LE MATEMATICHE
Vol. LV (2000) – Fasc. II, pp. 373–385

THE USUAL CASTELNUOVO'S ARGUMENT AND SPECIAL SUBHOMALOIDAL SYSTEMS OF QUADRICS

ALBERTO ALZATI - FRANCESCO RUSSO

Dedicated to Silvio Greco in occasion of his 60-th birthday.

We prove that a linearly normal n-dimensional variety $X_n \subset \mathbb{P}^r$, regular if $n \geq 2$, and of degree $d \leq 2(r-n)$ is the scheme-theoretic intersection of the quadrics through it by using the usual Castelnuovo's argument that $2r+1$ points in $\mathbb{P}^r$ in general linear position impose independent conditions to quadric hypersurfaces; if $d \leq 2(r-n)-1$, we apply the same argument to show that $H^0(\mathcal{I}_X(2))$ gives a special subhomaloidal system whose base locus is clearly X. If moreover $Sec(X) \subset \mathbb{P}^r$, then the linear system is homaloidal. We apply these results to show that some varieties $X_n \subset \mathbb{P}^{2n+1}$ have one apparent double point and we also study the relations between the linear system of quadrics defining a variety and the locus of secant lines passing through a general point of the ambient space.

The first author was partially supported by the M.U.R.S.T. of the Italian Government in the framework of the national Research Project "Geometry on algebraic varieties" (Cofin 2000).

The second author was partially supported by CNPq (Centro Nacional de Pesquisa), grant 300761/97-0.

Introduction.

A linear system of hypersurfaces of $\mathbb{P}^r$ is said *homaloidal* if it defines a birational map onto the image and *subhomaloidal* if the (closure of a) general fiber of the associated rational map is a linear projective space. It is said *special* if the base locus scheme of the linear system is a smooth irreducible subvariety $X \subset \mathbb{P}^r$. Special (sub)homaloidal systems of quadrics deserved great interest. Well known examples are given by quadrics through a rational normal curve of degree 4 (general representation of $\mathbb{G}(1, 3)$, see [17], [18]), by quadrics through a normal elliptic quintic curve (quadro-cubo transformation of $\mathbb{P}^4$, see [18]) or by quadrics through a Veronese surface in $\mathbb{P}^5$ (quadro-quadric involutory transformation of $\mathbb{P}^5$, see [18], [8]), by quadrics through a rational octic surface in $\mathbb{P}^6$ (see [19], [12]). These special homaloidal systems have also the property that the map they define is an isomorphism on $\mathbb{P}^r \setminus Sec(X)$. Several interesting examples of special subhomaloidal systems are discussed in [18], [8] and [12] and most of the examples of homaloidal systems are obtained from the subhomaloidal ones by restriction to a general linear space of dimension equal to the codimension of the generic fiber. Viceversa once we have a special homaloidal system and we know that the base locus can be extended non-trivially we could hope to find a special subhomaloidal system.

The main result of the paper is the generalization of the fact that the above mentioned varieties are scheme theoretically defined by the quadrics through them and of the fact that those linear systems of quadrics are subhomaloidal by giving a simple and direct geometric proof. In fact in theorem 1, and corollary 1, we prove that if the degree d of a linearly normal, regular if $n = dim(X) \geq 2$, variety $X \subset \mathbb{P}^r$ is such that $d \leq 2(r - n) - 1$, then the quadrics through X give a special subhomaloidal system whose base locus is exactly X. Moreover, if $Sec(X) \subset \mathbb{P}^r$, then this linear system is homaloidal, giving an embedding off $Sec(X)$. By the way we prove geometrically that a linearly normal variety $X \subset \mathbb{P}^r$, regular if $n \geq 2$, of degree $d \leq 2(r - n)$ is the scheme theoretic intersection of the quadrics through it, a result which is interesting in its own.

Actually, the above assumptions on X imply that it satisfies property N_p of Green for $p = 2(r - n) + 1 - d$, which in turn implies the quoted consequences (see [1], proposition 2). Here we have given a direct geometric proof, based on Castelnuovo's idea that a set of $2s + 1$ points in general linear position in $\mathbb{P}^s$ impose independent conditions to quadric hypersurfaces. We would like to remark that this is also the fundamental step in the proofs of results on the resolution of the ideal of a curve or of an algebraic variety defined by quadratic equations (see [10] and [11]), i.e. in showing that property N_p holds for some $p \geq 1$.

In the second section we study the fiber of a rational map defined by quadrics which scheme theoretically define a variety $X \subset \mathbb{P}^r$, showing some relations between the fiber of the map and the entry locus of the variety. In the third section we apply the above results to varieties with one apparent double point: these are n-dimensional varieties $X_n \subset \mathbb{P}^{2n+1}$ such that through the general point of $\mathbb{P}^{2n+1}$ there passes a unique secant line to X, or equivalently such that the projection from a general point of $\mathbb{P}^{2n+1}$ is a variety with only a double point as singularity. The first result shows that a variety $X_n \subset \mathbb{P}^{2n+1}$ scheme theoretically defined by quadrics has one apparent double point if the image of the rational map associated to $|H^0(\mathcal{I}_X(2))|$ has dimension $2n$; in this case the irreducible components of a general fiber of the associated rational map are secant lines to X and hence through a general point of $\mathbb{P}^{2n+1}$ there passes a unique secant line to X. In particular if $X_n \subset \mathbb{P}^{2n+1}$ is scheme theoretically defined by a subhomaloidal linear system of quadrics having as a general fiber a line, then X has one apparent double point. Then we apply this remark to verify that some interesting examples of varieties have one apparent double point; combining the above result with Theorem 1 we prove that a linearly normal regular variety $X_n \subset \mathbb{P}^{2n+1}$ with $Sec(X) = \mathbb{P}^{2n+1}$ and with degree $d \leq 2n+1$ has one apparent double point. Most of these results on varieties with one apparent double point are contained in [1], where more sophisticated methods were employed and where all varieties $X_n \subset \mathbb{P}^{2n+1}$ with one apparent double point and of degree $d \leq 2n+4$ were classified. We insist about the elementary and geometric nature of the arguments used here.

Acknowledgments. The idea of applying Castelnuovo's argument to subhomaloidal systems was suggested to us by Ciro Ciliberto some time ago. We would like to thank him for this and for many other suggestions and discussions, which helped and stimulated us very much during last years. We also thank Andrei Inshakov for some useful remarks on an old version of the paper.

1. Special subhomaloidal systems of quadrics.

We give a geometric and direct proof of the fact that quadrics through a linearly normal, regular if $dim(X) \geq 2$, variety $X \subset \mathbb{P}^r$, whose degree is sufficiently small with respect to codimension, define a special subhomaloidal system of quadrics. If $Sec(X) \subset \mathbb{P}^r$, then the associated rational map is an embedding off $Sec(X)$ and the linear system is homaloidal.

Theorem 1. *Let $X \subset \mathbb{P}^r$ be a smooth linearly normal variety such that $Sec(X) \subset \mathbb{P}^r$. Suppose $h^1(\mathcal{O}_X) = 0$, if dim(X) ≥ 2. Let $s = codim(X)$*

and let $d = deg(X)$. If $d \leq 2s - 1$, then the quadrics through X give a special homaloidal system, defining an isomorphism on $\mathbb{P}^r \setminus Sec(X)$.

We would like to remark that by virtue of Lemma 5 below varieties satisfying the hypothesis of the proposition are scheme theoretic intersection of the quadrics through them.

Let $X \subset \mathbb{P}^r$ be a smooth non-degenerate variety of degree $d = 3$, then X is either a cubic hypersurface or $s = codim(X) = 2$ (remember that for a non-degenerate variety $d \geq s + 1$). In the last case X is the Segre variety $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$ or one of its linear sections. In any case X has ideal generated by 3 quadrics giving a rational map $\phi : \mathbb{P}^r - - \to \mathbb{P}^2$, $r = 3, 4, 5$. To verify that the general fiber is a $\mathbb{P}^{r-2}$ it suffices to cut with a general $\mathbb{P}^2$, to remark that ϕ induces on this $\mathbb{P}^2$ a standard Cremona transformation so that a general $\mathbb{P}^2$ cuts a general fiber of ϕ in one point, giving the assertion. This fact can be generalized to arbitrary d in the same way using Theorem 1.

Corollary 1. *Let $X \subset \mathbb{P}^r$ be a smooth linearly normal variety. Suppose $h^1(\mathcal{O}_X) = 0$, if $dim(X) \geq 2$. Let $s = codim(X)$ and let $d = deg(X)$. If $d \leq 2s - 1$, then the quadrics through X define a special subhomaloidal system.*

To prove Theorem 1 we need some preliminary results, which we reproduce for the convenience of the reader.

Lemma 1. *Let $X \subset \mathbb{P}^r$ be a smooth algebraic variety scheme theoretically intersection of quadratic forms $F_0, \ldots, F_s$ and let $V =< F_0, \ldots, F_s >$. Let $p, q \in \mathbb{P}^r$ be two distinct points such that the line $< p, q >= L$ is not secant, or tangent, to X but it cuts X in a point h. Then the two points p and q are separated by the linear system $|V|$.*

Proof. The linear system $|V|_{|L}$ has at least a base point h. The residual part gives a base point free linear subsystem of $|\mathcal{O}_L(1)|$ so that it separates the points p and q. Otherwise this linear subsystem would be zero dimensional and would have a base point m, eventually coincident with h; the line L would be secant or tangent to X. $\square$

Lemma 2. *(generalized trisecant lemma) Let $C \subset \mathbb{P}^r$, $r \geq 3$, be an irreducible non-degenerate curve and let k, $1 \leq k \leq r - 2$, be an integer. Then the family of $(k + 2)$-secant k-planes to C has dimension at most k.*

Lemma 3. *Let $C \subset \mathbb{P}^r$, $r \geq 5$, be a non-degenerate irreducible curve and let $L \subset \mathbb{P}^r$ be a line such that $L \cap C = \emptyset$.*

Then, for a general hyperplane H through L, for every $p_1, \ldots, p_s$, $1 \leq s \leq r - 2$, in $H \cap C$ and for every $p, q \in L$, the points $p_1, \ldots, p_s, p, q$ are in *general linear position and the linear space $< p_1, \dots, p_s >$ does not contain other points of C.*

Moreover, given a point $p \in L$, for a general hyperplane H through L, there exist $p_1, \dots, p_{r-1} \in H \cap C$ such that $< p_1, \dots, p_{r-1} > \simeq \mathbb{P}^{r-2}$ does not pass through p.

Proof. To prove the first assertion we can assume $s = r - 2$. Suppose that for a general hyperplane H through L there exists $p_1, \dots, p_{r-2}, p, q$ such that $dim(< p_1, \dots, p_{r-2}, p, q >) \leq r - 2$. Fix a general $\mathbb{P}^{r-2}$ skew with L. Then taking on the curve $C' = \pi_L(C) \subset \mathbb{P}^{r-2}$ the section by a generic hyperplane $H' \subset \mathbb{P}^{r-2}$, necessarily of the form $\pi_L(H)$ with H general through L, the $r - 2$ points $\pi_L(p_1), \dots, \pi_L(p_{r-2})$ would not be in general linear position, in contradiction with the general position lemma.

If for a general hyperplane H through L, there exist $p_1, \dots, p_{r-2} \in H \cap C$ such that $< p_1, \dots, p_{r-2} >$ contains another point of C, then C would have an $(r - 2)$-dimensional family of $(r - 1)$-secant $(r - 3)$-planes, in contradiction with the generalized trisecant lemma.

Let us prove the last assertion. By Lemma 2 for general H through L there exist $p_1, \dots, p_{r-1} \in H \cap C$ such that $W_H =< p_1, \dots, p_{r-1} >$ is an hyperplane in H. Now fix a general $\mathbb{P}^{r-1}$ not passing through p: if every W_H passes through p, then the curve $\pi_p(C) = C' \subset \mathbb{P}^{r-1}$ would have an $(r - 2)$-dimensional family of $(r - 1)$-secant $(r - 3)$-planes, in contradiction with the generalized trisecant lemma. $\square$

Lemma 4. *Let $C \subset \mathbb{P}^r$, $r \geq 5$, be an irreducible linearly normal curve of degree $d \leq 2r - 3$. Let $p, q \in \mathbb{P}^r$ such that $L =< p, q > \cap C = \emptyset$. Then the points p and q are separated by the linear system $|H^0(\mathcal{I}_C(2))|$.*

Proof. Let H be a general hyperplane through L. By Lemma 3 there exist $p_1, \dots, p_{r-1} \in \Gamma = H \cap C$, such that $W =< p_1, \dots, p_{r-1} > \simeq \mathbb{P}^{r-2}$ does not pass through q (or viceversa through p).

We proceed in the proof by applying the idea of Castelnuovo to construct (reducible) quadrics containing Γ and satisfying the other properties. Let $\#\Gamma = r - 1 + k \leq 2r - 3$. Since H is general, the remaining k points $p_r, \dots, p_{r-1+k}$, $k \leq r - 2$, p and q are in general linear position in H by Lemma 3; hence there exists an hyperplane W_p in H containing p and $p_r, \dots, p_{r-1+k}$, but not q (or viceversa there exists the analogous W_q not containing p). Then $W \cdot W_p$ is a quadric in H containing $\Gamma \cup p$ but not q (or viceversa $W \cdot W_q$ contains $\Gamma \cup q$ but not p). Then there exists $Q \in H^0(\mathcal{I}_\Gamma(2))$ vanishing in p but not in q and viceversa. Since C is linearly normal, there exists $Q \in H^0(\mathcal{I}_C(2))$ with the same property. $\square$

Lemma 5. *Let $X \subset \mathbb{P}^r$ be a smooth linearly normal variety. Suppose $h^1(\mathcal{O}_X) = 0$, if $dim(X) \geq 2$. Let $s = codim(X)$ and let $d = deg(X)$. If $d \leq 2s$, then X is the scheme theoretic intersection of the quadrics through it.*

Proof. Let $p \in \mathbb{P}^r \setminus X$ and let M be a general $\mathbb{P}^s$ through p; applying arguments like in the above lemmas, one sees that cutting X with M we will obtain a collection $\Gamma = M \cap X$ of d points, $s+1 \leq d \leq 2s$ in $\mathbb{P}^s$ satisfying the following condition: s of them span a linear space $N = \mathbb{P}^{s-1}$ and the remaining $d-s$ a linear space $N' \simeq \mathbb{P}^{d-s-1}$ such that the points of each of these subsets do not belong to the linear space generated by the other subset. Then the usual Castelnuovo's argument (see Lemma 4) can be applied to show that Γ is the scheme theoretic intersection of the quadrics containing it; since the hypothesis imply that the restriction map $H^0(\mathcal{I}_X(2)) \to H^0(\mathcal{I}_\Gamma(2))$ is an isomorphism, p is not in the intersection of the quadrics containing X. We have proved that X is the set theoretic intersection of $H^0(\mathcal{I}_X(2))$; applying arguments like in the above lemmas to a point $p \in X$, a line L through p not tangent to X and a general $\mathbb{P}^s$ through L, since L cuts X transversally at most only in another point by the first part of the proof, the usual Castelnuovo's argument shows that L is not contained in the intersection of the tangent spaces in p to the quadrics containing X, i.e. X is scheme theoretically the intersection of the quadrics containing it. $\square$

Remark 1. The above results seem to be unknown. For example Lemma 5 implies that the rational surfaces $S \subset \mathbb{P}^6$ considered in [19], [6] and [12] are the scheme-theoretic intersection of the quadrics through them (see also [15] for a different proof of this fact). In the above mentioned papers either the problem was left open or solved by ad-hoc methods.

Proof of theorem 1. We want to show that every $p, q \in \mathbb{P}^r \setminus Sec(X)$ are separated by the linear system $|H^0(\mathcal{I}_X(2))|$. Let $L =< p, q >$ (if $p = q$ we will identify L with a line through p and we will leave to the reader the obvious modification of the argument in this case) be a line not secant or tangent to X. Let us suppose that $L \cap X$ consists of a single point. Since by Lemma 5 X is the scheme theoretic intersection of the quadrics containing it, applying Lemma 1 we obtain that the points p and q are separated by $|H^0(\mathcal{I}_X(2))|$.

We can assume $s \geq 4$ because, for $s \leq 3$, X is necessarily a quartic normal rational curve, a quintic normal elliptic curve or the Veronese surface in $\mathbb{P}^5$, for which the result is well known by classification of varieties of low degree (see for example [9] for $s = 2$ when $2s-1 = d = s+1$ and [13] for the case $s = 3$, i.e. $5 = 2s-1 \geq d \geq s+1 = 4$).

Let us assume $L \cap X = \emptyset$. Let M be a general linear space of dimension $s+1$ through L. Then $C = M \cap X$ is an irreducible curve of degree $d \leq$

$2(s+1)-3 = 2s-1$ and clearly $L \cap C = \emptyset$. The hypothesis imply also that C is linearly normal and that the restriction map $H^0(\mathbb{P}^r, \mathcal{I}_X(2)) \to H^0(\mathbb{P}^{s+1}, \mathcal{I}_C(2))$ induces an isomorphism. Since $s \geq 4$ Lemma 4 gives that the points p and q are separated by $|H^0(\mathcal{I}_C(2))|$ and hence by $|H^0(\mathcal{I}_X(2))|$.

Then $|H^0(\mathcal{I}_X(2))|$ is a special homaloidal system of quadrics defining an isomorphism on $\mathbb{P}^r \setminus Sec(X)$. $\square$

Proof of corollary 1. By the discussion before the statement of the corollary we can suppose $s \geq 3$. Let $k \leq r$ be the greatest integer such that the general linear section of X with a $\mathbb{P}^k$ is a smooth variety Y of dimension greater than or equal to 1 such that $Sec(Y) \subset \mathbb{P}^k$; since $s \geq 3$ we have $k \geq 4$. Then Y is a linearly normal variety of degree $d \leq 2s-1$ and such that $h^1(\mathcal{O}_Y) = 0$, if $dim(Y) \geq 2$ by Kodaira's vanishing theorem. Applying Theorem 1 to Y we see that a general fiber of the rational map given by $|H^0(\mathcal{I}_X(2))|$ cuts a general $\mathbb{P}^k$ in a point, giving the assertion. $\square$

2. Varieties which are scheme theoretic intersection of quadrics.

Let $X \subset \mathbb{P}^r$ be scheme theoretically defined by the quadrics through it. Assume, as usual, that X is smooth and non-degenerate. Let us consider the rational map Φ associated to the linear system $V = H^0(\mathbb{P}^r, \mathcal{I}_X(2))$, $dim(V) = \alpha + 1$. Let P be a generic point of $\mathbb{P}^r$ out of X and let $\underline{z}$ be its coordinates. Let us consider the Jacobian matrix of Φ, evaluated at P :

$$J_{\Phi|P} = \begin{bmatrix} gradF_{0|P} \\ gradF_{1|P} \\ \cdots\cdots\cdots \\ gradF_{\alpha|P} \end{bmatrix} = 2 \begin{bmatrix} \underline{z}_t A_0 \\ \underline{z}_t A_1 \\ \cdots\cdots \\ \underline{z}_t A_\alpha \end{bmatrix},$$

where t means transposition, and $A_0, \ldots, A_\alpha$ are the matrices of the corresponding quadrics. Note that $J_{\Phi|P}$ is a matrix of type $(\alpha+1, r+1)$. Let $\rho(P)$ be the rank of $J_{\Phi|P}$ and let $L_P = \{\underline{x} \in \mathbb{P}^r | \underline{z}_t A_i \underline{x} = 0, i \geq 1\}$.

Let us consider $\Phi_P := \overline{\Phi^{-1}[\Phi(P)]}$ and let us prove the following.

Lemma 6. *With the previous notation:* $T_P(\Phi_P) = L_P$ *and* $\dim(L_P) = r + 1 - \rho(P)$ *where* T_P *is the Zariski tangent space to* Φ_P *at* P.

Proof. Let us choose a coordinate system in $\mathbb{P}^\alpha$ such that $\Phi(P) \equiv (1 : 0 : \ldots : 0)$. Hence we have that $\Phi_P \cup X = \{\underline{x} \in \mathbb{P}^r | \underline{x}_t A_i \underline{x} = 0, i \geq 1\}$ and $T_P(\Phi_P) = \{\underline{x} \in \mathbb{P}^r | \underline{z}_t A_i \underline{x} = 0, i \geq 1\} = L_P$. Consider $J_{\Phi|P}$: the first row can not be a linear combination of the other ones, otherwise we would have: $\underline{z}_t A_0 = \sum \lambda_j \underline{z}_t A_j$; then $\underline{z}_t A_0 \underline{z} = \sum \lambda_j \underline{z}_t A_j \underline{z}$, but this is not possible since

$F_0(\underline{z}) \neq 0$ and $F_i(\underline{z}) = 0$ for $i \geq 1$. Therefore the rank of the matrix obtained from $J_{\Phi|P}$ by dropping out the first row is $\rho(P) - 1$; on the other hand this rank is the number of linearly independent hyperplanes defining L_P so that $\dim(L_P) = r - [\rho(P) - 1] = r + 1 - \rho(P)$. $\square$

Now, for any $P \in Sec(X) \setminus X$, let us define C_P as the (closure of the) set of points $Q \in \mathbb{P}^r$ such that the line $\overline{PQ}$ is a secant line for X. Obviously, C_P is never empty and the *entry locus of P with respect to X*,

$$\Sigma_P = \overline{\{x \in X \mid \exists y \in X \ y \neq x \ \text{ or } \ x = y : P \in \langle x, y \rangle\}},$$

is the intersection between X and C_P as sets.

We have essentially proved the next result.

Lemma 7. *For generic P the cone C_P is set-theoretically the intersection of $\Phi_P \cup X$ with $T_P(\Phi_P) = L_P$.*

Proof. As above we can assume that $\Phi(P) \equiv (1 : 0 : ..... : 0)$. Let Q be another point of $\mathbb{P}^r$, $Q \neq P$, and let $\underline{x}$ be its coordinates. To find the intersections between X and the line $\overline{PQ}$ we have to find the values $(\lambda : \mu)$ for which the following equations are satisfied:

$$(\lambda\underline{z} + \mu\underline{x})_t A_i (\lambda\underline{z} + \mu\underline{x}) = 0 \text{ for any } i \geq 0; \text{ i.e.}$$

$$\lambda^2(\underline{z}_t A_i \underline{z}) + 2\lambda\mu(\underline{z}_t A_i \underline{x}) + \mu^2(\underline{x}_t A_i \underline{x}) = 0 \text{ for any } i \geq 0; \ (*)$$

moreover $\overline{PQ}$ is a secant line to X if and only if all equations $(*)$ have the same solutions. This happens if and only if the following matrix

$$\begin{bmatrix} \underline{z}_t A_0 \underline{z} & \underline{z}_t A_0 \underline{x} & \underline{x}_t A_0 \underline{x} \\ 0 & \underline{z}_t A_1 \underline{x} & \underline{x}_t A_1 \underline{x} \\ 0 & \underline{z}_t A_2 \underline{x} & \underline{x}_t A_2 \underline{x} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ 0 & \underline{z}_t A_\alpha \underline{x} & \underline{x}_t A_\alpha \underline{x} \end{bmatrix}$$

has rank one and hence if and only if $\underline{z}_t A_i \underline{x} = \underline{x}_t A_i \underline{x} = 0$ for any $i \geq 1$ (recall that $\underline{z}_t A_0 \underline{z} \neq 0$). We have that

$$C_P = \{\underline{x} \in \mathbb{P}^r | \underline{z}_t A_i \underline{x} = \underline{x}_t A_i \underline{x} = 0, i \geq 1\}_{red},$$

that

$$\Phi_P \cup X = \{\underline{x} \in \mathbb{P}^r | \underline{x}_t A_i \underline{x} = 0, i \geq 1\}$$

and that

$$T_P(\Phi_P) = \{\underline{x} \in \mathbb{P}^r | \underline{z}_t A_i \underline{x} = 0, i \geq 1\} = L_P;$$

therefore $C_P = (\Phi_P \cup X) \cap L_P$ as sets. Moreover C_P is the intersection of the α cones: $\{\underline{z}_t A_i \underline{x} = \underline{x}_t A_i \underline{x} = 0\} i = 1, 2, \ldots, \alpha$, each of them being singular at P. $\square$

3. Varieties with one apparent double point.

We now apply the results of the previous sections to varieties with one apparent double point.

Proposition 1. *Let $X \subset \mathbb{P}^{2n+1}$ be a smooth, non-degenerate, variety of dimension n with $Sec(X) = \mathbb{P}^{2n+1}$, which is the scheme theoretic intersection of $\alpha + 1$ quadrics. Let $\Phi : \mathbb{P}^{2n+1} \backslash X \to \mathbb{P}^{\alpha}$ be the associated rational map and let $\rho(\Phi)$ be the rank of J_Φ at the generic point of $\mathbb{P}^{2n+1}$. If dim (Im $(\Phi)) = 2n$ (or equivalently $\rho(\Phi) = 2n + 1$), then X is a variety with one apparent double point. In particular if $X \subset \mathbb{P}^{2n+1}$ is scheme theoretically defined by a subhomaloidal linear systems of quadrics whose general fiber is a line, then X has one apparent double point.*

Proof. Recall that we can regard Φ as a regular map between smooth varieties, hence, for generic $P \in \mathbb{P}^{2n+1}$, Φ_P is smooth and of pure dimension $2n + 1 -$ dim (Im(Φ)). Moreover since it is defined by quadrics vanishing on X, Φ contracts secant lines to X to points. Hence in our hypothesis dim$(\Phi_P) = 1$ and every secant line passing through an arbitrary point $Q \in \mathbb{P}^{2n+1}$ is contained in Φ_Q. Since $Sec(X) = \mathbb{P}^{2n+1}$, there is at least a secant line to X passing through a general point $P \in \mathbb{P}^{2n+1}$, so that every irreducible component of a general fiber of Φ is a secant line to X and hence there passes a unique secant line to X through a general point of $\mathbb{P}^{2n+1}$, i.e. X is a variety with one apparent double point. Using the notations of the previous section, one could also argue as follows. By Lemma 6, L_P is a line tangent to Φ_P at P. As $Sec(X) = \mathbb{P}^{2n+1}$ there is at least a secant line to X, passing through P so that C_P is not empty and, by Lemma 7, C_P is contained in L_P. Therefore $C_P = L_P$ as schemes and for generic $P \in \mathbb{P}^{2n+1}$ there passes a unique secant line to X i.e. X is a variety with one apparent double point.

If X is scheme theoretically defined by a subhomaloidal system having a line as general fiber, then arguing as above, the general fiber is a secant line to X. $\square$

Corollary 2. *Let $X \subset \mathbb{P}^{2n+1}$ be a smooth linearly normal n-dimensional variety of degree $d \leq 2n + 1$ with $Sec(X) = \mathbb{P}^{2n+1}$. Suppose $h^1(\mathcal{O}_X) = 0$, if $n \geq 2$. Then X has one apparent double point.*

Proof. By corollary 1 we know that $|H^0(\mathcal{I}_X(2))|$ is a subhomaloidal linear system giving a rational map $\Phi : \mathbb{P}^{2n+1} - - \to \mathbb{P}H^0(\mathcal{I}_X(2)))$ which is not defined exactly along X and having as general fiber a projective space $\mathbb{P}^k$, with $k \geq 1$ since $Sec(X) = \mathbb{P}^{2n+1}$. To show that X is a variety with one apparent double point it is sufficient to prove that the general fiber of Φ is a line, necessarily secant to X. Cutting X with a general hyperplane H we obtain a variety $Y = X \cap H$ for which the hypothesis of Theorem 1 are satisfied; the restriction to H of Φ is then birational and the general fiber of Φ is a line. $\square$

As an application of the results we now prove that some varieties have one apparent double point.

Example 1. (*Smooth n-dimensional varieties of minimal degree $d = n+2$ in $\mathbb{P}^{2n+1}$, whose secant variety fill the whole space have one apparent double point*). We construct two series of varieties $X_n^{n+2} \subset \mathbb{P}^{2n+1}$, each of which has one apparent double point. These are the smooth n-dimensional rational normal scrolls in $\mathbb{P}^{2n+1}$ of degree $n+2$ $S(\underbrace{1,\ldots,1}_{n-2},2,2)$, $n \geq 2$, or $S(\underbrace{1,\ldots,1}_{n-1},3)$, $n \geq 1$. These varieties are linearly normal, have $h^1(\mathcal{O}) = 0$, their secant varieties fill the whole space and have degree $d = n+2 \leq 2(n+1)-1$; they have one apparent double point by corollary 2. Moreover, these are the only smooth $X_n^{n+2} \subset \mathbb{P}^{2n+1}$, whose secant varieties fill the whole space by a classical theorem of Bertini and Del Pezzo. For $n \geq 2$ these varieties can be realized as divisors of type $(2,1)$ on the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^n$ into $\mathbb{P}^{2n+1}$. For $n = 2$, $S(2,2)$ can be realized also as a divisor of type $(0,2)$ on the Segre 3-fold $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$. From the description of these varieties as divisors on the Segre variety $\mathbb{P}^1 \times \mathbb{P}^n$ into $\mathbb{P}^{2n+1}$, Edge showed geometrically that they have one apparent double point (see [7]).

We recall a construction, due to Edge and Babbage (see [7] and [3]) of two series of varieties $X_n \subset \mathbb{P}^{2n+1}$, $n \geq 1$ having one apparent double point.

Example 2. (*Edge varieties have one apparent double point*). Let $Y_{n+1} = \mathbb{P}^1 \times \mathbb{P}^n \subset \mathbb{P}^{2n+1}$ be the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^n$. Then Y_{n+1} is an $(n+1)$-dimensional subvariety of degree $n+1$. Let π be one of the $\mathbb{P}^n$'s of the ruling of Y_{n+1} and let Q_1 be a general quadric of $\mathbb{P}^{2n+1}$ containing π. Then the residual intersection of Q_1 with Y_{n+1} is an n-dimensional smooth variety X_n^{2n+1} of degree $2n+1$ in $\mathbb{P}^{2n+1}$ and it is a divisor of type $(1,2)$ on Y_{n+1}. Let now π_α and π_β be two fixed n-planes of the ruling of Y_{n+1}. If Q_2 is a general quadric through π_α and π_β, then the residual intersection of Q_2 with Y_{n+1} is an n-dimensional smooth variety X_n^{2n} of degree $2n$ and it is a divisor of type $(0,2)$ on Y_{n+1}, i.e. $X_n^{2n} \simeq \mathbb{P}^1 \times Q^{n-1}$. By applying corollary 2 we see that the above varieties have one apparent double point since they are linearly normal, have $h^1(\mathcal{O}_X) = 0$ and degree $d \leq 2(n+1)-1 = 2n+1$. Edge shows geometrically that these varieties have one apparent double point (see [7], I.2–I.5). The above construction gives, for $n = 1$, as X_1^3 the twisted cubic and as X_1^2 a pair of lines in a ruling of the quadric Y_1. This last one is not irreducible. For $n = 2$, X_2^5 is a Del Pezzo surface of degree 5 in $\mathbb{P}^5$, while the surface X_2^4 is a rational normal scroll $S(2,2)$ of degree 4 in $\mathbb{P}^5$. It is a classical theorem of Severi, recently completed in [16], that these two surfaces together with $S(1,3)$ are the unique surfaces with one apparent double point.

One could verify that some interesting examples of varieties $X_n \subset \mathbb{P}^{2n+1}$, even of degree $d > 2n+1$, have one apparent double point using the idea contained in the proof of corollary 2.

Example 3. (*Smooth 3-folds of degree 8 in $\mathbb{P}^7$ with one apparent double point*). Let $X = \mathbb{P}(\mathcal{E}) \subset \mathbb{P}^7$ be the scroll over $\mathbb{P}^2$ associated to the very ample vector bundle $\mathcal{E}$ of rank 2 given as an extension by the following exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{E} \to \mathcal{I}_{p_1,\dots,p_8}(4) \to 0,$$

where $p_1, \dots, p_8$ are points in $\mathbb{P}^2$ such that no 4 of them are collinear and no 8 of them lie on a conic. Such a vector bundle exists (see [13]). A general hyperplane section of X corresponding to a general section of $\mathcal{E}$ is a smooth octic rational surface Y, the embedding of the blow-up of $\mathbb{P}^2$ at the p_i's given by the quartics through the p_i's, is arithmetically Cohen-Macaulay and it is cut out by 7 quadrics, which define a Cremona transformation of $\mathbb{P}^6$ (see [12], [19] and also [15]). Hence X is a linearly normal 3-fold scheme theoretically intersection of the seven quadrics through it. In fact we know by Lemma 5 that X is the scheme theoretic intersection of the quadrics trough it. To show that $h^0(\mathcal{I}_X(2)) = 7$, we simply remark that cutting with a general $\mathbb{P}^4$ we obtain a set $\Gamma \subset \mathbb{P}^4$ of eight points in general linear position which impose independent conditions to quadrics by the usual Castelnuovo' s argument. Since X is linearly normal and $h^1(\mathcal{O}_X) = 0$, we have $h^0(\mathcal{I}_X(2)) = h^0(\mathcal{I}_\Gamma(2)) = h^0(\mathcal{O}_{\mathbb{P}^4}(2)) - 8 = 15 - 8 = 7$. We now prove that $Sec(X) = \mathbb{P}^7$ and that X has one apparent double point. Let $\Phi : \mathbb{P}^7 - - \to \mathbb{P}^6$ be the rational map given by the quadrics through X; Φ is defined outside X and contracts all the secant lines to X. Since restricting Φ to a general hyperplane H we obtain a Cremona transformation, the general fiber of Φ is a line and necessarily a secant line to X because Φ is defined by the quadrics through X.

Question 1. *Is it true that if X is a variety with one apparent double point, then X is scheme theoretically defined by a subhomaloidal linear system of quadrics such that the (closure of the) generic fibre is a line ?*

We do not know the answer. By the previous results we can say that if X is the scheme theoretic intersection of quadrics, then, for generic $P \in \mathbb{P}^{2n+1}$, Φ_P is smooth and it contains the unique secant lines to X passing through P.

In general, even if one dimensional, Φ_P could be the disjoint union of a finite number of secant lines to X.

REFERENCES

[1] A. Alzati - F. Russo, *Special subhomaloidal systems of quadrics and varieties with one apparent double point,* Math. Proc. Camb. Phil. Soc., 133 (2002).

[2] A. Alzati - F. Russo, *On the k-normality of projected algebraic varieties,* to appear on Bull. of the Brazilian Math. Soc..

[3] D. W. Babbage, *A series of rational loci with one apparent double point,* Proc. Camb. Phil. Soc., 27 (1931), pp. 399–403.

[4] A. Bertram - L. Ein - R. Lazarsfeld, *Vanishing theorems, a theorem of Severi, and the equations defining projective varieties,* J. of the A.M.S., 4 (1991), pp. 587–602.

[5] G. Castelnuovo, *Sui multipli di una serie lineare di gruppi di punti appartenente ad una curva algebrica,* Rend. Circ. Mat. Palermo, 7 (1893), pp. 89–110.

[6] B. Crauder - S. Katz, *Cremona transfromations with smooth base locus,* Amer. J. Math., 111 (1989), pp. 289–309.

[7] W.L. Edge, *The number of apparent double points of certain loci,* Proc. Cambridge Philos. Soc., 28 (1932), pp. 285–299.

[8] L. Ein - N. Shepherd-Barron, *Some special Cremona transformation,* Amer. J. Math., 11 (1989), pp. 783–800.

[9] D. Eisenbud - J. Harris, *On varieties of minimal degree,* Alg. Geometry Bowdin 1985, Proc. Symp. in Pure Math., 46 (1987), pp. 3–13.

[10] M. Green, *Koszul cohomology and the geometry of projective varieties,* J. Diff. Geometry, 19 (1984), pp. 125–171.

[11] M. Green - R. Lazarsfeld, *Some results on the syzygies of finite sets and algebraic curves,* Compositio Math., 67 (1988), pp. 301–314.

[12] K. Hulek - S. Katz - F.O. Schreyer, *Cremona transformations and syzygies,* Math. Z., 209 (1992), pp. 419–443.

[13] P. Ionescu, *Embedded projective varieties of small invariants,* L. N. M. 1056, pp. 142–186.

[14] P. Ionescu, *Embedded projective varieties of small invariants. III,* L. N. M. 1417, pp. 138–154.

[15] M. Mella - F. Russo, *On Cremona transformations whose fundamental locus is a smooth irreducible surface,* submitted.

[16] F. Russo, *On a theorem of Severi,* Math. Ann., 316 (2000), pp. 1–17.

[17] J.G. Semple, *On representations of the S_k's of S_n and of the Grassmann manifold $\mathbb{G}(k, n)$,* Proc. London. Math. Soc., 32 (1931), pp. 200–221.

[18] J.G. Semple - L. Roth, *Introduction to Algebraic Geometry,* Oxford University Press, 1949 and 1986.

[19] J.G. Semple - J.A. Tyrrell, *The $T_{2,4}$ of S_6 defined by a rational surface F^8,* Proc. Lond. Math. Soc., 20 (1970), pp. 205–221.

[20] F. Severi, *Intorno ai punti doppi impropri di una superficie generale dello spazio a quattro dimensioni e ai suoi punti tripli apparenti,* Rend. Circ. Mat. Palermo, 15 (1901), pp. 33–51.

[21] P. Vermeire, *Some results on secant varieties leading to a geometric flip construction,* Comp. Math., 125 (2001), pp. 263–282.

[22] F.L. Zak, *Tangents and secants of algebraic varieties,* Translations of Mathematical Monografhs, vol. 127, Amer. Math. Soc., 1993.

Alberto Alzati
Dipartimento di Matematica "F. Enriques",
Universitá degli Studi di Milano,
Via C. Saldini 50,
20133 Milano (ITALY)
email: alzati@mat.unimi.it

Francesco Russo
Departamento de Matematica,
Universidade Federal de Pernambuco,
Cidade Universitaria,
50670-901 Recife–PE (BRASIL)
email: frusso@dmat.ufpe.br