (p+2)$ matrix with linear entries has a linear resolution, we have that D_p and $D_{p'}$ belong to different CI-liaison classes provided $p \neq p'$ (See Corollary 3.9) and, by Theorem 5.3. they belong to the same G-liaison class.

6. Further Comments and Questions.

We end this paper with some comments and questions raised by this note.

Let $X \subset \mathbb{P}^{n+3}$ be a closed ACM subscheme of dimension $n > 0$. In Corollary 3.6 we have seen that if X is licci then the local cohomology groups $H^i_m(K \otimes_R I)$ vanish, $0 \leq i \leq n$. So, we are led to pose the following question which, to my knowledge, is still open:

Question 6.1. *Whether the converse of Corollary 3.5 is true, i.e., is a codimension 3 ACM scheme $X \subset \mathbb{P}^{n+3}$ licci if $H^i_m(K \otimes_R I) = 0$ for $0 \leq i \leq n$?*

Remark 6.2. In Theorem 3.4, we have seen that the graded modules $H^i_m(K \otimes_R I)$ are CI-liaison invariants. We will now show that $H^i_m(K \otimes_R I)$ are not G-liaison invariants. Indeed, we have the following example:

Denote by $D_t \subset \mathbb{P}^4$ the ACM curve defined by the maximal minors of a $t \times (t+2)$ matrix with linear entries. D_t has a t-linear resolution. According to Corollary 3.9, $H^0_m(K_{D_t} \otimes_R I(D_t))$ changes when t varies and it follows from Theorem 5.3 that D_t is glicci. Therefore, $H^0_m(K \otimes_R I)$ is not a G-liaison invariant.

In sections 4 and 5, we have determined huge families of ACM subschemes $X \subset \mathbb{P}^n$ which are glicci (see, Theorems 4.6, 4.10 and 4.11; and Corollaries 4.13-15). Hence, we are led to pose the following question which should be viewed as a generalization of Gaeta's Theorem (see section 5):

Question 6.3. Is there only one Gorenstein liaison class containing ACM schemes $X \subset \mathbb{P}^n$ of codimension c? or, equivalently, are all ACM subschemes $X \subset \mathbb{P}^n$ glicci?

Based on the results of section 4, as well as those in [13], [3], [4],[5], [6], [11] and [18], I would expect a yes answer to the last question. Notice that even in codimension 3, an affirmative answer to the above question will be a very interesting result. It will also be worthwhile to know if the following partial results are true:

Question 6.4. *Is any ACM curve $C_t \subset \mathbb{P}^n$ with a t-linear resolution:*

$$0 \longrightarrow R(-t-2)^{\frac{t^2+t}{2}} \longrightarrow R(-t-1)^{t^2+2t} \longrightarrow R(-t)^{\frac{t^2+3t+2}{2}} \longrightarrow I(C_t) \longrightarrow 0$$

glicci?

We know many examples of glicci, ACM curves $C_t \subset \mathbb{P}^4$ with a t-linear resolution. Indeed, any ACM curve $D_t \subset \mathbb{P}^4$ defined by the maximal minors of a $t \times (t+2)$ matrix with linear entries. D_t has a t-linear resolution and by [13]; Theorem 3.6 D_t is glicci. Nevertheless, not all ACM curves $C_t \subset \mathbb{P}^4$ with a t linear resolution are standard determinantal. In fact, the family of such determinantal curves has dimension $\leq 3t^2 + 6t - 3$ [13]; Proposition 10.3. On the other hand, each component of the Hilbert scheme of curves of degree $d(D_t) = \binom{t+3}{4} - \binom{t+2}{4}$ and genus $p_a(D_t) = (t-1)d(C_t) + 1 - \binom{t+3}{4}$ has dimension $\geq 5d(D_t) + 1 - p_a(D_t)$. Thus it is enough to take a value of t (for instance $t = 3, 4$) such that $3t^2 + 6t - 3 \leq 5d(D_t) + 1 - p_a(D_t)$.

The last question we would like to consider is the following one. Consider the subscheme $X_{p,q,r} \subset \mathbb{P}^n$ defined by the $r \times r$ minors of a $p \times q$, $r \leq min(p, q)$, homogeneous matrix. Assume that X has the expected codimension, i.e., $\operatorname{codim}(X) = (p - r + 1)(q - r + 1)$. It is well known that $X_{p,q,r}$ is an ACM scheme

Question 6.5. *Is $X_{p,q,r} \subset \mathbb{P}^n$ glicci?*

Example 6.6. Let V be the Veronese surface $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$, its ideal is defined by the 2×2 minors of the generic symmetric matrix:

$$A = \begin{pmatrix} X_0 & X_1 & X_2 \\ X_1 & X_3 & X_4 \\ X_2 & X_4 & X_5 \end{pmatrix},$$

and V is ACM. Therefore V is an ACM effective divisor on the rational normal scroll $S(0, 1, 2) \subset \mathbb{P}^5$ defined by the maximal minors of the matrix

$$\begin{pmatrix} X_0 & X_1 & X_2 \\ X_1 & X_3 & X_4 \end{pmatrix}.$$

Thus, V is glicci by [5]; Theorem 4.10 (i).

Finally I want to point out that a lot of work has been done on G-liaison of codimension c arithmetically Cohen-Macaulay schemes in $\mathbb{P}^n$ and some on G-liaison of codimension c non arithmetically Cohen-Macaulay schemes in $\mathbb{P}^n$. The study of G-liaison classes of non ACM schemes has not been addressed here. For results on G-liaison of non arithmetically Cohen-Macaulay schemes of codimesnion $c > 2$ in $\mathbb{P}^n$ the reader can see, for instance, [6],[3], [11], [13] and [14].

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