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PARTIAL GORENSTEIN IN CODIMENSION 3

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Dedicated to Silvio Greco in occasion of his 60-th birthday.

The goal of the paper is to build particular three codimensional arithmetically Cohen-Macaulay subschemes of $\mathbb{P}^r$, partial Gorenstein schemes, whose graded Betti numbers can be easily computed in terms of their combinatorial support. This approach permits to realize many Betti sequences of schemes with the same Hilbert function.

Introduction.

Graded Betti numbers are very refined projective invariants of a closed subscheme X of $\mathbb{P}^r$, strictly connected with its geometry. For instance the first of them represent the degrees of a minimal set of generators for the saturated homogeneous ideal of X.

It is known that the graded Betti numbers of a closed subscheme of $\mathbb{P}^r$ determine its Hilbert function. Vice versa if H is an admissible Hilbert function for a closed subscheme of $\mathbb{P}^r$, what are all the possible graded Betti numbers of closed subschemes of $\mathbb{P}^r$ whose Hilbert function is just H is indeed an hard question. Obviously, in order to try to solve this general problem, we can restrict the same question to particular classes of subschemes, for instance arithmetically Cohen-Macaulay (aCM) or arithmetically Gorenstein subschemes (aG) or schemes with low codimension.

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For 2-codimensional aCM or 3-codimensional aG subschemes the problem was completely solved (see papers by Gaeta [5], Stanley [12], Maggioni-Ragusa [8], Campanella [2], Diesel [4], De Negri-Valla [3], Ragusa-Zappalà [10], Geramita-Migliore [6], and many others). More precisely, if $X \subset \mathbb{P}^r$ is a closed scheme in a well specified class C and $\{\alpha_{ij}(X)\}$ is the associated Betti sequence, the set $\mathfrak{B}_C(H)$ of all possible Betti sequences which agree with H of subschemes in the same class C is a partially ordered set. In both of above cases it was proved the existence of the maximum and minimum for $\mathfrak{B}_{\mathrm{aCM}}(H)$ and $\mathfrak{B}_{\mathrm{aG}}(H)$, and it was found which possibilities between the maximum and the minimum are allowed.

If H is the Hilbert function of a c-codimensional aCM subscheme of $\mathbb{P}^r$, $c \geq 3$, $\mathfrak{B}_{\mathrm{aCM}}(H)$ still admits the maximum (see the extremal resolution described by Bigatti, Hulett and Pardue in [1], [7], [9]) but, in general, it does not admit the minimum (see the example of Evans in [7]), so it seems hard to describe the set $\mathfrak{B}_{\mathrm{aCM}}(H)$ in this more general situation, since we need to determine the minimal elements and which elements between the minimal and the maximum one are allowed.

In the paper [11] we built partial intersection schemes which permitted to find a lot of elements of $\mathfrak{B}_{\mathrm{aCM}}(H)$. In this paper we provide a construction of particular 3-codimensional aCM subschemes of $\mathbb{P}^r$, partial Gorenstein schemes, which generalize the partial intersection schemes and whose graded Betti numbers can be easily computed in terms of their combinatorial support. Partial Gorenstein permit us to realize a greatest family of Betti sequences than partial intersections, but we don't know yet if they are sufficient to fill up $\mathfrak{B}_{\mathrm{aCM}}(H)$, for any assigned H.

The structure of the paper is simple. In section 1 we define partial Gorenstein schemes, we prove that they are reduced aCM schemes and we compute their Hilbert function in terms of their support. In section 2 we compute all graded Betti numbers for such a scheme using *Gorenstein liaison*. Finally in section 3 we illustrate by an example how we can realize many Betti sequences corresponding to a same assigned Hilbert function (more than one can realize using partial intersection schemes).

1. Partial Gorenstein in codimension 3.

Throughout this paper k will denote an algebraically closed field, $\mathbb{P}^r$ the r-dimensional projective space over k, $R = k[x_0, \ldots, x_r] = \bigoplus_{n\in\mathbb{Z}} H^0(\mathcal{O}_{\mathbb{P}^r}(n))$.

If $V \subset \mathbb{P}^r$ is a subscheme, $I(V)$ will denote its defining ideal and $H_V(n) = \dim_k R_n - \dim_k(I(V))_n$ its Hilbert function. Moreover, if $V \subset \mathbb{P}^r$ is a c-codimensional aCM scheme with minimal free resolution

$$0 \to \oplus R(-j)^{\alpha_{cj}} \cdots \to \oplus R(-j)^{\alpha_{2j}} \to \oplus R(-j)^{\alpha_{1j}} \to I(V) \to 0$$

then the positive integers $\{\alpha_{ij}\}_j$ will denote the i-th graded Betti numbers.

In this section we construct suitable 3-codimensional aCM subschemes of $\mathbb{P}^r$ which generalize the partial intersection schemes studied in [11]. Also for such schemes we will be able to compute both Hilbert functions and graded Betti numbers.

In order to define these subschemes of $\mathbb{P}^r$, we need some elementary properties of particular posets.

Let $\mathcal{A} \subset \mathbb{N}^2$ be a 2-left segment (finite) with respect to the ordering $\leq$ induced by the usual order of $\mathbb{N}$ and denote

$$\mathbb{M}_{\mathcal{A}} = \{(H, U) \in \mathcal{A} \times (\mathbb{N}^2 \setminus \mathcal{A}) \mid \pi_1(H) = \pi_1(U)\}$$

where $\pi_i : \mathbb{N}^2 \to \mathbb{N}$ is the projection on the i-th component.

On $\mathbb{N}^2$ we define also the following ordering

$$(a, b) \preceq (c, d) \iff a \leq c, b \geq d.$$

Using these orderings we define the following ordering on $\mathbb{M}_{\mathcal{A}}$:

if $(H, U), (K, V) \in \mathbb{M}_{\mathcal{A}}$ we say $(H, U) \leq (K, V) \iff H \preceq K$ and $U \leq V$.

Definition 1.1. A finite left segment $\mathcal{F}$ of the poset $(\mathbb{M}_{\mathcal{A}}, \leq)$ will be called a *$\mathcal{A}$-left segment*.

Now let $\mathcal{F}$ be a $\mathcal{A}$-left segment; we denote

$$\begin{aligned} s &= \max\{\pi_1(U) \mid (H, U) \in \mathcal{F}\} \\ t &= \max\{\pi_2(U) \mid (H, U) \in \mathcal{F}\} \\ \bar{t} &= \max\{\pi_2(H) \mid (H, U) \in \mathcal{F}\}. \end{aligned}$$

If $\mathcal{A}$ is a 2-left segment, we define for every $H = (i, j) \in \mathcal{A}$

$$\begin{aligned} r_H &= \max\{m \in \mathbb{N} \mid (m, j) \in \mathcal{A}\}; \\ c_H &= \max\{n \in \mathbb{N} \mid (i, n) \in \mathcal{A}\}. \end{aligned}$$

Note that r_H depends only on j and c_H only on i, hence sometimes we will set $r(j) := r_H$ and $c(i) := c_H$. Moreover, since $\mathcal{A}$ is a left segment, we have

$$s = r(1) \geq r(2) \geq \ldots \geq r(\bar{t}); \ \bar{t} = c(1) \geq c(2) \geq \ldots \geq c(s).$$

In the sequel we will denote

$$\mathcal{R}_{\mathcal{A}} = \{(H, U) \in \mathcal{A} \times (\mathbb{N}^2 \setminus \mathcal{A})\} \mid \pi_2(H) = \pi_2(U), \pi_1(U) \leq s\}.$$

If $\mathcal{F}$ is a $\mathcal{A}$-left segment we set $\hat{\mathcal{F}} = \mathcal{F} \cup \mathcal{R}_{\mathcal{A}}$. Moreover, for every $H \in \mathcal{A}$ we set

$$u_H = \begin{cases} \max\{\pi_2(U) \mid (H, U) \in \mathcal{F}\} & \text{if such a set is not empty ,} \\ c_H & \text{otherwise.} \end{cases}$$

Note that, since $\mathcal{F}$ is a $\mathcal{A}$-left segment, if $K \preceq H$ then $u_K \geq u_H$.

Now, with the above notation, we choose two families of hyperplanes of $\mathbb{P}^r$ ($r \geq 3$) sufficiently generic (here we mean that every 3 of them meet on distinct 3-codimensional linear varieties):

$\{A_1, \dots, A_s\}$, $\{B_1, \dots, B_t\}$ whose defining forms are $\{z_1, \dots, z_s\}$ and $\{y_1, \dots, y_t\}$, respectively.

For any $(H, U) \in \hat{\mathcal{F}}$, $H = (h_1, h_2), U = (u_1, u_2)$, we denote $L_H = A_{h_1} \cap B_{h_2}$, $R_U = A_{u_1} \cap B_{u_2}$.

Now we are ready to define our schemes. Let $\hat{\mathcal{F}}$ be as above; for every $\alpha = (H, U) \in \hat{\mathcal{F}}$, set $V_\alpha = L_H \cap R_U$. Note that, because of the genericity of the hyperplanes and since either $\pi_1(H) = \pi_1(U)$ or $\pi_2(H) = \pi_2(U)$, V_α is a 3-codimensional linear variety of $\mathbb{P}^r$. Finally, the scheme

$$V_{\hat{\mathcal{F}}} = \bigcup_{\alpha \in \hat{\mathcal{F}}} V_\alpha$$

will be said a *partial Gorenstein* relative to $\mathcal{F}$ with respect to the families of hyperplanes $\{A_i\}$, $\{B_j\}$.

Remark 1.2. Note that, whenever $N \geq s + t - 1$ one can consider in $\mathbb{P}^N$ partial Gorenstein schemes with respect to families of hyperplanes whose defining forms are indeterminates. Such schemes will be called *standard partial Gorenstein*.

Proposition 1.3. *Every partial Gorenstein scheme is a generic h-plane section of a standard one relative to the same $\mathcal{A}$-left segment, for some h.*

Proof. Let $X \subset \mathbb{P}^r$ be a partial Gorenstein scheme relative to an $\mathcal{A}$-left segment $\mathcal{F}$ with respect to the families of hyperplanes whose defining forms are $\{z_1, \dots, z_s; y_1, \dots, y_t\}$. Let d be the dimension of the k-vector space U of R_1 generated by such forms. Write $R_1 = U \oplus V$. Let $\widetilde{R} = k[Z_1, \dots, Z_s; Y_1, \dots, Y_t; W_1, \dots, W_{r+1-d}]$ and define a surjective linear map

$\tau_1 : \widetilde{R}_1 \to R_1$ by $\tau_1(Z_i) = z_i$ for $1 \le i \le s$, $\tau_1(Y_j) = y_j$ for $1 \le j \le t$ and such that $\tau_1(W_1), \ldots, \tau_1(W_{r+1-d})$ is a basis for V. Let $\tau : \widetilde{R} \to R$ the surjective graded k-algebra map induced by τ_1. Note that $\ker\tau$ is minimally generated by $h = s + t - d$ linear forms. Then $\tau^{-1}(I(X))$ defines the standard partial Gorenstein scheme of $\mathbb{P}^N$, $N = s + t + r - d$, $\widetilde{X}$ relative to the same $\mathcal{F}$ with respect to the families of hyperplanes whose defining forms are the indeterminates $\{Z_1, \ldots, Z_s; Y_1, \ldots, Y_t\}$. $\square$

Remark 1.4. The name partial Gorenstein is justified since, when $\mathcal{F}$ is a $\mathcal{A}$-left segment such that $(H, U) \in \mathcal{F}$ whenever $H \in \mathcal{A}$ and $U \in (\mathbb{N}^2 \setminus \mathcal{A})$ and $\pi_2(U) \le t$, then a 3-partial Gorenstein with such a support (with respect to any families of hyperplanes) is indeed an arithmetically Gorenstein scheme (as we will see).

Remark 1.5. The 3-partial intersections defined in [11] are a particular case of the above partial Gorenstein schemes. Namely, if $\mathcal{B}$ is a 3-left segment (finite) of $\mathbb{N}^3$ and X is a partial intersection relative to $\mathcal{B}$ and with respect to some families of hyperplanes $\{A_i\}$, $\{B_j\}$ and $\{C_k\}$, if we set $a_i = \max\{\pi_i(\alpha) \mid \alpha \in \mathcal{B}\}$, we can consider the 2-left segment (rectangle) $\mathcal{R}$ generated by (a_2, a_3); now, define

$$\mathcal{F}_{\mathcal{B}} = \{(H, U) \in \mathbb{M}_{\mathcal{R}} \mid (\pi_1(H), \pi_2(H), \pi_2(U)) \in \mathcal{B}\}.$$

One can show easily that $\mathcal{F}_{\mathcal{B}}$ is a $\mathcal{R}$-left segment and the scheme $V_{\mathcal{F}_{\mathcal{B}}}$, with respect to the families of hyperplanes $\{C_{a_3}, \ldots, C_1; A_1, \ldots, A_{a_1}\}$ and $\{B_1, \ldots, B_{a_2}\}$ is just X.

In order to show that a partial Gorenstein scheme X with support on the $\mathcal{A}$-left segment $\mathcal{F}$ (as above) is a 3-codimensional aCM subscheme of $\mathbb{P}^r$ we define, for $i = 1, \ldots, s$

$$Y_i = \bigcup_{(i,j)\in\mathcal{A}} X \cap L_{(i,j)}.$$

Of course, $X = \bigcup_{i=1}^{s} Y_i$. If we set $H = (i, j)$ and

$$\mathcal{M}_H = \{B_{c_H+1}, \ldots, B_{u_H}, A_{r_H+1}, \ldots A_s\}$$

we have that

$$X \cap L_{(i,j)} = A_i \cap B_j \cap C_{(i,j)}$$

where $C_{(i,j)} = \bigcup_{C\in\mathcal{M}_{(i,j)}} C$. In the sequel we will denote

$$S_i = \bigcup_j (B_j \cap C_{(i,j)}).$$

Note that for all i $S_i \cap A_i = Y_i$.

Lemma 1.6. *S_i is a 2-partial intersection in $\mathbb{P}^r$ for every i.*

Proof. If $H = (i, j)$ and $K = (i, j + 1)$, we see that $\mathcal{M}_H \subseteq \mathcal{M}_K$: indeed, since $K \preceq H$ we have that $u_H \leq u_K$, and, on the other hand $r_H \geq r_K$. Because of that, it is not too hard to renumber the sets $\mathcal{M}_{(i,c(i))}$ and $\{B_j\}_{j=1,\dots,c(i)}$, so that the support of S_i becomes a 2-left segment. $\square$

As a consequence of the above lemma we have that every Y_i is a 2-partial intersection of A_i (or a degenerate 3-partial intersection); nevertheless, in general, $Y_i \neq X \cap A_i$.

Now, for every $H = (i, j) \in \mathcal{A}$, define

$$F_H = \prod_{p=r(j)+1}^{s} z_p \prod_{q=j+1}^{u_H} y_q.$$

Then, according to the result of Theorem 3.1 in [11], we see that $I(S_i)$ is generated by F_H for all $H \in \mathcal{A}$ with $\pi_1(H) = i$ and by $\prod_{q=1}^{c(i)} y_q$.

Lemma 1.7. *With the above notation, $I(S_i) \subseteq I(Y_n)$ for every $n = i, \dots s$.*

Proof. Observe, at first, that $I(Y_n) = (z_n) + I(S_n)$. Moreover, since $n \geq i$, $c(n) \leq c(i)$; therefore $\prod_{q=1}^{c(n)} y_q$, which is in $I(S_n)$, divides $\prod_{q=1}^{c(i)} y_q$. Hence $\prod_{q=1}^{c(i)} y_q \in I(S_n) \subseteq I(Y_n)$. Finally, take F_H with $H = (i, j) \in \mathcal{A}$. If $n > r(j)$ then z_n divides $F_H = \prod_{p=r(j)+1}^{s} z_p \prod_{q=j+1}^{u_H} y_q$, therefore $F_H \in I(Y_n)$. We are left with the case when $n \leq r(j)$. In this case $K = (n, j) \in \mathcal{A}$ and, since $n \geq i$, $H \preceq K$, hence $u_K \leq u_H$. This implies that F_K divides F_H, and, since $F_K \in I(S_n)$, we can conclude that $F_H \in I(Y_n)$. $\square$

We are ready to prove that a partial Gorenstein is an aCM scheme and to give a set of generators for its defining ideal.

Theorem 1.8. *Let X be a partial Gorenstein subscheme of $\mathbb{P}^r$ relative to the $\mathcal{A}$-left segment $\mathcal{F}$ with respect to the families of hyperplanes $\{A_i\}$, $\{B_j\}$, whose defining forms are, respectively, $\{z_i\}$, $\{y_j\}$. Let Y_i and S_i as before and denote $X_n = Y_1 \cup \dots \cup Y_n$ for $n = 1, \dots, s$. Then the following sequence of graded R-modules*

$$0 \longrightarrow I(Y_n)(-(n-1)) \xrightarrow{f} I(X_n) \xrightarrow{\varphi} I(X_{n-1})/(f) \longrightarrow 0$$

is exact, where $f = \prod_{i=1}^{n-1} z_i$ *and* φ *is the natural map. Moreover*

$$I(X_n) = I(S_1) + z_1 I(S_2) + z_1 z_2 I(S_3) + \cdots + z_1 \ldots z_{n-1} I(S_n) + (z_1 \ldots z_n).$$

In particular, X *is a* 3*-codimensional aCM (reduced) subscheme of* $\mathbb{P}^r$.

Proof. We use induction on n. The case $n = 1$ is trivial since $I(X_1) = I(Y_1) = I(S_1) + (z_1)$. If $n > 1$ we show that the sequence

$$0 \longrightarrow I(Y_n)(-(n-1)) \xrightarrow{f} I(X_n) \xrightarrow{\varphi} I(X_{n-1})/(f) \longrightarrow 0$$

is exact. The only thing to verify is the surjectivity of φ. Take an element $\alpha \in I(X_{n-1})/(f)$; by inductive hypothesis there is

$$\beta \in I(S_1) + z_1 I(S_2) + z_1 z_2 I(S_3) + \cdots + z_1 \ldots z_{n-2} I(S_{n-1})$$

such that $\alpha = \beta + \lambda \prod_{i=1}^{n-1} z_i$. Now $\beta = \sum_{p=1}^{n-1} \prod_{q=1}^{p-1} z_q f_p$, with $f_p \in I(S_p)$. By the previous lemma $I(S_p) \subseteq I(Y_n)$, for $p = 1, \ldots, n-1$; hence, $f_p \in I(Y_n)$, for $p = 1, \ldots, n-1$. Therefore $\beta \in I(Y_n)$ and, as $\beta \in I(X_{n-1})$, we have $\beta \in I(X_{n-1}) \cap I(Y_n) = I(X_n)$. Thus φ is surjective.

The exactness of the above sequence implies

$$I(X_n) = \sum_{p=1}^{n-1} \left(\prod_{q=1}^{p-1} z_q \right) I(S_p) + \left(\prod_{q=1}^{n-1} z_q \right) I(Y_n) = \sum_{p=1}^{n-1} \left(\prod_{q=1}^{p-1} z_q \right) I(S_p)$$

$$+ \left(\prod_{q=1}^{n-1} z_q \right) [(z_n) + I(S_n)] = \sum_{p=1}^{n-1} \left(\prod_{q=1}^{p-1} z_q \right) I(S_p) + (\prod_{q=1}^{n} z_q) + \left(\prod_{q=1}^{n-1} z_q \right) I(S_n)$$

$$= \sum_{p=1}^{n} \left(\prod_{q=1}^{p-1} z_q \right) I(S_p) + (\prod_{q=1}^{n} z_q).$$

Now, the fact that X is a 3-codimensional aCM subscheme of $\mathbb{P}^r$ follows from the above exact sequence using induction on n. □

Corollary 1.9. *Let* $X \subset \mathbb{P}^r$ *be a partial Gorenstein scheme and* $\widetilde{X} \subset \mathbb{P}^N$ *a standard partial Gorenstein such that* X *is a generic* h*-plane section of it. Let* $\widetilde{R}$ *be the coordinate ring of* $\mathbb{P}^N$. *If* $\widetilde{\mathbf{F}}_\bullet$ *is an* $\widetilde{R}$*-free resolution of* $I(\widetilde{X})$ *then* $\widetilde{\mathbf{F}}_\bullet \otimes_{\widetilde{R}} R$ *is an* R*-free resolution of* $I(X)$.

Proof. Just use that X is a generic h-plane section of $\widetilde{X}$ and that both are aCM schemes. $\square$

Now we compute the Hilbert function of a partial Gorenstein X of $\mathbb{P}^r$ in terms of its support $\mathcal{F}$. If $\mathcal{A}$ is the 2-left segment underlying $\mathcal{F}$, we denote $d_H = c(H) + \pi_1(H) - \pi_2(H) - 1$, for all $H \in \mathcal{A}$; moreover we define $\varphi : \mathcal{A} \to \mathbb{N}_0$ the map $\varphi(H) = |\{U \in \mathbb{N}^2 \mid (H, U) \in \hat{\mathcal{F}}\}|$. Finally, for all $H \in \mathcal{A}$ we set

$$\varphi_H(n) = \begin{cases} 1 & \text{for } d_H \leq n \leq d_H + \varphi(H) - 1 \\ 0 & \text{otherwise} \end{cases}$$

for every $n \in \mathbb{N}_0$.

Proposition 1.10. *If X is a partial Gorenstein subscheme of $\mathbb{P}^r$ relative to the $\mathcal{A}$-left segment $\mathcal{F}$, then*

$$\Delta^{r-2} H(n) = |\{H \in \mathcal{A} \mid d_H \leq n \leq d_H + \varphi(H) - 1\}|$$

or equivalently

$$\Delta^{r-2} H(n) = \sum_{H \in \mathcal{A}} \varphi_H(n).$$

Proof. For every $H = (i, j) \in \mathcal{A}$ we set $Y_H = X \cap L_H$. Since

$$\Delta^{r-2} H_{Y_H}(n) = \begin{cases} 1 & \text{for } 0 \leq n \leq \varphi(H) - 1 \\ 0 & \text{otherwise.} \end{cases}$$

we see that $\varphi_H(n) = \Delta^{r-2} H_{Y_H}(n - d_H)$, for every n. On the other hand, since $Y_i = \bigcap_{\pi_1(H)=i} Y_H$ is a partial intersection, by Lemma 1.5 of [11], we have

$$\Delta^{r-2} H_{Y_i}(n) = \sum_{\pi_1(H)=i} H_{Y_H}(n - c(H) + \pi_2(H)).$$

Using Lemma 1.6 we get

$$\Delta^{r-2} H_X(n) = \sum_{i=1}^{s} \Delta^{r-2} H_{Y_i}(n - i + 1),$$

hence

$$\begin{aligned} \Delta^{r-2} H_X(n) &= \sum_{i=1}^{s} \sum_{j=1}^{c(H)} \Delta^{r-2} H_{Y_H}(n - i + 1 - c(H) + j) \\ &= \sum_{H \in \mathcal{A}} \Delta^{r-2} H_{Y_H}(n - d_H) = \sum_{H \in \mathcal{A}} \varphi_H(n). \qquad \square \end{aligned}$$

2. Betti numbers for Partial Gorenstein schemes.

In this section we want to compute the graded Betti numbers for partial Gorenstein subschemes of $\mathbb{P}^r$ in terms of their support.

Let $X \subseteq \mathbb{P}^r$ be a partial Gorenstein scheme whose support is the $\mathcal{A}$-left segment $\mathcal{F}$ and relative to the families of hyperplanes $\{A_i\}$, $\{B_j\}$ with defining forms, respectively, $\{z_i\}$, $\{y_j\}$ and let s and t be as in the previous section. We define *Gorenstein completion* of $\mathcal{F}$ the $\mathcal{A}$-left segment

$$\mathcal{F}^G =< (s, 1; s, t) >= \{(H, U) \in \mathbb{M}_{\mathcal{A}} \mid \pi_2(U) \leq t\}.$$

Note that the terminology is justified by the fact that if X is a partial Gorenstein relative to $\mathcal{F}$ with respect to the families of hyperplanes $\{A_i : z_i = 0 \mid 1 \leq i \leq s\}$ and $\{B_j : y_j = 0 \mid 1 \leq j \leq t\}$, $Z = V_{\hat{\mathcal{F}}^G}$ is a 3-codimensional arithmetically Gorenstein scheme. Namely it is the intersection between two 2-codimensional arithmetically Cohen-Macaulay schemes V_1 and V_2 geometrically linked in a complete intersection, where V_1 is just the partial intersection with support on $\mathcal{A}$, relative to the same families of hyperplanes, and V_2 is the linked scheme to V_1 in the complete intersection $(\prod_{i=1}^{s} z_i, \prod_{j=1}^{t} y_j)$. Precisely $I(Z)$ is generated by the following set of polynomials

$$\Gamma = \begin{cases} \gamma \ = \prod_{i=1}^{s} z_i; \\ \gamma_a = \prod_{i=1}^{a-1} z_i \prod_{j=1}^{c(a)} y_j \text{ for every } a \text{ such that c(a)} < \text{c(a-1)}; \\ \gamma_a^* = \prod_{i=a+1}^{s} z_i \prod_{j=c(a)+1}^{t} y_j \text{ for every } a \text{ such that c(a)} > \text{c(a+1)} \end{cases}$$

this follows from results in [11] (here we let $c(0) = \bar{t} + 1$ and $c(s + 1) = 0$). If $t > \bar{t}$, Γ is minimal. If $t = \bar{t}$, then γ is multiple of $\gamma^*_{r(t)}$ and γ_1 is multiple of γ^*_s and $\Gamma \setminus \{\gamma, \gamma_1\}$ is minimal.

Now we introduce in $\mathcal{F}^G$ the following ordering:

$$(K, V) \leq_g (H, U) \iff (K, V) \leq (H, U) \text{ and } r(K) = r(H).$$

Finally, we set $\mathcal{F}^* = \mathcal{F}^G \setminus \mathcal{F}$ and

$$\mathcal{F}^*_{min} = \{\alpha \in \mathcal{F}^* \mid \alpha \text{ is minimal with respect to } \leq_g\}.$$

Now we associate to every $\alpha = (a, b; a, c) \in \mathcal{F}^G$ the following form

$$P_\alpha = \prod_{i=r(b)+1}^{s} z_i \prod_{j=b+1}^{c-1} y_j \prod_{k=1}^{a-1} z_k.$$

In the following we let

$$[a, b] = \{n \in \mathbb{Z} \mid a \leq n \leq b\}$$

Lemma 2.1. *Let* $\alpha = (H, U), \beta = (K, V) \in \mathcal{F}^G$*; then* P_β *divides* P_α *if and only if* $\beta \leq_g \alpha$.

Proof. For the if part it is enough to prove that

a) $[r(K) + 1, s] \subseteq [r(H) + 1, s]$;
b) $[\pi_2(K) + 1, \pi_2(V) - 1] \subseteq [\pi_2(H) + 1, \pi_2(U) - 1]$;
c) $[1, \pi_1(K) - 1] \subseteq [1, \pi_1(H) - 1]$.

The inclusion a) follows since $r(H) = r(K)$. Since $K \preceq H$ we have $\pi_2(K) \geq \pi_2(H)$, and since $V \leq U$ we have $\pi_2(V) \leq \pi_2(U)$; this implies b). Now, since $\pi_1(K) \leq \pi_1(H)$, c) follows.

Vice versa, if P_β divides P_α, each linear form appearing in P_β should appear also in P_α (up to scalars), but this implies, by the generality of the forms, b) and $[1, \pi_1(K) - 1] \cup [r(K) + 1, s] \subseteq [1, \pi_1(H) - 1] \cup [r(H) + 1, s]$. Now, since $\pi_1(H) \notin [1, \pi_1(H) - 1] \cup [r(H) + 1, s]$, an easy check shows that a) and c) should hold. From c) one gets immediately $\pi_1(K) \leq \pi_1(H)$ and from a) we see that $r(K) \geq r(H)$. If $[\pi_2(K) + 1, \pi_2(V) - 1] = \emptyset$ then $\pi_2(K) = \pi_2(V) - 1$, this implies (using $\pi_1(K) \leq \pi_1(H)$) $\pi_2(K) = c(K) \geq c(H) \geq \pi_2(H)$. If $[\pi_2(K) + 1, \pi_2(V) - 1] \neq \emptyset$ b) implies again $\pi_2(K) \geq \pi_2(H)$. Therefore, $r(K) \leq r(H)$, thus $r(K) = r(H)$. $\square$

Proposition 2.2. *With the above terminology, the ideal* $I(X)$ *is generated by* $\{P_\alpha \mid \alpha \in \mathcal{F}^*\} \cup \Gamma$.

Proof. If m is an integer, $1 \leq m \leq s$, in the previous section we defined the 2-partial intersection S_m and in Theorem 1.8 we proved that

$$I(X) = \sum_{p=1}^{s+1} \left(\prod_{q=1}^{p-1} z_q \right) I(S_p)$$

(here we set $I(S_{s+1}) = R$).

Since $I(Z) \subseteq I(X)$ then $\Gamma \subset I(X)$. Moreover, $\alpha = (H, U) \in \mathcal{F}^*$ implies that $1 \leq \pi_1(H) \leq s$. If we set $m = \pi_1(H)$ then $\prod_{q=1}^{m-1} z_q$ divides P_α. Now we claim that $P_\alpha / \prod_{q=1}^{m-1} z_q \in I(S_m)$. Let $V = \min\{W \mid (H, W) \in \mathcal{F}^*\}$ and $\beta = (H, V)$. Then, with the notation of the previous section, we see that $P_\beta / \prod_{q=1}^{m-1} z_q = F_H \in I(S_m)$. Since $V \leq U$, by Lemma 2.1, P_β divides P_α, so $P_\alpha / \prod_{q=1}^{m-1} z_q \in I(S_m)$ i.e. $P_\alpha \in I(X)$.

On the other hand $I(S_m)$ is generated by $\{F_H \mid H \in \mathcal{A}, \pi_1(H) = m\}$ and by $\prod_{q=1}^{c(m)} y_q$, for $1 \leq m \leq s$. Let $H \in \mathcal{A}$ with $\pi_1(H) = m$; if $u_H = t$ then $\prod_{q=1}^{m-1} z_q F_H$ is multiple of γ_a^* for $a = r(H)$; if $u_H < t$ then $\alpha = (H; \pi_1(H), u_H + 1) \in \mathcal{F}^*$ and $\prod_{q=1}^{m-1} z_q F_H$ is multiple of P_α. Finally, $\prod_{q=1}^{m-1} z_q \prod_{q=1}^{c(m)} y_q$ is multiple of γ_a where $a = \min\{i \mid c(i) = c_H\}$ $\square$

So we are ready to state the main results of this section.

Theorem 2.3. *Let $X \subset \mathbb{P}^r$ be a partial Gorenstein. Then a set of minimal generators for $I(X)/I(Z)$ is*

$$\{P_\alpha + I(Z) \mid \alpha \in \mathcal{F}^*_{min}\}.$$

Proof. Using Corollary 1.9 we can assume that X is standard. Now, by Lemma 2.1 the set $\{P_\alpha + I(Z) \mid \alpha \in \mathcal{F}^*_{min}\}$ generates $I(X)/I(Z)$; so we have only to prove the minimality. Let $\alpha = (a, b; a, c) \in \mathcal{F}^*_{min}$ and say Z_α the linear variety defined by $I(Z_\alpha) = (z_a, y_b, y_c, z_{r(b)})$. An easy check shows that $I(Z) \subseteq I(Z_\alpha)$. Then $P_\alpha + I(Z) \notin I(Z_\alpha)/I(Z)$. In fact, if $P_\alpha + I(Z)$ were in $I(Z_\alpha) + I(Z)$, then $P_\alpha \in I(Z_\alpha)$ therefore a linear factor of P_α should stay in $I(Z_\alpha)$. Now, since $P_\alpha = \prod_{i=r(b)+1}^{s} z_i \prod_{j=b+1}^{c-1} y_j \prod_{k=1}^{a-1} z_k$, this is impossible.

Now let us consider $\beta \in \mathcal{F}^*_{min}$, $\beta \neq \alpha$; we would like to show that $P_\beta + I(Z) \in I(Z_\alpha) + I(Z)$ (this is enough to complete the proof). Namely, say $\beta = (u, v; u, w)$, if $\beta \not\leq \alpha$ we have three possibilities: $u > a$ or $v < b$ or $w > c$. In the first case z_a divides P_β; in the second case y_b divides P_β; in the third case y_c divides P_β; otherwise if $\beta \leq \alpha$ then $r(b) \neq r(v)$ and, since $(u, v) \preceq (a, b)$, more precisely we have $r(b) > r(v)$, this implies $z_{r(b)}$ divides P_β. $\square$

Theorem 2.4. *Let $X \subset \mathbb{P}^r$ be a partial Gorenstein scheme. Then a set of minimal generators for $I(X)$ is given by $\{P_\alpha \mid \alpha \in \mathcal{F}^*_{min}\}$ and by the following elements of Γ:*

1. *$\gamma \iff (a, c(a); a, c(a) + 1) \in \mathcal{F}$ for $1 \leq a \leq s$;*
2. *$\gamma_a \iff$ for some b with $r(b) = s$ $(a, b; a, c(a) + 1) \in \mathcal{F}$;*
3. *$\gamma_a^* \iff (1, c(a); 1, t) \in \mathcal{F}$ or $c(a) = t$.*

Proof. Again we can assume that X is standard. By Proposition 2.2 and the previous theorem we have that $\{P_\alpha \mid \alpha \in \mathcal{F}^*_{min}\} \cup \Gamma$ is a set of generators for $I(X)$. Moreover the previous theorem ensure us that the $P_\alpha' s$ are minimal in such a set.

1. If $\alpha = (a, c(a); a, c(a) + 1) \notin \mathcal{F}$ for some a, $1 \leq a \leq s$, and $c(a) = t$ our element is multiple, as we saw, of an element of Γ. If $c(a) < t$ then

$\alpha \in \mathcal{F}^*$ and our element is multiple of P_α. Vice versa, let us suppose that $(a, c(a); a, c(a)+1) \in \mathcal{F}$ for every $1 \le a \le s$. Denote $J = (y_1, \dots, y_t)$; of course, $\gamma \notin J$. On the other hand, the assumption says that each P_α with $\alpha \in \mathcal{F}^*_{min}$ contains some y_j as a factor, hence $P_\alpha \in J$; moreover, since by assumption $c(a) < t$, we see that both the γ_a's and the γ_a^*'s contain some y_i as a factor and consequently stay in J.

2. If there is b with $r(b) = s$ such that $\alpha = (a, b; a, c(a)+1) \notin \mathcal{F}$ we have again two possibilities: either $c(a) = t$ or $\alpha \in \mathcal{F}^*$. In the first case we have $t = \bar{t}$, $a = 1$ and γ_1 is multiple of γ_s^* as we saw previously. In the second case γ_a is multiple of P_α. Vice versa let us suppose that for every b with $r(b) = s$ $(a, b; a, c(a)+1) \in \mathcal{F}$ for some a such that $1 \le a \le s$ and $c(a) < c(a-1)$. Let us consider the ideal $J = (z_a, \dots, z_s, y_{c(a)+1}, \dots, y_t)$. Of course, $\gamma_a \notin J$; one easily sees that $\gamma \in J$, $\gamma_u \in J$ for all $u \neq a$ and $\gamma_u^* \in J$ for all u. Finally, take any $\beta = (u, v; u, w) \in \mathcal{F}^*_{min}$. If $u > a$ then z_a divides P_β hence $P_\beta \in J$. If $u < a$ then $c(u) > c(a)$; in this case if $[v+1, w-1] = \emptyset$ then $v = c(u)$, $w = c(u)+1$ hence $r(v)+1 \le a$, thus again z_a divides P_β and consequently $P_\beta \in J$; if $[v+1, w-1] \neq \emptyset$ then $c(a)+1 \le w-1$ therefore $y_{w-1} \in J$ hence $P_\beta \in J$. If $u = a$ either $r(v) < s$, in this case z_s divides P_β hence $P_\beta \in J$; or $r(v) = s$ in this case, the hypothesis implies that $(a, v; a, c(a)+1) \notin \mathcal{F}^*$, then $w > c(a)+1$ and the conclusion follows as before.

3. Let us suppose $(1, c(a); 1, t) \notin \mathcal{F}$ and $c(a) < t$; this means $\beta = (1, c(a); 1, t) \in \mathcal{F}^*$ and consequently P_β divides γ_a^*. Vice versa, consider the ideal $J = (z_1, \dots, z_a, y_1, \dots, y_{c(a)})$. Clearly, $\gamma_a^* \notin J$ and every element of Γ different from γ_a^* is in J. Let now $\beta = (u, v; u, w) \in \mathcal{F}^*_{min}$. If $u > 1$, z_1 divides P_β hence $P_\beta \in J$; if $r(v) < a$ then z_a divides P_β hence $P_\beta \in J$. Therefore, we are left with the case $u = 1$ and $r(v) \ge a$; of course, for the conclusion it is enough to show that $[v+1, w-1] \cap [1, c(a)] \neq \emptyset$. If it were empty either $[v+1, w-1] = \emptyset$ or $v+1 > c(a)$. In the first case it follows that $v = c(1) = \bar{t}$ and $w = \bar{t}+1$; but $r(v) \ge a$ will imply $r(v) = a$ and consequently $c(a) = \bar{t}$. Now, by hypothesis such a β cannot stay in $\mathcal{F}^*$. Finally, if $v+1 < c(a)$, since $r(v) \ge a$ we get $v = c(a)$. Now, again applying the hypothesis we see that such a β cannot stay in $\mathcal{F}^*$.

□

For computing the first graded Betti numbers it is convenient to use the following terminology, which comes from the previous theorem:

$$v_\alpha = a - r(b) + c - b \text{ for all } \alpha = (a, b; a, c) \in \mathcal{F}^G$$

$N_{\mathcal{F}} = \{a \in [1, s] \mid c(a) < c(a-1)$ and for some b with $r(b) = s$ $(a, b; a, c(a)+1) \in \mathcal{F}\}$;
$N_{\mathcal{F}^*} = \{a \in [1, s] \mid c(a) > c(a+1)$ and $(1, c(a); 1, t) \in \mathcal{F}$ or $c(a) = t\}$.

Corollary 2.5. *Let $X \subset \mathbb{P}^r$ be a partial Gorenstein. Then the first graded Betti numbers for $I(X)$ are:*

$s + v_\alpha - 2$ *for all* $\alpha \in \mathcal{F}^*_{min}$
$a + c(a) - 1$ *for all* $a \in N_{\mathcal{F}}$
$s + t - a - c(a)$ *for all* $a \in N_{\mathcal{F}^*}$

and s if $(a, c(a); a, c(a)+1) \in \mathcal{F}$ for $1 \le a \le s$.

Now our aim is computing the degrees of the second syzygies of a partial Gorenstein in terms of its support.

Observe that, since $\mathcal{F}$ is a left segment of $\mathbb{M}_{\mathcal{A}}$, $\mathcal{F}^* = \mathcal{F}^G \setminus \mathcal{F}$ has the following property:

$$\alpha \in \mathcal{F}^*,\ \beta \in \mathcal{F}^G \text{ and } \beta \ge \alpha \Rightarrow \beta \in \mathcal{F}^*$$

i.e. $\mathcal{F}^*$ is a right segment inside $\mathcal{F}^G$.

Observe that, by definition,

$$Y = V_{\mathcal{F}^*} = \bigcup_{(H,U)\in\mathcal{F}^*} L_H \cap R_U$$

is the Gorenstein linked scheme of $V_{\hat{\mathcal{F}}}$ in $V_{\hat{\mathcal{F}}^G}$. We want to describe a minimal set of generators for the ideal $I(Y)/I(Z)$.

Describe first $V_{\mathcal{F}^*}$ in an other way. Let $\mathcal{A}^* = (\{1, \ldots, s\} \times \{1, \ldots, t\}) \setminus \mathcal{A}$ and

$$\mathcal{A}^*_j = \{U \in \mathcal{A}^* \mid \exists\ H \text{ with } (H, U) \in \mathcal{F}^* \text{ and } \pi_2(H) = j\}.$$

Now define for every $1 \le j \le \bar{t}$

$$T_j = \bigcup_{U\in\mathcal{A}^*_j} R_U \text{ and } Y_j = T_j \cap B_j.$$

Clearly, we have $Y = \bigcup_{1\le j\le\bar{t}} Y_j$.

Now, working as in Lemma 1.6, we see that every T_j is a 2-partial intersection. Moreover, $T_j \supseteq T_{j+1}$: indeed, if $U \in \mathcal{A}^*_{j+1}$ there exists an element H such that $(H, U) \in \mathcal{F}^*$ and $\pi_2(H) = j+1$; now, if $H = (i, j+1)$ and

$H' = (i, j)$ then H' is in $\mathcal{A}$, since $\mathcal{A}$ is a left segment, $(H', U) \geq (H, U)$. Therefore, by the property of $\mathcal{F}^*$, $(H', U) \in \mathcal{F}^*$, which means that $U \in \mathcal{A}_j^*$.

For every $H = (i, j) \in \mathcal{A}$ we define

$$G_H = \prod_{p=i+1}^{r(j)} z_p \prod_{q=u_H+1}^{t} y_q;$$

then, according to the result of Theorem 3.1 in [11], we see that $I(T_j)$ is generated by G_H for all $H \in \mathcal{A}$ with $\pi_2(H) = j$ and by $\prod_{p=1}^{r(j)} z_p$.

Thus, similarly as in Theorem 1.8, one can show

Theorem 2.6. *With the above notation*

$$I(Y) = I(T_1) + y_1 I(T_2) + y_1 y_2 I(T_3) + \cdots + y_1 \ldots y_{n-1} I(T_n) + (y_1 \ldots y_n).$$

Let us associate to every $\alpha = (a, b; a, c) \in \mathcal{F}^G$ the following form

$$Q_\alpha = \prod_{i=a+1}^{r(b)} z_i \prod_{j=1}^{b-1} y_j \prod_{k=c+1}^{t} y_k.$$

Because of the following lemma it is convenient to introduce in $\mathcal{F}^G$ the following new ordering

$$(H, U) \leq_s (K, V) \Leftrightarrow \begin{cases} (H, U) \leq (K, V) & \text{if } \pi_1(K) = r(K) \\ (H, U) \leq_g (K, V) & \text{if } \pi_1(K) < r(K) \end{cases}$$

Lemma 2.7. *Let $\alpha = (H, U), \beta = (K, V) \in \mathcal{F}^G$; then Q_β divides Q_α if and only if $\beta \geq_s \alpha$.*

Proof. Just use the same techniques as in Lemma 2.1. □

Proposition 2.8. *With the above terminology, the ideal $I(Y)$ is generated by $\{Q_\alpha \mid \alpha \in \mathcal{F}\} \cup \Gamma$.*

Proof. Just use a similar computation as in Proposition 2.2 and the definition of the polynomials Q_α. □

Now set

$$\mathcal{F}_{max} = \{\alpha \in \mathcal{F} \mid \alpha \text{ is maximal with respect to } \leq_s\}.$$

We can state the following result

Theorem 2.9. *Let $Y = V_{\mathcal{F}^*}$ as above. Then a set of minimal generators for $I(Y)/I(Z)$ is*

$$\{Q_\alpha + I(Z) \mid \alpha \in \mathcal{F}_{max}\}.$$

Proof. It works as in Theorem 2.3. $\square$

Corollary 2.10. *Let $X \subset \mathbb{P}^r$ be a partial Gorenstein with support on a $\mathcal{A}$-left segment $\mathcal{F}$. Then the degrees of the second syzygies of $I(X)$ are:*

$$s + 1 + v_\alpha \qquad \textit{for all } \alpha \in \mathcal{F}_{max}.$$

Proof. Let Y be the scheme linked to X in the aG scheme $Z = V_{\hat{\mathcal{F}}G}$; since the second syzygy of Z is $s + t$, according to the previous theorem, the degrees of the minimal second syzygies are $s + t-$ deg Q_α where α runs over $\mathcal{F}_{max}$. Now a simple computation gives the formula. $\square$

3. An example.

In this final section we apply previous constructions and the related properties to obtain as many graded Betti numbers as possible with respect to an assigned Hilbert function of a 3-codimensional aCM subscheme of $\mathbb{P}^r$. The example that we explore is relevant in the recent literature since was the first example, used by G. Evans, to show that in codimension ≥ 3 there exists a sequence β, obtained by the maximum in $\mathcal{B}_{\text{aCM}}(H)$ by some "deleting", which is not in $\mathcal{B}_{\text{aCM}}(H)$ i.e. there is not an aCM scheme whose graded Betti sequence is β.

So let consider the following Hilbert function (of 11 points of $\mathbb{P}^3$):

$$H: \ 1\ 4\ 8\ 10\ 11 \ \rightarrow$$

or

$$\Delta H = \varphi : 1\ 3\ 4\ 2\ 1\ 0 \ \rightarrow$$

we want to find as many different graded Betti numbers as possible which agree with such an Hilbert function.

According to our definition in section 1, if X is a partial Gorenstein subscheme of $\mathbb{P}^3$ with Hilbert function H there exist two planes π_1, π_2 and two (disjoint) subschemes $X_1 \subset \pi_1, X_2 \subset \pi_2$ of X such that $X = X_1 \cup X_2$ and, if $\varphi_1 = \Delta H_{X_1}, \varphi_2 = \Delta H_{X_2}$, are the O-sequences associated to X_1 and X_2, respectively, then

$$(*) \qquad \varphi(n) = \varphi_1(n) + \varphi_2(n-1)$$

for all n. Now, in this example, there are just few O-sequences φ_1 and φ_2 for which $(*)$ holds; precisely, the only possibilities are

1) $\varphi_1 = (1, 2, 3, 2, 1)$; $\varphi_2 = (1, 1)$.
2) $\varphi_1 = (1, 2, 3, 1, 1)$; $\varphi_2 = (1, 1, 1)$.
3) $\varphi_1 = (1, 2, 3, 1)$; $\varphi_2 = (1, 1, 1, 1)$.
4) $\varphi_1 = (1, 2, 2, 2, 1)$; $\varphi_2 = (1, 2)$.
5) $\varphi_1 = (1, 2, 2, 1, 1)$; $\varphi_2 = (1, 2, 1)$.
6) $\varphi_1 = (1, 2, 2, 1)$; $\varphi_2 = (1, 2, 1, 1)$.
7) $\varphi_1 = (1, 2, 2)$; $\varphi_2 = (1, 2, 2, 1)$.

Note that $1) - 6)$ are linear decompositions of φ, i.e. $\varphi_2 \leq \varphi_1$, as were defined in [11], therefore there are partial intersection schemes which arise from such decompositions (see [11]). For the $7-$th decomposition we cannot have a partial intersection scheme but we are able to build partial Gorenstein scheme from it. Nevertheless, from decomposition 2) and 5) we can construct two partial Gorenstein schemes whose graded Betti numbers cannot be reached just using partial intersection schemes.

From decomposition 1) we can construct 4 partial Gorenstein schemes with different graded Betti numbers. Take in $\mathbb{N}^2$ the 2-left segment $\mathcal{A} =< (2, 3) >$ and in $\mathbb{M}_{\mathcal{A}}$ the left segments

$$\mathcal{F}_1 =< (1, 3; 1, 8), (1, 2; 1, 6), (1, 1; 1, 4), (2, 3; 2, 5) >;$$
$$\mathcal{F}_2 =< (1, 3; 1, 8), (1, 1; 1, 5), (2, 3; 2, 5) >;$$
$$\mathcal{F}_3 =< (1, 2; 1, 7), (1, 1; 1, 4), (2, 3; 2, 5) >;$$
$$\mathcal{F}_4 =< (1, 1; 1, 6), (2, 3; 2, 5) > .$$

Let $X_1 = V_{\widehat{\mathcal{F}_1}}$, $X_2 = V_{\widehat{\mathcal{F}_2}}$, $X_3 = V_{\widehat{\mathcal{F}_3}}$, $X_4 = V_{\widehat{\mathcal{F}_4}}$; for these schemes we have the following graded Betti numbers:
for I_{X_1} we have

$$\alpha_{12} = 2, \alpha_{13} = 3, \alpha_{14} = 1, \alpha_{15} = 1;$$
$$\alpha_{23} = 1, \alpha_{24} = 5, \alpha_{25} = 2, \alpha_{26} = 2;$$
$$\alpha_{35} = 2, \alpha_{36} = 1, \alpha_{37} = 1;$$

for I_{X_2} we have

$$\alpha_{12} = 2, \alpha_{13} = 3, \alpha_{15} = 1;$$
$$\alpha_{23} = 1, \alpha_{24} = 4, \alpha_{25} = 1, \alpha_{26} = 2;$$
$$\alpha_{35} = 1, \alpha_{36} = 1, \alpha_{37} = 1;$$

for I_{X_3} we have

$$\alpha_{12} = 2, \alpha_{13} = 3, \alpha_{14} = 1;$$
$$\alpha_{23} = 1, \alpha_{24} = 5, \alpha_{25} = 1, \alpha_{26} = 1;$$
$$\alpha_{35} = 2, \alpha_{37} = 1;$$

for I_{X_4} we have

$$\alpha_{12} = 2, \alpha_{13} = 3;$$
$$\alpha_{23} = 1, \alpha_{24} = 4, \alpha_{26} = 1;$$
$$\alpha_{35} = 1, \alpha_{37} = 1.$$

Note that indeed all these schemes are partial intersection schemes; moreover we can get from this decomposition other partial Gorenstein schemes (precisely 8 more) but all produce one of the above sets of graded Betti numbers. The scheme I_{X_4} gives one of the two sets of graded Betti numbers which appear in the Evans example.

From decomposition 2) we can construct 7 partial Gorenstein schemes all of these but one produces a set of graded Betti numbers already obtained by the first decomposition. To produce a scheme with a new set of graded Betti numbers take $\mathcal{A} =< (1, 3), (2, 1) >$ and in $\mathbb{M}_{\mathcal{A}}$ the left segment

$$\mathcal{F}_5 =< (1, 3; 1, 7), (2, 1; 2, 4) >.$$

The partial Gorenstein scheme $X = V_{\widehat{\mathcal{F}_5}}$, has the following graded Betti numbers:

$$\alpha_{12} = 2, \alpha_{13} = 3, \alpha_{15} = 1;$$
$$\alpha_{23} = 1, \alpha_{24} = 4, \alpha_{26} = 2;$$
$$\alpha_{36} = 1, \alpha_{37} = 1;$$

(such graded Betti numbers cannot be obtained by p.i. schemes).

The decomposition 3) gives 6 different partial Gorenstein schemes but no new set of graded Betti numbers. From decomposition 4) arise two new sets of graded Betti numbers. Take $\mathcal{A} =< (2, 2) >$ and in $\mathbb{M}_{\mathcal{A}}$ the left segments

$$\mathcal{F}_6 =< (1, 1; 1, 6), (2, 2; 2, 4), (2, 1; 2, 3) >;$$
$$\mathcal{F}_7 =< (1, 2; 1, 7), (1, 1; 1, 5), (2, 2; 2, 4), (2, 1; 2, 3) >.$$

The partial Gorenstein schemes (indeed p.i.) $X_1 = V_{\widehat{\mathcal{F}_6}}$ and $X_2 = V_{\widehat{\mathcal{F}_7}}$ have the following graded Betti numbers:
for I_{X_1}

$$\alpha_{12} = 2, \alpha_{13} = 2, \alpha_{14} = 1;$$

$\alpha_{24} = 5, \alpha_{25} = 1, \alpha_{26} = 1;$

$\alpha_{35} = 2, \alpha_{37} = 1.$

for I_{X_2}

$\alpha_{12} = 2, \alpha_{13} = 2, \alpha_{14} = 1, \alpha_{15} = 1;$

$\alpha_{24} = 5, \alpha_{25} = 2, \alpha_{26} = 2;$

$\alpha_{35} = 2, \alpha_{36} = 1, \alpha_{37} = 1.$

Decomposition 5) produces some of the previous sets of graded Betti numbers plus two new sets. To get these take first $\mathcal{A} =< (2, 2) >$ and in $\mathbb{M}_{\mathcal{A}}$ the left segment

$\mathcal{F}_8 =< (1, 2; 1, 7), (1, 1; 1, 4), (2, 1; 2, 4) >;$

the partial Gorenstein scheme $X = V_{\widehat{\mathcal{F}_8}}$ has the following graded Betti numbers:

$\alpha_{12} = 2, \alpha_{13} = 2, \alpha_{15} = 1;$

$\alpha_{24} = 4, \alpha_{26} = 2;$

$\alpha_{36} = 1, \alpha_{37} = 1.$

(also this appears in the Evans example). Now take $\mathcal{A} =< (1, 4), (2, 2) >$ and in $\mathbb{M}_{\mathcal{A}}$ the left segment

$\mathcal{F}_9 =< (1, 4; 1, 8), (1, 3; 1, 5), (2, 1; 2, 4) >;$

the partial Gorenstein scheme $X = V_{\widehat{\mathcal{F}_9}}$ has the following graded Betti numbers:

$\alpha_{12} = 2, \alpha_{13} = 2, \alpha_{15} = 1;$

$\alpha_{24} = 4, \alpha_{25} = 1, \alpha_{26} = 2;$

$\alpha_{35} = 1, \alpha_{36} = 1, \alpha_{37} = 1.$

(even these graded Betti numbers cannot be reached by p.i. schemes). Finally, decompositions 6) and 7) give 3 and 2, respectively, partial Gorenstein schemes but no new set of graded Betti numbers.

REFERENCES

[1] A.M. Bigatti, *Upper Bounds for the Betti Numbers of a Given Hilbert Function,* Comm. in Alg., 21 - 7 (1993), pp. 2317–2334.

[2] G. Campanella, *Standard bases of perfect homogeneous polynomial ideals of height 2,* J. of Alg., 101 - 1 (1986), pp. 47–60.

[3] E. De Negri - G. Valla, *The h-vector of a Gorenstein codimension three domain,* Nagoya Math. J., 138 (1995), pp. 113–140.

[4] S. Diesel, *Irreducibility and dimension theorems for families of height 3 Gorenstein algebras,* Pac. J. of Math., 172 - 4 (1996), pp. 365–397.

[5] F. Gaeta, *Nuove ricerche sulle curve sghembe algebriche di residuale finito e sui gruppi di punti del piano,* Ann. Mat. Pura Appl., 31 (4) (1950), pp. 1–64.

[6] A. Geramita - J. Migliore, *Reduced Gorenstein codimension three subschemes of projective space,* Proc. A. M. S., 125 - 4 (1997), pp. 943–950.

[7] H.A. Hulett, *Maximum Betti Numbers of Homogeneous Ideals with a Given Hilbert Function,* Comm. in Alg., 21 - 7 (1993), pp. 2335–2350.

[8] R. Maggioni - A. Ragusa, *Construction of smooth curves of $\mathbb{P}^3$ with assigned Hilbert function and generators' degrees,* Le Matematiche, 42 (1987), pp. 195–209.

[9] K. Pardue, *Deformation classes of graded modules and maximal Betti numbers,* Illinois J. of Math., 40 (1996), pp. 564–585.

[10] A. Ragusa - G. Zappalà, *Properties of* 3*-codimensional Gorenstein schemes,* Comm. in Alg., 29 - 1(2001), pp. 303–318.

[11] A. Ragusa - G. Zappalà, *Partial intersections and graded Betti numbers,* to appear on Beiträge zur Algebra und Geometrie.

[12] R. Stanley, *Hilbert functions of graded algebras,* Adv. in Math., 28 (1978), pp. 57–83.

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