

ON LOCALLY FACTORIAL FANO FOURFOLDS OF PICARD NUMBER TWO

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We classify the locally factorial Fano fourfolds of Picard number two with a hypersurface Cox ring that admit an effective action of a three-dimensional torus.

1. Introduction

This article contributes to the classification of Fano varieties admitting an action of an algebraic torus. An intensely studied example class is given by the toric Fano varieties, i.e. the torus action has a full-dimensional orbit. In this setting, the classification is entirely combinatorial and basically runs in terms of the associated Fano polytopes; for instance, the smooth toric Fano varieties are classified up to dimension nine [3, 20–22].

We focus on the case of Picard number two. In this situation, Kleinschmidt [19] gave a complete description of all smooth toric varieties, which leads in particular to complete classifications of the Fano ones in any dimension. Via linear Gale duality, Kleinschmidt's approach can be turned into a study of two-dimensional combinatorial structures, see [7, Prop. 1.11]. The latter point of view applies as well to torus actions of higher complexity, i.e. higher maximal orbit codimension [14, 15, 17]. This allows for instance to extend Kleinschmidt's description to smooth varieties with a torus action of complexity one

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and gives complete classifications of smooth Fano varieties with torus action of complexity one in any dimension [12]. Further work in this spirit concerns smooth intrinsic quadrics, general arrangement varieties and intrinsic Grassmannians [11, 15, 24].

In the present article we leave the smooth case and consider more generally locally factorial varieties, meaning that every Weil divisor is a Cartier divisor. Whereas in the toric case smoothness and local factoriality coincide, the latter setting turns out to be much more general for torus actions of complexity one; for instance, the varieties need not be log terminal any more and we find infinite series of non-isomorphic Fanos in fixed dimensions. We settle the case of dimension four, complexity one and a Cox ring defined by a single relation. In this situation, Cox ring theory [2] allows us to realize X as a hypersurface in a toric variety Z and closely relates the geometry of X to that of the ambient Z . Our main result considerably extends the corresponding one in the smooth case [12, Thm. 1.2].

Main Theorem 1.1. *There are 447 sporadic cases and 106 infinite series of locally factorial Fano fourfolds of Picard number two coming with an effective action of a three-dimensional torus and a Cox ring defined by a single relation.*

Our varieties in question are uniquely determined by the generator degrees and the relation in their Cox ring. *Classification lists 5.1 to 5.11 provide the complete and redundancy free presentation of the specifying data for the Main Theorem 1.1.* A data file containing the complete classification data is also available at [6].

For the proof of Theorem 1.1 we distinguish two main cases. In the first case we assume that the hypersurface X is an ample divisor on its ambient toric variety Z . There, we provide a smoothing procedure via Bertini's theorem which allows us to infer first constraints on the relevant invariants from the classification of smooth Fano fourfolds of Picard number two with a hypersurface Cox ring [18, Thm. 1.1]. The situation becomes more involved when X does not define an ample divisor on Z . In this situation, we have to classify case by case according to the possible constellations of the Cox ring generator degrees in the effective cone, making heavy use of the combinatorial description of varieties with a torus action of complexity one from [14, 17], see also [2, Sec. 3.4].

The article is organized as follows. Section 2 serves to provide the necessary background on Cox rings. In Section 3 we establish two general facts, essentially supporting our classification: First, Proposition 3.1 shows that in our setting we always have a torsion-free Picard group and second, Proposition 3.2 supplies us with an explicit smoothing procedure. An example case of the proof of Theorem 1.1 is presented in Section 4. The full elaboration will appear in [4]. Finally, in Section 5 we present all the data of the classification.

2. Background on Cox rings

By a *Mori dream space* we mean an irreducible, normal, projective complex variety X with finitely generated divisor class group $\text{Cl}(X)$ and finitely generated Cox ring $\mathcal{R}(X)$. We give a brief summary on the combinatorial approach [2, 8, 13] to Mori dream spaces, adapted to our needs. By a *K -graded affine algebra*, where K is a finitely generated abelian group, we mean an affine \mathbb{C} -algebra R coming with a direct sum decomposition into \mathbb{C} -vector subspaces

$$R = \bigoplus_{w \in K} R_w$$

such that $R_w R_{w'} \subseteq R_{w+w'}$ holds for all $w, w' \in K$. An element $f \in R$ is called *homogeneous* if $f \in R_w$ holds for some $w \in K$. In that case w is the *degree* of f and we write $w = \text{deg}(f)$. Geometrically, we have the affine variety \bar{X} with R as its algebra of global functions and the quasitorus H with K as its character group:

$$\bar{X} = \text{Spec } R, \quad H = \text{Spec } \mathbb{C}[K].$$

The K -grading of R defines an algebraic action of H on \bar{X} . By $K_{\mathbb{Q}} := K \otimes_{\mathbb{Z}} \mathbb{Q}$ we denote the \mathbb{Q} -vector space associated with K .

(i) The *effective cone* of R is

$$\text{Eff}(R) := \text{cone}(w \in K; R_w \neq 0) \subseteq K_{\mathbb{Q}}.$$

(ii) For $x \in \bar{X}$ we have the *orbit cone*

$$\omega_x := \text{cone}(w \in K; f(x) \neq 0 \text{ for some } f \in R_w) \subseteq K_{\mathbb{Q}}.$$

(iii) For $w \in \text{Eff}(R)$ we have the *GIT-cone*

$$\lambda_w := \bigcap_{x \in \bar{X}, w \in \omega_x} \omega_x \subseteq K_{\mathbb{Q}}.$$

The K -grading of R is called *pointed* if $R_0 = \mathbb{C}$ holds and $\text{Eff}(R)$ contains no line. The effective cone, as well as orbit cones and GIT-cones are convex polyhedral cones and there are only finitely many of them. The GIT-cones form a (quasi-) fan $\Lambda(R)$ in $K_{\mathbb{Q}}$ called the *GIT-fan* of R , having the effective cone $\text{Eff}(R)$ as its support. We recall Cox’s quotient presentation [9] for projective toric varieties.

Construction 2.1. Let $S = \mathbb{C}[T_1, \dots, T_r]$ and consider a pointed K -grading on S , such that the variables T_1, \dots, T_r are homogeneous. Write $w_i := \text{deg}(T_i) \in K$ for

the generator degrees, also when considered in $K_{\mathbb{Q}}$. We denote the *grading map* by

$$Q: \mathbb{Z}^r \rightarrow K, \quad e_i \mapsto w_i.$$

We have the action of the quasitorus H on the affine toric variety \bar{Z} , where

$$H := \text{Spec } \mathbb{C}[K], \quad \bar{Z} := \text{Spec } S = \mathbb{C}^r.$$

We assume that any $r - 1$ of the degrees w_1, \dots, w_r generate K as a group, i.e. the K -grading is *almost free*. Moreover we assume that the *moving cone*

$$\text{Mov}(S) := \bigcap_{i=1}^r \text{cone}(w_j; j \neq i) \subseteq K_{\mathbb{Q}}$$

is of full dimension. Fix a GIT-cone $\tau \in \Lambda(S)$ with $\tau^\circ \subseteq \text{Mov}(S)^\circ$. There is the H -invariant open set of *semistable points* \hat{Z} and the corresponding *good quotient* Z :

$$\hat{Z} := \bar{Z}^{ss}(\tau) = \{x \in \bar{Z}; \tau \subseteq \omega_x\}, \quad Z := \hat{Z} // H.$$

The quotient variety Z is a projective toric variety of dimension $r - \dim(K_{\mathbb{Q}})$ with divisor class group $\text{Cl}(Z) = K$ and Cox ring $\mathcal{R}(Z) = S$.

The following construction produces Mori dream spaces as hypersurfaces in projective toric varieties. A K -graded algebra R is called *K -factorial*, or the K -grading of R is called *factorial*, if R is integral and every homogeneous non-zero non-unit is a product of K -primes. A *K -prime* is a homogeneous non-zero non-unit $f \in R$ with the property that $f \mid gh$ for homogeneous $g, h \in R$ implies that $f \mid g$ or $f \mid h$ holds.

Construction 2.2 (See [2, Sec. 3.2, 3.3] and [18, Constr. 4.1, Rem. 4.2]). In the setting of Construction 2.1 fix $0 \neq \mu \in K$ and $g \in S_\mu$ and set

$$R_g := S/\langle g \rangle, \quad \bar{X}_g := V(g) \subseteq \bar{Z}, \quad \hat{X}_g := \bar{X}_g \cap \hat{Z}, \quad X_g := \hat{X}_g // H \subseteq Z.$$

Then the factor algebra R_g inherits a K -grading from S and the quotient $X_g \subseteq Z$ is a closed subvariety. Moreover, there is a GIT-cone $\lambda \in \Lambda(R_g)$ with

$$\hat{X}_g = \bar{X}_g^{ss}(\lambda) = \{x \in \bar{X}_g; \lambda \subseteq \omega_x\}.$$

We assume that R_g is integral and normal with $R_g^* = \mathbb{C}^*$, the induced K -grading is factorial and T_1, \dots, T_r define a minimal system of pairwise non-associated K -primes in R_g . Then X_g is a normal, projective variety with dimension, divisor class group and Cox ring given by

$$\dim(X_g) = \dim(Z) - 1, \quad \text{Cl}(X_g) = K, \quad \mathcal{R}(X_g) = R_g.$$

Moreover, the cones of effective, movable, semiample and ample divisor classes of X_g are given in the rational divisor class group $\text{Cl}(X_g)_{\mathbb{Q}} = K_{\mathbb{Q}}$ by

$$\begin{aligned} \text{Eff}(X_g) &= \text{Eff}(R_g), & \text{Mov}(R_g) &:= \text{Mov}(X_g) = \text{Mov}(S), \\ \text{SAmple}(X_g) &= \lambda, & \text{Ample}(X_g) &= \lambda^{\circ}. \end{aligned}$$

The *minimal ambient toric variety* $Z_g \subseteq Z$ of X is the minimal open toric subvariety containing X . For the respective ample cones, we have

$$\tau^{\circ} = \text{Ample}(Z) \subseteq \text{Ample}(Z_g) = \text{Ample}(X_g) = \lambda^{\circ}.$$

Proposition 2.3 (See [2, Prop. 3.3.3.2]). *Let $X = X_g$ as in Construction 2.2. Then the anticanonical class of X is given in $K = \text{Cl}(X)$ by*

$$-\mathcal{K} = w_1 + \cdots + w_r - \mu.$$

There is a decomposition of $X = X_g$ into locally closed subsets as follows. Denote by γ the positive orthant in \mathbb{Q}^r . A face $\gamma_0 \preceq \gamma$ is called an \bar{X}_g -face if there is $x \in \bar{X}_g$ with

$$x_i \neq 0 \iff e_i \in \gamma_0.$$

The orbit cones of \bar{X}_g are precisely the cones $Q(\gamma_0)$, where γ_0 is an \bar{X}_g -face. An X -face is an \bar{X}_g -face γ_0 with $\lambda^{\circ} \subseteq Q(\gamma_0)^{\circ}$. Write $\text{rlv}(X)$ for the set of X -faces. We have

$$X = \bigcup_{\gamma_0 \in \text{rlv}(X)} X(\gamma_0), \quad X(\gamma_0) := \{x \in \bar{X}_g; x_i \neq 0 \iff e_i \in \gamma_0\} // H.$$

Proposition 2.4 (See [2, Cor. 3.3.1.8]). *For $X = X_g$ as in Construction 2.2 the following hold.*

- (i) X is \mathbb{Q} -factorial if and only if the cone $\lambda = \text{SAmple}(X) \subseteq K_{\mathbb{Q}}$ is full-dimensional.
- (ii) X is locally factorial if and only if for every X -face $\gamma_0 \preceq \gamma$, the group K is generated by $Q(\gamma_0 \cap \mathbb{Z}^r)$.

For locally factorial X of Picard number two, Proposition 2.4 (ii) in particular yields the following two statements.

Lemma 2.5 (See [18, Lemma 5.6]). *Let $X = X_g$ as in Construction 2.2. Assume X is locally factorial and of Picard number two. Let $1 \leq i, j \leq r$ with $\lambda \subseteq \text{cone}(w_i, w_j)$. Then either w_i, w_j generate K as a group, or g has precisely one monomial of the form $T_i^{l_i} T_j^{l_j}$, where $l_i + l_j > 0$.*

Let $X = X_g$ from Construction 2.2 be of Picard number two. Then we decompose the effective cone into the convex sets

$$\text{Eff}(R_g) = \lambda^- \cup \lambda^\circ \cup \lambda^+,$$

where λ^- and λ^+ are the convex polyhedral cones not intersecting λ° and the intersection $\lambda^+ \cap \lambda^-$ consists only of the origin.

Lemma 2.6 (See [18, Lemma 5.7]). *Let $X = X_g$ as in Construction 2.2. Assume X is locally factorial and of Picard number two. Let $1 \leq i < j < k \leq r$. Then w_i, w_j, w_k generate K as a group, provided that one of the following holds:*

- (i) $w_i, w_j \in \lambda^-, w_k \in \lambda^+$ and g has no monomial of the form $T_k^{l_k}$,
- (ii) $w_i \in \lambda^-, w_j, w_k \in \lambda^+$ and g has no monomial of the form $T_i^{l_i}$,
- (iii) $w_i \in \lambda^-, w_j \in \lambda^\circ, w_k \in \lambda^+$.

Moreover, if (iii) holds, then g has a monomial of the form $T_j^{l_j}$ where l_j is divisible by the order of the factor group $K/\langle w_i, w_k \rangle$.

We turn to rational varieties with a complexity one torus action. For the general theory see [14, 15, 17], also [2, Sec. 3.4]. Here we focus on the case of hypersurface Cox rings.

Definition 2.7. A Mori dream space X with divisor class group $\text{Cl}(X) = K$ has a hypersurface Cox ring if there is an irredundant K -graded presentation

$$\mathcal{R}(X) = \mathbb{C}[T_1, \dots, T_r]/\langle g \rangle,$$

where irredundant means that the ideal $\langle g \rangle$ contains no element $T_i - h_i$ with $h_i \in \mathbb{C}[T_1, \dots, T_r]$ not depending on T_i .

Proposition 2.8. *For a Mori dream space X with a hypersurface Cox ring the following are equivalent:*

- (i) X admits a torus action of complexity one.
- (ii) The Cox ring of X has an irredundant $\text{Cl}(X)$ -graded presentation

$$\mathcal{R}(X) = \mathbb{C}[T_1, \dots, T_r]/\langle g \rangle,$$

where g is a trinomial consisting of pairwise coprime monomials.

Proof. We write $K = \text{Cl}(X)$ for the divisor class group and $R = \mathcal{R}(X)$ for the Cox ring of X . Assume that (i) holds. Then by [2, Thm. 4.4.1.6] there is an irredundant K -graded presentation

$$R = \mathbb{C}[T_1, \dots, T_r] / \langle g_1, \dots, g_t \rangle,$$

such that the variables define pairwise non-associated K -prime generators and the polynomials $g_1, \dots, g_t \in \mathbb{C}[T_1, \dots, T_r]$ are homogeneous trinomials, each one consisting of pairwise coprime monomials. Moreover, since X has a hypersurface Cox ring, there is an irredundant K -graded presentation

$$R = \mathbb{C}[T_1, \dots, T_{r'}] / \langle g \rangle.$$

The graded isomorphism between these two presentations of R lifts to a graded isomorphism $\mathbb{C}[T_1, \dots, T_r] \rightarrow \mathbb{C}[T_1, \dots, T_{r'}]$, see [16, Lemma 2.4]. This yields $r = r'$ and $t = 1$.

Now assume that (ii) holds. Using [2, Constr. 3.2.4.2] we construct the *gradiator* of g . This is the maximal effective grading of $\mathbb{C}[T_1, \dots, T_r]$ for which the variables T_1, \dots, T_r and the polynomial g are homogeneous. Geometrically, this grading yields an effective action of a quasitorus H_g on X . The coprimeness of the monomials of g guarantees that the quasitorus H_g contains a torus T of dimension $\dim(T) = \dim(X) - 1$. Thus X admits an effective torus action of complexity one. □

In the situation of Proposition 2.8 we have the following easy to apply criterion to check for factoriality of the Cox ring.

Theorem 2.9 (See [14, Thm. 1.1]). *For $l_1, l_2, l_3 \in \mathbb{Z}_{\geq 0}^r$ assume that the monomials $T^{l_1}, T^{l_2}, T^{l_3} \in \mathbb{C}[T_1, \dots, T_r]$ are pairwise coprime. Then the ring*

$$R = \mathbb{C}[T_1, \dots, T_r] / \langle T^{l_1} + T^{l_2} + T^{l_3} \rangle$$

is a unique factorization domain if and only if the integers $\gcd(l_1)$, $\gcd(l_2)$ and $\gcd(l_3)$ are pairwise coprime.

We conclude this section by quoting two results from [18] that we will use in various proofs throughout the paper.

Proposition 2.10 (See [18, Prop. 2.4]). *Let $X = X_g$ as in Construction 2.2. Then we have*

$$\mu \in \bigcap_{1 \leq i < j \leq r} \text{cone}(w_k; k \neq i, k \neq j) \subseteq K_{\mathbb{Q}}.$$

Proposition 2.11 (See [18, Prop. 2.8]). *Let K be a finitely generated abelian group and R an affine algebra with a pointed K -grading. Consider a K -graded presentation*

$$R = \mathbb{C}[T_1, \dots, T_r] / \langle g_1, \dots, g_s \rangle$$

such that T_1, \dots, T_r define non-constant elements in R . Assume that there are a GIT-cone $\lambda \in \Lambda(R)$ of dimension at least two and an index i with $\deg(T_i) \in \lambda^\circ$.

- (i) *There exists a j such that g_j has a monomial $T_i^{l_i}$ with $l_i \in \mathbb{Z}_{\geq 0}$.*
- (ii) *There exists a j such that $\deg(g_j) = l_i \deg(T_i)$ holds with $l_i \in \mathbb{Z}_{\geq 0}$.*
- (iii) *If $s = 1$ holds, then, $\deg(T_k)$ generates a ray of $\Lambda(R)$ whenever $k \neq i$.*

3. Picard group and smoothability

In this section we establish two general facts, which are essential for our proof of Theorem 1.1. The first is Proposition 3.1, which shows that in our setting the Picard group is always torsion-free. The second is Proposition 3.2, which in particular gives rise to an explicit smoothing procedure in case our variety X is an ample divisor on its ambient toric variety Z ; see also Remark 5.13.

Proposition 3.1. *Let $X = X_g$ as in Construction 2.2. Assume that X is \mathbb{Q} -factorial, Fano, of Picard number two and admits a torus action of complexity one. Then the Picard group $\text{Pic}(X)$ is torsion-free.*

Proof. We adopt the notation of Construction 2.2. By [2, Cor. 3.3.1.6] we have the identity

$$\text{Pic}(X) = \bigcap_{\gamma_0 X\text{-face}} Q(\text{lin}(\gamma_0) \cap \mathbb{Z}^r).$$

It therefore suffices to show that there is a two-dimensional X -face. By Proposition 2.8 we may assume that $g \in S$ is a trinomial consisting of pairwise coprime monomials. We write ρ_1, \dots, ρ_s for the rays generated by the generator degrees w_1, \dots, w_r . Reordering them if necessary, we may assume that the effective cone $\text{Eff}(R)$ of $R := R_g$ is spanned by ρ_1 and ρ_s . We distinguish two cases. First, assume $\mu = \deg(g)$ is contained in $\text{Eff}(R)^\circ$. In this case we can find generator degrees w_i, w_j that satisfy the following conditions:

- (i) $\lambda^\circ \subseteq \text{cone}(w_i, w_j)$.
- (ii) $\mu \in \text{cone}(w_i, w_j)^\circ$.
- (iii) g does not contain a monomial of the form $T_i^{l_i} T_j^{l_j}$.

To see this, consider $\kappa := \text{cone}(\mu) + \lambda$. Then κ is a two-dimensional cone. Suitably renumbering, we may assume $\text{Eff}(R) = \text{cone}(w_1, w_r)$ and $\kappa = \text{cone}(w_i, w_j)$ such that we have

$$\text{Eff}(R) = \text{cone}(w_1, w_i) \cup \kappa^\circ \cup \text{cone}(w_j, w_r)$$

as a disjoint union and $\mu \notin \text{cone}(w_j, w_r)$. Moreover, Construction 2.2 says $\lambda \subseteq \text{Mov}(R)$ and thus we can assume $w_j \neq w_r$. Now, each of the pairs w_1, w_j and w_1, w_r satisfies (i) and (ii). There are unique rational numbers l_1, l'_1, l_j, l_r with

$$\mu = l_1 w_1 + l_r w_r = l'_1 w_1 + l_j w_j.$$

As μ lies in the interior of $\text{cone}(w_1, w_j)$, all these numbers are positive. Since g has pairwise coprime monomials, only one of the pairs w_1, w_j and w_1, w_r can stem from a monomial of g and the other satisfies (iii) and hence defines the desired X -cone.

Now assume that μ lies on one of the bounding rays of $\text{Eff}(R)$. We may assume that $\mu \in \rho_1$ holds. Since g is a trinomial and its monomials are pairwise coprime, the ray ρ_1 contains at least three generator degrees. If ρ_1 contains four or more generator degrees, then there is $w_i \in \rho_1$ such that g does not contain a monomial of the form $T_i^{l_i}$. Choose any $w_j \in \rho_s$. Then the face $\gamma_0 := \text{cone}(e_i, e_j)$ is again a two-dimensional X -face. Now assume that ρ_1 contains exactly three generator degrees, say w_1, w_2 and w_3 . The ample cone λ of X is of the form $\lambda = \rho_k + \rho_{k+1}$ for some $k = 1, \dots, s - 1$. If $\lambda \neq \rho_1 + \rho_2$, then we take w_i and w_j from each of the bounding rays of λ . The face $\gamma_0 := \text{cone}(e_i, e_j)$ is again a two-dimensional X -face.

It remains to consider the case $\lambda = \rho_1 + \rho_2$. In this case we will not succeed in explicitly constructing a two-dimensional X -face. Instead we show that X has at most log terminal singularities. Torsion-freeness of $\text{Pic}(X)$ then follows by [23, Prop. 2.1.2]. By applying a unimodular transformation, we achieve that ρ_1 is the ray generated by e_1 . We write $w_1 = (a_1, 0)$, $w_2 = (a_2, 0)$ and $w_3 = (a_3, 0)$. Switching the roles of w_1, w_2 and w_3 if necessary, we may assume that $a_1 \geq a_2 \geq a_3$ holds. Homogeneity of g yields $l_1 a_1 = l_2 a_2 = l_3 a_3$. As X is Fano, it's anticanonical class is ample. By Proposition 2.3, this means

$$(1 - l_1)w_1 + w_2 + w_3 + w_4 + \dots + w_r \in \lambda^\circ = (\rho_1 + \rho_2)^\circ.$$

The point $w := w_4 + \dots + w_r$ is contained in the cone $\rho_2 + (-\rho_1)$. Thus, for the sum to lie in the interior of λ , we must have $(1 - l_1)w_1 + w_2 + w_3 \in \rho_1$. This is equivalent to $(l_1 - 1)a_1 < a_2 + a_3$. Since a_1 is at least as big as a_2 and a_3 , this yields $l_1 = 2$. Homogeneity of g thus yields $a_2 = 2a_1/l_2$ and $a_3 = 2a_1/l_3$. With this, the exponents l_2 and l_3 satisfy the following inequality:

$$1 < \frac{2}{l_2} + \frac{2}{l_3}.$$

Since $a_2 \geq a_3$, we have $l_3 \geq l_2$. Moreover, both exponents are at least two by irredundancy of the presentation of R . The triple (l_1, l_2, l_3) is therefore one of the following:

$$\begin{aligned} (l_1, l_2, l_3) &= (2, 2, y), & (l_1, l_2, l_3) &= (2, 3, 3), \\ (l_1, l_2, l_3) &= (2, 3, 4), & (l_1, l_2, l_3) &= (2, 3, 5), \end{aligned}$$

where $y \geq 2$. Thus, by [1, Thm. 3.13] the variety X has at most log terminal singularities. With this, we are in the situation of [23, Prop. 2.1.2], which tells us that the Picard group $\text{Pic}(X)$ is torsion-free. \square

Proposition 3.2. *Let $X = X_g$ as in Construction 2.2 be locally factorial and of Picard number two. Assume that*

$$\mu \in \text{SAmple}(X) \cap \text{Mov}(X)^\circ.$$

Then there is a non-empty open subset $U \subseteq S_\mu$ such that for all $h \in U$ the variety $X_h \subseteq Z$ is smooth with divisor class group $\text{Cl}(X_h) = K$ and Cox ring $\mathcal{R}(X_h) = R_h$.

The remainder of this section is devoted to the proof of Proposition 3.2. We adopt the notation of Construction 2.1 and Construction 2.2. A homogeneous polynomial $h \in S_\mu$ is called *spread*, if every monomial $T^\nu \in S$ of degree $\mu = \deg(h)$ is a convex combination of monomials of h . We say that R_h is spread, if h is spread, see [18, Def. 4.3]. Here we identify a monomial $T^\nu = T_1^{\nu_1} \cdots T_r^{\nu_r}$ with its exponent vector $\nu \in \mathbb{Q}^r$. If $h, h' \in S_\mu$ are spread, then the minimal ambient toric varieties Z_h of X_h and $Z_{h'}$ of $X_{h'}$ coincide. Thus Z_h does not depend on the particular choice of h .

Definition 3.3 (See [18, Def. 4.18]). In the setting of Construction 2.1 and Construction 2.2 we call $Z_\mu := Z_h$ with $h \in S_\mu$ spread the μ -*minimal ambient toric variety*.

The following two Propositions, originally [18, Prop. 4.11] and [18, Cor. 4.19], are essential to the proof of Proposition 3.2.

Proposition 3.4 (See [18, Cor. 4.19]). *In the setting of Construction 2.2, assume $\text{rank}(K) = 2$ and that $Z_\mu \subseteq Z$ is smooth. If $\mu \in \tau$ holds, then μ is basepoint free. Moreover, then there is a non-empty open subset of polynomials $g \in S_\mu$ such that X_g is smooth.*

Proposition 3.5 (See [18, Prop. 4.11]). *In the setting of Construction 2.2 assume that K is of rank at most $r - 4$ and torsion-free, there is $g \in S_\mu$ such that T_1, \dots, T_r define primes in R_g , we have $\mu \in \tau^\circ$ and μ is basepoint free on Z . Then there is a non-empty open subset of polynomials $g \in S_\mu$ such that R_g is a UFD.*

For the rest of this section it is assumed that we have $X = X_g$ as in Construction 2.2 and that X is locally factorial and of Picard number two. For any homogeneous $h \in S_\mu$ we denote by $\lambda_h \in \Lambda(R_h)$ the smallest GIT-cone that contains τ . Note that local factoriality of X in particular implies \mathbb{Q} -factoriality. Thus by Proposition 2.4 (i) the cone λ is full-dimensional.

Lemma 3.6. *Let $h \in S_\mu$ such that each monomial of g is also a monomial of h . Then $\lambda \subseteq \lambda_h$ holds.*

Proof. The cone $\lambda_h \in \Lambda(R_h)$ is the smallest GIT-cone that contains τ . Thus in the case $\tau = \lambda$ there is nothing to show. So we may assume that $\tau \subsetneq \lambda$ holds. We write $\lambda = \text{cone}(w_i, w_j)$ and $\tau = \text{cone}(w_k, w_l)$. Since τ is a proper subset of λ , one of its ray generators is contained in the interior of λ , say $w_k \in \lambda^\circ$. By Proposition 2.11 the degree w_k is the only generator degree that is contained in the interior of λ . Moreover, μ lies on the ray of w_k and the relation g contains a monomial of the form $T_k^{l_k}$. Since h contains each monomial of g , it also contains the monomial $T_k^{l_k}$. Therefore, each projected \bar{X}_h -face $Q(\gamma_0)$, that contains τ , necessarily also contains generator degrees on both sides of w_k . Since w_k is the only generator degree in the interior of λ , the cone $Q(\gamma_0)$ also contains λ . This implies the assertion. \square

Proposition 3.7. *The μ -minimal ambient toric variety $Z_\mu \subseteq Z$ is smooth.*

Proof. Let $h \in S_\mu$ be spread such that each monomial of g is also a monomial of h and let $\gamma_0 \preceq \gamma$ with $\lambda_h^\circ \subseteq Q(\gamma_0)^\circ$. Write $\gamma_0 = \text{cone}(e_{i_1}, \dots, e_{i_m})$. By Proposition 2.4 (ii) we have to show that either w_{i_1}, \dots, w_{i_m} generate K as a group, or γ_0 is not an \bar{X}_h -face. Assume that w_{i_1}, \dots, w_{i_m} do not generate K . We show that γ_0 is not an \bar{X}_h -face. By Lemma 3.6, we have $\lambda^\circ \subseteq Q(\gamma_0)^\circ$. Since λ is of full dimension, γ_0 is at least two-dimensional. In particular we have $m \geq 2$. None of the degrees w_{i_1}, \dots, w_{i_m} lies in λ° : If one of them did, say $w_{i_1} \in \lambda^\circ$, then by Proposition 2.11 it is the only generator degree in the interior of λ and g contains a monomial of the form $T_{i_1}^{l_{i_1}}$. Moreover, in this case we have $m \geq 3$. We may assume that w_{i_2} and w_{i_3} each lie in one of the bounding rays of $Q(\gamma_0)$. The degrees w_{i_2}, w_{i_3} do not generate K as a group. Thus, since X is locally factorial, by Proposition 2.4 (ii) the cone spanned by e_{i_2} and e_{i_3} is not an \bar{X} -face. This means that g contains a monomial of the form $T_{i_2}^{l_{i_2}} T_{i_3}^{l_{i_3}}$. But then the cone spanned by e_{i_1}, e_{i_2} and e_{i_3} is an \bar{X} -face and thus $w_{i_1}, w_{i_2}, w_{i_3}$ generate K . A contradiction. Thus none of the degrees w_{i_1}, \dots, w_{i_m} lies in λ° . We may assume that the generator degrees are sorted in such a way that for all $v \in \lambda^\circ$ we have $\det(v, w_{i_j}) < 0$ if $j \leq k$ and $\det(v, w_{i_j}) > 0$ if $j > k$ for some fixed $1 \leq k \leq m$. We show that either $k = 1$ or $k + 1 = m$ holds. If $1 < k$ and $k + 1 < m$, then we have $m \geq 4$ and

neither $w_{i_1}, w_{i_{m-1}}$, nor w_{i_2}, w_{i_m} generate K . By local factoriality of X and Proposition 2.4 (ii), the relation g then contains monomials of the form $T_{i_1}^{l_{i_1}} T_{i_{m-1}}^{i_{m-1}}$ and $T_{i_2}^{l_{i_2}} T_{i_m}^{i_m}$. Thus $\gamma'_0 = \text{cone}(w_{i_1}, w_{i_2}, w_{i_{m-1}}, w_{i_m})$ is an \bar{X} -face and $w_{i_1}, w_{i_2}, w_{i_{m-1}}, w_{i_m}$ generate K . A contradiction. We may thus assume that $k = 1$ holds. If $m = 2$, then g contains a monomial of the form $T_{i_1}^{l_{i_1}} T_{i_2}^{l_{i_2}}$ and this is the only monomial in S_μ only depending on these two variables. Therefore γ_0 is not an \bar{X}_h -face. If $m > 2$, then by Proposition 2.4 (ii), the relation g contains a monomial of the form $T_{i_1}^{l_{i_1}}$ and this is the only monomial in S_μ only depending on the variables T_{i_1}, \dots, T_{i_m} . Thus also in this case γ_0 is not an \bar{X}_h -face. \square

Lemma 3.8. *If $\mu \in \lambda$ holds, then we also have $\mu \in \tau$.*

Proof. In case τ and λ coincide, there is nothing to show. So we assume that $\tau \subsetneq \lambda$ holds. Write $\lambda = \text{cone}(w_i, w_j)$ and $\tau = \text{cone}(w_k, w_l)$. Since τ is a proper subset of λ , one of the generator degrees in its bounding rays lies in the interior of λ , say $w_k \in \lambda^\circ$. By Proposition 2.11, it is the only generator degree that lies in λ° and g contains a monomial of the form $T_k^{l_k}$. This shows that $\mu \in \tau$ holds. \square

Proposition 3.9. *Assume that $\mu \in \lambda \cap \text{Mov}(R_g)^\circ$ holds. Then there is a non-empty open subset $U \subseteq S_\mu$ such that R_h is a UFD for each $h \in U$.*

Proof. By Lemma 3.8 the relation degree μ is contained in the cone τ . We distinguish two cases. First assume that $\mu \in \tau^\circ$ holds. As Z_μ is smooth by Proposition 3.7, the class μ is basepoint free by Proposition 3.4. We can thus apply Proposition 3.5, which yields the assertion. Now assume that $\mu \in \partial\tau$ holds. Let $\tau_\mu \in \Lambda(S)$ the unique GIT-cone that contains μ in its interior, ie. τ_μ is the bounding ray of τ containing μ . We write $Z(\tau_\mu) := \bar{Z}^{ss}(\tau_\mu)//H$ for the projective toric variety associated with τ_μ as in Construction 2.1. We show that μ is basepoint free on $Z(\tau_\mu)$. Note that μ is semiample. Thus by [10, Thm. 6.3.12] it suffices to show that μ is Cartier on $Z(\tau_\mu)$. By [2, Cor. 3.3.1.6] this is the case if and only if

$$\mu \in \bigcap_{\gamma_0 \in \text{rlv}(Z(\tau_\mu))} Q(\gamma_0 \cap \mathbb{Z}^r)$$

holds. Let $\gamma_0 \in \text{rlv}(Z(\tau_\mu))$. If $Q(\gamma_0)$ is two-dimensional, then $\lambda^\circ \subseteq Q(\gamma_0)^\circ$ holds and by Proposition 3.7 we have $Q(\gamma_0 \cap \mathbb{Z}^r) = K$. So assume that $Q(\gamma_0)$ is one-dimensional, ie. $Q(\gamma_0) = \tau_\mu$. We distinguish two cases. First assume $\mu \in \lambda^\circ$. Then by Proposition 2.11, τ_μ contains a single generator degree w_k and μ is a multiple of w_k . Thus in this case $\mu \in Q(\gamma_0 \cap \mathbb{Z}^r)$ holds. Now we assume that $\mu \in \partial\lambda$ holds. Then τ_μ is one of the bounding rays of λ .

Write $\gamma_0 = \text{cone}(e_{i_1}, \dots, e_{i_m})$ and let w_k a generator degree in the other bounding ray of μ . If $w_{i_1}, \dots, w_{i_m}, w_k$ generate K , then μ is a linear combination of w_{i_1}, \dots, w_{i_m} and thus $\mu \in Q(\gamma_0 \cap \mathbb{Z}^r)$ holds. If they do not generate K , then by Proposition 2.4 (ii) the relation g contains a single monomial only depending on $T_{i_1}, \dots, T_{i_m}, T_k$. Since μ is contained in τ_μ , this monomial can not depend on T_k . Thus μ is again a linear combination of the degrees w_{i_1}, \dots, w_{i_m} and thus $\mu \in Q(\gamma_0 \cap \mathbb{Z}^r)$ holds. This shows that μ is basepoint free on $Z(\tau_\mu)$. We again apply Proposition 3.5, which yields the assertion. \square

Proof of Proposition 3.2. The set $U_1 \subseteq S_\mu$ of polynomials h such that R_h is a UFD is open and non-empty by Proposition 3.9. The set $U_2 \subseteq S_\mu$ of polynomials h such that T_1, \dots, T_r form a minimal system of non-associated K -prime generators in R_h is open by [18, Prop. 4.10] and since R_g has that property, the set U_2 is non-empty. Finally, the set $U_3 \subseteq S_\mu$ of polynomials h such that X_h is smooth is open and non-empty by Proposition 3.4 and Proposition 3.7. Now for any h in the intersection

$$U = U_1 \cap U_2 \cap U_3$$

the affine K -algebra R_h is a UFD and the variables T_1, \dots, T_r define pairwise non-associated primes in R_h . Being a UFD implies that R_h is normal and that the K -grading is factorial. The grading is also pointed, as this is inherited from S . In particular this implies that $R_h^* = \mathbb{C}^*$ holds. We are thus in the situation of Construction 2.2. So R_h is the Cox ring of the smooth projective variety X_h . \square

4. Proof of the main result: An example case

We enter the proof of Theorem 1.1. We restrict the discussion to an example case, see Theorem 4.3. However, we will still encounter all types of arguments of the full proof. The complete elaboration appears in [4]. We start by fixing the setting.

Setting 4.1. Let X be a locally factorial Fano fourfold of Picard number two with a hypersurface Cox ring $R = \mathcal{R}(X)$. Write $K = \text{Cl}(X)$ and let

$$R = \mathbb{C}[T_1, \dots, T_7]/\langle g \rangle$$

an irredundant K -graded presentation of R such that the variables T_1, \dots, T_7 define pairwise non-associated K -prime generators of R . We have $X = X_g$ as in Construction 2.2. We assume that X is of complexity one. By Proposition 3.1 the grading group K is torsion-free and we identify $K = \mathbb{Z}^2$. In this case the notions of K -factoriality and factoriality coincide, see [2, Thm. 3.4.1.11]. Thus, the ring R is a unique factorization domain. By Proposition 2.8 and Theorem 2.9 we may thus assume that g satisfies the following conditions.

(C1) The relation g is of the form

$$g = T^{l_1} + T^{l_2} + T^{l_3}$$

with $l_1, l_2, l_3 \in \mathbb{Z}_{\geq 0}^7$ such that each variable T_1, \dots, T_7 divides at most one monomial of g .

(C2) The integers $\gcd(l_1)$, $\gcd(l_2)$ and $\gcd(l_3)$ are pairwise coprime.

We turn to the grading map Q of R . Write $w_i := Q(e_i) = \deg(T_i)$ and $\mu := \deg(g)$ for the degrees in K , also when regarded in $K_{\mathbb{Q}}$. Suitably ordering w_1, \dots, w_7 we ensure

$$\det(w_i, w_j) \geq 0$$

whenever $i \leq j$. We call (Q, g) the *specifying data* of X . Some of the degrees w_i may share a common ray. We denote by s the number of distinct rays ρ_1, \dots, ρ_s generated by the degrees w_1, \dots, w_7 ,

$$s := \#\{\text{cone}(w_i); i = 1, \dots, 7\}.$$

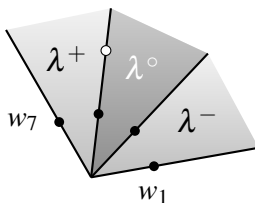
Moreover, we denote the number of generator degrees w_j contained in the ray ρ_i by n_i . We have $s \leq 7$ and

$$n_1 + \dots + n_s = 7.$$

Up to reversing order, the tuple (n_1, \dots, n_s) is invariant under automorphisms of \mathbb{Z}^2 . We call it the *degree constellation* of X . Each ray ρ_j in the GIT-fan $\Lambda(R)$ is of the form $\rho_j = \text{cone}(w_i)$ for some w_i , but the converse may not hold. As X is locally factorial, it is in particular \mathbb{Q} -factorial. By Proposition 2.4 (i) this means that the cone $\lambda = \text{SAmple}(X)$ is full-dimensional. As a GIT-cone in $K_{\mathbb{Q}} = \mathbb{Q}^2$, the cone λ is the intersection of two projected \bar{X}_g -faces and thus each bounding ray of λ contains at least one of the degrees w_i . We decompose the effective cone $\text{Eff}(R)$ into the three convex sets

$$\text{Eff}(X) = \lambda^- \cup \lambda^\circ \cup \lambda^+,$$

where λ^- and λ^+ are the convex polyhedral cones not intersecting $\lambda^\circ = \text{Ample}(X)$ and the intersection $\lambda^+ \cap \lambda^-$ consists only of the origin. Each of the cones λ^+ and λ^- contains at least two of the generator degrees w_1, \dots, w_7 . However, λ^+ as well as λ^- may be one-dimensional. The following picture illustrates the situation for the case $s = 4$.



The black dots represent the generator degrees w_1, \dots, w_7 . The white dot represents the relation degree μ . In this example the cones λ^+ and λ^- are full-dimensional.

Remark 4.2. Given specifying data (Q, g) and (Q', g') , we need criteria to decide computationally whether the varieties $X(Q, g)$ and $X(Q', g')$ are isomorphic. Here we can use [5, Prop. 3.4]: If $X(Q, g)$ and $X(Q', g')$ are isomorphic, then g and g' coincide up to permutation of variables.

Setting 4.1 divides the proof of Theorem 1.1 into six cases, according to the number s of rays spanned by the degrees w_1, \dots, w_7 . Here we treat the case $s = 3$ as an example, showing basically all arguments used in the full proof, see [4].

Theorem 4.3. *The tables from 5.2, 5.3 and 5.4 provide specifying data (Q, g) for 223 sporadic cases and 4 infinite series of locally factorial Fano fourfolds of Picard number two and complexity one with a hypersurface Cox ring and $s = 3$. Moreover, any locally factorial Fano fourfold of Picard number two and complexity one with a hypersurface Cox ring and $s = 3$ is isomorphic to precisely one $X(Q, g)$ with (Q, g) from these tables.*

Let us see that each entry (Q, g) from the tables 5.1 to 5.11 indeed defines a locally factorial Fano fourfold of Picard number two and complexity one with a hypersurface Cox ring. Endow $S = \mathbb{C}[T_1, \dots, T_7]$ with the grading given by $Q = [w_1, \dots, w_7]$ via $\deg(T_i) = w_i$ and let Z the quotient associated with the smallest GIT cone in $\Lambda(S)$ that contains

$$\kappa = w_1 + \dots + w_7 - \deg(g)$$

in its interior, see Construction 2.1. Set $X := X_g \subseteq Z$ as in Construction 2.2. It is a four dimensional projective variety. By Theorem 2.9 the \mathbb{C} -algebra R_g is a unique factorization domain. Thus $\mathcal{R}(X) = R_g$ holds. In particular X has a hypersurface Cox ring and Picard number two. By Proposition 2.3 it is Fano and by Proposition 2.8 it is of complexity one. Proposition 2.4 guarantees that X is locally factorial. Taking the specifying data of X we again arrive at (Q, g) .

Another task is to show that distinct entries from tables 5.1 to 5.11 define non-isomorphic varieties. This can be done with Remark 4.2 and comparing invariants such as \mathcal{K}^4 .

We now turn to the proof that any locally factorial Fano fourfold of Picard number two and complexity one with a hypersurface Cox ring and $s = 3$ is isomorphic to some $X(Q, g)$ with (Q, g) from tables 5.2, 5.3 and 5.4. We divide the proof into the two general cases

$$\mu \in \text{SAmple}(X), \quad \mu \notin \text{SAmple}(X).$$

The case $\mu \in \text{SAmple}(X)$ will be treated in Proposition 4.6. In Proposition 4.7 we treat the case $\mu \notin \text{SAmple}(X)$. We start with some general observations that will be used throughout the proof.

Lemma 4.4. *In the situation of Setting 4.1, for each ray ρ_j , at most two of the generator degrees w_i contained in ρ_j are non-primitive lattice points.*

Proof. Assume by contradiction that ρ_j contains three non-primitive generator degrees w_{i_1}, w_{i_2} and w_{i_3} . Applying a unimodular transformation if necessary, we may assume that ρ_j is the ray generated by the first standard basis vector. Write

$$w_{i_1} = (a_{i_1}, 0), \quad w_{i_2} = (a_{i_2}, 0), \quad w_{i_3} = (a_{i_3}, 0).$$

As the degrees $w_{i_1}, w_{i_2}, w_{i_3}$ are non-primitive, we have $a_{i_1}, a_{i_2}, a_{i_3} > 1$. By Proposition 2.11 the ray ρ_j does not lie in the interior of λ . Thus there is a generator degree w_k such that λ is contained in the two-dimensional cone $\tau_m = \text{cone}(w_{i_m}, w_k)$ for $m = 1, 2, 3$. Since w_{i_m} is not primitive, the degrees w_{i_m} and w_k do not generate K as a group. Lemma 2.5 thus tells us that the relation g contains monomials of the form $T_{i_m}^{l_m} T_k^{l_k(m)}$. As the monomials of g are pairwise coprime, at least two of the exponents $l_k(m)$ must be zero. By homogeneity of g we obtain $l_k(1) = l_k(2) = l_k(3) = 0$ and μ lies in ρ_j . In particular, we have

$$\mu = l_{i_1} w_{i_1} = l_{i_2} w_{i_2} = l_{i_3} w_{i_3}.$$

By the condition (C2) from Setting 4.1, the integers $l_{i_1}, l_{i_2}, l_{i_3}$ are pairwise coprime. Moreover, they are all bigger than one by irredundancy of the presentation of R . Let p a prime divisor of l_{i_1} . Then p must divide both a_{i_2} and a_{i_3} . In particular, a_{i_2} and a_{i_3} are not coprime. Thus the three degrees w_{i_2}, w_{i_3} and w_k do not generate K as a group. By Lemma 2.6 the relation g must therefore contain a monomial of the form $w_k^{l_k}$. This is a contradiction to the position of μ . \square

Lemma 4.5. *In the situation of Setting 4.1, assume that $\mu \in \lambda^+ \setminus \lambda$ holds. Then the following hold.*

- (i) *The cone λ is regular and every generator degree lying on its boundary is primitive.*

(ii) All generator degrees contained in λ^- coincide. In particular, $n_1 \geq 2$ holds and $\lambda^- = \rho_1$ is a bounding ray of λ .

Proof. We prove (i). Let $1 \leq i < j \leq 7$ such that $\lambda = \text{cone}(w_i, w_j)$ holds. Since the relation degree μ is not contained in λ , the relation g does not contain a monomial of the form $T_i^{l_i} T_j^{l_j}$. By Lemma 2.5, the generator degrees w_i and w_j generate K as a group.

We prove (ii). Let $1 \leq i < j \leq 7$ such that $\lambda = \text{cone}(w_i, w_j)$ holds. By (i) we may assume that $w_i = (1, 0)$ and $w_j = (0, 1)$ holds. We write $w_1 = (a_1, -b_1)$ for some $a_1, b_1 \in \mathbb{Z}_{\geq 0}$. Applying Lemma 2.5 to the generator degrees w_1 and w_j shows that $a_1 = 1$ holds. So in order to verify item (ii) it suffices to show that $b_1 = 0$ holds. For this we first show that $\text{Eff}(R)$ contains a lattice point $v \in \mathbb{Z}^2$ of the form $v = (-a, 1)$ for some $a \in \mathbb{Z}_{\geq 1}$. Note that w_5, w_6, w_7 do not lie in λ . Write $w_k = (-a_k, b_k)$ for $k = 5, 6, 7$. If $b_k = 1$ holds for one of those, then we have found such a point v . Otherwise we have $b_5, b_6, b_7 > 1$. Thus $\det(w_i, w_k) = b_k > 1$ holds and by Lemma 2.5, the relation g contains monomials of the form $T_i^{k_i} T_5^{k_5}$, $T_i^{l_i} T_6^{l_6}$ and $T_i^{m_i} T_7^{m_7}$. Since only at most one monomial of g is divisible by T_i , we conclude that two of the exponents k_i, l_i and m_i must be zero. Homogeneity of g thus implies that two of the generator degrees w_5, w_6, w_7 lie on the ray through μ . Let $v = (v_1, v_2) \in \mathbb{Z}^2$ denote the primitive lattice vector on this ray. We apply Lemma 2.6 to w_i and the two generator degrees on the ray through μ to infer $v_2 = \det(w_i, v) = 1$. Thus v is a lattice point of the desired form. From $w_1 \in \lambda^-$ and $v \in \lambda^+$ we infer

$$0 < \det(w_1, v) = 1 - ab_1.$$

As a is positive, this inequality can only be fulfilled by $b_1 = 0$. Hence $w_1 = (1, 0)$ holds, which proves the assertion. \square

Proposition 4.6. *Let X be as in Setting 4.1 with $s = 3$ and assume that $\mu \in \lambda$ holds. Then X is isomorphic to $X(Q, g)$ with specifying data (Q, g) appearing in Classification list 5.2.*

Proof. We divide the proof into the two cases $\mu \in \text{Mov}(R)^\circ$ and $\mu \in \partial\text{Mov}(R)$.

Case: $\mu \in \text{Mov}(R)^\circ$. We are in the situation of Proposition 3.2. Thus, for a general polynomial $h \in \mathbb{C}[T_1, \dots, T_7]$ of degree $\deg(h) = \mu$, the projective variety X_h is smooth with divisor class group $\text{Cl}(X_h) = K$ and Cox ring $\mathcal{R}(X_h) = R_h$. Moreover, by Proposition 2.3 X_h is Fano. Thus, X_h is a smooth Fano fourfold of Picard number two with a spread hypersurface Cox ring. In particular, up to unimodular equivalence, the grading matrix $Q = (w_1, \dots, w_7)$ together with the relation degree $\mu = \deg(g)$ appear in the classification list in [18, Thm. 1.1].

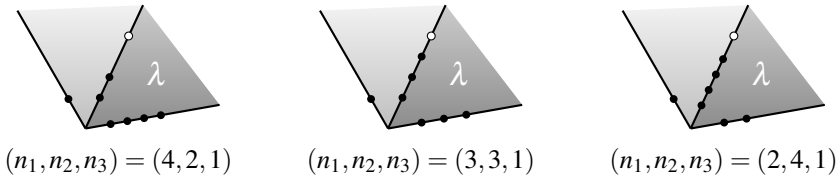
For each such entry (Q, μ) with $s = 3$ we determine all trinomials g of degree $\deg(g) = \mu$ that satisfy the conditions (C1) and (C2) from Setting 4.1 and filter the resulting list by isomorphism. This yields the specifying data no. 19 to 80 in Classification list 5.2.

Case: $\mu \in \partial\text{Mov}(R)$. The relation degree μ is contained in one of the rays ρ_1, ρ_2, ρ_3 . Reversing the order of the variables if necessary, we may assume that $\lambda = \rho_1 + \rho_2$ holds. There are the two cases $\mu \in \rho_1$ and $\mu \in \rho_2$. Here we exemplarily treat the case $\mu \in \rho_2$.

Case: $\mu \in \partial\text{Mov}(R)$, $\mu \in \rho_2$. The ray ρ_2 is a bounding ray of $\text{Mov}(R)$. Thus in this configuration the cones λ and $\text{Mov}(R)$ coincide. With the formula for $\text{Mov}(R)$ given in Construction 2.1 we obtain $n_3 = 1$ and $n_1 \geq 2$. Moreover, Proposition 2.10 yields $n_2 \geq 2$. Applying Lemma 2.6 to the generator degrees w_1, w_2, w_7 shows that the cone $\text{Eff}(R)$ is regular and that w_7 is primitive. We may thus assume that $\text{Eff}(R)$ is the positive quadrant and that $w_7 = (0, 1)$ holds. Applying Lemma 2.6 to the triples (w_1, w_5, w_6) and (w_2, w_5, w_6) shows that the primitive generator v of ρ_2 is of the form $v = (c, 1)$ for some $c \geq 1$. We obtain

$$w_1 = w_2 = (1, 0), \quad w_5 = (a_5c, a_5), \quad w_6 = (a_6c, a_6),$$

where $a_5, a_6, c \in \mathbb{Z}_{\geq 1}$ and a_5, a_6 are coprime. We may assume that $a_6 \geq a_5$ holds. There are three possible degree constellations (n_1, n_2, n_3) for X , displayed in the following pictures.



As an example we consider the degree constellation $(n_1, n_2, n_3) = (3, 3, 1)$.

Case: $\mu \in \partial\text{Mov}(R)$, $\mu \in \rho_2$, $(n_1, n_2, n_3) = (3, 3, 1)$. We apply Lemma 2.6 to the triple (w_3, w_5, w_6) to obtain $w_3 = (1, 0)$. By Lemma 4.4 at least one of w_4, w_5, w_6 is primitive. We may assume that w_4 is primitive. Grading matrix and relation degree are given by

$$Q = \begin{bmatrix} 1 & 1 & 1 & c & a_5c & a_6c & 0 \\ 0 & 0 & 0 & 1 & a_5 & a_6 & 1 \end{bmatrix}, \quad \mu = (kc, k),$$

for some $k \geq 2$. If $a_5 > 1$ holds, then by Lemma 2.6, applied to the tuple (w_1, w_5) , the relation g has a monomial of the form $T_5^{l_5}$ with $l_5 \geq 2$. By ho-

mogeneity of g we have $k = l_5 a_5$. On the other hand, if $a_5 = 1$ holds, then clearly k is a multiple of a_5 . The same holds for a_6 . Thus we have

$$k = l_5 a_5 = l_6 a_6$$

with $l_5 \geq l_6 \geq 2$. By Proposition 2.3 the anticanonical class $-\mathcal{K}$ of X is given by

$$-\mathcal{K} = \begin{bmatrix} 3 + (1 + a_5 + a_6 - k)c \\ 2 + a_5 + a_6 - k \end{bmatrix}.$$

From X being Fano, ie. $-\mathcal{K} \in \lambda$, we infer the inequalities

$$k \leq 1 + a_5 + a_6, \tag{4.6.1}$$

$$c \leq 2. \tag{4.6.2}$$

We distinguish three cases, depending on the values of a_5 and a_6 .

Case: $\mu \in \partial \text{Mov}(R)$, $\mu \in \rho_2$, $(n_1, n_2, n_3) = (3, 3, 1)$, $a_5 = a_6 = 1$. Equation 4.6.1 yields the bound $k \leq 3$. Grading matrix and relation degree are given by

$$Q = \begin{bmatrix} 1 & 1 & 1 & c & c & c & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}, \quad \mu = (kc, k),$$

with $2 \leq k \leq 3$ and $1 \leq c \leq 2$. For the possible values of k and c we check each homogeneous trinomial g of degree $\text{deg}(g) = \mu$ for the conditions (C1) and (C2) from Setting 4.1 and filter by isomorphism. Depending on the values of k and c this leads to the following specifying data from Classification list 5.2:

(k, c)	(2, 1)	(3, 1)	(2, 2)	(3, 2)
ID	149	150-151	152-153	154-158

Case: $\mu \in \partial \text{Mov}(R)$, $\mu \in \rho_2$, $(n_1, n_2, n_3) = (3, 3, 1)$, $a_5 = 1$, $a_6 > 1$. Equation 4.6.1 yields $a_6 = 2$ and $l_6 = 2$. Grading matrix and relation degree are given by

$$Q = \begin{bmatrix} 1 & 1 & 1 & c & c & 2c & 0 \\ 0 & 0 & 0 & 1 & 1 & 2 & 1 \end{bmatrix}, \quad \mu = (4c, 4).$$

For the two values of c we check each homogeneous trinomial g of degree $\text{deg}(g) = \mu$ for the conditions (C1) and (C2) from Setting 4.1 and filter by isomorphism. For $c = 1$ this leads to specifying data no. 159 and 160. For $c = 2$ we get the specifying data no. 161 to no. 166.

Case: $\mu \in \partial\text{Mov}(R)$, $\mu \in \rho_2$, $(n_1, n_2, n_3) = (3, 3, 1)$, $a_5, a_6 > 1$. We have $a_5 = l_6$, $a_6 = l_5$ and $k = a_5 a_6$. As $a_5 < a_6$ are coprime, Equation 4.6.1 yields $a_5 = 2$ and $a_6 = 3$ and we have $k = 6$. Grading matrix and relation degree are thus given by

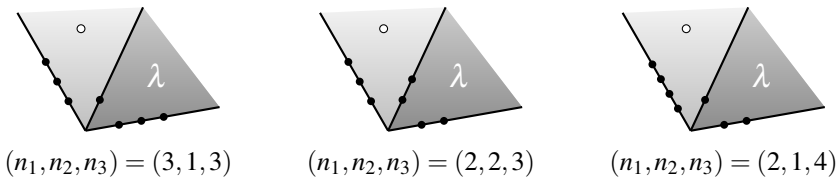
$$Q = \begin{bmatrix} 1 & 1 & 1 & c & 2c & 3c & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 & 1 \end{bmatrix}, \quad \mu = (6c, 6).$$

For the two values of c we check each homogeneous trinomial g of degree $\deg(g) = \mu$ for the conditions (C1) and (C2) from Setting 4.1 and filter by isomorphism. For $c = 1$ this leads to specifying data no. 167 to 180. For $c = 2$ we get the specifying data no. 181 to no. 219. □

Proposition 4.7. *Let X be as in Setting 4.1 with $s = 3$ and assume that $\mu \notin \lambda$ holds. Then X is isomorphic to $X(Q, g)$ with specifying data (Q, g) appearing in Classification list 5.3 or in Classification list 5.4.*

Proof. Reversing the order of the variables if necessary, we may assume that $\lambda = \rho_1 + \rho_2$ holds. By assumption μ is not contained in λ , so we have $\mu \in (\rho_2 + \rho_3) \setminus \rho_2$. By Lemma 4.5 we have $n_1 \geq 2$ and $w_1 = w_2$ is the primitive point in ρ_1 . Moreover, by Proposition 2.10 we have $n_3 \geq 3$. Applying Lemma 2.6 to the triple (w_1, w_6, w_7) shows that the cone $\text{Eff}(R)$ is regular. We may thus assume that $\text{Eff}(R)$ is the positive quadrant in \mathbb{Q}^2 and that $w_1 = w_2 = (1, 0)$ holds. We distinguish the two cases $\mu \in \rho_3$ and $\mu \in (\rho_2 + \rho_3)^\circ$. We exemplarily treat the case $\mu \in (\rho_2 + \rho_3)^\circ$.

Case: $\mu \in (\rho_2 + \rho_3)^\circ$. By Proposition 2.10 we have $n_3 \geq 3$. Applying Lemma 2.6 to the triples (w_1, w_2, w_i) with $w_i \in \rho_2 \cup \rho_3$ and the triples (w_j, w_6, w_7) , where $w_j \in \rho_1$, shows that all generator degrees w_1, \dots, w_7 are primitive. In particular we have $w_5 = w_6 = w_7 = (0, 1)$. The primitive point $v \in \rho_2$ is of the form $v = (a, 1)$ for some $a \geq 1$. There are three possible degree constellations (n_1, n_2, n_3) for X , displayed in the following pictures.



We consider the degree constellation $(n_1, n_2, n_3) = (3, 1, 3)$ as an example.

Case: $\mu \in (\rho_2 + \rho_3)^\circ$, $(n_1, n_2, n_3) = (3, 1, 3)$. Grading matrix and anticanonical

class of X are given by

$$Q = \begin{bmatrix} 1 & 1 & 1 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}, \quad -\mathcal{K} = \begin{bmatrix} 3+a-\mu_1 \\ 4-\mu_2 \end{bmatrix}.$$

From X being Fano, ie. $-\mathcal{K} \in \lambda^\circ$, we infer the inequalities

$$\mu_2 \leq 3, \tag{4.7.1}$$

$$\mu_1 \leq 2 - (3 - \mu_2)a. \tag{4.7.2}$$

In particular, we have $2 \leq \mu_2 \leq 3$. We first consider the case $\mu_2 = 2$. As μ_1 is positive, Equation 4.7.2 yields $a = 1$ and $\mu_1 = 1$. Grading matrix and relation degree are thus given by

$$Q = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}, \quad \mu = (1,2).$$

Up to isomorphism this leads to specifying data no. 227 and 228. Now consider the case $\mu_2 = 3$. Then Equation 4.7.2 yields $1 \leq \mu_1 \leq 2$. We first consider the case $\mu = (1,3)$. The relation g is a trinomial with pairwise coprime monomials. Due to the position of λ , each monomial of g is divisible by one of T_1, \dots, T_4 . If T_4 divides a monomial of g , then by homogeneity we have $a = 1$. Up to isomorphism this yields specifying data no. 229. If T_4 does not appear in g , then each monomial of the relation is divisible by precisely one of T_1, \dots, T_3 . Moreover, by the same argument, each monomial is divisible by precisely one of T_5, \dots, T_7 . Thus up to permutation of variables, the relation g is of the form

$$g = T_1T_5^3 + T_2T_6^3 + T_3T_7^3.$$

Any choice for $a \geq 1$ yields valid specifying data. This is series S1. Finally we consider the case $\mu = (2,3)$. Again, if T_4 divides a monomial of g , then homogeneity yields $a \leq 2$. For the two possible values of a we check each homogeneous trinomial g of degree $\deg(g) = \mu$ for the conditions (C1) and (C2) from Setting 4.1 and filter by isomorphism. For $a = 1$ we get specifying data no. 230 to 232. For $a = 2$ we get specifying data no. 233 and 234. If T_4 does not appear in g , then up to permutation of variables, the relation g is of the form

$$g = T_1^2T_5^3 + T_2^2T_6^3 + T_3^2T_7^3$$

Any choice for $a \geq 1$ yields valid specifying data. This is series S2. □

5. Classification lists

Here we provide a detailed presentation of our classification results. Let us briefly recall the background. Each locally factorial Fano fourfold X of Picard number two and complexity one with a hypersurface Cox ring can be encoded by the degree matrix Q , that means the list $[w_1, \dots, w_7]$ of degrees of the Cox ring generators in $\text{Cl}(X) = \mathbb{Z}^2$, and the defining trinomial g . Each such variety X is isomorphic to precisely one $X(Q, g)$ with specifying data $Q = [w_1, \dots, w_7]$ and g appearing in the Classification lists 5.1 to 5.11. Here $X(Q, g) = X_g$ is the variety from Construction 2.2 associated with the \mathbb{Z}^2 -graded \mathbb{C} -algebra R_g , where the grading is given by $\text{deg}(T_i) = w_i$.

To make the classification easier to navigate, we split it into several lists, each one containing the specifying data for a given number s of rays generated by the degrees $[w_1, \dots, w_7]$, with either $\mu \in \lambda$ or $\mu \notin \lambda$. Moreover, in the case $\mu \notin \lambda$ we separate the sporadic cases from the infinite series of specifying data. Apart from the specifying data (Q, g) , the classification lists also contain the relation degree $\mu = \text{deg}(g)$, the anticanonical class $-\mathcal{K} \in \mathbb{Z}^2$ and, for the sporadic cases, also the anticanonical degree \mathcal{K}^4 . A data file containing the complete classification data is also available at [6].

Classification list 5.1. *Locally factorial Fano fourfolds of Picard number two with a hypersurface Cox ring and an effective three-torus action: Specifying data for the case with $s = 2$.*

ID	$[w_1, \dots, w_7]$	μ	$-\mathcal{K}$	\mathcal{K}^4	g
1	$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$	(1,1)	(3,2)	432	$T_1T_6 + T_2T_5 + T_4T_7$
2	$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$	(2,1)	(2,2)	256	$T_1^2T_6 + T_2T_3T_7 + T_4^2T_5$
3					$T_1^2T_6 + T_3^2T_7 + T_4^2T_5$
4	$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$	(3,1)	(1,2)	80	$T_1^3T_7 + T_3^3T_5 + T_4^3T_6$
5					$T_1^3T_5 + T_2^2T_4T_6 + T_3^3T_7$
6	$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$	(1,2)	(3,1)	270	$T_1T_6^2 + T_2T_7^2 + T_3T_5^2$
7	$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$	(3,2)	(1,1)	26	$T_1^3T_7^2 + T_2^3T_6^2 + T_3^3T_4T_5^2$
8					$T_1^3T_5^2 + T_3^3T_6^2 + T_4^3T_7^2$
9	$\begin{bmatrix} 1 & 1 & 1 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$	(6,0)	(1,2)	16	$T_1^3T_3^3 + T_2^5T_4 + T_5^2$
10					$T_1T_4^5 + T_2^5T_3 + T_5^2$
11	$\begin{bmatrix} 1 & 1 & 1 & 2 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$	(6,0)	(2,2)	64	$T_1^4T_2T_3 + T_4^3 + T_5^2$
12					$T_1^5T_3 + T_4^3 + T_5^2$
13					$T_1^3T_2^2T_3 + T_4^3 + T_5^2$

ID	$[w_1, \dots, w_7]$	μ	$-\mathcal{K}$	\mathcal{K}^4	g
14	$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$	(4, 0)	(2, 2)	128	$T_1 T_3^3 + T_2^3 T_4 + T_5^2$
15	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$	(2, 0)	(3, 2)	432	$T_1 T_3 + T_2 T_5 + T_4^2$
16	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$	(3, 0)	(2, 2)	192	$T_1^2 T_4 + T_2^2 T_5 + T_3^3$
17	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$	(4, 0)	(1, 2)	32	$T_1^3 T_2 + T_3^3 T_4 + T_5^4$
18	$\begin{bmatrix} 1 & 1 & 2 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$	(6, 0)	(1, 3)	54	$T_1^5 T_2 + T_3^3 + T_4^2$

Classification list 5.2. *Locally factorial Fano fourfolds of Picard number two with a hypersurface Cox ring and an effective three-torus action: Specifying data for the cases with $s = 3$ and $\mu \in \lambda$.*

ID	$[w_1, \dots, w_7]$	μ	$-\mathcal{K}$	\mathcal{K}^4	g
19 20	$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$	(1, 1)	(2, 2)	416	$T_2^2 T_7 + T_1 T_5 + T_3 T_6$ $T_3 T_4 T_7 + T_1 T_6 + T_2 T_5$
21 22	$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$	(1, 2)	(2, 1)	163	$T_2 T_3^2 T_7^2 + T_1 T_5^2 + T_4 T_6^2$ $T_3^3 T_7^2 + T_2 T_5^2 + T_4 T_6^2$
26 27	$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$	(1, 1)	(1, 2)	464	$T_4^3 T_7 + T_2 T_5 + T_3 T_6$ $T_3^2 T_4 T_7 + T_1 T_6 + T_2 T_5$
31 32	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$	(2, 1)	(3, 2)	352	$T_1 T_4 T_6 + T_3^2 T_7 + T_2 T_5$ $T_1^2 T_7 + T_4^2 T_6 + T_3 T_5$
33 34	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$	(3, 2)	(2, 1)	81	$T_1^3 T_7^2 + T_2^2 T_4 T_6^2 + T_3 T_5^2$ $T_3^3 T_6^2 + T_4^3 T_7^2 + T_1 T_5^2$
44 45	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$	(1, 1)	(3, 1)	432	$T_1 T_5^2 + T_3 T_6 + T_4 T_7$ $T_2 T_7 + T_3 T_5 + T_4 T_6$
46 47	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$	(2, 2)	(2, 3)	272	$T_1^2 T_5 T_7 + T_2 T_4 T_6 + T_3^2$ $T_1^2 T_6 T_7 + T_2^2 T_5^2 + T_3 T_4$
51 52	$\begin{bmatrix} 1 & 1 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 1 & 1 \end{bmatrix}$	(4, 4)	(1, 2)	34	$T_2^4 T_6 T_7^3 + T_1 T_3^3 T_5 + T_4^2$ $T_2^4 T_5^3 T_6 + T_1^3 T_3 T_7^3 + T_4^2$
72	$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$	(2, 2)	(1, 3)	216	$T_1^2 T_4 T_7 + T_2^2 T_5 T_6 + T_3^2$
73	$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 1 & 1 & 1 \end{bmatrix}$	(2, 4)	(1, 2)	64	$T_1^2 T_4 T_6^3 + T_2^2 T_5 T_7^3 + T_3^2$

ID	$[w_1, \dots, w_7]$	μ	$-\mathcal{K}$	\mathcal{K}^4	g
77	$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$	(2, 2)	(2, 2)	192	$T_1 T_3 T_7^2 + T_2^2 T_5 T_6 + T_4^2$
78	$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$	(3, 3)	(1, 1)	18	$T_1^3 T_5 T_6^2 + T_2 T_3^2 T_7^3 + T_4^3$
79	$\begin{bmatrix} 1 & 1 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$	(4, 2)	(1, 2)	48	$T_1^3 T_2 T_5^2 + T_3^4 T_6 T_7 + T_4^2$
80	$\begin{bmatrix} 1 & 1 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 1 & 1 \end{bmatrix}$	(4, 4)	(1, 1)	12	$T_1 T_2^3 T_7^4 + T_3^4 T_5^3 T_6 + T_4^2$
149	$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$	(2, 2)	(2, 4)	352	$T_1^2 T_5 T_6 + T_2^2 + T_3 T_4$
150	$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$	(3, 3)	(1, 3)	99	$T_1^3 T_5 T_6^2 + T_2^3 + T_3 T_4^2$
151					$T_1^3 T_5 T_6 T_7 + T_2 T_4^2 + T_3^3$
152	$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & 2 & 2 & 1 & 1 & 1 \end{bmatrix}$	(2, 4)	(2, 5)	304	$T_1^2 T_5 T_6^3 + T_2 T_4 + T_3^2$
153					$T_1^2 T_5 T_6^2 T_7 + T_2 T_3 + T_4^2$
159	$\begin{bmatrix} 1 & 1 & 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 & 1 & 1 & 1 \end{bmatrix}$	(4, 4)	(1, 3)	66	$T_1^4 T_5 T_7^3 + T_2 T_3^3 + T_4^2$
160					$T_1^4 T_5 T_6 T_7^2 + T_2^3 T_3 + T_4^2$
223	$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$	(4, 0)	(3, 2)	160	$T_1 T_3^3 + T_2^3 T_4 + T_5^2$
224	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$	(3, 0)	(3, 2)	240	$T_1^2 T_4 + T_2^2 T_5 + T_3^3$
225	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$	(2, 0)	(4, 2)	480	$T_1 T_3 + T_2 T_5 + T_4^2$
226	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$	(2, 0)	(5, 2)	624	$T_1 T_3 + T_2 T_5 + T_4^2$

$Q = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$	$\mu = (2, 1)$	$-\mathcal{K} = (1, 2)$	$\mathcal{K}^4 = 224$		
ID	g	ID	g	ID	g
23	$T_1^2 T_3 T_7 + T_2^2 T_6 + T_4^2 T_5$	24	$T_3^3 T_7 + T_2^2 T_5 + T_4^2 T_6$	25	$T_4^3 T_7 + T_1 T_3 T_5 + T_2^2 T_6$

$Q = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$	$\mu = (1, 2)$	$-\mathcal{K} = (1, 1)$	$\mathcal{K}^4 = 98$		
ID	g	ID	g	ID	g
28	$T_1^4 T_3 T_7^2 + T_2 T_5^2 + T_4 T_6^2$	29	$T_1^3 T_2^2 T_7^2 + T_3 T_5^2 + T_4 T_6^2$	30	$T_4^5 T_7^2 + T_2 T_6^2 + T_3 T_5^2$

$Q = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$	$\mu = (3, 1)$	$-\mathcal{K} = (1, 1)$	$\mathcal{K}^4 = 38$		
ID	g	ID	g	ID	g
35	$T_1^3 T_7^4 + T_2^3 T_6 + T_3 T_4^2 T_5$	36	$T_2^3 T_5 + T_3^3 T_7 + T_4^3 T_6$	37	$T_1 T_3^2 T_5^2 + T_2^3 T_6 + T_4^3 T_7$
38	$T_1^3 T_7^4 + T_2^3 T_5 + T_4^3 T_6$	39	$T_1^2 T_3 T_6^3 + T_2^3 T_7 + T_4^3 T_5$		

$Q = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$			$\mu = (2, 1)$	$-\mathcal{K} = (2, 1)$	$\mathcal{K}^4 = 192$
ID	g	ID	g	ID	g
40	$T_1 T_4 T_5^2 + T_2^2 T_7 + T_3^2 T_6$	41	$T_1^2 T_7^3 + T_2 T_4 T_6 + T_3^2 T_5$	42	$T_2^2 T_7 + T_3^2 T_5 + T_4^2 T_6$
43	$T_1^2 T_6^3 + T_3^2 T_7 + T_4^2 T_5$				
$Q = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$			$\mu = (3, 3)$	$-\mathcal{K} = (1, 2)$	$\mathcal{K}^4 = 51$
ID	g	ID	g	ID	g
48	$T_2^3 T_5 T_6^2 + T_1 T_4^2 T_7 + T_3^3$	49	$T_2^3 T_5^2 T_7 + T_1^2 T_4 T_6^2 + T_3^3$	50	$T_1^3 T_6 T_7^2 + T_2^3 T_5^3 + T_3 T_4 T_6^2$
$Q = \begin{bmatrix} 1 & 1 & 2 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 3 & 1 & 1 & 1 \end{bmatrix}$			$\mu = (6, 6)$	$-\mathcal{K} = (1, 2)$	$\mathcal{K}^4 = 17$
ID	g	ID	g	ID	g
53	$T_1 T_2^5 T_3 T_6 T_7^4 + T_3^3 + T_4^2$	54	$T_1 T_2^5 T_3^3 T_6 T_7^2 + T_3^3 + T_4^2$	55	$T_1 T_2^5 T_3 T_5 T_7^5 + T_3^3 + T_4^2$
56	$T_1^2 T_2^4 T_3 T_6^5 + T_3^3 + T_4^2$	57	$T_1^6 T_3^5 T_6 + T_3^3 + T_4^2$	58	$T_1^2 T_2^4 T_3^3 T_5^3 + T_3^3 + T_4^2$
59	$T_1^6 T_2^5 T_6 T_7^3 + T_3^3 + T_4^2$	60	$T_1^6 T_2^4 T_6 T_7 + T_3^3 + T_4^2$	61	$T_1^3 T_2^3 T_2^2 T_6 T_7^3 + T_3^3 + T_4^2$
62	$T_1^4 T_2^2 T_3^3 T_6 T_7^2 + T_3^3 + T_4^2$	63	$T_1^5 T_2 T_3^4 T_6^2 + T_3^3 + T_4^2$	64	$T_1^5 T_2 T_3^2 T_6^2 T_7^2 + T_3^3 + T_4^2$
65	$T_1^3 T_2^3 T_3^2 T_6 + T_3^3 + T_4^2$	66	$T_1^3 T_2^3 T_3^2 T_7^4 + T_3^3 + T_4^2$	67	$T_1 T_2^5 T_7^6 + T_3^3 + T_4^2$
68	$T_1^2 T_2^4 T_3^4 T_6 T_7 + T_3^3 + T_4^2$	69	$T_1 T_2^5 T_3^3 T_6^3 + T_3^3 + T_4^2$	70	$T_1^3 T_2^3 T_3 T_6 T_7^4 + T_3^3 + T_4^2$
71	$T_1^3 T_2^3 T_3^2 T_6^2 T_7^2 + T_3^3 + T_4^2$				
$Q = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 1 & 1 & 1 \end{bmatrix}$			$\mu = (2, 6)$	$-\mathcal{K} = (1, 1)$	$\mathcal{K}^4 = 8$
ID	g	ID	g	ID	g
74	$T_1^2 T_3^5 T_7 + T_2^2 T_4 T_6^5 + T_5^2$	75	$T_1^2 T_3^5 T_6 + T_2^2 T_4^3 T_7^3 + T_5^2$	76	$T_1^2 T_6^3 T_7^3 + T_2^2 T_4^3 T_5^3 + T_5^2$
$Q = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$			$\mu = (2, 2)$	$-\mathcal{K} = (3, 4)$	$\mathcal{K}^4 = 378$
ID	g	ID	g	ID	g
81	$T_1 T_2 T_7 + T_3^2 + T_4 T_5$	82	$T_1^2 T_6^2 + T_2 T_3 + T_4 T_5$	83	$T_1^2 T_6 T_7 + T_3 T_5 + T_4^2$
84	$T_1^2 T_6 T_7 + T_2 T_4 + T_3 T_5$				
$Q = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$			$\mu = (3, 3)$	$-\mathcal{K} = (2, 3)$	$\mathcal{K}^4 = 144$
ID	g	ID	g	ID	g
85	$T_1^2 T_3 T_7^2 + T_2^2 T_4 + T_3^3$	86	$T_1^3 T_6^2 T_7 + T_2 T_5^2 + T_3 T_4^2$	87	$T_1^3 T_6 T_7^2 + T_3^3 + T_4 T_5^2$
88	$T_1 T_5^2 T_7 + T_2^2 + T_3 T_4^2$	89	$T_1^3 T_6^3 + T_2^2 T_4 + T_3^2 T_5$	90	$T_1^2 T_4 T_6 T_7 + T_2^2 T_3 + T_5^3$
91	$T_1^3 T_6^2 T_7 + T_2 T_3 T_5 + T_4^3$				
$Q = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$			$\mu = (4, 4)$	$-\mathcal{K} = (1, 2)$	$\mathcal{K}^4 = 20$
ID	g	ID	g	ID	g
92	$T_1^3 T_4 T_6^3 + T_2 T_5^3 + T_3^4$	93	$T_1^3 T_2 T_6^2 T_7 + T_3^4 + T_4 T_5^3$	94	$T_1^4 T_6 T_7^3 + T_2 T_5^3 + T_3^3 T_4$

$Q = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$				$\mu = (4,4)$	$-\mathcal{K} = (1,2)$	$\mathcal{K}^4 = 20$
ID	g	ID	g	ID	g	
95	$T_1^4 T_6^3 T_7 + T_2 T_3^3 + T_4^2 T_5^2$	96	$T_1^2 T_2^2 T_6 T_7 + T_3^3 T_5 + T_4^4$	97	$T_1 T_3^3 T_7 + T_2^3 T_4 + T_3^4$	
98	$T_1^4 T_6 T_7^3 + T_2 T_3^3 + T_4^4$	99	$T_1^4 T_6^2 T_7^2 + T_2^2 T_4 + T_3 T_5^3$	100	$T_1^4 T_6^3 T_7 + T_2 T_3^3 T_4 + T_5^4$	
101	$T_1^4 T_6^4 + T_2 T_3^3 + T_4 T_5^3$					
$Q = \begin{bmatrix} 1 & 1 & 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 1 & 2 & 1 & 1 \end{bmatrix}$				$\mu = (4,4)$	$-\mathcal{K} = (2,3)$	$\mathcal{K}^4 = 96$
ID	g	ID	g	ID	g	
102	$T_1^3 T_4 T_6 T_7^2 + T_2^2 T_3 + T_5^2$	103	$T_1 T_2^3 T_7 + T_3^3 T_4 + T_5^2$	104	$T_1^4 T_2^3 T_7 + T_2 T_3^2 T_4 + T_5^2$	
105	$T_1^4 T_6 T_7^3 + T_3^3 T_4 + T_5^2$	106	$T_1^3 T_4 T_6^3 + T_2^2 T_3 + T_5^2$	107	$T_1^2 T_3^2 T_6 T_7 + T_2 T_4^3 + T_5^2$	
$Q = \begin{bmatrix} 1 & 1 & 1 & 1 & 3 & 0 & 0 \\ 0 & 1 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}$				$\mu = (6,6)$	$-\mathcal{K} = (1,2)$	$\mathcal{K}^4 = 10$
ID	g	ID	g	ID	g	
108	$T_1^5 T_2 T_7^3 + T_3 T_4^5 + T_5^2$	109	$T_1 T_4^5 T_7 + T_2^5 T_3 + T_5^2$	110	$T_1^5 T_3 T_6^3 T_7^2 + T_2 T_4^5 + T_5^2$	
111	$T_1^6 T_5^2 T_7 + T_2^2 T_3^3 T_4 + T_5^2$	112	$T_1^5 T_3 T_6^3 + T_2^2 T_4^3 + T_5^2$	113	$T_1^4 T_2^2 T_6 T_7^3 + T_3^3 T_4^3 + T_5^2$	
114	$T_1^3 T_3^3 T_6^2 T_7 + T_2^2 T_4^3 + T_5^2$	115	$T_1^6 T_6^3 T_7^3 + T_2 T_3 T_4^4 + T_5^2$	116	$T_1^5 T_4 T_6^4 T_7 + T_2^2 T_3^3 + T_5^2$	
117	$T_1^4 T_2^2 T_6^3 T_7 + T_2 T_4^4 + T_5^2$	118	$T_1^6 T_5^2 T_7 + T_3^5 T_4 + T_5^2$	119	$T_1^6 T_6^3 T_7^3 + T_2 T_4^5 + T_5^2$	
120	$T_1^2 T_2^4 T_6 T_7 + T_3^3 T_4 + T_5^2$	121	$T_1^3 T_2^3 T_3^3 + T_3 T_4^5 + T_5^2$	122	$T_1^3 T_2^3 T_6 T_7^2 + T_2 T_3^5 + T_5^2$	
123	$T_1^6 T_5^2 T_7 + T_3^3 T_4^3 + T_5^2$	124	$T_1^5 T_3 T_6 T_7^4 + T_2 T_4^5 + T_5^2$	125	$T_1^6 T_6^3 T_7^3 + T_2^2 T_3^3 T_4 + T_5^2$	
126	$T_1^6 T_6 T_7^3 + T_2^4 T_3 T_4 + T_5^2$	127	$T_1^5 T_4 T_6^2 T_7^3 + T_2^2 T_3^3 + T_5^2$	128	$T_1^2 T_4^4 T_6 T_7 + T_2^2 T_3^3 + T_5^2$	
129	$T_1 T_4^5 T_7 + T_2^2 T_3^3 + T_5^2$					
$Q = \begin{bmatrix} 1 & 1 & 1 & 2 & 3 & 0 & 0 \\ 0 & 1 & 1 & 2 & 3 & 1 & 1 \end{bmatrix}$				$\mu = (6,6)$	$-\mathcal{K} = (2,3)$	$\mathcal{K}^4 = 48$
ID	g	ID	g	ID	g	
130	$T_1^4 T_3^2 T_6 T_7^3 + T_4^3 + T_5^2$	131	$T_1^4 T_2 T_3 T_6 T_7^3 + T_4^3 + T_5^2$	132	$T_1^3 T_2^2 T_3 T_6^3 + T_4^3 + T_5^2$	
133	$T_1^2 T_2 T_3^3 T_6 T_7 + T_4^3 + T_5^2$	134	$T_1 T_2^4 T_3 T_6 + T_4^3 + T_5^2$	135	$T_1^4 T_2 T_3 T_6^4 + T_4^3 + T_5^2$	
136	$T_1 T_2^3 T_3^3 T_7 + T_4^3 + T_5^2$	137	$T_1^2 T_2^4 T_6 T_7 + T_4^3 + T_5^2$	138	$T_2^2 T_3 + T_4^3 + T_5^2$	
139	$T_1^2 T_2 T_3^3 T_6^2 + T_4^3 + T_5^2$	140	$T_1^5 T_3 T_6 T_7^4 + T_4^3 + T_5^2$	141	$T_1^3 T_2^2 T_6 T_7^2 + T_4^3 + T_5^2$	
142	$T_1^5 T_3 T_6^3 + T_4^3 + T_5^2$	143	$T_1^6 T_5^2 T_7 + T_4^3 + T_5^2$	144	$T_1^4 T_2 T_3 T_6^2 T_7^2 + T_4^3 + T_5^2$	
145	$T_1^5 T_2 T_6^2 T_7^3 + T_4^3 + T_5^2$	146	$T_1 T_3^5 T_6 + T_4^3 + T_5^2$	147	$T_1^3 T_2 T_3^2 T_6^2 T_7 + T_4^3 + T_5^2$	
148	$T_1^2 T_2^2 T_3^2 T_6 T_7 + T_4^3 + T_5^2$					
$Q = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & 2 & 2 & 1 & 1 & 1 \end{bmatrix}$				$\mu = (3,6)$	$-\mathcal{K} = (1,3)$	$\mathcal{K}^4 = 54$
ID	g	ID	g	ID	g	
154	$T_1^3 T_2^2 T_6 T_7^3 + T_2^2 T_4 + T_3^3$	155	$T_1^3 T_2^2 T_7^4 + T_2^3 + T_3^2 T_4$	156	$T_1^3 T_5^4 T_6 T_7 + T_2^3 + T_3^2 T_4$	
157	$T_1^3 T_6 T_7^3 + T_2^2 T_3 + T_4^3$	158	$T_1^3 T_2^2 T_6^2 T_7^2 + T_2 T_3^2 + T_4^3$			

$Q = \begin{bmatrix} 1 & 1 & 1 & 2 & 0 & 0 & 0 \\ 0 & 2 & 2 & 4 & 1 & 1 & 1 \end{bmatrix}$			$\mu = (4, 8)$	$-\mathcal{K} = (1, 3)$	$\mathcal{K}^4 = 36$
ID	g	ID	g	ID	g
161	$T_1^4 T_5^5 T_6 T_7^2 + T_2 T_3^3 + T_4^2$	162	$T_1^4 T_5 T_6^4 T_7^3 + T_2^3 T_3 + T_4^2$	163	$T_1^4 T_5^3 T_6^5 + T_2 T_3^3 + T_4^2$
164	$T_1^4 T_5^2 T_6^3 T_7^3 + T_2^3 T_3 + T_4^2$	165	$T_1^4 T_5^2 T_7 + T_2 T_3^3 + T_4^2$	166	$T_1^4 T_5^5 T_6 T_7 + T_2 T_3^3 + T_4^2$
$Q = \begin{bmatrix} 1 & 1 & 2 & 3 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 1 & 1 & 1 \end{bmatrix}$			$\mu = (6, 6)$	$-\mathcal{K} = (1, 3)$	$\mathcal{K}^4 = 33$
ID	g	ID	g	ID	g
167	$T_1^6 T_5 T_6^2 T_7^3 + T_3^3 + T_4^2$	168	$T_1^6 T_6^5 T_7 + T_3^3 + T_4^2$	169	$T_1 T_2^5 T_7 + T_3^3 + T_4^2$
170	$T_1^3 T_2^3 T_6^2 T_7 + T_3^3 + T_4^2$	171	$T_1^5 T_2 T_3^3 T_6 T_7 + T_3^3 + T_4^2$	172	$T_1^5 T_2 T_3^3 T_6^3 + T_3^3 + T_4^2$
173	$T_1^2 T_2^3 T_5 T_7 + T_3^3 + T_4^2$	174	$T_1^6 T_5^4 T_6 T_7 + T_3^3 + T_4^2$	175	$T_1^5 T_2 T_5^4 T_6 + T_3^3 + T_4^2$
$Q = \begin{bmatrix} 1 & 1 & 2 & 3 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 1 & 1 & 1 \end{bmatrix}$			$\mu = (6, 6)$	$-\mathcal{K} = (1, 3)$	$\mathcal{K}^4 = 33$
ID	g	ID	g	ID	g
176	$T_1^5 T_2 T_5^5 + T_3^3 + T_4^2$	177	$T_1^4 T_2^2 T_3 T_6^3 + T_3^3 + T_4^2$	178	$T_1^5 T_2 T_5 T_6^2 T_7^2 + T_3^3 + T_4^2$
179	$T_1^4 T_2^2 T_3 T_6 T_7^2 + T_3^3 + T_4^2$	180	$T_1^3 T_2^3 T_3 T_6 T_7 + T_3^3 + T_4^2$		
$Q = \begin{bmatrix} 1 & 1 & 2 & 3 & 0 & 0 & 0 \\ 0 & 2 & 4 & 6 & 1 & 1 & 1 \end{bmatrix}$			$\mu = (6, 12)$	$-\mathcal{K} = (1, 3)$	$\mathcal{K}^4 = 18$
ID	g	ID	g	ID	g
181	$T_1 T_2^5 T_5^3 + T_3^3 + T_4^2$	182	$T_1^6 T_6^5 T_7^3 + T_3^3 + T_4^2$	183	$T_1^6 T_5^5 T_6^2 T_7^5 + T_3^3 + T_4^2$
184	$T_1^5 T_2 T_3^3 T_6^5 T_7^3 + T_3^3 + T_4^2$	185	$T_1^5 T_2 T_3^3 T_7 + T_3^3 + T_4^2$	186	$T_1^5 T_2 T_3^3 T_6^7 + T_3^3 + T_4^2$
187	$T_1^3 T_2^3 T_3^3 T_6 T_7^3 + T_3^3 + T_4^2$	188	$T_1^4 T_2^2 T_3 T_6 T_7^6 + T_3^3 + T_4^2$	189	$T_1^6 T_5^4 T_6^3 T_7^5 + T_3^3 + T_4^2$
190	$T_1^3 T_2^3 T_3^4 T_6 T_7 + T_3^3 + T_4^2$	191	$T_1^4 T_2^2 T_3^5 T_6^3 + T_3^3 + T_4^2$	192	$T_1^5 T_2 T_3^6 T_6^2 T_7^3 + T_3^3 + T_4^2$
193	$T_1^3 T_2^3 T_3^5 T_7 + T_3^3 + T_4^2$	194	$T_1^6 T_3^4 T_7 + T_3^3 + T_4^2$	195	$T_1^6 T_5^3 T_6^3 T_7^3 + T_3^3 + T_4^2$
196	$T_1^6 T_5 T_6^{10} T_7 + T_3^3 + T_4^2$	197	$T_1^5 T_2 T_3^5 T_6 T_7^2 + T_3^3 + T_4^2$	198	$T_1^2 T_2^3 T_3 T_6^3 + T_3^3 + T_4^2$
199	$T_1^6 T_5 T_6^7 T_7^4 + T_3^3 + T_4^2$	200	$T_1^6 T_5^9 T_6 T_7^2 + T_3^3 + T_4^2$	201	$T_1^4 T_2^2 T_3^2 T_6^3 T_7^3 + T_3^3 + T_4^2$
202	$T_1^4 T_2^2 T_3^5 T_6^2 T_7 + T_3^3 + T_4^2$	203	$T_1^6 T_5^5 T_6 T_7^6 + T_3^3 + T_4^2$	204	$T_1^5 T_2 T_3^3 T_6^4 T_7^3 + T_3^3 + T_4^2$
205	$T_1^5 T_2 T_3^8 T_7^2 + T_3^3 + T_4^2$	206	$T_1^3 T_2^3 T_3^2 T_7^4 + T_3^3 + T_4^2$	207	$T_1^5 T_2 T_3^4 T_5^5 T_7 + T_3^3 + T_4^2$
208	$T_1^5 T_2 T_7^{10} + T_3^3 + T_4^2$	209	$T_1^5 T_2 T_3^2 T_4^4 T_7^3 + T_3^3 + T_4^2$	210	$T_1^5 T_2 T_3^2 T_6^6 T_7 + T_3^3 + T_4^2$
211	$T_1 T_2^5 T_6 T_7 + T_3^3 + T_4^2$	212	$T_1^5 T_2 T_3^8 T_6 T_7 + T_3^3 + T_4^2$	213	$T_1^4 T_2^2 T_3 T_6^3 T_7^4 + T_3^3 + T_4^2$
214	$T_1^2 T_2^4 T_3 T_6^2 T_7 + T_3^3 + T_4^2$	215	$T_1^4 T_2^2 T_3 T_6^7 + T_3^3 + T_4^2$	216	$T_1^5 T_2 T_6^5 T_7^5 + T_3^3 + T_4^2$
$Q = \begin{bmatrix} 1 & 1 & 2 & 3 & 0 & 0 & 0 \\ 0 & 2 & 4 & 6 & 1 & 1 & 1 \end{bmatrix}$			$\mu = (6, 12)$	$-\mathcal{K} = (1, 3)$	$\mathcal{K}^4 = 18$
ID	g	ID	g	ID	g
217	$T_1^5 T_2 T_3^6 T_6^4 + T_3^3 + T_4^2$	218	$T_1^6 T_5 T_6^8 T_7^3 + T_3^3 + T_4^2$	219	$T_1^3 T_2^3 T_3^2 T_6^2 T_7^2 + T_3^3 + T_4^2$
$Q = \begin{bmatrix} 1 & 1 & 1 & 2 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$			$\mu = (6, 0)$	$-\mathcal{K} = (3, 2)$	$\mathcal{K}^4 = 80$
ID	g	ID	g	ID	g
220	$T_1^4 T_2 T_3 + T_4^3 + T_5^2$	221	$T_1^5 T_3 + T_4^3 + T_5^2$	222	$T_1^3 T_2^3 T_3 + T_4^3 + T_5^2$

Classification list 5.3. *Locally factorial Fano fourfolds of Picard number two with a hypersurface Cox ring and an effective three-torus action: Specifying data for the sporadic cases with $s = 3$ and $\mu \notin \lambda$.*

ID	$[w_1, \dots, w_7]$	μ	$-\mathcal{K}$	\mathcal{K}^4	g
227 228	$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$	(1, 2)	(3, 2)	352	$T_1T_5^2 + T_2T_6^2 + T_3T_7^2$ $T_1T_5^2 + T_2T_6^2 + T_4T_7$
229	$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$	(1, 3)	(3, 1)	140	$T_1T_5^3 + T_2T_6^3 + T_4T_7^2$
230 231 232	$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$	(2, 3)	(2, 1)	65	$T_1^2T_5^3 + T_2^2T_6^3 + T_3T_4T_7^2$ $T_1^2T_5^3 + T_2^2T_6^3 + T_4^2T_7$ $T_1^2T_5^3 + T_2T_3T_6^3 + T_4^2T_7^2$
233 234	$\begin{bmatrix} 1 & 1 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$	(2, 3)	(3, 1)	122	$T_1^2T_5^3 + T_2^2T_6^3 + T_4T_7^2$ $T_1^2T_5^3 + T_2T_3T_6^3 + T_4T_7^2$
235 236	$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$	(1, 3)	(3, 2)	208	$T_1T_5^3 + T_2T_6^3 + T_4T_7^2$ $T_1T_5^3 + T_3T_6^2 + T_4T_7^2$
237 238 239 240	$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$	(1, 4)	(2, 1)	29	$T_1T_4^4 + T_2T_6^4 + T_3T_7^3$ $T_1T_4^3T_5 + T_2T_6^4 + T_3T_7^3$ $T_1T_4^2T_5^2 + T_2T_6^4 + T_3T_7^3$ $T_1T_4^4 + T_2T_5^4 + T_3T_6T_7^2$
241	$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 2 & 3 \end{bmatrix}$	(0, 6)	(3, 2)	80	$T_4^5T_5 + T_6^3 + T_7^2$

Classification list 5.4. *Locally factorial Fano fourfolds of Picard number two with a hypersurface Cox ring and an effective three-torus action: Specifying data for the infinite series with $s = 3$.*

ID	$[w_1, \dots, w_7]$	μ	$-\mathcal{K}$	g
S1	$\begin{bmatrix} 1 & 1 & 1 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$	(1, 3)	$(a+2, 1)$	$T_1T_5^3 + T_2T_6^3 + T_3T_7^3$ $a \geq 1$
S2	$\begin{bmatrix} 1 & 1 & 1 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$	(2, 3)	$(a+1, 1)$	$T_1^2T_5^3 + T_2^2T_6^3 + T_3^2T_7^3$ $a \geq 1$
S3 S4	$\begin{bmatrix} 1 & 1 & a & a & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$	$(a, 4)$	$(a+2, 1)$	$T_1^{a-l}T_2^lT_5^4 + T_3T_6^3 + T_4T_7^3$ $a \geq 1, 0 \leq l \leq a/2$ $T_1^aT_5^4 + T_2^aT_6^4 + T_4T_7^3$ $a \geq 1, a$ odd

Classification list 5.5. *Locally factorial Fano fourfolds of Picard number two with a hypersurface Cox ring and an effective three-torus action: Specifying data for the cases with $s = 4$ and $\mu \in \lambda$.*

ID	$[w_1, \dots, w_7]$	μ	$-\mathcal{K}$	\mathcal{K}^4	g
242	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$	(3,1)	(2,1)	113	$T_2T_4^2T_7 + T_3^3T_6 + T_1T_5^2$
243					$T_3^3T_6 + T_4^3T_7 + T_1T_5^2$
244	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 2 & 1 & 1 \end{bmatrix}$	(2,2)	(3,5)	433	$T_1T_5 + T_2T_4 + T_3^2$
245	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & 2 & 3 & 1 & 1 \end{bmatrix}$	(3,6)	(2,5)	145	$T_1T_5^2 + T_2^2T_3 + T_4^3$
246	$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 & 1 & 1 & 1 \end{bmatrix}$	(2,4)	(2,3)	144	$T_1^2T_6T_7^3 + T_2T_3T_5^2 + T_4^2$
247					$T_1T_3T_6^3 + T_2^2T_5T_7 + T_4^2$
248	$\begin{bmatrix} 1 & 1 & 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 3 & 1 & 1 & 1 \end{bmatrix}$	(4,6)	(1,2)	22	$T_1^4T_5T_7^5 + T_2T_3^3T_6^2 + T_4^2$
249					$T_1T_3^3T_6^3 + T_2^2T_5T_7 + T_4^2$
250					$T_1^3T_2T_7^5 + T_3^4T_5T_6 + T_4^2$
251					$T_1^4T_6^3T_7^3 + T_2^3T_3T_5^2 + T_4^2$
252	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$	(2,0)	(4,1)	431	$T_1T_6 + T_2T_4 + T_3^2$
253					$T_1T_6 + T_2T_3 + T_4T_5$
254	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$	(4,0)	(2,1)	62	$T_1^2T_6^2 + T_2^3T_4 + T_3^3T_5$
255	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 2 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$	(3,0)	(4,1)	376	$T_2^2T_3 + T_4^3 + T_1T_6$
256					$T_2T_5^2 + T_3^2T_4 + T_1T_6$
257					$T_2T_3T_5 + T_4^3 + T_1T_6$
258	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 3 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$	(4,0)	(4,1)	341	$T_2^3T_4 + T_3^2T_5^2 + T_1T_6$
259					$T_2^3T_3 + T_4T_5^3 + T_1T_6$
260					$T_2^4 + T_3T_4^3 + T_1T_6$
261					$T_2T_3^2T_5 + T_4^4 + T_1T_6$
262	$\begin{bmatrix} 1 & 1 & 1 & 1 & 3 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$	(6,0)	(2,1)	31	$T_1^3T_6^3 + T_2T_3T_4^4 + T_5^2$
263					$T_1^3T_6^3 + T_3^5T_4 + T_5^2$
264					$T_1^3T_6^3 + T_2T_3^3T_4^2 + T_5^2$

Classification list 5.6. *Locally factorial Fano fourfolds of Picard number two with a hypersurface Cox ring and an effective three-torus action: Specifying data for the sporadic cases with $s = 4$ and $\mu \notin \lambda$.*

ID	$[w_1, \dots, w_7]$	μ	$-\mathcal{K}$	\mathcal{K}^4	g
279 280	$\begin{bmatrix} 1 & 1 & 1 & 4 & 2 & 3 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 & 1 \end{bmatrix}$	(6, 6)	(6, 1)	117	$T_1^2 T_4 T_7^5 + T_5^3 + T_6^2$ $T_1 T_2 T_4 T_7^5 + T_5^3 + T_6^2$
281	$\begin{bmatrix} 1 & 1 & 1 & 5 & 2 & 3 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 & 1 \end{bmatrix}$	(6, 6)	(7, 1)	157	$T_1 T_4 T_7^5 + T_5^3 + T_6^2$
282	$\begin{bmatrix} 1 & 1 & 1 & 6 & 2 & 3 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 & 1 \end{bmatrix}$	(6, 6)	(8, 1)	203	$T_4 T_7^5 + T_5^3 + T_6^2$
341 342	$\begin{bmatrix} 1 & 1 & 1 & 10 & 4 & 6 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 & 1 \end{bmatrix}$	(12, 6)	(11, 1)	322	$T_1^2 T_4 T_7^5 + T_5^3 + T_6^2$ $T_1 T_2 T_4 T_7^5 + T_5^3 + T_6^2$
343	$\begin{bmatrix} 1 & 1 & 1 & 11 & 4 & 6 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 & 1 \end{bmatrix}$	(12, 6)	(12, 1)	387	$T_1 T_4 T_7^5 + T_5^3 + T_6^2$
344	$\begin{bmatrix} 1 & 1 & 1 & 12 & 4 & 6 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 & 1 \end{bmatrix}$	(12, 6)	(13, 1)	458	$T_4 T_7^5 + T_5^3 + T_6^2$
345	$\begin{bmatrix} 1 & 1 & 1 & 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 1 \end{bmatrix}$	(4, 4)	(3, 1)	68	$T_1^3 T_2 T_6^4 + T_3^2 T_4 T_7^3 + T_5^2$
346	$\begin{bmatrix} 1 & 1 & 1 & 3 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 1 \end{bmatrix}$	(4, 4)	(4, 1)	114	$T_1^3 T_2 T_6^4 + T_3 T_4 T_7^3 + T_5^2$
347 348	$\begin{bmatrix} 1 & 1 & 1 & 4 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 1 \end{bmatrix}$	(4, 4)	(5, 1)	172	$T_1^3 T_2 T_6^4 + T_4 T_7^3 + T_5^2$ $T_1^2 T_2 T_3 T_6^4 + T_4 T_7^3 + T_5^2$
349	$\begin{bmatrix} 1 & 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$	(3, 3)	(3, 1)	102	$T_1^2 T_2 T_6^3 + T_3 T_4 T_7^2 + T_5^3$
350 351	$\begin{bmatrix} 1 & 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$	(3, 3)	(4, 1)	171	$T_1^2 T_2 T_6^3 + T_4 T_7^2 + T_5^3$ $T_1 T_2 T_3 T_6^3 + T_4 T_7^2 + T_5^3$
352	$\begin{bmatrix} 1 & 1 & 2 & 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$	(5, 4)	(3, 1)	29	$T_1 T_6^4 + T_2 T_4^2 T_7^2 + T_3 T_5^3$
353	$\begin{bmatrix} 1 & 1 & 3 & 3 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$	(6, 4)	(4, 1)	38	$T_1 T_2 T_6^4 + T_3 T_5^3 + T_4^2 T_7^2$
364 365	$\begin{bmatrix} 1 & 1 & 4 & 4 & 2 & 3 & 0 \\ 0 & 0 & 1 & 1 & 2 & 3 & 1 \end{bmatrix}$	(6, 6)	(9, 2)	144	$T_1^2 T_3 T_7^5 + T_5^3 + T_6^2$ $T_1 T_2 T_3 T_7^5 + T_5^3 + T_6^2$
366	$\begin{bmatrix} 1 & 1 & 5 & 5 & 2 & 3 & 0 \\ 0 & 0 & 1 & 1 & 2 & 3 & 1 \end{bmatrix}$	(6, 6)	(11, 2)	176	$T_1 T_3 T_7^5 + T_5^3 + T_6^2$
367	$\begin{bmatrix} 1 & 1 & 6 & 6 & 2 & 3 & 0 \\ 0 & 0 & 1 & 1 & 2 & 3 & 1 \end{bmatrix}$	(6, 6)	(13, 2)	208	$T_3 T_7^5 + T_5^3 + T_6^2$
368	$\begin{bmatrix} 1 & 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$	(4, 4)	(3, 1)	32	$T_1^3 T_2 T_6^4 + T_3 T_4 T_7^2 + T_5^4$
369	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 2 & 1 & 1 \end{bmatrix}$	(2, 4)	(3, 2)	128	$T_1 T_3 T_6^3 + T_2 T_4 T_7^3 + T_5^2$
370	$\begin{bmatrix} 1 & 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 2 & 1 & 1 \end{bmatrix}$	(2, 4)	(5, 2)	192	$T_3 T_6^3 + T_4 T_7^3 + T_5^2$
378 379	$\begin{bmatrix} 1 & 1 & 2 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$	(4, 4)	(3, 1)	32	$T_1^2 T_3 T_7^3 + T_4^4 + T_5^3 T_6$ $T_1 T_2 T_3 T_7^3 + T_4^4 + T_5^3 T_6$
380	$\begin{bmatrix} 1 & 1 & 3 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$	(4, 4)	(4, 1)	44	$T_1 T_3 T_7^3 + T_4^4 + T_5^3 T_6$
381	$\begin{bmatrix} 1 & 1 & 4 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$	(4, 4)	(5, 1)	56	$T_3 T_7^3 + T_4^4 + T_5^3 T_6$

ID	$[w_1, \dots, w_7]$	μ	$-\mathcal{K}$	\mathcal{K}^4	g
399	$\begin{bmatrix} 1 & 1 & 5 & 1 & 1 & 3 & 0 \\ 0 & 0 & 1 & 1 & 1 & 3 & 1 \end{bmatrix}$	(6, 6)	(6, 1)	34	$T_1 T_3 T_7^5 + T_4^5 T_5 + T_6^2$
400					$T_1 T_3 T_7^5 + T_4^3 T_5^3 + T_6^2$
401	$\begin{bmatrix} 1 & 1 & 6 & 1 & 1 & 3 & 0 \\ 0 & 0 & 1 & 1 & 1 & 3 & 1 \end{bmatrix}$	(6, 6)	(7, 1)	40	$T_3 T_7^5 + T_4^5 T_5 + T_6^2$
402					$T_3 T_7^5 + T_4^3 T_5^3 + T_6^2$
403	$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 & 1 & 1 \end{bmatrix}$	(2, 6)	(2, 1)	14	$T_2^2 T_5^5 T_6 + T_1 T_3 T_7^5 + T_4^2$
404					$T_2^2 T_5^3 T_6^3 + T_1 T_3 T_7^5 + T_4^2$

$Q = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$		$\mu = (1, 3)$	$-\mathcal{K} = (2, 1)$	$\mathcal{K}^4 = 83$	
ID	g	ID	g	ID	g
265	$T_3^2 T_4 T_7^4 + T_1 T_5^3 + T_2 T_6^3$	266	$T_2^4 T_7^3 + T_1 T_5^3 + T_4 T_6^2$	267	$T_2^4 T_7^3 + T_1 T_5^3 + T_3 T_6^3$
268	$T_2^3 T_3 T_7^3 + T_1 T_6^3 + T_4 T_5^2$	269	$T_1^2 T_2^2 T_7^3 + T_3 T_6^3 + T_4 T_5^2$	270	$T_1 T_5^3 + T_3 T_6^3 + T_4^2 T_7$

$Q = \begin{bmatrix} 1 & 1 & 1 & 2 & 2 & 3 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 & 1 \end{bmatrix}$		$\mu = (6, 6)$	$-\mathcal{K} = (4, 1)$	$\mathcal{K}^4 = 55$	
ID	g	ID	g	ID	g
271	$T_1 T_2 T_4^2 T_7^4 + T_5^3 + T_6^2$	272	$T_1^4 T_4 T_7^5 + T_5^3 + T_6^2$	273	$T_1^3 T_2 T_4 T_7^5 + T_5^3 + T_6^2$
274	$T_1^2 T_2^2 T_4 T_7^5 + T_5^3 + T_6^2$	275	$T_1^2 T_2 T_3 T_4 T_7^5 + T_5^3 + T_6^2$		

$Q = \begin{bmatrix} 1 & 1 & 1 & 3 & 2 & 3 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 & 1 \end{bmatrix}$		$\mu = (6, 6)$	$-\mathcal{K} = (5, 1)$	$\mathcal{K}^4 = 83$	
ID	g	ID	g	ID	g
276	$T_1^3 T_4 T_7^5 + T_5^3 + T_6^2$	277	$T_1^2 T_2 T_4 T_7^5 + T_5^3 + T_6^2$	278	$T_1 T_2 T_3 T_4 T_7^5 + T_5^3 + T_6^2$

$Q = \begin{bmatrix} 1 & 1 & 1 & 3 & 4 & 6 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 & 1 \end{bmatrix}$		$\mu = (12, 6)$	$-\mathcal{K} = (4, 1)$	$\mathcal{K}^4 = 35$	
ID	g	ID	g	ID	g
283	$T_1^2 T_2 T_4^3 T_7^3 + T_5^3 + T_6^2$	284	$T_1 T_2 T_3 T_4^3 T_7^3 + T_5^3 + T_6^2$	285	$T_1^5 T_2 T_4^2 T_7^4 + T_5^3 + T_6^2$
286	$T_1^3 T_2^3 T_4^2 T_7^4 + T_5^3 + T_6^2$	287	$T_1^4 T_2 T_3 T_4^2 T_7^4 + T_5^3 + T_6^2$	288	$T_1^3 T_2^2 T_3 T_4^2 T_7^4 + T_5^3 + T_6^2$
289	$T_1^9 T_4 T_7^5 + T_5^3 + T_6^2$	290	$T_1^8 T_2 T_4 T_7^5 + T_5^3 + T_6^2$	291	$T_1^7 T_2^2 T_4 T_7^5 + T_5^3 + T_6^2$
292	$T_1^6 T_2^3 T_4 T_7^5 + T_5^3 + T_6^2$	293	$T_1^5 T_2^4 T_4 T_7^5 + T_5^3 + T_6^2$	294	$T_1^7 T_2 T_3 T_4 T_7^5 + T_5^3 + T_6^2$
295	$T_1^6 T_2^2 T_3 T_4 T_7^5 + T_5^3 + T_6^2$	296	$T_1^5 T_2^3 T_3 T_4 T_7^5 + T_5^3 + T_6^2$	297	$T_1^4 T_2^4 T_3 T_4 T_7^5 + T_5^3 + T_6^2$
298	$T_1^5 T_2^2 T_3^2 T_4 T_7^5 + T_5^3 + T_6^2$	299	$T_1^4 T_2^3 T_3^2 T_4 T_7^5 + T_5^3 + T_6^2$	300	$T_1^3 T_2^3 T_3^2 T_4 T_7^5 + T_5^3 + T_6^2$

$Q = \begin{bmatrix} 1 & 1 & 1 & 4 & 4 & 6 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 & 1 \end{bmatrix}$		$\mu = (12, 6)$	$-\mathcal{K} = (5, 1)$	$\mathcal{K}^4 = 58$	
ID	g	ID	g	ID	g
301	$T_1^3 T_2 T_4^2 T_7^4 + T_5^3 + T_6^2$	302	$T_1^2 T_2 T_3 T_4^2 T_7^4 + T_5^3 + T_6^2$	303	$T_1^8 T_4 T_7^5 + T_5^3 + T_6^2$

$Q = \begin{bmatrix} 1 & 1 & 1 & 4 & 4 & 6 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 & 1 \end{bmatrix}$				$\mu = (12, 6)$	$-\mathcal{K} = (5, 1)$	$\mathcal{K}^4 = 58$
ID	g	ID	g	ID	g	
304	$T_1^7 T_2 T_4 T_7^5 + T_5^3 + T_6^2$	305	$T_1^6 T_2^2 T_4 T_7^5 + T_5^3 + T_6^2$	306	$T_1^5 T_2^3 T_4 T_7^5 + T_5^3 + T_6^2$	
$Q = \begin{bmatrix} 1 & 1 & 1 & 4 & 4 & 6 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 & 1 \end{bmatrix}$				$\mu = (12, 6)$	$-\mathcal{K} = (5, 1)$	$\mathcal{K}^4 = 58$
ID	g	ID	g	ID	g	
307	$T_1^4 T_2^4 T_4 T_7^5 + T_5^3 + T_6^2$	308	$T_1^6 T_2 T_3 T_4 T_7^5 + T_5^3 + T_6^2$	309	$T_1^5 T_2^2 T_3 T_4 T_7^5 + T_5^3 + T_6^2$	
310	$T_1^4 T_2^3 T_3 T_4 T_7^5 + T_5^3 + T_6^2$	311	$T_1^4 T_2^2 T_3^2 T_4 T_7^5 + T_5^3 + T_6^2$	312	$T_1^3 T_2^3 T_3^2 T_4 T_7^5 + T_5^3 + T_6^2$	
$Q = \begin{bmatrix} 1 & 1 & 1 & 5 & 4 & 6 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 & 1 \end{bmatrix}$				$\mu = (12, 6)$	$-\mathcal{K} = (6, 1)$	$\mathcal{K}^4 = 87$
ID	g	ID	g	ID	g	
313	$T_1 T_2 T_2^4 T_7^4 + T_5^3 + T_6^2$	314	$T_1^7 T_4 T_7^5 + T_5^3 + T_6^2$	315	$T_1^6 T_2 T_4 T_7^5 + T_5^3 + T_6^2$	
316	$T_1^5 T_2^2 T_4 T_7^5 + T_5^3 + T_6^2$	317	$T_1^4 T_2^3 T_4 T_7^5 + T_5^3 + T_6^2$	318	$T_1^5 T_2 T_3 T_4 T_7^5 + T_5^3 + T_6^2$	
319	$T_1^4 T_2^2 T_3 T_4 T_7^5 + T_5^3 + T_6^2$	320	$T_1^3 T_2^3 T_3 T_4 T_7^5 + T_5^3 + T_6^2$	321	$T_1^3 T_2^2 T_3^2 T_4 T_7^5 + T_5^3 + T_6^2$	
$Q = \begin{bmatrix} 1 & 1 & 1 & 6 & 4 & 6 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 & 1 \end{bmatrix}$				$\mu = (12, 6)$	$-\mathcal{K} = (7, 1)$	$\mathcal{K}^4 = 122$
ID	g	ID	g	ID	g	
322	$T_1^6 T_4 T_7^5 + T_5^3 + T_6^2$	323	$T_1^5 T_2 T_4 T_7^5 + T_5^3 + T_6^2$	324	$T_1^4 T_2^2 T_4 T_7^5 + T_5^3 + T_6^2$	
325	$T_1^3 T_2^3 T_4 T_7^5 + T_5^3 + T_6^2$	326	$T_1^4 T_2 T_3 T_4 T_7^5 + T_5^3 + T_6^2$	327	$T_1^3 T_2^2 T_3 T_4 T_7^5 + T_5^3 + T_6^2$	
328	$T_1^2 T_2^2 T_3^2 T_4 T_7^5 + T_5^3 + T_6^2$					
$Q = \begin{bmatrix} 1 & 1 & 1 & 7 & 4 & 6 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 & 1 \end{bmatrix}$				$\mu = (12, 6)$	$-\mathcal{K} = (8, 1)$	$\mathcal{K}^4 = 163$
ID	g	ID	g	ID	g	
329	$T_1^5 T_4 T_7^5 + T_5^3 + T_6^2$	330	$T_1^4 T_2 T_4 T_7^5 + T_5^3 + T_6^2$	331	$T_1^3 T_2^2 T_4 T_7^5 + T_5^3 + T_6^2$	
332	$T_1^3 T_2 T_3 T_4 T_7^5 + T_5^3 + T_6^2$	333	$T_1^2 T_2^2 T_3 T_4 T_7^5 + T_5^3 + T_6^2$			
$Q = \begin{bmatrix} 1 & 1 & 1 & 8 & 4 & 6 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 & 1 \end{bmatrix}$				$\mu = (12, 6)$	$-\mathcal{K} = (9, 1)$	$\mathcal{K}^4 = 210$
ID	g	ID	g	ID	g	
334	$T_1^4 T_4 T_7^5 + T_5^3 + T_6^2$	335	$T_1^3 T_2 T_4 T_7^5 + T_5^3 + T_6^2$	336	$T_1^2 T_2^2 T_4 T_7^5 + T_5^3 + T_6^2$	
337	$T_1^2 T_2 T_3 T_4 T_7^5 + T_5^3 + T_6^2$					
$Q = \begin{bmatrix} 1 & 1 & 1 & 9 & 4 & 6 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 & 1 \end{bmatrix}$				$\mu = (12, 6)$	$-\mathcal{K} = (10, 1)$	$\mathcal{K}^4 = 263$
ID	g	ID	g	ID	g	
338	$T_1^3 T_4 T_7^5 + T_5^3 + T_6^2$	339	$T_1^2 T_2 T_4 T_7^5 + T_5^3 + T_6^2$	340	$T_1 T_2 T_3 T_4 T_7^5 + T_5^3 + T_6^2$	

$Q = \begin{bmatrix} 1 & 1 & 2 & 2 & 2 & 3 & 0 \\ 0 & 0 & 1 & 1 & 1 & 2 & 3 & 1 \end{bmatrix}$				$\mu = (6, 6)$	$-\mathcal{K} = (5, 2)$	$\mathcal{K}^4 = 80$
ID	g	ID	g	ID	g	
354	$T_3^2 T_4 T_7^3 + T_5^3 + T_6^2$	355	$T_1 T_2 T_3^2 T_7^4 + T_5^3 + T_6^2$	356	$T_1^2 T_3 T_4 T_7^4 + T_5^3 + T_6^2$	
357	$T_1 T_2 T_3 T_4 T_7^4 + T_5^3 + T_6^2$	358	$T_1^4 T_3 T_7^5 + T_5^3 + T_6^2$	359	$T_1^3 T_2 T_3 T_7^5 + T_5^3 + T_6^2$	
360	$T_1^2 T_2^2 T_3 T_7^5 + T_5^3 + T_6^2$					
$Q = \begin{bmatrix} 1 & 1 & 3 & 3 & 2 & 3 & 0 \\ 0 & 0 & 1 & 1 & 2 & 3 & 1 \end{bmatrix}$				$\mu = (6, 6)$	$-\mathcal{K} = (7, 2)$	$\mathcal{K}^4 = 112$
ID	g	ID	g	ID	g	
361	$T_3 T_4 T_7^4 + T_5^3 + T_6^2$	362	$T_1^3 T_3 T_7^5 + T_5^3 + T_6^2$	363	$T_1^2 T_2 T_3 T_7^5 + T_5^3 + T_6^2$	
$Q = \begin{bmatrix} 1 & 1 & 1 & 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}$				$\mu = (4, 6)$	$-\mathcal{K} = (2, 1)$	$\mathcal{K}^4 = 12$
ID	g	ID	g	ID	g	
371	$T_1^3 T_2 T_6^6 + T_3^3 T_4 T_7^2 + T_5^2$	372	$T_1 T_3^3 T_6^3 + T_2 T_4^3 T_7^3 + T_5^2$	373	$T_1^3 T_3 T_6^5 + T_2 T_4^3 T_7^3 + T_5^2$	
$Q = \begin{bmatrix} 1 & 1 & 2 & 2 & 3 & 0 & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}$				$\mu = (6, 6)$	$-\mathcal{K} = (3, 1)$	$\mathcal{K}^4 = 16$
ID	g	ID	g	ID	g	
374	$T_1 T_2 T_3^2 T_6^4 + T_4^3 T_7^3 + T_5^2$	375	$T_1^4 T_3 T_6^5 + T_4^3 T_7^3 + T_5^2$	376	$T_1^3 T_2 T_3 T_6^5 + T_4^3 T_7^3 + T_5^2$	
377	$T_1^2 T_2^2 T_3 T_6^5 + T_4^3 T_7^3 + T_5^2$					
$Q = \begin{bmatrix} 1 & 1 & 2 & 1 & 1 & 3 & 0 \\ 0 & 0 & 1 & 1 & 1 & 3 & 1 \end{bmatrix}$				$\mu = (6, 6)$	$-\mathcal{K} = (3, 1)$	$\mathcal{K}^4 = 16$
ID	g	ID	g	ID	g	
382	$T_3^3 T_7^3 + T_4^5 T_5 + T_6^2$	383	$T_1 T_2 T_3^2 T_7^4 + T_4^5 T_5 + T_6^2$	384	$T_1 T_2 T_3^2 T_7^4 + T_4^3 T_5^3 + T_6^2$	
385	$T_1^4 T_3 T_7^5 + T_4^5 T_5 + T_6^2$	386	$T_1^4 T_3 T_7^5 + T_4^3 T_5^3 + T_6^2$	387	$T_1^3 T_2 T_3 T_7^5 + T_4^5 T_5 + T_6^2$	
388	$T_1^3 T_2 T_3 T_7^5 + T_4^3 T_5^3 + T_6^2$	389	$T_1^2 T_2^2 T_3 T_7^5 + T_4^5 T_5 + T_6^2$	390	$T_1^2 T_2^2 T_3 T_7^5 + T_4^3 T_5^3 + T_6^2$	
$Q = \begin{bmatrix} 1 & 1 & 3 & 1 & 1 & 3 & 0 \\ 0 & 0 & 1 & 1 & 1 & 3 & 1 \end{bmatrix}$				$\mu = (6, 6)$	$-\mathcal{K} = (4, 1)$	$\mathcal{K}^4 = 22$
ID	g	ID	g	ID	g	
391	$T_1^3 T_3 T_7^5 + T_4^5 T_5 + T_6^2$	392	$T_1^3 T_3 T_7^5 + T_4^3 T_5^3 + T_6^2$	393	$T_1^2 T_2 T_3 T_7^5 + T_4^5 T_5 + T_6^2$	
394	$T_1^2 T_2 T_3 T_7^5 + T_4^3 T_5^3 + T_6^2$					
$Q = \begin{bmatrix} 1 & 1 & 4 & 1 & 1 & 3 & 0 \\ 0 & 0 & 1 & 1 & 1 & 3 & 1 \end{bmatrix}$				$\mu = (6, 6)$	$-\mathcal{K} = (5, 1)$	$\mathcal{K}^4 = 28$
ID	g	ID	g	ID	g	
395	$T_1^2 T_3 T_7^5 + T_4^5 T_5 + T_6^2$	396	$T_1^2 T_3 T_7^5 + T_4^3 T_5^3 + T_6^2$	397	$T_1 T_2 T_3 T_7^5 + T_4^5 T_5 + T_6^2$	
398	$T_1 T_2 T_3 T_7^5 + T_4^3 T_5^3 + T_6^2$					

$Q = \begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 & 1 & 1 \end{bmatrix}$						$\mu = (2, 6)$	$-\mathcal{K} = (3, 1)$	$\mathcal{K}^4 = 20$	
ID	g	ID	g	ID	g				
405	$T_1^2 T_5^5 T_6 + T_3 T_7^5 + T_4^2$	406	$T_1^2 T_5^3 T_6^3 + T_3 T_7^5 + T_4^2$	407	$T_1 T_2 T_5^6 + T_3 T_7^5 + T_4^2$				
408	$T_1 T_2 T_5^5 T_6 + T_3 T_7^5 + T_4^2$	409	$T_1 T_2 T_5^4 T_6^2 + T_3 T_7^5 + T_4^2$	410	$T_1 T_2 T_5^3 T_6^3 + T_3 T_7^5 + T_4^2$				

Classification list 5.7. *Locally factorial Fano fourfolds of Picard number two with a hypersurface Cox ring and an effective three-torus action: Specifying data for the infinite series with $s = 4$.*

ID	$[w_1, \dots, w_7]$	$\begin{bmatrix} \mu \\ -\mathcal{K} \end{bmatrix}$	g	
S18	$\begin{bmatrix} 1 & 1 & a & a & c & c & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$	$\begin{matrix} (a, 4) \\ (a+2c+2, 1) \end{matrix}$	$T_1^{(a-4c)} T_5^4 + T_2^{(a-4c)} T_6^4 + T_4 T_7^3$	$c \geq 1, a > 4c,$ $a \text{ odd}$
S22	$\begin{bmatrix} 1 & 1 & a & a & 2 & 3 & 0 \\ 0 & 0 & 1 & 1 & 2 & 3 & 1 \end{bmatrix}$	$\begin{matrix} (6, 6) \\ (2a+1, 2) \end{matrix}$	$T_1^5 T_2 T_7^6 + T_5^3 + T_6^2$	$a \geq 2$
S25	$\begin{bmatrix} 1 & 1 & a & a & c & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$	$\begin{matrix} (4c, 4) \\ (2a-3c+2, 1) \end{matrix}$	$T_1^{(a-4c)} T_3 T_6^3 + T_2^{(a-4c)} T_4 T_7^3 + T_5^4$	$c \geq 1, a > 4c$
S26	$\begin{bmatrix} 1 & 1 & 4a & 4a & a & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$	$\begin{matrix} (4a, 4) \\ (5a+2, 1) \end{matrix}$	$T_1^{(4a-l)} T_2^l T_6^4 + T_4 T_7^3 + T_5^4$	$a \geq 1, l \text{ odd},$ $0 \leq l < 2c$
S27	$\begin{bmatrix} 1 & 1 & a & a & c & 0 & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}$	$\begin{matrix} (2c, 6) \\ (2a-c+2, 1) \end{matrix}$	$T_1^{(2c-a)} T_3 T_6^5 + T_2^{(2c-a)} T_4 T_7^5 + T_5^2$	$a, c \geq 1,$ $c-1 \leq a < 2c$
S28	$\begin{bmatrix} 1 & 1 & a & a & a & 0 & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}$	$\begin{matrix} (2a, 6) \\ (a+2, 1) \end{matrix}$	$T_1^{(2a-l)} T_2^l T_6^6 + T_3 T_4 T_7^4 + T_5^2$	$a \geq 1, l \text{ odd},$ $1 \leq l \leq a$
S29	$\begin{bmatrix} 1 & 1 & 2a & 2a & a & 0 & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}$	$\begin{matrix} (2a, 6) \\ (3a+2, 1) \end{matrix}$	$T_5^2 + T_1^{(2a-l)} T_2^l T_6^6 + T_4 T_7^5$	$a \geq 1, l \text{ odd},$ $1 \leq l \leq a$
S30	$\begin{bmatrix} 1 & 1 & a & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$	$\begin{matrix} (4, 4) \\ (a+1, 1) \end{matrix}$	$T_1^3 T_2 T_7^4 + T_4^3 T_5 + T_6^4$	$a \geq 2$
S34	$\begin{bmatrix} 1 & 1 & a & c & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$	$\begin{matrix} (c, 4) \\ (a+2, 1) \end{matrix}$	$T_1^c T_5^4 + T_2^c T_6^4 + T_4 T_7^3$	$c \geq 1, a > c,$ $c \text{ odd}$

$Q = \begin{bmatrix} 1 & 1 & 1 & a & 2 & 3 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 & 1 \end{bmatrix}$						$\mu = (6, 6)$	$-\mathcal{K} = (a+2, 1)$	
ID	g	ID	g	ID	g			
S5	$T_1^5 T_2 T_7^6 + T_5^3 + T_6^2,$ $a \geq 2$	S6	$T_1^4 T_2 T_3 T_7^6 + T_5^3 + T_6^2,$ $a \geq 2$	S7	$T_1^3 T_2^2 T_3 T_7^6 + T_5^3 + T_6^2,$ $a \geq 2$			

$Q = \begin{bmatrix} 1 & 1 & 1 & a & 4 & 6 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 & 1 \end{bmatrix}$			$\mu = (12, 6)$	$-\mathcal{K} = (a+1, 1)$	
ID	g	ID	g	ID	g
S8	$T_1^{11}T_2T_7^6+T_5^3+T_6^2,$ $a \geq 3$	S9	$T_1^7T_2^5T_7^6+T_5^3+T_6^2,$ $a \geq 3$	S10	$T_1^{10}T_2T_3T_7^6+T_5^3+T_6^2,$ $a \geq 3$
S11	$T_1^9T_2^2T_3T_7^6+T_5^3+T_6^2,$ $a \geq 3$	S12	$T_1^8T_2^3T_3T_7^6+T_5^3+T_6^2,$ $a \geq 3$	S13	$T_1^7T_2^4T_3T_7^6+T_5^3+T_6^2,$ $a \geq 3$
S14	$T_1^6T_2^5T_3T_7^6+T_5^3+T_6^2,$ $a \geq 3$	S15	$T_1^7T_2^3T_3^2T_7^6+T_5^3+T_6^2,$ $a \geq 3$	S16	$T_1^5T_2^5T_3^2T_7^6+T_5^3+T_6^2,$ $a \geq 3$
S17	$T_1^5T_2^4T_3^3T_7^6+T_5^3+T_6^2,$ $a \geq 3$				

$Q = \begin{bmatrix} 1 & 1 & a & a & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$			$\mu = (a+3, 4)$	$-\mathcal{K} = (a+1, 1)$
ID	g	ID	g	
S19	$T_3T_5^3+T_4T_6^3+T_1^{(a+3-1)}T_2^lT_7^4,$ $a \geq 2, 0 \leq l \leq (a+3)/2$	S20	$T_1^{(a-1)}T_5^4+T_3T_6^3+T_2^{(a+3)}T_7^4,$ $a \geq 2, a \text{ even}$	
S21	$T_1^{(a-1)}T_5^4+T_3T_6^3+T_2^3T_4T_7^3,$ $a \geq 2$			

$Q = \begin{bmatrix} 1 & 1 & a & a & c & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$			$\mu = (a, 4)$	$-\mathcal{K} = (a+c+2, 1)$
ID	g	ID	g	
S23	$T_1^{(a-4c)}T_5^4+T_2^aT_6^4+T_4T_7^3,$ $c \geq 1, a > 4c$	S24	$T_1^{(a-1)}T_2^lT_5^4+T_3T_6^3+T_4T_7^3,$ $c \geq 1, a > 4c, 0 \leq l \leq a/2$	

$Q = \begin{bmatrix} 1 & 1 & a & 1 & 1 & 3 & 0 \\ 0 & 0 & 1 & 1 & 1 & 3 & 1 \end{bmatrix}$			$\mu = (6, 6)$	$-\mathcal{K} = (a+1, 1)$
ID	g	ID	g	
S31	$T_1^5T_2T_7^6+T_4^5T_5+T_6^2,$ $a \geq 2$	S32	$T_1^5T_2T_7^6+T_4^3T_5^3+T_6^2,$ $a \geq 2$	
S33	$T_1^3T_2^3T_7^6+T_4^5T_5+T_6^2,$ $a \geq 2$			

$Q = \begin{bmatrix} 1 & 1 & a+1 & a & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$			$\mu = (a+1, 4)$	$-\mathcal{K} = (a+2, 1)$
ID	g	ID	g	
S35	$T_1^{(a+1)}T_5^4+T_2T_4T_6^3+T_3T_7^3,$ $a \geq 1$	S36	$T_1^{(a+1)}T_5^4+T_2^{(a+1)}T_6^4+T_3T_7^3,$ $a \geq 1, a \text{ even}$	

Classification list 5.8. *Locally factorial Fano fourfolds of Picard number two with a hypersurface Cox ring and an effective three-torus action: Specifying data for the cases with $s = 5$ and $\mu \in \lambda$.*

ID	$[w_1, \dots, w_7]$	μ	$-\mathcal{K}$	\mathcal{K}^4	g
411					$T_2^3 T_3 T_7^2 + T_1 T_5^3 + T_4^2$
412	$\begin{bmatrix} 1 & 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 3 & 2 & 1 & 1 \end{bmatrix}$	(4, 6)	(2, 3)	65	$T_2^3 T_3 T_6 T_7 + T_1 T_5^3 + T_4^2$
413					$T_2^4 T_6 T_7 + T_1 T_5^3 + T_4^2$
414					$T_2^2 T_3^2 T_6 T_7 + T_1 T_5^3 + T_4^2$

Classification list 5.9. *Locally factorial Fano fourfolds of Picard number two with a hypersurface Cox ring and an effective three-torus action: Specifying data for the sporadic cases with $s = 5$ and $\mu \notin \lambda$.*

ID	$[w_1, \dots, w_7]$	μ	$-\mathcal{K}$	\mathcal{K}^4	g
415					$T_3^4 T_4 T_7^3 + T_1 T_2 T_6^4 + T_5^2$
416	$\begin{bmatrix} 1 & 1 & 1 & 2 & 3 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 1 \end{bmatrix}$	(6, 4)	(3, 1)	50	$T_1 T_2 T_6^4 + T_4^3 T_7 + T_5^2$
417					$T_1^5 T_2 T_7^4 + T_3 T_4 T_6^3 + T_5^2$
418					$T_1^3 T_2^3 T_7^4 + T_3 T_4 T_6^3 + T_5^2$
419					$T_3^3 T_4 T_7^3 + T_1 T_2 T_6^4 + T_5^2$
420	$\begin{bmatrix} 1 & 1 & 1 & 3 & 3 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 1 \end{bmatrix}$	(6, 4)	(4, 1)	90	$T_1^5 T_2 T_7^4 + T_4 T_6^3 + T_5^2$
421					$T_1^4 T_2 T_3 T_7^4 + T_4 T_6^3 + T_5^2$
422					$T_1^3 T_2^2 T_3 T_7^4 + T_4 T_6^3 + T_5^2$
423					$\begin{bmatrix} 1 & 1 & 1 & 4 & 3 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 1 \end{bmatrix}$
424	$\begin{bmatrix} 1 & 1 & 1 & 5 & 3 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 1 \end{bmatrix}$	(6, 4)	(6, 1)	206	$T_1 T_2 T_6^4 + T_3 T_4 T_7^3 + T_5^2$
425	$\begin{bmatrix} 1 & 1 & 1 & 6 & 3 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 1 \end{bmatrix}$	(6, 4)	(7, 1)	282	$T_1 T_2 T_6^4 + T_4 T_7^3 + T_5^2$
426	$\begin{bmatrix} 1 & 1 & 2 & 2 & 4 & 1 & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}$	(8, 6)	(3, 1)	14	$T_1 T_2 T_6^6 + T_3^3 T_4 T_7^2 + T_5^2$
427					$T_2^2 T_4^3 T_7^3 + T_1 T_3 T_6^5 + T_5^2$
428	$\begin{bmatrix} 1 & 1 & 3 & 3 & 5 & 1 & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}$	(10, 6)	(4, 1)	18	$T_1^2 T_3 T_6^5 + T_2 T_4^3 T_7^3 + T_5^2$
429	$\begin{bmatrix} 1 & 1 & 4 & 4 & 6 & 1 & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}$	(12, 6)	(5, 1)	22	$T_1^3 T_3 T_6^5 + T_4^3 T_7^3 + T_5^2$
430					$T_1^2 T_2 T_3 T_6^5 + T_4^3 T_7^3 + T_5^2$
431	$\begin{bmatrix} 1 & 1 & 4 & 4 & 7 & 2 & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}$	(14, 6)	(5, 1)	20	$T_2^2 T_4^3 T_7^3 + T_3 T_6^5 + T_5^2$
432					$T_1 T_2 T_4^3 T_7^3 + T_3 T_6^5 + T_5^2$

ID	$[w_1, \dots, w_7]$	μ	$-\mathcal{K}$	\mathcal{K}^4	g
433	$\begin{bmatrix} 1 & 1 & 5 & 5 & 8 & 2 & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}$	(16, 6)	(6, 1)	24	$T_1 T_3 T_6^5 + T_2 T_4^3 T_7^3 + T_5^2$
434	$\begin{bmatrix} 1 & 1 & 6 & 6 & 9 & 2 & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}$	(18, 6)	(7, 1)	28	$T_1^2 T_3 T_6^5 + T_4^3 T_7^3 + T_5^2$
435					$T_1 T_2 T_3 T_6^5 + T_4^3 T_7^3 + T_5^2$
436	$\begin{bmatrix} 1 & 1 & 7 & 7 & 11 & 3 & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}$	(22, 6)	(8, 1)	30	$T_2 T_4^3 T_7^3 + T_3 T_6^5 + T_5^2$
437	$\begin{bmatrix} 1 & 1 & 8 & 8 & 12 & 3 & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}$	(24, 6)	(9, 1)	34	$T_1 T_3 T_6^5 + T_4^3 T_7^3 + T_5^2$
438	$\begin{bmatrix} 1 & 1 & 10 & 10 & 15 & 4 & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}$	(30, 6)	(11, 1)	40	$T_3 T_6^5 + T_4^3 T_7^3 + T_5^2$
439	$\begin{bmatrix} 1 & 1 & 3 & 2 & 3 & 0 & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}$	(6, 6)	(4, 1)	22	$T_1^3 T_3 T_6^5 + T_4^3 T_7^3 + T_5^2$
440					$T_1^2 T_2 T_3 T_6^5 + T_4^3 T_7^3 + T_5^2$
441	$\begin{bmatrix} 1 & 1 & 4 & 2 & 3 & 0 & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}$	(6, 6)	(5, 1)	28	$T_1^2 T_3 T_6^5 + T_4^3 T_7^3 + T_5^2$
442					$T_1 T_2 T_3 T_6^5 + T_4^3 T_7^3 + T_5^2$
443	$\begin{bmatrix} 1 & 1 & 5 & 2 & 3 & 0 & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}$	(6, 6)	(6, 1)	34	$T_1 T_3 T_6^5 + T_4^3 T_7^3 + T_5^2$
444	$\begin{bmatrix} 1 & 1 & 6 & 2 & 3 & 0 & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}$	(6, 6)	(7, 1)	40	$T_3 T_6^5 + T_4^3 T_7^3 + T_5^2$
445	$\begin{bmatrix} 1 & 1 & 3 & 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}$	(4, 6)	(4, 1)	24	$T_1 T_3 T_6^5 + T_2 T_4^3 T_7^3 + T_5^2$
446	$\begin{bmatrix} 1 & 1 & 4 & 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}$	(4, 6)	(5, 1)	30	$T_2 T_4^3 T_7^3 + T_3 T_6^5 + T_5^2$
447	$\begin{bmatrix} 1 & 1 & 2 & 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}$	(4, 6)	(3, 1)	18	$T_1^3 T_2 T_6^5 + T_3 T_4^2 T_7^3 + T_5^2$

Classification list 5.10. *Locally factorial Fano fourfolds of Picard number two with a hypersurface Cox ring and an effective three-torus action: Specifying data for the infinite series with $s = 5$.*

ID	$[w_1, \dots, w_7]$	$\begin{bmatrix} \mu \\ -\mathcal{K} \end{bmatrix}$	g
S39	$\begin{bmatrix} 1 & 1 & a & a & b & c & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$	$\begin{matrix} (a, 4) \\ (a+c+b+2, 1) \end{matrix}$	$T_1^{(a-4b)} T_5^4 + T_2^{(a-4c)} T_6^4 + T_4 T_7^3,$ $b > c \geq 1, a > 4b, a \text{ odd}$
S42	$\begin{bmatrix} 1 & 1 & a & a & b & c & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$	$\begin{matrix} (4a, 4) \\ (c+b-2a+2, 1) \end{matrix}$	$T_5^4 + T_1^{(4b-a-3c)} T_3 T_6^3 + T_2^{(4b-a)} T_4 T_7^3,$ $b > c \geq 1, b > 2c-1, 3b-c-1 \leq a < 4b-3c$
S43	$\begin{bmatrix} 1 & 1 & 2a+2 & 2a+2 & a+1 & a & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$	$\begin{matrix} (4a+4, 4) \\ (2a+3, 1) \end{matrix}$	$T_1^3 T_2 T_6^4 + T_3 T_4 T_7^2 + T_5^4,$ $a \geq 1$
S44	$\begin{bmatrix} 1 & 1 & 4a & 4a & a & b & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$	$\begin{matrix} (4a, 4) \\ (5a+b+2, 1) \end{matrix}$	$T_5^4 + T_1^{(4a-4b-1)} T_2 T_6^4 + T_4 T_7^3,$ $a > b \geq 1, 0 < l < 2a+2b, l \text{ odd}$

ID	$[w_1, \dots, w_7]$	$\begin{bmatrix} \mu \\ -\mathcal{K} \end{bmatrix}$	g
S45	$\begin{bmatrix} 1 & 1 & a & a & a+2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}$	$(2a+4, 6)$ $(a+1, 1)$	$T_5^2 + T_3 T_4 T_6^4 + T_1^{(2a+4-l)} T_2^l T_7^6$, $a \geq 1, 0 < l \leq a+2, l \text{ odd}$
S49	$\begin{bmatrix} 1 & 1 & a & a & b & c & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}$	$(2b, 6)$ $(2a-b+c+2, 1)$	$T_5^2 + T_1^{(2b-5c-a)} T_3 T_6^5 + T_2^{(2b-a)} T_4 T_7^5$, $c \geq 1, b \geq 4c, b/3 < a < 2b-5c$
S53	$\begin{bmatrix} 1 & 1 & a & a & a & b & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}$	$(2a, 6)$ $(a+b+2, 1)$	$T_5^2 + T_1^{(2a-6b-l)} T_2^l T_6^6 + T_3 T_4 T_7^4$, $b \geq 1, a > 3b, 0 < l \leq a-3b, l \text{ odd}$
S54	$\begin{bmatrix} 1 & 1 & 2a & 2a & a & b & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}$	$(2a, 6)$ $(b+3a+2, 1)$	$T_5^2 + T_1^{(2a-l)} T_2^l T_6^6 + T_4 T_7^5$, $b \geq 1, a > 3b, 0 < l \leq a, l \text{ odd}$
S55	$\begin{bmatrix} 1 & 1 & a+1 & a & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$	$(a+3, 4)$ $(a+2, 1)$	$T_1^{(a-1)} T_5^4 + T_4 T_6^3 + T_2^2 T_3 T_7^3$, $a \geq 2$
S56	$\begin{bmatrix} 1 & 1 & a+2 & a & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$	$(a+3, 4)$ $(a+3, 1)$	$T_1^{(a-1)} T_5^4 + T_4 T_6^3 + T_2 T_3 T_7^3$, $a \geq 2$
S57	$\begin{bmatrix} 1 & 1 & a+3 & a & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$	$(a+3, 4)$ $(a+4, 1)$	$T_1^{(a-1-l)} T_2^l T_5^4 + T_4 T_6^3 + T_3 T_7^3$, $a \geq 2, 0 \leq l \leq (a-1)/2$
S58	$\begin{bmatrix} 1 & 1 & a & b & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$	$(b+3, 4)$ $(a+1, 1)$	$T_1^{(b-1)} T_5^4 + T_4 T_6^3 + T_2^{(b+3)} T_7^4$, $a > b \geq 2, b \text{ even}$
S59	$\begin{bmatrix} 1 & 1 & 2a+1 & 2a & a & a & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$	$(4a+1, 4)$ $(2a+2, 1)$	$T_1 T_5^4 + T_2 T_6^4 + T_3 T_4 T_7^2$, $a \geq 1$
S60	$\begin{bmatrix} 1 & 1 & a & b & c & c & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$	$(a, 4)$ $(2c+b+2, 1)$	$T_1^{(a-4c)} T_5^4 + T_2^{(a-4c)} T_6^4 + T_3 T_7^3$, $c \geq 1, b > 2c-1, 4c < a \leq 1+b+2c, a \text{ odd}$
S61	$\begin{bmatrix} 1 & 1 & a & b & c & c & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$	$(b, 4)$ $(a+2c+2, 1)$	$T_1^{(b-4c)} T_5^4 + T_2^{(b-4c)} T_6^4 + T_4 T_7^3$, $c \geq 1, b > 4c, a > b, b \text{ odd}$
S64	$\begin{bmatrix} 1 & 1 & a & b & c & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$	$(b, 4)$ $(a+c+2, 1)$	$T_1^{(b-4c)} T_5^4 + T_2^b T_6^4 + T_4 T_7^3$, $c \geq 1, b > 4c, b < a \leq 1+b+c, b \text{ odd}$
S67	$\begin{bmatrix} 1 & 1 & a & b & c & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$	$(4c, 4)$ $(a+b-3c+2, 1)$	$T_5^4 + T_1^{(4c-a)} T_3 T_6^3 + T_2^{(4c-b)} T_4 T_7^3$, $c \geq 1, a > b > 4c$
S68	$\begin{bmatrix} 1 & 1 & a & 4b & b & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$	$(4b, 4)$ $(a+b+2, 1)$	$T_5^4 + T_1^{(4b-l)} T_2^l T_6^4 + T_4 T_7^3$, $b \geq 1, 4b < a \leq 5b+1, 0 < l < 2b, l \text{ odd}$
S69	$\begin{bmatrix} 1 & 1 & 2a & a-1 & a & 0 & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}$	$(2a, 6)$ $(1+2a, 1)$	$T_1 T_2 T_4^2 T_7^4 + T_3 T_6^5 + T_5^2$, $a \geq 2$
S70	$\begin{bmatrix} 1 & 1 & 2b & a & b & 0 & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}$	$(2a, 6)$ $(3b-a+2, 1)$	$T_5^2 + T_3 T_6^5 + T_1^{(2b-a-l)} T_2^l T_4 T_7^5$, $a, b \geq 1, b-1 \leq a < 2b$
S71	$\begin{bmatrix} 1 & 1 & a & b & c & 0 & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}$	$(2c, 6)$ $(a+b-c+2, 1)$	$T_5^2 + T_1^{(2c-a)} T_3 T_6^5 + T_2^{(2c-b)} T_4 T_7^5$, $c \geq 1, c-1 \leq b < 2c, b < a < 2c$

ID	$[w_1, \dots, w_7]$	$\begin{bmatrix} \mu \\ -\mathcal{K} \end{bmatrix}$	g
S72	$\begin{bmatrix} 1 & 1 & a & 2 & 3 & 0 & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}$	$(6, 6)$ $(a+1, 1)$	$T_1^5 T_2 T_6^6 + T_4^3 T_7^3 + T_5^2,$ $a \geq 3$
S73	$\begin{bmatrix} 1 & 1 & a+1 & a-1 & a & 0 & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}$	$(2a, 6)$ $(a+2, 1)$	$T_1^{(2a-1)} T_2^l T_6^6 + T_3 T_4 T_7^4 + T_5^2,$ $a \geq 2, 0 < l \leq a, l \text{ odd}$
S74	$\begin{bmatrix} 1 & 1 & 2b & a & b & 0 & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}$	$(2b, 6)$ $(a+b+2, 1)$	$T_5^2 + T_1^{(2b-1)} T_2^l T_6^6 + T_3 T_7^5,$ $a, b \geq 1, b-1 \leq a < 2b, 0 < l \leq b, l \text{ odd}$
S75	$\begin{bmatrix} 1 & 1 & a & 2b & b & 0 & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}$	$(2b, 6)$ $(a+b+2, 1)$	$T_5^2 + T_1^{(2b-1)} T_2^l T_6^6 + T_4 T_7^5,$ $b \geq 1, a > 2b, 0 < l \leq b, l \text{ odd}$

$Q = \begin{bmatrix} 1 & 1 & a & a & b & c & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$		$\mu = (a+3c, 4)$	$-\mathcal{K} = (a-2c+b+2, 1)$
ID	g	ID	g
S37	$T_1^{(a+3c-4b)} T_5^4 + T_3 T_6^3 + T_2^3 T_4 T_7^3,$ $b > c \geq 1, b \geq 2c-1, a > 4b-3c$	S38	$T_1^{(a+3c-4b)} T_5^4 + T_3 T_6^3 + T_2^{(a+3c)} T_7^4,$ $b > c \geq 1, b \geq 2c-1, a > 4b-3c, a \text{ or } c \text{ odd}$
$Q = \begin{bmatrix} 1 & 1 & 4a-3b & 4a-3b & a & b & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$		$\mu = (4a, 4)$	$-\mathcal{K} = (5a-5b+2, 1)$
ID	g	ID	g
S40	$T_5^4 + T_3 T_6^3 + T_1^{(3b-1)} T_2^l T_4 T_7^3,$ $a > b \geq 1, a \geq 2b-1, 0 \leq l \leq 3b/2$	S41	$T_5^4 + T_3 T_6^3 + T_1^{(4a-1)} T_2^l T_7^4,$ $a > b \geq 1, a \geq 2b-1, 0 < l < 2a, l \text{ odd}$
$Q = \begin{bmatrix} 1 & 1 & 2a-5b & 2a-5b & a & b & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}$		$\mu = (2a, 6)$	$-\mathcal{K} = (3a-9b+2, 1)$
ID	g	ID	g
S46	$T_5^2 + T_3 T_6^5 + T_1^{(10b-2a-1)} T_2^l T_4^2 T_7^4,$ $b \geq 1, 4b-1 \leq a < 5b$	S47	$T_5^2 + T_3 T_6^5 + T_1^{(5b-1)} T_2^l T_4 T_7^5,$ $b \geq 1, a \geq 4b-1, 0 \leq l \leq 5b/2$
S48	$T_5^2 + T_3 T_6^5 + T_1^{(2a-1)} T_2^l T_7^6,$ $b \geq 1, a \geq 4b-1, 0 < l \leq a, l \text{ odd}$		
$Q = \begin{bmatrix} 1 & 1 & 2a+2 & 2a+2 & 3a+3 & a & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}$		$\mu = (6a+6, 6)$	$-\mathcal{K} = (2a+3, 1)$
ID	g	ID	g
S50	$T_1^5 T_2 T_6^6 + T_3^2 T_4 T_7^3 + T_5^2,$ $a \geq 1$	S51	$T_1^5 T_2 T_6^6 + T_3^3 T_7^3 + T_5^2,$ $a \geq 1$
S52	$T_1^3 T_2^3 T_6^6 + T_3^2 T_4 T_7^3 + T_5^2,$ $a \geq 1$		

$Q = \begin{bmatrix} 1 & 1 & a & b & c & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \quad \mu = (a, 4) \quad -\mathcal{K} = (c+b+2, 1)$			
ID	g	ID	g
S62	$T_1^{(a-4c)}T_5^4+T_3T_6^3+T_2^{(a-b)}T_4T_7^3,$ $c \geq 1, b > 3c-1, 4c < a \leq 1+b+c$	S63	$T_1^{(a-4c)}T_5^4+T_2^aT_6^4+T_3T_7^3,$ $c \geq 1, b > 3c-1, 4c < a \leq 1+b+c, a \text{ odd}$
$Q = \begin{bmatrix} 1 & 1 & 4b & a & b & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \quad \mu = (4b, 4) \quad -\mathcal{K} = (a+b+2, 1)$			
ID	g	ID	g
S65	$T_5^4+T_3T_6^3+T_1^{(4b-a-1)}T_2^lT_4T_7^3,$ $b \geq 1, 3b-1 \leq a < 4b, 0 \leq l \leq (4b-a)/2$	S66	$T_5^4+T_1^{(4b-1)}T_2^lT_6^4+T_3T_7^3,$ $b \geq 1, 3b-1 \leq a < 4b, 0 < l < 2b, l \text{ odd}$

Classification list 5.11. *Locally factorial Fano fourfolds of Picard number two with a hypersurface Cox ring and an effective three-torus action: Specifying data for the infinite series with $s = 6$.*

ID	$[w_1, \dots, w_7]$	$\begin{bmatrix} \mu \\ -\mathcal{K} \end{bmatrix}$	g
S78	$\begin{bmatrix} 1 & 1 & b+3d & b & c & d & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$	$(b+3d, 4)$ $(b+d+c+2, 1)$	$T_1^{(b-4c+3d-l)}T_2^lT_5^4+T_4T_6^3+T_3T_7^3,$ $c > d \geq 1, c \geq 2d-1, b > 4c-3d,$ $0 \leq l \leq (b-4c+3d)/2$
S83	$\begin{bmatrix} 1 & 1 & a & b & c & d & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$	$(b, 4)$ $(d+c+a+2, 1)$	$T_1^{(b-4c)}T_5^4+T_2^{(b-4d)}T_6^4+T_4T_7^3,$ $c > d \geq 1, b > 4c, a > b, b \text{ odd}$
S88	$\begin{bmatrix} 1 & 1 & 4c & 4c-3d & c & d & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$	$(4c, 4)$ $(5c-2d+2, 1)$	$T_3T_7^3+T_4T_6^3+T_5^4,$ $c > d \geq 1, c \geq 2d-1$
S93	$\begin{bmatrix} 1 & 1 & a & 4c & c & d & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$	$(4c, 4)$ $(d+c+a+2, 1)$	$T_5^4+T_1^{(4c-4d-l)}T_2^lT_6^4+T_4T_7^3,$ $c > d \geq 1, a > 4c, 0 < l < 2c-2d, l \text{ odd}$
S102	$\begin{bmatrix} 1 & 1 & 2c & 2c-5d & c & d & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}$	$(2c, 4)$ $(3c-4d+2, 3)$	$T_3T_7^3+T_4T_6^5+T_5^2,$ $d \geq 1, c > 3, c \geq 4d-1$
S105	$\begin{bmatrix} 1 & 1 & 2c-b & b & c & d & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}$	$(2c, 4)$ $(c+d+2, 3)$	$T_5^2+T_1^{(2c-6d-l)}T_2^lT_6^6+T_3T_4T_7^4,$ $d \geq 1, c > 3d, b > 1, c-d-1 \leq b < c,$ $0 < l \leq c-3d, l \text{ odd}$
S106	$\begin{bmatrix} 1 & 1 & a & 2c & c & d & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}$	$(2c, 4)$ $(d+c+a+2, 3)$	$T_5^2+T_1^{(2c-6d-l)}T_2^lT_6^6+T_4T_7^5,$ $d \geq 1, c > 3d, a > 2c, 0 < l \leq c-3d, l \text{ odd}$

$Q = \begin{bmatrix} 1 & 1 & a & b & c & d & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \quad \mu = (a+3d, 4) \quad -\mathcal{K} = (c+b-2d+2, 1)$			
ID	g	ID	g
S76	$T_1^{(a-4c+3d)}T_5^4+T_3T_6^3+T_2^{(a-b+3d)}T_4T_7^3,$ $b>c>d\geq 1, a>4c-3d,$ $a>b-3d, a\leq 1+b+c-2d$	S77	$T_1^{(a-4c+3d)}T_5^4+T_3T_6^3+T_2^{(a+3d)}T_7^4,$ $b>c>d\geq 1, 4c-3d<a\leq 1+b+c-2d,$ $a+3d$ odd
$Q = \begin{bmatrix} 1 & 1 & a & b & c & d & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \quad \mu = (a, 4) \quad -\mathcal{K} = (b+d+c+2, 1)$			
ID	g	ID	g
S79	$T_1^{(a-4c)}T_5^4+T_2^{(a-b-3d)}T_4T_6^3+T_3T_7^3,$ $b>c>d\geq 1, a>4c, a>b+3d,$ $a\leq 1+b+c+d$	S80	$T_1^{(a-4c)}T_5^4+T_2^{(a-4d)}T_6^4+T_3T_7^3,$ $c>d\geq 1, b\geq 3c-d, 4c<a\leq 1+b+c+d,$ a odd
$Q = \begin{bmatrix} 1 & 1 & a & b & c & d & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \quad \mu = (b+3d, 4) \quad -\mathcal{K} = (c+a-2d+2, 1)$			
ID	g	ID	g
S81	$T_1^{(b-4c+3d)}T_5^4+T_4T_6^3+T_2^{(b+3d-a)}T_3T_7^3,$ $c>d\geq 1, b>4c-3d, b<a<b+3d$	S82	$T_1^{(b-4c+3d)}T_5^4+T_4T_6^3+T_2^{(b+3d)}T_7^4,$ $c>d\geq 1, c\geq 2d-1, b>4c-3d,$ $a>b, b+3d$ odd
$Q = \begin{bmatrix} 1 & 1 & 4c-3d & b & c & d & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \quad \mu = (4c, 4) \quad -\mathcal{K} = (c+b-2d+2, 1)$			
ID	g	ID	g
S84	$T_5^4+T_3T_6^3+T_1^{(4c-b-l)}T_2^lT_4T_7^3,$ $c>d\geq 1, 3c-d-1\leq b<4c-3d$	S85	$T_5^4+T_3T_6^3+T_1^{(4c-l)}T_2^lT_7^4,$ $d\geq 1, c>2d-1, 3c-d-1\leq b<4c-3d,$ $0<l<2c, l$ odd
$Q = \begin{bmatrix} 1 & 1 & a & b & c & d & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \quad \mu = (4c, 4) \quad -\mathcal{K} = (d+b+a-3c+2, 1)$			
ID	g	ID	g
S86	$T_5^4+T_1^{(4c-a-3d)}T_3T_6^3+T_2^{(4c-b)}T_4T_7^3,$ $c>d\geq 1, 3c-d-1\leq b<4c, a>4c-3d$	S87	$T_5^4+T_1^{(4c-b-3d)}T_4T_6^3+T_2^{(4c-a)}T_3T_7^3,$ $d\geq 1, c>2d-1, 3c-d-1\leq b<4c-3d,$ $b<a<4c$
$Q = \begin{bmatrix} 1 & 1 & 4c & b & c & d & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \quad \mu = (4c, 4) \quad -\mathcal{K} = (b+d+c+2, 1)$			
ID	g	ID	g
S89	$T_5^4+T_1^{(4c-b-3d-l)}T_2^lT_4T_6^3+T_3T_7^3,$ $d\geq 1, c>2d-1, 3c-d-1\leq b<4c-3d$	S90	$T_5^4+T_1^{(4c-4d-l)}T_2^lT_6^4+T_3T_7^3,$ $c>d\geq 1, 3c-d-1\leq b<4c,$ $0<l<2c-2d, l$ odd
$Q = \begin{bmatrix} 1 & 1 & a & 4c-3d & c & d & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \quad \mu = (4c, 4) \quad -\mathcal{K} = (c+a-2d+2, 1)$			
ID	g	ID	g
S91	$T_5^4+T_4T_6^3+T_1^{(4c-a-l)}T_2^lT_3T_7^3,$ $c>d\geq 1, c\geq 2d-1, 4c-3d<a<4c$	S92	$T_5^4+T_4T_6^3+T_1^{(4c-l)}T_2^lT_7^4,$ $c, d\geq 1, c\geq 2d-1, a>4c-3d,$ $0<l<2c, l$ odd

$Q = \begin{bmatrix} 1 & 1 & a & 2c-5d & c & d & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix} \quad \mu = (2c, 4) \quad -\mathcal{K} = (c-4d+a+2, 3)$	
<p>ID g</p> <p>S94 $T_5^2+T_4T_6^5+T_1^{(2c-2a-l)}T_2^lT_3^2T_7^4,$ $d \geq 1, c > 3d, 4d-1 \leq c < 5d,$ $2c-5d < a < c, 0 < l \leq c-a, l \text{ odd}$</p> <p>S96 $T_5^2+T_4T_6^5+T_1^{(2c-l)}T_2^lT_7^6,$ $d \geq 1, c > 3, c \geq 4d-1, a > 2c-5d,$ $0 < l \leq c, l \text{ odd}$</p>	<p>ID g</p> <p>S95 $T_5^2+T_4T_6^5+T_1^{(2c-a-l)}T_2^lT_3T_7^5,$ $d \geq 1, c > 3, c \geq 4d-1, 2c-5d < a < 2c,$ $0 \leq l \leq (2c-a)/2$</p>

$Q = \begin{bmatrix} 1 & 1 & 2c-5d & b & c & d & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix} \quad \mu = (2c, 4) \quad -\mathcal{K} = (c-4d+b+2, 3)$	
<p>ID g</p> <p>S97 $T_5^2+T_3T_6^5+T_1^{(2c-2b-l)}T_2^lT_4^2T_7^4,$ $d \geq 1, c > 4d-1, c-d-1 \leq b < c,$ $b < 2c-5d, 0 < l \leq c-b, l \text{ odd}$</p>	<p>ID g</p> <p>S98 $T_5^2+T_3T_6^5+T_1^{(2c-b-l)}T_2^lT_4T_7^5,$ $d \geq 1, c > 3d, c-d-1 \leq b < 2c-5d,$ $0 \leq l \leq (2c-b)/2$</p>

$Q = \begin{bmatrix} 1 & 1 & 2c-5d & b & c & d & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix} \quad \mu = (2c, 4) \quad -\mathcal{K} = (c-4d+b+2, 3)$	
<p>ID g</p> <p>S99 $T_5^2+T_3T_6^5+T_1^{(2c-l)}T_2^lT_7^6,$ $d \geq 1, c > 4d-1, b > 1, c-d-1 \leq b < 2c-5d,$ $0 < l \leq c \text{ odd}$</p>	

$Q = \begin{bmatrix} 1 & 1 & a & b & c & d & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix} \quad \mu = (2c, 4) \quad -\mathcal{K} = (d+b+a-c+2, 3)$	
<p>ID g</p> <p>S100 $T_5^2+T_1^{(2c-a-5d)}T_3T_6^5+T_2^{(2c-b)}T_4T_7^5,$ $d \geq 1, c > 3d, c-d-1 \leq b < a < 2c-5d$</p>	<p>ID g</p> <p>S101 $T_5^2+T_1^{(2c-b-5d)}T_4T_6^5+T_2^{(2c-a)}T_3T_7^5,$ $d \geq 1, c > 4d-1, b > 1, c-d-1 \leq b < 2c-5d,$ $b < a < 2c$</p>

$Q = \begin{bmatrix} 1 & 1 & 2c & b & c & d & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix} \quad \mu = (2c, 4) \quad -\mathcal{K} = (b+d+c+2, 3)$	
<p>ID g</p> <p>S103 $T_5^2+T_1^{(2c-b-5d-l)}T_2^lT_4T_6^5+T_3T_7^5,$ $d \geq 1, c > 4d-1, c-d-1 \leq b < 2c-5d,$ $0 \leq l \leq (2c-b-5d)/2$</p>	<p>ID g</p> <p>S104 $T_5^2+T_1^{(2c-6d-l)}T_2^lT_6^6+T_3T_7^5,$ $d \geq 1, c > 3d, b > 1, c-d-1 \leq b < 2c,$ $0 < l \leq c-3d, l \text{ odd}$</p>

Finally, let us compare our results with existing classifications.

Remark 5.12. The 447 sporadic cases from Classification lists 5.1 to 5.11 encompass in particular the smooth Fano fourfolds with hypersurface Cox ring of Picard number two and torus action of complexity one. The following table translates their ID's in the present classification to the cases of [12, Thm. 1.2].

Theorem 1.2 in [12]	ID	Theorem 1.2 in [12]	ID
1	84 (5.2)	5: $m = 1, a = 0$	228 (5.3)
2	20 (5.2)	7: $m = 1$	253 (5.5)
4.A: $m = 1, c = -1$	45 (5.2)	10: $m = 2$	244 (5.5)
4.A: $m = 1, c = 0$	1 (5.1)	11: $m = 2, a_2 = 1$	225 (5.2)
4.B: $m = 1$	44 (5.2)	11: $m = 2, a_2 = 2$	226 (5.2)
4.C: $m = 1$	6 (5.1)	12: $m = 2$	15 (5.1)

Remark 5.13. At least 268 varieties from our classification admit a *one-parameter smoothing* to a smooth Fano fourfold of Picard number two. Here, by a one-parameter smoothing of X we mean a flat morphism of varieties $\varphi: \mathcal{X} \rightarrow \mathbb{C}$ such that $\mathcal{X}_0 := \varphi^{-1}(0)$ is isomorphic to X and there is a non-empty open subset $U \subseteq \mathbb{C}$ such that $\mathcal{X}_t := \varphi^{-1}(t)$ is smooth for all $t \in U$. The procedure to explicitly construct such a smoothing goes as follows: Let $X = X(Q, g)$ with (Q, g) from the lists 5.1 to 5.11 and assume that up to a unimodular transformation the data $Q = [w_1, \dots, w_7]$ and $\mu = \deg(g)$ appears in [18, Thm. 1.1]. Then there is a homogeneous spread polynomial h of degree $\deg(h) = \mu$ such that X_h is a smooth Fano fourfold with general hypersurface Cox ring. We extend the action of $H = (\mathbb{C}^*)^2$ on \mathbb{C}^7 given by the grading map Q to \mathbb{C}^8 by letting H act trivially on the last coordinate. We set

$$\bar{\mathcal{Z}} = \mathbb{C}^8, \quad \hat{\mathcal{Z}} = \bar{\mathcal{Z}}^{ss}(\tau), \quad \mathcal{Z} = \hat{\mathcal{Z}} // H,$$

where $\tau \in \Lambda(\mathbb{C}[T_1, \dots, T_7, T])$ is the unique GIT-cone that contains the anti-canonical class $-\mathcal{K}_X$ in its interior. Moreover we set

$$\bar{\mathcal{X}} := V((1-T)g + Th) \subseteq \mathbb{C}^8, \quad \hat{\mathcal{X}} = \bar{\mathcal{X}} \cap \hat{\mathcal{Z}}, \quad \mathcal{X} = \hat{\mathcal{X}} // H.$$

The projection $\text{pr}: \hat{\mathcal{X}} \rightarrow \mathbb{C}$ to the last coordinate is H -invariant and thus factors through a morphism $\varphi: \mathcal{X} \rightarrow \mathbb{C}$. We have $X \cong \varphi^{-1}(0)$ and φ is a smoothing of X with fiber over $t = 1$ isomorphic to X_h . In the following table, for each entry (Q, μ) from the table in [18, Thm. 1.1] we list the IDs of the varieties $X(Q, g)$ from the present classification that admit such an explicit smoothing to a smooth Fano fourfold of Picard number two with a general hypersurface Cox ring and data (Q, μ) .

[18, Thm. 1.1]	IDs	[18, Thm. 1.1]	IDs
1	1	34	81 - 84
2	2, 3	35	85 - 91
3	4, 5	36	92 - 101
4	6	37	102 - 107
5	-	38	108 - 129
6	7, 8	39	130 - 148
7	19, 20	40	149
8	21, 22	41	150, 151
9	23 - 25	42	152, 153
10	-	43	154 - 158
11	26, 27	44	159, 160
12	28 - 30	45	161 - 166
13	227, 228	46	167 - 180
14	230 - 232; S2: $a = 1$	47	181 - 219
15	265 - 270	48	244
16	31, 32	49	245
17	33, 34	50	246, 247
18	35 - 39	51	248 - 251
19	40 - 43	52	411 - 414
20	44, 45	53	252, 253
21	242, 243	54	254
22	46, 47	55	255 - 257
23	48 - 50	56	258 - 261
24	51, 52	57	262 - 264
25	53 - 71	58	9, 10
26	72	59	11 - 13
27	73	60	220 - 222
28	74 - 76	61	14
29	77	62	223
30	78	63	16
31	79	64	224
32	80	65	15
33	415 - 418	66	225
		67	226

With the smoothing procedure from above one obtains a one-parameter smoothing of the variety no. 17 in Classification list 5.1 to $X = Y \times \mathbb{P}^1$, where $Y \subseteq \mathbb{P}^4$ is a smooth quartic. The specifying data of X is missing from [18, Thm. 1.1].

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