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ON THE COHOMOLOGY AND GENUS OF PROJECTIVE CURVES

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Dedicated to Silvio Greco in occasion of his 60-th birthday.

We discuss recent results on the possible pairs of degree and genus of projective curves and on the related problem of bounding the cohomology of curves.

1. Introduction.

This is the slightly expanded version of the talk given at the Catania conference on Commutative Algebra and Algebraic Geometry in honour of Silvio Greco.

Let $\mathbb{P}^n$ be the n-dimensional projective space over an algebraically closed field of characteristic zero. By a curve $C \subset \mathbb{P}^n$ we will always understand a closed subscheme which is locally Cohen-Macaulay of pure dimension 1, thus in particular without embedded points. The curve is called non-degenerate if it is not contained in a hyperplane of $\mathbb{P}^n$. The most important invariants of C are its degree and its (arithmetic) genus. However, a more precise description of the curve C requires some knowledge of its Hartshorne-Rao module. Its importance is particularly highlighted in Liaison theory (cf. [21]).

Since every module of finite length is (up to degree shift) the Hartshorne-Rao module of a curve, the investigation of its structure is only meaningful in particular situations. Actually, it often suffices to know some estimates of the "size" of the Hartshorne-Rao module. Thus, it is of fundamental importance to answer the following questions.

Problems. Let $C \subset \mathbb{P}^n$ be a non-degenerate curve of degree d and (arithmetic) genus g.

(1) What pairs (d, g) can occur?
(2) Are there (optimal) upper estimates for the Rao function $j \mapsto h^1(\mathcal{I}_C(j))$?

The first problem is very classical for irreducible, reduced spaced curves. However, recently it has become clear that one should consider the problems above in more generality for at least two reasons. In Liaison theory it is not possible to restrict the attention to integral curves. In order to obtain a satisfactory theory one has to study curves in the generality above. For example, the minimal curves in an even liaison class are often neither irreducible nor reduced nor locally complete intersections. Moreover, even if one wants to study families of integral curves the special members are often not of this type. The Hilbert scheme $H^n_{d,g}$ of locally Cohen-Macaulay curves in $\mathbb{P}^n$ of degree d and genus g is the right environment for the investigation of families of curves. Of course, the first problem above just asks: When does the Hilbert scheme $H^n_{d,g}$ contain a non-degenerate curve.

As one might expect, the most detailed answers to the problems above are known for space curves. We discuss them in the next section.

Curves of higher codimension are considered in Sections 3 and 4. The first generalizations to curves of higher codimension have been obtained in [1] by considering curves whose general hyperplane section is non-degenerate. Note that this is not an assumption in the case of space curves, but for $n \geq 4$. This assumption and the related results are discussed in Section 3 while the case of arbitrary curves is treated in Section 4. The results indicate that the problems become more difficult if the codimension of the curves becomes larger.

Of course, it is also interesting to investigate the problems above for special classes of curves. This is briefly mentioned in the final section.

2. Space curves.

Let $C \subset \mathbb{P}^3$ be a non-degenerate curve. Thus its degree d is at least 2. If C has degree 2 then it is either a pair of two skew lines or a double line. Double lines are completely described by Ferrand's construction [8]. In particular, it turns out that there is a curve of degree 2 and genus g if and only if $g \leq -1$. For curves of higher degree we have.

Proposition 2.1. *There is a non-degenerate curve $C \subset \mathbb{P}^3$ of degree $d \geq 3$ and genus g if and only if $g \leq \binom{d-2}{2}$.*

This result has been proved several times [30], [28], [13]. Note that the result is still true if the field K has positive characteristic by [13]. Hartshorne's approach consists in a combination of Castelnuovo's classical method and the following restriction result.

Theorem 2.2. (Hartshorne, [13]). *If $C \subset \mathbb{P}^3$ is a non-degenerate curve of degree $d \geq 3$ then its general hyperplane section $C \cap H$ is non-degenerate (in $H \cong \mathbb{P}^2$) as well.*

Actually, this statement is not true over fields of positive characteristic. Hartshorne classified the counterexamples. They are certain multiple lines (cf. [13], Theorem 2.1).

Using Hartshorne's restriction theorem, Martin-Deschamps and Perrin could derive optimal upper bounds for the Hartshorne-Rao module of space curves. Their result implies immediately Proposition 2.1.

Theorem 2.3. (Martin-Deschamps, Perrin, [18]).

(a) *Let $C \subset \mathbb{P}^3$ be a non-degenerate curve of degree $d \geq 3$. Then we have:*

$$h^1(\mathcal{I}_C(j)) \leq \rho_3(j) \quad \textit{for all } j \in \mathbb{Z}$$

where $\rho_3 : \mathbb{Z} \to \mathbb{Z}$ is the function defined by

$$\rho_3(j) = \begin{cases} 0 & \textit{if } j \leq -\binom{d-2}{2} + g \\ \binom{d-2}{2} - g + j & \textit{if } -\binom{d-2}{2} + g \leq j \leq 0 \\ \binom{d-2}{2} - g & \textit{if } 0 \leq j \leq d-2 \\ \binom{d-1}{2} - g - j & \textit{if } d-2 \leq j \leq \binom{d-1}{2} - g \\ 0 & \textit{if } \binom{d-1}{2} - g \leq j \end{cases}$$

(b) *For every pair (d, g) of integers such that $d \geq 3$ and $g \leq \binom{d-2}{2}$ there is an* extremal curve*, i.e. a non-degenerate curve $C \subset \mathbb{P}^3$ of degree d and genus g such that $h^1(\mathcal{I}_C(j)) = \rho_3(j)$ for all $j \in \mathbb{Z}$.*

Remark 2.4. The statement above remains true for curves of degree 2 if we restrict the pairs $(2, g)$ in (b) by $g \leq -1$. Note, that every curve of degree 2 is extremal according to the description of the Hartshorne-Rao module of double lines by Migliore in [20].

After having established Theorem 2.3 the family of extremal curves has been studied in a subsequent paper by Martin-Deschamps and Perrin. Their result shows that the Hilbert scheme is almost never reduced.

Theorem 2.5. (Martin-Deschamps, Perrin, [19]). *The extremal curves in the Hilbert scheme $H^3_{d,g}$ form an irreducible component which is not reduced if $d \geq 6$ and $g \leq \binom{d-3}{2}$.*

We refer to [19] for the more precise results in the cases excluded above.

Remark 2.6.

(i) The component of the extremal curves seems to play a particular role in its Hilbert scheme. It is an open problem if the Hilbert scheme $H^3_{d,g}$ of locally Cohen-Macaulay curves is connected. Since semicontinuity does not provide obstructions to deform a curve in $H^3_{d,g}$ to an extremal curve, one attack to this problem consists in showing that in fact every curve can be deformed to an extremal curve. For positive results in this direction we refer to [14] where also further references can be found.

(ii) In order to prove Theorem 2.5, Martin-Deschamps and Perrin computed the Hartshorne-Rao module and the defining equations of an extremal curve. They showed in [19] that every extremal curve $C \subset \mathbb{P}^3$ is a minimal curve in its even liaison class, its Hartshorne-Rao module is (up to changes of coordinates) $R/(x_0, x_1, F, G)(a-1)$ where $a := \binom{d-2}{2} - g$ and $\{x_0, x_1, F, G\}$ is a regular sequence such that $\deg F = a$ and $\deg G = a + d - 2$, and its homogeneous ideal has in case $a \geq 2$ the shape
$$I_C = (x_0^2, x_0x_1, x_1h, x_0G - x_1hF)$$
where $h \in K[x_1, x_2, x_3]$ is a non-trivial form of degree $d-2$.

A geometric characterization of extremal curves has been obtained by Ellia.

Theorem 2.7. (Ellia, [6]). *Let $C \subset \mathbb{P}^3$ be a non-degenerate curve of degree $d \geq 5$. Then C is an extremal curve if and only if C contains a planar subcurve of degree $d-1$.*

Nollet has shown in [25] that the bound in Theorem 2.3 can be improved if we exclude the extremal curves. The curves achieving the improved bound are called *sub-extremal curves*. As a consequence one has.

Theorem 2.8. (Nollet, [25]). *Let $C \subset \mathbb{P}^3$ be a non-degenerate curve of degree $d \geq 4$. If there is an integer j such that*
$$h^1(\mathcal{I}_C(j)) = \rho_3(j) > 0$$
then C is an extremal curve.

Nollet has also shown that a curve is sub-extremal if and only if it can be obtained from an extremal curve by an elementary biliasion on a quadric of height 1. Moreover, if $d \geq 7$ then a sub-extremal curve contains a planar subcurve of degree $d-2$.

In [26] Notari and Sabadini have established optimal bounds for the Rao function of space curves of degree $d \geq 3$ with the property that the largest degree of a planar subcurve is $d-p$ where $1 \leq p \leq \frac{d}{2}$. If one of these curves (of suitable degree) achieves the bound then it can be obtained from an extremal curve by an elementary biliasion on a quadric of height $p-1$.

Families of space curves whose Rao function is almost as large as the one of a sub-extremal curve are studied in [3].

3. Curves with non-degenerate hyperplane section.

In this section we will describe the first generalizations of Theorem 2.3 to curves in $\mathbb{P}^n$, $n \geq 3$.

Consider the following condition on a curve $C \subset \mathbb{P}^n$:

$(*)$ $C \cap H \subset H \cong \mathbb{P}^{n-1}$ is non-degenerate for a general hyperplane $H \subset \mathbb{P}^n$.

Observe that a curve satisfying this condition must have degree $\geq n$. Moreover, there are the following bounds.

Theorem 3.1. (Chiarli, Greco, Nagel, [1]).

(a) *Let $C \subset \mathbb{P}^n$ be a curve having Property $(*)$. Then we have*

$$h^1(\mathcal{I}_C(j)) \leq \rho_n(j) \quad \textit{for all } j \in \mathbb{Z}$$

where $\rho_n : \mathbb{Z} \to \mathbb{Z}$ is the function defined by

$$\rho_n(j) = \begin{cases} 0 & \textit{if} \quad j \leq -\binom{d-n+1}{2} + g \\ \binom{d-n+1}{2} - g + j & \textit{if} \quad -\binom{d-n+1}{2} + g \leq j \leq 0 \\ \binom{d-n+1}{2} - g & \textit{if} \quad 0 \leq j \leq d-n+1 \\ \binom{d-n+2}{2} - g - j & \textit{if} \quad d-n+1 \leq j \leq \binom{d-n+2}{2} - g \\ 0 & \textit{if} \quad \binom{d-n+2}{2} - g \leq j. \end{cases}$$

(b) *For every pair (d, g) of integers such that $d \geq n$ and $g \leq \binom{d-n+1}{2}$ there is an* extremal curve subject to Condition $(*)$, *i.e. a non-degenerate curve $C \subset \mathbb{P}^n$ of degree d and genus g such that $h^1(\mathcal{I}_C(j)) = \rho_n(j)$ for all $j \in \mathbb{Z}$ and C has Property $(*)$.*

Remark 3.2.

(i) Recall that all non-degenerate space curves have Property $(*)$ by Hartshorne's Restriction theorem 2.2. Hence the theorem above specializes to Theorem 2.3 for $n = 3$.
(ii) Extremal curves subject to condition $(*)$ were just called extremal curves in [1]. However, we want to reserve the name extremal curves for later use in the next section.

In spite of the Restriction theorem for space curves one might wonder about a similar result for curves of higher codimension. However, the situation is more complicated. For curves of codimension 3 there is the following restriction result.

Theorem 3.3. (Chiarli, Greco, Nagel, [2]). *Let $C \subset \mathbb{P}^4$ be a non-degenerate curve of degree $d \geq 5$. Then C has Property $(*)$ if and only if C does not contain a planar subcurve of $d - 1$.*

Of course, one direction is clear. If C contains a planar subcurve of $d - 1$ then its general hyperplane section is degenerate. To show the converse is the difficult part.

Remark 3.4. The corresponding statement is false for curves of degree 4. Counterexamples are described in [2], Remark 4.13. They are certain multiple lines whose generic embedding dimension is 3.

It seems unlikely that there is a similarly clean statement characterizing Property $(*)$ if $n \geq 5$.

The proof of the last result is based on the one hand on the Socle lemma in [16], which captures a great deal of a technique introduced by Strano in [31], and on the other hand on the following result.

Theorem 3.5. (Chiarli, Greco, Nagel, [2]). *Let $C \subset \mathbb{P}^n$ be a non-degenerate curve of degree $d \leq 2r - 3$. Then C contains a planar subcurve of degree r if and only if the general hyperplane section of C contains a subscheme of degree r spanning a line and does not contain a collinear subscheme of degree $> r$.*

In general, it is not possible to weaken the assumption $d \leq 2r - 3$ (cf. [2], Remark 4.5).

Remark 3.6. The last result implies that an extremal curve $C \subset \mathbb{P}^n$ subject to Condition $(*)$ of degree d contains a planar subcurve of degree $d - n + 2$ provided $d \geq 2n - 1$.

Fixing the Rao functions in non-negative degrees, Notari and Spreafico introduced the following curves.

Definition 3.7. A curve $C \subset \mathbb{P}^n$ of degree $d \geq n+1$ is called *quasi-extremal* if it satisfies Condition $(*)$ and if

$$h^1(\mathcal{I}_C(j)) = \rho_n(j) \quad \text{for all} j \geq 0.$$

For such curves they showed.

Theorem 3.8. (Notari, Spreafico, [27]).

(a) *Every quasi-extremal curve $C \subset \mathbb{P}^n$ of degree d contains a planar subcurve of degree $d-n+2$.*
(b) *The graph of the Rao function of a quasi-extremal curve $C \subset \mathbb{P}^n$ is in negative degrees a polygonal of increasing slope $\leq n-2$.*
(c) *For every pair (d, g) of integers such that $d \geq n+1$ and $g \leq \binom{d-n+1}{2}$ and for every numerical function ρ satisfying* (b) *there is a quasi-extremal curve $C \subset \mathbb{P}^n$ of degree d, genus g and with Rao function ρ.*

Part (a) of this result improves Remark 3.6. Part (c) shows that for extremal curves subject to Condition $(*)$ the analogue of Theorem 2.8 is false if $n \geq 4$.

Equations of some quasi-extremal curves have been described in [27], Lemma 3.3. A geometric construction of quasi-extremal curves is described in [23], Theorem 5.6. By taking a certain union of a plane curve of degree $d-n+1$ and a rope of degree $n-1$, which is supported on a line, one gets a quasi-extremal curve in $\mathbb{P}^n$.

According to [27], Corollary 2.4, the graded Betti numbers of a quasi-extremal curve are known. This information allows to compute the Hartshorne-Rao module of extremal curves subject to condition $(*)$.

Proposition 3.9. *Let $C \subset \mathbb{P}^n$ be an extremal curve subject to Condition $(*)$ of degree $d \geq n+1$. Then the Hartshorne-Rao module $M(C)$ of C is (up to a change of coordinates) isomorphic to*

$$R/(x_0, \ldots, x_{n-2}, F, G)(a-1)$$

where $a := \binom{d-n+1}{2} - g$ and $\{x_0, \ldots, x_{n-2}, F, G\}$ is a regular sequence such that $\deg F = a$ *and* $\deg G = a + d + 1 - n$.

Proof. Put $S = K[x_{n-1}, x_n] \cong R/(x_0, \ldots, x_{n-2})$. It follows from [27], Corollary 2.4 that the K-dual of $M(C)$ has the form

$$M(C)^\vee \cong S/(F, G_1, \ldots, G_{n-2})(a+d-n)$$

where $\deg F = a$ and $\deg G_i = a + d + 1 - n$ for all $i = 1, \dots, n-2$. Hence, $M(C)^\vee$ has a free resolution of the shape

$$0 \to P \to S(-a) \oplus S^\beta(-a-d-1+n) \to M(C)^\vee(-a-d+n) \to 0$$

where $1 \leq \beta \leq n-2$ and P is a free S-module of rank β. Since C is extremal subject to Condition $(*)$ we know the Hilbert function of $M(C)$. This implies $\beta = 1$ and $P = S(-2a-d-1+n)$. Now, the claim follows easily.

In particular, we see for the Hartshorne-Rao module of these curves that $M(C) \cong M(C)^\vee(d-n+1)$, i.e. the Hartshorne-Rao module is self-dual up to a degree shift.

4. Curves with maximal cohomology.

In [1] it has been shown that a non-degenerate curve $C \subset \mathbb{P}^4$ of degree $d \geq 3$ can have every genus $g \leq \binom{d-2}{2} - 1$. Therefore, the bounds in Theorem 3.1 cannot be true for all curves in $\mathbb{P}^4$. The goal of this section is to describe and to discuss upper bounds which hold for all curves in $\mathbb{P}^n$. Proofs of the results will appear in [24].

If a curve of degree 2 is not arithmetically Cohen-Macaulay then it is a pair of two skew lines or a double line. Thanks to the results in [23] we do not only know the possible Rao functions but even all the possible Hartshorne-Rao modules of a curve of degree 2. Thus, here we can restrict ourselves to curves of degree $d \geq 3$.

Proposition 4.1. *Let $C \subset \mathbb{P}^n$ be a non-degenerate curve of degree $d \geq 3$. Then we have*

$$h^1(\mathcal{I}_C(j)) \leq \rho_n^{ex}(j) \quad \text{for all } j \in \mathbb{Z}$$

where $\rho_n^{ex} : \mathbb{Z} \to \mathbb{Z}$ is the function defined by

$$\rho_n^{ex}(j) = \begin{cases} 0 & \text{if } j \leq -\binom{d-2}{2} + g \\ \binom{d-2}{2} - g + j & \text{if } -\binom{d-2}{2} + g \leq j \leq 0 \\ \binom{d-2}{2} - g - (n-3) & \text{if } 1 \leq j \leq d-2 \\ \binom{d-1}{2} - g - (n-3) - j & \text{if } d-2 \leq j \leq \binom{d-1}{2} - g - (n-3) \\ 0 & \text{if } \binom{d-1}{2} - g - (n-3) \leq j \end{cases}$$

Observe that $\rho_3^{ex} = \rho_3$. Hence this result is a generalization of Theorem 2.3 (a).

The last result immediately implies.

Corollary 4.2. *Let $C \subset \mathbb{P}^n$ be a non-degenerate curve of degree $d \geq 3$. Then the arithmetic genus g of C satisfies*

$$g \leq \binom{d-2}{2} - (n-3).$$

As for space curves a non-degenerate curve $C \subset \mathbb{P}^n$ is called *extremal* if

$$h^1(\mathcal{I}_C(j)) = \rho_n^{ex}(j) \quad \text{for all } j \in \mathbb{Z}.$$

The question is if the bound on the genus is optimal and if extremal curves do exist if $n \geq 4$. This is taken care of by the following construction of certain multiple lines.

Construction 4.3. *Let $D \subset \mathbb{P}^n$ be a planar curve of degree $d-1$ which is supported on the line L. Let $a \geq 0$ be an integer. Then there is an exact sequence defining the curve C*

$$0 \to \mathcal{I}_C \to \mathcal{I}_D \to \mathcal{O}_L(a+n-4) \to 0.$$

Using this sequence it is easy to compute the genus of C. This provides the first part of the following result. The second one requires more care.

Theorem 4.4.

(a) *A non-degenerate curve $C \subset \mathbb{P}^n$ of degree $d \geq 3$ can have every genus $g \leq g_{max} := \binom{d-2}{2} - (n-3)$.*
(b) *For every pair (d, g) of integers such that $d \geq 3$ and$g \leq g_{max}$ there is an extremal curve $C \subset \mathbb{P}^n$.*

Remark 4.5. Contrary to the situation for space curves, extremal curves of $\mathbb{P}^n$ of maximal genus $g = g_{max}$ are not arthmetically Cohen-Macaulay if $n \geq 4$.

The construction above does not provide all extremal curves. Nevertheless, we have.

Proposition 4.6. *If $C \subset \mathbb{P}^n$ is an extremal curve of degree $d \geq 5$ then it contains a planar subcurve of degree $d-1$.*

Remark 4.7.

(i) The converse is not true, i.e. Ellia's characterization of extremal space curves (cf. Theorem 2.7) cannot be generalized to $\mathbb{P}^n$ with $n \geq 4$. There are curves in $\mathbb{P}^n$, $n \geq 4$, of degree d which do contain a planar subcurve of degree $d-1$, but whose cohomology is smaller than the one of extremal curves.

(ii) Nollet's Theorem 2.8 cannot be generalized either to $n \geq 4$. This also follows from a careful study of the construction above.

The last observations stress the fact that the situation for space curves is simpler than for curves of codimension ≥ 3. This is also reflected in the following result.

Theorem 4.8. *Let $C \subset \mathbb{P}^n$ be a non-degenerate curve of degree $d \geq 5$ and genus $g < g_{max}$. Then C is an extremal curve if and only if its Hartshorne-Rao module is (up to change of coordinates) isomorphic to $R/(x_2, \ldots, x_n, f \cdot (x_0, x_1)^{n-3}, h)(\binom{d-2}{2} - g - 1)$ where $\{f, h, x_2, \ldots, x_n\}$ is a regular sequence.*

The Hartshorne-Rao module of extremal curves can also be described if $g = g_{max}$. Moreover, the graded Betti numbers of an extremal curve are computed in [24].

5. Remarks on reduced and integral curves.

We have seen in Sections 2 and 4 that Problems (1) and (2) posed in the introduction are solved. However, the situation changes if one asks these problems for classes of curves having some of the following properties: reduced, irreducible, smooth, linearly normal, not lying on a hypersurface of degree $< s$. We discuss some of the related results beginning with Problem (1).

The question of the possible pairs (d, g) of non-degenerate, integral, smooth curves in $\mathbb{P}^n$ is very classical. In 1882 Halphen and M. Noether received the Steiner Prize for their great treatises devoted to this problem in case $n = 3$. Halphen gave a complete description of the occurring pairs (d, g), but the first correct proof was given by Gruson and Peskine in [9]. The result is also true over fields of positive characteristic according to Hartshorne [11]. Later on Rathmann [29] answered the question for curves in $\mathbb{P}^4$ and $\mathbb{P}^5$ while Ciliberto [4] found the solution for curves in $\mathbb{P}^6$. It should be noted that the existence problem becomes increasingly more difficult if n grows because the curves have to be constructed on various surfaces. A complete description of the possible pairs (d, g) for curves in $\mathbb{P}^n$ is known in case $g \leq \frac{(d-1)^2}{8}$ due to [5].

For reduced and for reduced connected curves in $\mathbb{P}^3$, Tedeschi [32] described all pairs (d, g) in the case that the genus g exceeds the classical Castelnuovo's bound.

For problem (2) much less is known. Note that for a reduced curve $C \subset \mathbb{P}^n$ we always have

$$h^1(\mathcal{I}_C(j)) = 0 \quad \text{if} j < 0.$$

Asking for the vanishing of $H^1(\mathcal{I}_C(j))$ in positive degrees means essentially searching for upper bounds of the Castelnuovo-Mumford regularity of the curve. In this respect the beautiful results of L. Gruson, R. Lazarsfeld, C. Peskine in [10] say that

$$H^1(\mathcal{I}_C(j)) = 0 \quad \text{if} \quad j \geq \deg C - 1 \quad \text{and} \quad C \quad \text{is reduced}$$

and

$$H^1(\mathcal{I}_C(j)) = 0 \quad \text{if} \quad j \geq \deg C + 1 - n \quad \text{and} \quad C \quad \text{is integral.}$$

It seems even more difficult to obtain good bounds for the dimension of $H^1(\mathcal{I}_C(j))$ for all $j \in \mathbb{Z}$. Bounds depending on d, g and the index of speciality are established by Ellia and Sols in [7]. They are optimal for space curves, but not if $n \geq 4$.

It is also very interesting to consider Problems (1) and (2) for the class of integral curves not lying on a hypersurface of degree $< s$. Then Problem (1) is open even for space curves in the so-called range B (cf. [12]).

With respect to problem (2) one expects that the cohomology becomes smaller if s grows. However, very little is known in this respect. But there are bounds for arbitrary space curves due to Miro-Roig and Nollet (cf. [22]).

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