LE MATEMATICHE
Vol. LXXX (2025) – Issue II, pp. 701–727
doi: 10.4418/2025.80.2.10

LOCAL AND GLOBAL SOLVABILITY IN ATTRACTION-REPULSION CHEMOTAXIS SYSTEMS WITH L^2-INITIAL DATA

Y. CHIYO - T. SAGA - T. YOKOTA

This paper deals with the attraction-repulsion chemotaxis system

$$\begin{cases} u_t = a\Delta u - \chi\nabla\cdot(u\nabla v) + \xi\nabla\cdot(u\nabla w), & x\in\Omega,\ t>0,\\ \tau v_t = b\Delta v + \alpha u - \beta v, & x\in\Omega,\ t>0,\\ 0 = c\Delta w + \gamma u - \delta w, & x\in\Omega,\ t>0 \end{cases}$$

under homogeneous Neumann initial-boundary conditions, where $\Omega\subset\mathbb{R}^n$ ($n\le 3$) is a smoothly bounded domain and $a,b,c,\chi,\xi,\alpha,\beta,\gamma,\delta>0$ and $\tau\in\{0,1\}$ are constants. The purpose of the present paper is to construct a local solution of this system for any L^2-initial data without additional conditions on χ and ξ by using the theory for abstract evolution equations and to extend the local solution globally in the repulsion-dominant case by relying on a priori estimates.

1. Introduction

Problem. In this paper we apply the theory for abstract evolution equations (Yagi [22]) to establish local and global existence of solutions to the Neumann

Received on February 10, 2025

AMS 2010 Subject Classification: 35K51, 35A01, 35Q92, 35B45

Keywords: abstract evolution equations, L^2-initial data, local existence, attraction-repulsion chemotaxis systems

Y.C. is partially supported by JSPS KAKENHI Grant Number JP24K22838.

T.Y. is partially supported by JSPS KAKENHI Grant Number JP21K03278.

initial-boundary value problem

$$\begin{cases} u_t = a\Delta u - \chi \nabla \cdot (u \nabla v) + \xi \nabla \cdot (u \nabla w), & x \in \Omega,\ t > 0, \\ \tau v_t = b\Delta v + \alpha u - \beta v, & x \in \Omega,\ t > 0, \\ 0 = c\Delta w + \gamma u - \delta w, & x \in \Omega,\ t > 0, \\ \nabla u \cdot \nu = \nabla v \cdot \nu = \nabla w \cdot \nu = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x),\ \tau v(x,0) = \tau v_0(x), & x \in \Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^n$ ($n \leq 3$) is a bounded domain with smooth boundary $\partial\Omega$, $\tau \in \{0,1\}$ and $a,b,c,\chi,\xi,\alpha,\beta,\gamma,\delta > 0$ are constants, and ν is the outward normal vector to $\partial\Omega$. This attraction-repulsion chemotaxis system was proposed by Luca et al. [13] to describes aggregation of *Microglia* in Alzheimer's disease. In this system the functions u,v and w idealize the cell density, the concentrations of the chemoattractant and chemorepellent, respectively.

The system consisting of the first and second equations in (1) with $\xi = 0$ is well-known as the classical Keller–Segel system ([12]), which has been studied by many researchers. For example, Osaki and Yagi [20] showed local and global solvability of generalized systems including

$$\begin{cases} u_t = a\Delta u - \chi \nabla \cdot (u \nabla v), & x \in \Omega,\ t > 0, \\ v_t = b\Delta v + \alpha u - \beta v, & x \in \Omega,\ t > 0 \end{cases} \tag{2}$$

in the one-dimensional setting by rewriting (2) as the evolution equation

$$\frac{dU}{dt} + AU = F(U), \quad U = (u,v),$$

where A is a suitable linear operator, and F is some nonlinear operator which contains chemotaxis terms. After that, many types of Keller–Segel systems have been investigated (see [1, 2, 14–19, 21], for instance).

Going back to the attraction-repulsion chemotaxis system (1), we can find a lot of literatures dealing with the case that the initial data is regular. Also, some related models with nonlinear production or saturation were studied by [6–9]. On the other hand, there are few studies on the setting that the regularity of initial data is low. For example, Heihoff [10] showed existence of global-in-time classical solutions in the two-dimensional fully parabolic case and in at most three-dimensional parabolic–elliptic–elliptic case when the initial data u_0 is a positive Radon measure or belongs to $L^k(\Omega)$ for some $k \in (1,2)$. In this direction, to the best of our knowledge, the literature [10] is the only work for attraction-repulsion chemotaxis systems. However, since the proof in [10] is based on the global-in-time approximate solutions corresponding to smooth initial data, it is imposed that $\frac{\chi\alpha}{b} - \frac{\xi\gamma}{c} \le 0$ with $b = c = 1$. Thus, in this paper we consider the question whether we can establish local existence in (1) with L^2-initial data as in [20] without any condition on the sign of $\frac{\chi\alpha}{b} - \frac{\xi\gamma}{c}$.

Main results. We define function spaces $H^2_{\mathrm{N}}(\Omega)$ and $\mathcal{H}^4_{\mathrm{NN}}(\Omega)$ as

$$\begin{aligned} H^2_{\mathrm{N}}(\Omega) &:= \{\varphi \in H^2(\Omega) \mid \nabla\varphi\cdot\nu = 0 \quad \text{on } \partial\Omega\}, \\ \mathcal{H}^4_{\mathrm{NN}}(\Omega) &:= \{\varphi \in H^2_{\mathrm{N}}(\Omega) \mid \Delta\varphi \in H^2_{\mathrm{N}}(\Omega)\}. \end{aligned}$$

We now state theorems which guarantee local and global existence in (1).

Theorem 1.1 (Local existence; $\tau = 0$). *Let $\Omega \subset \mathbb{R}^n$ $(n \le 3)$ be a bounded domain with C^2-boundary. Let $\tau = 0$. Then for all nonnegative data $u_0 \in L^2(\Omega)$ satisfying $u_0 \not\equiv 0$, there exists $T_{u_0} > 0$ such that* (1) *with $\tau = 0$ admits a unique local solution (u, v, w) such that $u, v, w \ge 0$ in $\overline{\Omega} \times [0, T_{u_0}]$ and that*

$$\begin{cases} u \in C^0([0,T_{u_0}];L^2(\Omega)) \cap C^0((0,T_{u_0}];H^2_{\mathrm{N}}(\Omega)) \cap C^1((0,T_{u_0}];L^2(\Omega)), \\ v, w \in C^0([0,T_{u_0}];H^2_{\mathrm{N}}(\Omega)) \cap C^0((0,T_{u_0}];\mathcal{H}^4_{\mathrm{NN}}(\Omega)). \end{cases}$$

Theorem 1.2 (Global existence; $\tau = 0$). *Let $\Omega \subset \mathbb{R}^n$ $(n \le 3)$ be as in Theorem* 1.1. *Let $\tau = 0$. Assume that*

$$\frac{\chi\alpha}{b} - \frac{\xi\gamma}{c} < 0. \tag{3}$$

Then for all nonnegative data $u_0 \in L^2(\Omega)$ satisfying $u_0 \not\equiv 0$, there exists a unique triplet (u, v, w) of nonnegative functions

$$\begin{cases} u \in C^0([0,\infty);L^2(\Omega)) \cap C^0((0,\infty);H^2_{\mathrm{N}}(\Omega)) \cap C^1((0,\infty);L^2(\Omega)), \\ v, w \in C^0([0,\infty);H^2_{\mathrm{N}}(\Omega)) \cap C^0((0,\infty);\mathcal{H}^4_{\mathrm{NN}}(\Omega)), \end{cases}$$

which solves the problem (1) *with $\tau = 0$.*

Theorem 1.3 (Local existence; $\tau = 1$). *Let $\Omega \subset \mathbb{R}^n$ $(n \le 3)$ be as in Theorem* 1.1. *Let $\tau = 1$. Then for all nonnegative data $u_0 \in L^2(\Omega)$ and $v_0 \in H^2_{\mathrm{N}}(\Omega)$ satisfying $u_0, v_0 \not\equiv 0$, there exists $T_{u_0,v_0} > 0$ such that* (1) *with $\tau = 1$ admits a unique local solution (u, v, w) such that $u, v, w \ge 0$ in $\overline{\Omega} \times [0, T_{u_0,v_0}]$ and that*

$$\begin{cases} u \in C^0([0,T_{u_0,v_0}];L^2(\Omega)) \cap C^0((0,T_{u_0,v_0}];H^2_{\mathrm{N}}(\Omega)) \cap C^1((0,T_{u_0,v_0}];L^2(\Omega)), \\ v \in C^0([0,T_{u_0,v_0}];H^2_{\mathrm{N}}(\Omega)) \cap C^0((0,T_{u_0,v_0}];\mathcal{H}^4_{\mathrm{NN}}(\Omega)) \cap C^1((0,T_{u_0,v_0}];H^2_{\mathrm{N}}(\Omega)), \\ w \in C^0([0,T_{u_0,v_0}];H^2_{\mathrm{N}}(\Omega)) \cap C^0((0,T_{u_0,v_0}];\mathcal{H}^4_{\mathrm{NN}}(\Omega)). \end{cases}$$

Theorem 1.4 (Global existence; $\tau = 1$)**.** *Let $\Omega \subset \mathbb{R}^n$ $(n \le 3)$ be a bounded domain with C^3-boundary. Let $\tau = 1$. Assume that*

$$C_{\mathrm{MR}}^{\frac{1}{3}} \frac{\chi\alpha}{b} - \frac{\xi\gamma}{c} < 0, \tag{4}$$

where $C_{\mathrm{MR}} > 0$ is a constant given in Lemma 2.7 below. Then for all nonnegative data $u_0 \in L^2(\Omega), v_0 \in H^2_{\mathrm{N}}(\Omega)$ satisfying $u_0, v_0 \ne 0$, there exists a unique triplet (u,v,w) of nonnegative functions

$$\begin{cases} u \in C^0([0,\infty);L^2(\Omega)) \cap C^0((0,\infty);H^2_{\mathrm{N}}(\Omega)) \cap C^1((0,\infty);L^2(\Omega)), \\ v \in C^0([0,\infty);H^2_{\mathrm{N}}(\Omega)) \cap C^0((0,\infty);\mathcal{H}^4_{\mathrm{NN}}(\Omega)) \cap C^1((0,\infty);H^2_{\mathrm{N}}(\Omega)), \\ w \in C^0([0,\infty);H^2_{\mathrm{N}}(\Omega)) \cap C^0((0,\infty);\mathcal{H}^4_{\mathrm{NN}}(\Omega)), \end{cases}$$

which solves the problem (1) *with $\tau = 1$.*

We can provide further information about solvability of (1). To state this precisely we note that if Ω is of class C^3, then $\mathcal{H}^4_{\mathrm{NN}}(\Omega) \subset H^3(\Omega)$. Thus, for the global-in-time solution (u,v,w) obtained by Theorem 1.2 (resp. Theorem 1.4) and for all $\varepsilon > 0$, by the Sobolev embedding theorem, we have $u(\varepsilon) \in C^0(\overline{\Omega})$ and $v(\varepsilon) \in W^{1,\infty}(\Omega)$. Thus, in view of [5, Theorem 3.5] (resp. [3, Theorem 1.1 with $n \in \{1,2\}$]), the following corollary holds.

Corollary 1.5. *In Theorem* 1.2 *assume further that Ω is of class C^3. In Theorem* 1.4 *suppose further that $n \in \{1,2\}$. Then for $\tau \in \{0,1\}$ the unique triplet (u,v,w) satisfies that $u,v,w \ge 0$ in $\overline{\Omega} \times [0,\infty)$ and that*

$$\begin{cases} u \in C^{2,1}(\overline{\Omega} \times (0,\infty)), \\ v,w \in \bigcap_{\vartheta > n} C^0((0,\infty);W^{1,\vartheta}(\Omega)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)), \end{cases}$$

which solves the problem (1) *classically.*

Strategies for proving local and global existence. The strategies for proving Theorems 1.1 and 1.3 are based on Yagi [22]. Specifically, in the proof of Theorem 1.1 we rewrite the problem (1) with $\tau = 0$ as the evolution equation in $L^2(\Omega)$,

$$\frac{du}{dt} + A_1 u = F(u),$$

where

$$\begin{aligned} &A_1 u := -a\Delta u + u, \\ &F(u) := u - \chi\nabla\cdot(u\nabla v) + \xi\nabla\cdot(u\nabla w) \\ &\text{with} \quad v := \alpha(-b\Delta + \beta I)^{-1}u, \quad w := \gamma(-c\Delta + \delta I)^{-1}u. \end{aligned}$$

Also, in the proof of Theorem 1.3 we formulate (1) with $\tau = 1$ as the evolution equation in $L^2(\Omega) \times H^2_{\mathrm{N}}(\Omega)$,

$$\frac{dU}{dt} + AU = F(U),$$

where

$$\begin{aligned}
&A := \begin{pmatrix} -a\Delta + I & 0 \\ -\alpha I & -b\Delta + \beta I \end{pmatrix}, \\
&F(U) := \begin{pmatrix} u - \chi \nabla \cdot (u \nabla v) + \xi \nabla \cdot (u \nabla w) \\ 0 \end{pmatrix}, \quad U = (u, v) \\
&\text{with} \quad w := \gamma(-c\Delta + \delta I)^{-1} u.
\end{aligned}$$

Then we show local existence by applying an abstract local existence result (Lemma 2.1) to these equations. In order to construct global solutions we invoke an abstract global existence result (Lemma 2.2). To see this in the proof of Theorem 1.2 we derive uniform-in-time boundedness of $\|u(t)\|^2_{L^2(\Omega)}$. Also, in the proof of Theorem 1.4 we first construct a differential inequality for $\zeta \|u(t)\|^2_{L^2(\Omega)} + \|A_2 v(t)\|^2_{L^2(\Omega)}$ with some $\zeta > 1$ and $A_2 v := -b\Delta v + \beta v$. We next employ the maximal Sobolev regularity as in [11, Proof of Proposition 3.2] to show uniform-in-time boundedness of $\zeta \|u(t)\|^2_{L^2(\Omega)} + \|A_2 v(t)\|^2_{L^2(\Omega)}$.

Plan of the paper. This paper is organized as follows. In Section 2 we collect some results on local and global existence in the abstract Cauchy problem for semilinear evolution equations in a Banach space, and recall some properties of sectorial operators in some Hilbert spaces, as well as give basic inequalities. Sections 3 and 4 are devoted to the proofs of local and global existence in the parabolic–elliptic–elliptic case ($\tau = 0$) and the parabolic–parabolic–elliptic case ($\tau = 1$), respectively.

Throughout this paper, we denote by c_j generic positive constants, which will be sometimes specified by $c_j(\varepsilon)$ depending on small parameters $\varepsilon > 0$.

2. Preliminaries

In this section we first focus on the abstract Cauchy problem for semilinear evolution equations,

$$\begin{cases} \dfrac{dU}{dt} + AU = F(U), \quad t > 0, \\ U(0) = U_0 \end{cases} \tag{5}$$

in a Banach space X defined over the complex field $\mathbb{C}$. Here A is a sectorial operator in X with angle $\omega_A \in (0, \frac{\pi}{2})$ satisfying

$$\sigma(A) \subset \Sigma_\omega := \{\lambda \in \mathbb{C} \mid |\arg \lambda| < \omega\}, \quad \omega_A < \omega < \frac{\pi}{2}, \tag{6}$$

and

$$\|(\lambda I - A)^{-1}\| \le \frac{M_\omega}{|\lambda|}, \quad \lambda \notin \Sigma_\omega, \quad \omega_A < \omega < \frac{\pi}{2}, \tag{7}$$

where $\sigma(A)$ is the spectrum of A and $M_\omega \ge 1$ is some constant. Meanwhile $F : D(A^\eta) \to X$ is a nonlinear operator with $\eta \in [0,1)$.

We now recall results on local- and global-in-time existence in the problem (5) (see [22, Theorem 4.4 and Corollary 4.3 (pp. 188–189)]).

Lemma 2.1. *Let A satisfy* (6) *and* (7)*. Assume that F fulfills*

$$\begin{aligned} \|F(U) - F(V)\|_X \le \varphi(\|U\|_X + \|V\|_X) \\ \cdot \left[\|A^\eta(U-V)\|_X + (\|A^\eta U\|_X + \|A^\eta V\|_X)\|U-V\|_X\right] \end{aligned} \tag{8}$$

for all $U, V \in D(A^\eta)$ with some $\eta \in [0,1)$, where φ is some increasing continuous function. Then for any $U_0 \in X$, the problem (5) *possesses a unique local solution*

$$U \in C^0([0,T_{U_0}];X) \cap C^0((0,T_{U_0}];D(A)) \cap C^1((0,T_{U_0}];X),$$

where $T_{U_0} > 0$ depends only on the norm $\|U_0\|_X$.

Lemma 2.2. *Under the assumption of Lemma* 2.1*, let $U_0 \in X$. Assume that any local solution U of* (5) *in the function space*

$$C^0([0,T_U];X) \cap C^0((0,T_U];D(A)) \cap C^1((0,T_U];X)$$

satisfies the estimate

$$\|U(t)\|_X \le C_{U_0}, \quad 0 \le t \le T_U$$

with some constant $C_{U_0} > 0$ independent of $T_U > 0$. Then the problem (5) *admits a unique global solution U in the function space*

$$C^0([0,\infty);X) \cap C^0((0,\infty);D(A)) \cap C^1((0,\infty);X).$$

We next give some lemmas to construct a local solution of the problem (1). In the following four lemmas we deal with complex-valued functions. We define elliptic operators A_1, A_2, A_3 in $L^2(\Omega)$ as follows:

$$A_1 u := -a\Delta u + u, \quad u \in D(A_1) := H^2_{\mathrm{N}}(\Omega),$$

$$A_2 v := -b\Delta v + \beta v, \quad v \in D(A_2) := H^2_{\mathrm{N}}(\Omega),$$

$$A_3 w := -c\Delta w + \delta w, \quad w \in D(A_3) := H^2_{\mathrm{N}}(\Omega).$$

The next lemma states a characterization of fractional powers of elliptic operators in $L^2(\Omega)$ (see [22, Theorems 2.2 and 16.7 (pp. 59 and 547)] for the proof).

Lemma 2.3. *Let $\Omega \subset \mathbb{R}^n$ $(n \in \mathbb{N})$ be a bounded domain with C^2-boundary. Then for $i = 1,2,3$, the operators A_i defined as above are sectorial in $L^2(\Omega)$ satisfying* (6) *and* (7). *Moreover,*

$$D(A_i^\theta) = [L^2(\Omega), H^2_{\mathrm{N}}(\Omega)]_\theta = \begin{cases} H^{2\theta}(\Omega) & \text{if} \quad 0 \le \theta < \dfrac{3}{4}, \\ H^{2\theta}_{\mathrm{N}}(\Omega) & \text{if} \quad \dfrac{3}{4} < \theta \le 1 \end{cases}$$

with norm equivalence

$$C^{-1}\|u\|_{H^{2\theta}(\Omega)} \le \|A_i^\theta u\|_{L^2(\Omega)} \le C\|u\|_{H^{2\theta}(\Omega)}, \quad u \in D(A_i^\theta), \tag{9}$$

where $C > 0$ is a constant.

We also regard A_2 and A_3 as operators from $\mathcal{H}^4_{\mathrm{NN}}(\Omega) = D(A_i^2)$ into $D(A_i) = H^2_{\mathrm{N}}(\Omega)$ $(i \in \{2,3\})$. Then it follows from Lemma 2.3 that for $i \in \{2,3\}$,

$$D(A_i^\theta) = \begin{cases} \mathcal{H}^{2\theta}_{\mathrm{N}}(\Omega) = \{v \in H^2_{\mathrm{N}}(\Omega) \mid \Delta v \in H^{2(\theta-1)}(\Omega)\} & \text{if } 1 \le \theta < \dfrac{7}{4}, \\ \mathcal{H}^{2\theta}_{\mathrm{NN}}(\Omega) = \{v \in H^2_{\mathrm{N}}(\Omega) \mid \Delta v \in H^{2(\theta-1)}_{\mathrm{N}}(\Omega)\} & \text{if } \dfrac{7}{4} < \theta \le 2. \end{cases}$$

We next consider sectorial operators in the product space $L^2(\Omega) \times H^2_{\mathrm{N}}(\Omega)$. The following lemma can be proved similarly to the proof of [22, Theorem 2.16 (p. 82)].

Lemma 2.4. *Let $\Omega \subset \mathbb{R}^n$ $(n \in \mathbb{N})$ be a bounded domain with C^2-boundary. Then the matrix operator A defined as*

$$A := \begin{pmatrix} A_1 & 0 \\ -\alpha I & A_2 \end{pmatrix}, \quad D(A) := D(A_1) \times D(A_2) \tag{10}$$

is a sectorial operator in $L^2(\Omega) \times H^2_{\mathrm{N}}(\Omega)$ with angle $0 < \omega_A < \frac{\pi}{2}$ satisfying (6) *and* (7), *where $I \in \mathcal{L}(D(A_1); H^2_{\mathrm{N}}(\Omega))$ is an identity operator.*

Let A_{D} be a diagonal operator in $L^2(\Omega) \times H^2_{\mathrm{N}}(\Omega)$ defined as

$$A_{\mathrm{D}} := \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \quad D(A_{\mathrm{D}}) := D(A_1) \times D(A_2). \tag{11}$$

Then A_{D} is a selfadjoint operator and $A_{\mathrm{D}}^\theta = \mathrm{diag}\{A_1^\theta, A_2^\theta\}$. In particular,

$$D(A_{\mathrm{D}}^\theta) = \begin{cases} \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \,\middle|\, u \in H^{2\theta}(\Omega) \text{ and } v \in \mathcal{H}_{\mathrm{N}}^{2(\theta+1)}(\Omega) \right\} & \text{if } 0 \le \theta < \frac{3}{4}, \\ \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \,\middle|\, u \in H_{\mathrm{N}}^{2\theta}(\Omega) \text{ and } v \in \mathcal{H}_{\mathrm{NN}}^{2(\theta+1)}(\Omega) \right\} & \text{if } \frac{3}{4} < \theta \le 1. \end{cases}$$

The next lemma has been established in [22, Proposition 12.1 (p. 421)].

Lemma 2.5. *Let A be the matrix operator defined as* (10) *and A_{D} the matrix operator defined as* (11)*. Then*

$$D(A^\theta) = D(A_{\mathrm{D}}^\theta), \quad \theta \in [0,1]$$

with norm equivalence

$$C^{-1} \left\| A_{\mathrm{D}}^\theta U \right\|_{L^2(\Omega) \times H^2_{\mathrm{N}}(\Omega)} \le \left\| A^\theta U \right\|_{L^2(\Omega) \times H^2_{\mathrm{N}}(\Omega)} \le C \left\| A_{\mathrm{D}}^\theta U \right\|_{L^2(\Omega) \times H^2_{\mathrm{N}}(\Omega)} \tag{12}$$

for all $U \in L^2(\Omega) \times H^2_{\mathrm{N}}(\Omega)$, where $C > 0$ is a constant.

Finally, we present some inequalities. The following inequality plays an important role in deriving local-in-time existence.

Lemma 2.6. *Let $\Omega \subset \mathbb{R}^n$ $(n \le 3)$ be a bounded domain with C^2-boundary. Let $\varepsilon \in (0, \frac{1}{2})$ and*

$$\theta := \begin{cases} \dfrac{1+\varepsilon}{2} & \textit{if } n \in \{1,2\}, \\ \dfrac{3}{4} + \dfrac{\varepsilon}{2} & \textit{if } n = 3. \end{cases} \tag{13}$$

Then there exists a constant $C > 0$ such that

$$\|\nabla \cdot (u \nabla v)\|_{L^2(\Omega)} \le C \|u\|_{H^{2\theta}(\Omega)} \|v\|_{H^2(\Omega)}, \quad u \in H^{2\theta}(\Omega),\ v \in H^2(\Omega). \tag{14}$$

Proof. Let $u \in H^{2\theta}(\Omega)$ and $v \in H^2(\Omega)$. Then we note that

$$\|\nabla \cdot (u \nabla v)\|_{L^2(\Omega)} \le \|\nabla u \cdot \nabla v\|_{L^2(\Omega)} + \|u \Delta v\|_{L^2(\Omega)}. \tag{15}$$

From the Sobolev embedding theorem, we infer that $H^{2\theta}(\Omega) \hookrightarrow L^\infty(\Omega)$ and hence,

$$\|u\Delta v\|_{L^2(\Omega)} \le \|u\|_{L^\infty(\Omega)}\|\Delta v\|_{L^2(\Omega)} \le c_1\|u\|_{H^{2\theta}(\Omega)}\|v\|_{H^2(\Omega)}. \tag{16}$$

In order to obtain the estimate (14) it is sufficient to show

$$\|\nabla u\cdot\nabla v\|_{L^2(\Omega)} \le c_2\|u\|_{H^{2\theta}(\Omega)}\|v\|_{H^2(\Omega)}. \tag{17}$$

We first consider the case $n=1$. From the Sobolev embedding $H^1(\Omega)\hookrightarrow L^\infty(\Omega)$ we have

$$\begin{aligned}\|\nabla u\cdot\nabla v\|_{L^2(\Omega)} &\le \|\nabla u\|_{L^2(\Omega)}\|\nabla v\|_{L^\infty(\Omega)}\\ &\le c_3\|u\|_{H^1(\Omega)}\|\nabla v\|_{H^1(\Omega)}\\ &\le c_4\|u\|_{H^{2\theta}(\Omega)}\|v\|_{H^2(\Omega)},\end{aligned}$$

which means (17). In order to show the cases $n=2,3$ we note that $2\theta=\frac{n}{2}+\varepsilon$ by (13). Therefore, since $2\theta-1<\frac{n}{2}$, again by the Sobolev embedding theorem, it follows that

$$H^{2\theta-1}(\Omega)\hookrightarrow L^r(\Omega), \quad r\in\left(2,\frac{n}{1-\varepsilon}\right),$$

which implies

$$H^{2\theta}(\Omega)\hookrightarrow W^{1,r}(\Omega), \quad r\in\left(2,\frac{n}{1-\varepsilon}\right). \tag{18}$$

Let us consider the case $n=2$. Since for all $r\in(2,\frac{2}{1-\varepsilon})$ there exists $q\in(2,\infty)$ such that $\frac{1}{q}+\frac{1}{r}=\frac{1}{2}$, we observe that

$$\|\nabla u\cdot\nabla v\|_{L^2(\Omega)} \le \|\nabla u\|_{L^r(\Omega)}\|\nabla v\|_{L^q(\Omega)}. \tag{19}$$

Applying the Sobolev embeddings (18) and $H^1(\Omega)\hookrightarrow L^q(\Omega)$ for $q\in(2,\infty)$ to (19), we obtain

$$\begin{aligned}\|\nabla u\cdot\nabla v\|_{L^2(\Omega)} &\le c_5\,\|u\|_{W^{1,r}(\Omega)}\,\|\nabla v\|_{H^1(\Omega)}\\ &\le c_6\|u\|_{H^{2\theta}(\Omega)}\|v\|_{H^2(\Omega)},\end{aligned}$$

which means (17) in the case $n=2$. We finally consider the case $n=3$. The Sobolev embedding theorem derives

$$H^1(\Omega)\hookrightarrow L^q(\Omega), \quad q\in(2,6). \tag{20}$$

Moreover, we see that (19) holds with $q\in(3,\infty)$ and $r\in(3,\frac{3}{1-\varepsilon})$ satisfying $\frac{1}{q}+\frac{1}{r}=\frac{1}{2}$. Therefore, as in the case $n=2$, the inequality (17) is shown by (18), (19) and (20). Thus, combining (16) and (17) with (15), we arrive at the conclusion. □

We next deal with real-valued functions in the following two lemmas, which will be used to show a priori estimates for solutions. Let $T \in (0,\infty]$. Then we consider the inhomogeneous linear heat equation

$$\begin{cases} z_t = b\Delta z - \widetilde{\beta} z + \alpha f & \text{in } \Omega \times (0,T), \\ \nabla z \cdot \nu = 0 & \text{on } \partial\Omega \times (0,T), \\ z(\cdot,0) = z_0 & \text{in } \Omega, \end{cases} \tag{21}$$

where Ω is a bounded domain with C^3-boundary, $f \in L^3(0,T;L^3(\Omega))$ and $\widetilde{\beta} > 0$. The next lemma gives an estimate for solutions of (21) (see [11, Lemma 2.1] for the proof).

Lemma 2.7. *Let $\Omega \subset \mathbb{R}^n$ $(n \in \mathbb{N})$ be a bounded domain with C^3-boundary and let $z_0 \in W^{2,3}(\Omega)$ with $\nabla z_0 \cdot \nu = 0$ on $\partial\Omega$. If $f \in L^3(0,T;L^3(\Omega))$, then the solution z of (21) belongs to $L^3(0,T;W^{2,3}(\Omega))$ and satisfies*

$$\int_0^T \|\Delta z(t)\|_{L^3(\Omega)}^3\,dt \le C_{\mathrm{MR}} \left(\|\Delta z_0\|_{L^3(\Omega)}^3 + \left(\frac{\alpha}{b}\right)^3 \int_0^T \|\Delta f(t)\|_{L^3(\Omega)}^3\,dt \right),$$

where $C_{\mathrm{MR}} > 0$ is a constant independent of T. Moreover, assume that $t_0 \in (0,T)$ and $z(t_0) \in W^{2,3}(\Omega)$ with $\nabla z(t_0) \cdot \nu = 0$ on $\partial\Omega$. Then

$$\int_{t_0}^T \|\Delta z(t)\|_{L^3(\Omega)}^3\,dt \le C_{\mathrm{MR}} \left(\|\Delta z(t_0)\|_{L^3(\Omega)}^3 + \left(\frac{\alpha}{b}\right)^3 \int_{t_0}^T \|\Delta f(t)\|_{L^3(\Omega)}^3\,dt \right).$$

We finally state the Gagliardo–Nirenberg type inequality (see [4, Lemma 2.3]).

Lemma 2.8. *Let $\Omega \subset \mathbb{R}^n$ $(n \in \mathbb{N})$ be a bounded domain with C^2-boundary. Then there exists a constant $C_{\mathrm{GN}} > 0$ such that*

$$\|\varphi\|_{L^2(\Omega)}^2 \le C_{\mathrm{GN}} \left(\|\nabla\varphi\|_{L^2(\Omega)}^{2\kappa} \|\varphi\|_{L^1(\Omega)}^{2(1-\kappa)} + \|\varphi\|_{L^1(\Omega)}^2 \right), \quad \varphi \in H^1(\Omega),$$

where the power κ is determined as

$$\kappa := \frac{1}{2}\left(\frac{1}{n} + \frac{1}{2}\right)^{-1} \in (0,1).$$

3. The parabolic–elliptic–elliptic case $\tau = 0$

3.1. Local existence

In this subsection we prove Theorem 1.1. Let $u_0 \in L^2(\Omega)$ be a nonnegative function satisfying $u_0 \not\equiv 0$. The proof of Theorem 1.1 consists of two steps:

Step 1. Let $X := L^2(\Omega) = L^2(\Omega;\mathbb{C})$. To begin with, we formulate the problem (1) with $\tau = 0$ as the Cauchy problem for an abstract evolution equation in the Hilbert space X. Firstly, since $\tau = 0$, we see from the second and third equations (1) with $\tau = 0$ that

$$v = \alpha(-b\Delta + \beta I)^{-1}u, \tag{22}$$

$$\|v\|_{H^2(\Omega)} \le c_1\|u\|_{L^2(\Omega)}, \tag{23}$$

$$w = \gamma(-c\Delta + \delta I)^{-1}u, \tag{24}$$

$$\|w\|_{H^2(\Omega)} \le c_2\|u\|_{L^2(\Omega)}, \tag{25}$$

where I is the identity operator in X. Secondly, let us define a nonlinear operator F as

$$F(u,v,w) := u - \chi\nabla\cdot(u\nabla v) + \xi\nabla\cdot(u\nabla w).$$

Then from (22) and (24), we can regard $F(u,v,w)$ as $F(u)$. Hence we can rewrite the first equation in (1) as

$$\frac{du}{dt} + A_1u = F(u), \tag{26}$$

where $A_1 : D(A_1) \subset L^2(\Omega) \to L^2(\Omega)$ is defined as

$$A_1u := -a\Delta u + u, \quad D(A_1) := H^2_{\mathrm{N}}(\Omega).$$

Then Lemma 2.3 ensures that A_1 satisfies (6) and (7). To use Lemma 2.1 we next confirm that F fulfills the condition (8). Let θ be a constant defined in (13) and take $u_1, u_2 \in D(A_1^\theta)$. For $i \in \{1,2\}$, we set

$$v_i := \alpha(-b\Delta + \beta I)^{-1}u_i$$

and

$$w_i := \gamma(-c\Delta + \delta I)^{-1}u_i.$$

Then we can compute as

$$\begin{aligned}\|F(u_1) - F(u_2)\|_{L^2(\Omega)} &= \|[u_1 - \chi\nabla\cdot(u_1\nabla v_1) + \xi\nabla\cdot(u_1\nabla w_1)] \\ &\qquad - [u_2 - \chi\nabla\cdot(u_2\nabla v_2) + \xi\nabla\cdot(u_2\nabla w_2)]\|_{L^2(\Omega)} \\ &\le \chi\|\nabla\cdot(u_1\nabla v_1) - \nabla\cdot(u_2\nabla v_2)\|_{L^2(\Omega)} \\ &\quad + \xi\|\nabla\cdot(u_1\nabla w_1) - \nabla\cdot(u_2\nabla w_2)\|_{L^2(\Omega)} \\ &\quad + \|u_1 - u_2\|_{L^2(\Omega)}.\end{aligned} \tag{27}$$

Rewriting the first term on the right-hand side of (27), we have

$$\begin{aligned}
&\chi\|\nabla\cdot(u_1\nabla v_1)-\nabla\cdot(u_2\nabla v_2)\|_{L^2(\Omega)}\\
&=\chi\|\nabla\cdot(u_1\nabla v_1)-\nabla\cdot(u_1\nabla v_2)+\nabla\cdot(u_1\nabla v_2)-\nabla\cdot(u_2\nabla v_2)\|_{L^2(\Omega)}\\
&=\chi\|\nabla\cdot[u_1\nabla(v_1-v_2)]+\nabla\cdot[(u_1-u_2)\nabla v_2]\|_{L^2(\Omega)}.
\end{aligned}$$

Due to (14), we infer that

$$\begin{aligned}
&\chi\|\nabla\cdot(u_1\nabla v_1)-\nabla\cdot(u_2\nabla v_2)\|_{L^2(\Omega)}\\
&\le\chi\|\nabla\cdot[u_1\nabla(v_1-v_2)]\|_{L^2(\Omega)}+\chi\|\nabla\cdot[(u_1-u_2)\nabla v_2]\|_{L^2(\Omega)}\\
&\le c_3\chi\|u_1\|_{H^{2\theta}(\Omega)}\|v_1-v_2\|_{H^2(\Omega)}+c_3\chi\|u_1-u_2\|_{H^{2\theta}(\Omega)}\|v_2\|_{H^2(\Omega)}. \qquad (28)
\end{aligned}$$

Similarly, we can obtain

$$\begin{aligned}
&\xi\|\nabla\cdot(u_1\nabla w_1)-\nabla\cdot(u_2\nabla w_2)\|_{L^2(\Omega)}\\
&\le c_3\xi\|u_1\|_{H^{2\theta}(\Omega)}\|w_1-w_2\|_{H^2(\Omega)}+c_3\xi\|u_1-u_2\|_{H^{2\theta}(\Omega)}\|w_2\|_{H^2(\Omega)}. \qquad (29)
\end{aligned}$$

Therefore plugging (28) and (29) into (27) yields

$$\begin{aligned}
&\|F(u_1)-F(u_2)\|_{L^2(\Omega)}\\
&\le c_3\chi\|u_1\|_{H^{2\theta}(\Omega)}\|v_1-v_2\|_{H^2(\Omega)}+c_3\chi\|u_1-u_2\|_{H^{2\theta}(\Omega)}\|v_2\|_{H^2(\Omega)}\\
&\quad+c_3\xi\|u_1\|_{H^{2\theta}(\Omega)}\|w_1-w_2\|_{H^2(\Omega)}+c_3\xi\|u_1-u_2\|_{H^{2\theta}(\Omega)}\|w_2\|_{H^2(\Omega)}\\
&\quad+\|u_1-u_2\|_{H^{2\theta}(\Omega)}. \qquad (30)
\end{aligned}$$

Here, noting that $\|v_1-v_2\|_{H^2(\Omega)}\le c_1\|u_1-u_2\|_{L^2(\Omega)}$, $\|v_2\|_{H^2(\Omega)}\le c_1\|u_2\|_{L^2(\Omega)}$ and $\|w_1-w_2\|_{H^2(\Omega)}\le c_2\|u_1-u_2\|_{L^2(\Omega)}$, $\|w_2\|_{H^2(\Omega)}\le c_2\|u_2\|_{L^2(\Omega)}$ by (23) and (25), respectively, we see from (30) that

$$\begin{aligned}
&\|F(u_1)-F(u_2)\|_{L^2(\Omega)}\\
&\le c_3c_1\chi\left(\|u_1\|_{H^{2\theta}(\Omega)}\|u_1-u_2\|_{L^2(\Omega)}+\|u_1-u_2\|_{H^{2\theta}(\Omega)}\|u_2\|_{L^2(\Omega)}\right)\\
&\quad+c_3c_2\xi\left(\|u_1\|_{H^{2\theta}(\Omega)}\|u_1-u_2\|_{L^2(\Omega)}+\|u_1-u_2\|_{H^{2\theta}(\Omega)}\|u_2\|_{L^2(\Omega)}\right)\\
&\quad+\|u_1-u_2\|_{H^{2\theta}(\Omega)}\\
&=c_3(c_1\chi+c_2\xi)\left(\|u_1\|_{H^{2\theta}(\Omega)}\|u_1-u_2\|_{L^2(\Omega)}+\|u_1-u_2\|_{H^{2\theta}(\Omega)}\|u_2\|_{L^2(\Omega)}\right)\\
&\quad+\|u_1-u_2\|_{H^{2\theta}(\Omega)}.
\end{aligned}$$

This in conjunction with (9) leads to

$$
\begin{aligned}
&\|F(u_1)-F(u_2)\|_{L^2(\Omega)}\\
&\le c_4\Big(\|A_1^\theta u_1\|_{L^2(\Omega)}\|u_1-u_2\|_{L^2(\Omega)}+\|A_1^\theta(u_1-u_2)\|_{L^2(\Omega)}\|u_2\|_{L^2(\Omega)}\\
&\qquad +\|A_1^\theta(u_1-u_2)\|_{L^2(\Omega)}\Big)\\
&\le c_4\Big(1+\|u_1\|_{L^2(\Omega)}+\|u_2\|_{L^2(\Omega)}\Big)\\
&\quad\cdot\Big[\|A_1^\theta(u_1-u_2)\|_{L^2(\Omega)}+\Big(\|A_1^\theta u_1\|_{L^2(\Omega)}+\|A_1^\theta u_2\|_{L^2(\Omega)}\Big)\|u_1-u_2\|_{L^2(\Omega)}\Big],
\end{aligned}
$$

whence, from Lemma 2.1, the problem (26) with $U_0=u_0$ possesses a unique local solution

$$u\in C^0([0,T_{u_0}];L^2(\Omega))\cap C^0((0,T_{u_0}];H^2_{\mathrm{N}}(\Omega))\cap C^1((0,T_{u_0}];L^2(\Omega))$$

with some $T_{u_0}>0$. Also, from (22) and (24), we can verify that

$$v,w\in C^0([0,T_{u_0}];H^2_{\mathrm{N}}(\Omega))\cap C^0((0,T_{u_0}];\mathcal{H}^4_{\mathrm{NN}}(\Omega)). \tag{31}$$

Consequently, we obtain a unique local solution (u,v,w) of (1).

Step 2. We next prove that

$$u(t),v(t),w(t)\ge 0\quad \text{for all } t\in[0,T_{u_0}].$$

Let us first observe that u,v and w are real-valued functions. Since the complex conjugate $(\overline{u},\overline{v},\overline{w})$ of (u,v,w) is also a local solution of the problem (1) with the same initial data u_0, uniqueness of solutions yields

$$u(t)=\overline{u(t)},\ v(t)=\overline{v(t)},\ w(t)=\overline{w(t)}\quad \text{for all } t\in[0,T_{u_0}],$$

so that u,v and w are real-valued functions. Thereafter, we deal with the function spaces $L^2(\Omega)=L^2(\Omega;\mathbb{R})$ and $H^2_{\mathrm{N}}(\Omega)=H^2_{\mathrm{N}}(\Omega;\mathbb{R})$. Let us introduce a cut-off function H defined by

$$H(s):=\begin{cases}\dfrac{1}{2}s^2 & \text{for } s\in(-\infty,0),\\ 0 & \text{for } s\in[0,\infty),\end{cases} \tag{32}$$

and set

$$\Phi(t):=\int_\Omega H(u(t))\,dx\quad \text{for } t\in[0,T_{u_0}].$$

Then, from the first equation in (1), we have

$$\Phi'(t)=\big(H'(u),a\Delta u-\chi\nabla\cdot(u\nabla v)+\xi\nabla\cdot(u\nabla w)\big)_{L^2(\Omega)} \tag{33}$$

for all $t \in (0, T_{u_0}]$. Employing the Green formula and (32), we see that

$$(H'(u), a\Delta u)_{L^2(\Omega)} = -a\int_\Omega \nabla H'(u)\cdot\nabla u\,dx = -a\int_\Omega |\nabla H'(u)|^2\,dx \tag{34}$$

and

$$\begin{aligned}-(H'(u), \chi\nabla\cdot(u\nabla v))_{L^2(\Omega)} &= \chi\int_\Omega u\nabla H'(u)\cdot\nabla v\,dx\\ &= \chi\int_\Omega H'(u)\nabla H'(u)\cdot\nabla v\,dx\\ &= -\frac{\chi}{2}\int_\Omega (H'(u))^2\Delta v\,dx.\end{aligned}$$

Since (31) implies

$$\|\Delta v(t)\|_{L^2(\Omega)} \le c_5 \quad \text{for all } t \in (0, T_{u_0}],$$

using the Hölder inequality and the Young inequality, we can compute as

$$\begin{aligned}-(H'(u), \chi\nabla\cdot(u\nabla v))_{L^2(\Omega)} &\le \frac{\chi}{2}\|H'(u)\|^2_{L^4(\Omega)}\|\Delta v\|_{L^2(\Omega)}\\ &\le \frac{\chi}{2}c_5\|H'(u)\|_{H^1(\Omega)}\|H'(u)\|_{L^2(\Omega)}\\ &\le \frac{a}{2}\|H'(u)\|^2_{H^1(\Omega)} + c_6\|H'(u)\|^2_{L^2(\Omega)}.\end{aligned} \tag{35}$$

Similarly, it follows that

$$(H'(u), \xi\nabla\cdot(u\nabla w))_{L^2(\Omega)} \le \frac{a}{2}\|H'(u)\|^2_{H^1(\Omega)} + c_7\|H'(u)\|^2_{L^2(\Omega)}. \tag{36}$$

Thus, combining (34), (35) and (36) with (33) and recalling (32), we have

$$\begin{aligned}\Phi'(t) &\le a\|H'(u)\|^2_{L^2(\Omega)} + c_6\|H'(u)\|^2_{L^2(\Omega)} + c_7\|H'(u)\|^2_{L^2(\Omega)}\\ &\le (a + c_6 + c_7)\int_\Omega u^2\,dx\\ &= 2(a + c_6 + c_7)\Phi(t).\end{aligned}$$

Hence the inequality $\Phi(t) \le \Phi(0)e^{2(a+c_6+c_7)t}$ holds. Since $\Phi(0) = 0$ by the nonnegativity $u_0 \ge 0$, it follows that $\Phi(t) \equiv 0$ i.e.

$$u(t) \ge 0 \quad \text{for all } t \in [0, T_{u_0}],$$

which along with applications of the comparison principle to the second and third equations in (1) asserts that

$$v(t), w(t) \ge 0 \quad \text{for all } t \in [0, T_{u_0}].$$

This completes the proof of Theorem 1.1. □

3.2. Global existence

Let $u_0 \in L^2(\Omega)$ be a nonnegative function satisfying $u_0 \neq 0$ and let $U = (u,v,w)$ be any local solution of the problem (1) with $\tau = 0$ on $[0,T_U]$ with $T_U > 0$ such that

$$\begin{cases} 0 \le u \in C^0([0,T_U];L^2(\Omega)) \cap C^0((0,T_U];H^2_{\mathrm{N}}(\Omega)) \cap C^1((0,T_U];L^2(\Omega)), \\ 0 \le v,w \in C^0([0,T_U];H^2_{\mathrm{N}}(\Omega)) \cap C^0((0,T_U];\mathcal{H}^4_{\mathrm{NN}}(\Omega)). \end{cases}$$

We will prove an L^2-estimate for u, which together with Lemma 2.2 enables us to verify global existence in (1) with $\tau = 0$.

Proposition 3.1. *Let $\tau = 0$. Assume* (3)*. Then there exists a constant $C > 0$ such that C is independent of T_U and*

$$\|u(t)\|_{L^2(\Omega)} \le C \quad \text{for all } t \in [0,T_U].$$

In order to prove Proposition 3.1 we give two lemmas which are valid for $\tau \in \{0,1\}$. We first recall that u has the mass conservation law.

Lemma 3.2. *Let $\tau \in \{0,1\}$. Then the first component of the solution satisfies that*

$$\|u(t)\|_{L^1(\Omega)} = \|u_0\|_{L^1(\Omega)} \quad \text{for all } t \in [0,T_U].$$

Proof. Integrating the first equation in (1) over Ω, by the Green formula, we have

$$\frac{d}{dt}\int_\Omega u\,dx = \int_\Omega [a\Delta u - \chi\nabla\cdot(u\nabla v) + \xi\nabla\cdot(u\nabla w)]\,dx = 0,$$

so that we arrive at the conclusion. □

The next lemma plays an important role in deriving a priori estimate for u.

Lemma 3.3. *Let $\tau \in \{0,1\}$. Then the first and third components of the solution satisfy that for all $\varepsilon > 0$,*

$$\int_\Omega w^2\,dx \le \varepsilon \int_\Omega u^2\,dx + c(\varepsilon) \quad \text{on } (0,T_U] \tag{37}$$

with some $c(\varepsilon) > 0$.

Proof. Noticing that the second equation in the problem (1) is not used in the proof of [5, Lemma 3.3], we can similarly see that (37) holds for $\tau \in \{0,1\}$. □

Now we proceed to the proof of Proposition 3.1.

Proof of Proposition 3.1. Let $t \in (0, T_U)$. Multiplying the first equation in (1) by u and integrating it over Ω, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2\,dx = -a\int_\Omega |\nabla u|^2\,dx - \frac{\chi}{2}\int_\Omega u^2\Delta v\,dx + \frac{\xi}{2}\int_\Omega u^2\Delta w\,dx. \tag{38}$$

Substituting the second equation in (1) into the second term on the right-hand side of (38) and using the nonnegativity of v, we observe that

$$-\frac{\chi}{2}\int_\Omega u^2\Delta v\,dx = \frac{\chi}{2b}\int_\Omega u^2(-\beta v + \alpha u)\,dx \le \frac{\chi\alpha}{2b}\int_\Omega u^3\,dx. \tag{39}$$

Moreover, the third equation in (1) yields

$$\frac{\xi}{2}\int_\Omega u^2\Delta w\,dx = \frac{\xi}{2c}\int_\Omega u^2(\delta w - \gamma u)\,dx. \tag{40}$$

Combining (39) and (40) with (38), we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega u^2\,dx + a\int_\Omega |\nabla u|^2\,dx &\le \frac{\chi\alpha}{2b}\int_\Omega u^3\,dx + \frac{\xi}{2c}\int_\Omega (\delta u^2 w - \gamma u^3)\,dx \\ &= \frac{1}{2}\left(\frac{\chi\alpha}{b} - \frac{\xi\gamma}{c}\right)\int_\Omega u^3\,dx + \frac{\xi\delta}{2c}\int_\Omega u^2 w\,dx. \end{aligned}\tag{41}$$

We next deal with the term $\frac{\xi\delta}{2c}\int_\Omega u^2 w\,dx$. Take $\varepsilon_1 > 0$ which will be fixed later. Firstly employing the Hölder inequality, and secondly applying Lemma 3.3 with $\varepsilon = (\frac{\varepsilon_1}{2})^3$, we can compute as

$$\begin{aligned}\frac{\xi\delta}{2c}\int_\Omega u^2 w\,dx &\le \frac{\xi\delta}{2c}\left(\int_\Omega u^3\,dx\right)^{\frac{2}{3}}\left(\int_\Omega w^3\,dx\right)^{\frac{1}{3}} \\ &\le \frac{\xi\delta}{2c}\left(\int_\Omega u^3\,dx\right)^{\frac{2}{3}}\left[\left(\frac{\varepsilon_1}{2}\right)^3\int_\Omega u^3\,dx + c_1(\varepsilon_1)\right]^{\frac{1}{3}} \\ &\le \frac{\xi\delta}{4c}\varepsilon_1\int_\Omega u^3\,dx + \frac{\xi\delta}{2c}c_1(\varepsilon_1)^{\frac{1}{3}}\left(\int_\Omega u^3\,dx\right)^{\frac{2}{3}}.\end{aligned}$$

Using the Young inequality, we see that

$$\frac{\xi\delta}{2c}\int_\Omega u^2 w\,dx \le \frac{\xi\delta}{2c}\varepsilon_1\int_\Omega u^3\,dx + c_2(\varepsilon_1). \tag{42}$$

Hence a combination of (41) and (42) implies that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega u^2\,dx + a\int_\Omega |\nabla u|^2\,dx \\ &\le \frac{1}{2}\left(\frac{\chi\alpha}{b} - \frac{\xi\gamma}{c}\right)\int_\Omega u^3\,dx + \frac{\xi\delta}{2c}\varepsilon_1\int_\Omega u^3\,dx + c_2(\varepsilon_1) \\ &= \frac{1}{2}\left(\frac{\chi\alpha}{b} - \frac{\xi\gamma}{c} + \frac{\xi\delta}{c}\varepsilon_1\right)\int_\Omega u^3\,dx + c_2(\varepsilon_1).\end{aligned}$$

Here, due to the condition (3), we can pick $\varepsilon_1 > 0$ satisfying $\varepsilon_1 < \frac{c}{\xi\delta}(\frac{\xi\gamma}{c} - \frac{\chi\alpha}{b})$, that is, $\frac{\chi\alpha}{b} - \frac{\xi\gamma}{c} + \frac{\xi\delta}{c}\varepsilon_1 < 0$. Then we arrive at

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2\,dx + a\int_\Omega |\nabla u|^2\,dx \le c_2. \tag{43}$$

We next estimate the term $a\int_\Omega |\nabla u|^2\,dx$. Thanks to Lemmas 2.8 and 3.2, we see that

$$\begin{aligned}
\|u(t)\|^2_{L^2(\Omega)} &\le c_3\left(\|\nabla u(t)\|^{2\kappa}_{L^2(\Omega)}\|u(t)\|^{2(1-\kappa)}_{L^1(\Omega)} + \|u(t)\|^2_{L^1(\Omega)}\right)\\
&= c_3\left(\|\nabla u(t)\|^{2\kappa}_{L^2(\Omega)}\|u_0\|^{2(1-\kappa)}_{L^1(\Omega)} + \|u_0\|^2_{L^1(\Omega)}\right)\\
&\le c_4\left(\|\nabla u(t)\|^{2\kappa}_{L^2(\Omega)} + 1\right)\\
&\le c_5\left(\|\nabla u(t)\|_{L^2(\Omega)} + 1\right)^{2\kappa},
\end{aligned}$$

where $\kappa = \frac{1}{2}(\frac{1}{n} + \frac{1}{2})^{-1} \in (0,1)$. Thus we deduce that

$$c_6\|u(t)\|^{\frac{2}{\kappa}}_{L^2(\Omega)} - 1 \le \|\nabla u(t)\|^2_{L^2(\Omega)}, \tag{44}$$

which together with (43) leads to

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L^2(\Omega)} + ac_6\left(\|u(t)\|^2_{L^2(\Omega)}\right)^{\frac{1}{\kappa}} \le c_7.$$

Upon an ODE comparison argument this inequality warrants that

$$\|u(t)\|^2_{L^2(\Omega)} \le \max\left\{\left(\frac{c_7}{ac_6}\right)^{\kappa}, \|u_0\|^2_{L^2(\Omega)}\right\} \quad \text{for all } t \in [0, T_U],$$

which proves the conclusion. □

We are now in the position to complete the proof of Theorem 1.2.

Proof of Theorem 1.2. Combining Proposition 3.1 with Lemma 2.2, we arrive at the conclusion of Theorem 1.2. □

4. The parabolic–parabolic–elliptic case $\tau = 1$

4.1. Local existence

In this subsection we show Theorem 1.3. Let $u_0 \in L^2(\Omega)$ and $v_0 \in H^2_{\mathrm{N}}(\Omega)$ be nonnegative functions satisfying $u_0, v_0 \ne 0$. The proof of Theorem 1.3 is divided into two steps:

Step 1. Let $X := L^2(\Omega) \times H^2_{\mathrm{N}}(\Omega) = L^2(\Omega;\mathbb{C}) \times H^2_{\mathrm{N}}(\Omega;\mathbb{C})$ with inner product

$$(U_1,U_2)_X := (u_1,u_2)_{L^2(\Omega)} + (v_1,v_2)_{H^2(\Omega)}, \quad U_1 = (u_1,v_1),\ U_2 = (u_2,v_2) \in X. \tag{45}$$

We begin with formulating the problem (1) with $\tau = 1$ as an abstract problem of the form (5) in the Hilbert space X. Firstly, standard elliptic regularity theory tells us that

$$w = \gamma(-c\Delta + \delta I)^{-1}u, \tag{46}$$

$$\|w\|_{H^2(\Omega)} \le c_1 \|u\|_{L^2(\Omega)}, \tag{47}$$

where I is an identity operator. Secondly, let us define a nonlinear operator F as

$$F(u,v,w) := \begin{pmatrix} u - \chi\nabla\cdot(u\nabla v) + \xi\nabla\cdot(u\nabla w) \\ 0 \end{pmatrix}.$$

Then by (46) we can regard F as a nonlinear operator of two variables u and v. Therefore we can rewrite the first and second equations in (1) with $\tau = 1$ as

$$\frac{dU}{dt} + AU = F(U), \tag{48}$$

where $A : D(A) \subset X \to X$, $D(A) := H^2_{\mathrm{N}}(\Omega) \times \mathcal{H}^4_{\mathrm{NN}}(\Omega)$, $U = (u,v)$ and

$$A := \begin{pmatrix} A_1 & 0 \\ -\alpha I & A_2 \end{pmatrix} = \begin{pmatrix} -a\Delta + I & 0 \\ -\alpha I & -b\Delta + \beta I \end{pmatrix}.$$

Then Lemma 2.4 implies that A is a sectorial operator satisfying (6) and (7). We next demonstrate that the operator F fulfills (8). Let θ be the constant defined in (13) and take $U_1 = (u_1,v_1), U_2 = (u_2,v_2) \in D(A^\theta)$. Then, by virtue of Lemma 2.5, we have $D(A^\theta) = D(A^\theta_{\mathrm{D}})$, where A_{D} is the diagonal operator defined in (11). Similarly to (30), we can obtain

$$\begin{aligned}
&\|F(U_1) - F(U_2)\|_X \\
&\le c_2\chi\|u_1\|_{H^{2\theta}(\Omega)}\|v_1 - v_2\|_{H^2(\Omega)} + c_2\chi\|u_1 - u_2\|_{H^{2\theta}(\Omega)}\|v_2\|_{H^2(\Omega)} \\
&\quad + c_2\xi\|u_1\|_{H^{2\theta}(\Omega)}\|w_1 - w_2\|_{H^2(\Omega)} + c_2\xi\|u_1 - u_2\|_{H^{2\theta}(\Omega)}\|w_2\|_{H^2(\Omega)} \\
&\quad + \|u_1 - u_2\|_{H^{2\theta}(\Omega)},
\end{aligned}$$

where $w_i := \gamma(-c\Delta + \delta I)^{-1}u_i$ for $i \in \{1,2\}$. Applying (9) with $i = 1$ and using (45) and (47), we observe that

$$\begin{aligned}
&\|F(u_1) - F(u_2)\|_X \\
&\le c_3\left[\|A_1^\theta u_1\|_{L^2(\Omega)}\|U_1 - U_2\|_X + \|A_1^\theta(u_1 - u_2)\|_{L^2(\Omega)}\|U_2\|_X\right] \\
&\quad + c_4\left[\|A_1^\theta u_1\|_{L^2(\Omega)}\|u_1 - u_2\|_{L^2(\Omega)} + \|A_1^\theta(u_1 - u_2)\|_{L^2(\Omega)}\|u_2\|_{L^2(\Omega)}\right] \\
&\quad + \|A_1^\theta(u_1 - u_2)\|_{L^2(\Omega)}.
\end{aligned} \tag{49}$$

Here, for $i \in \{1,2\}$, we infer from (12) and (45) that

$$\|A_1^\theta u_i\|_{L^2(\Omega)} \le \|A_1^\theta u_i\|_{L^2(\Omega)} + \|A_2^\theta v_i\|_{H^2(\Omega)} \le c_5 \left\|A_{\mathrm{D}}^\theta U_i\right\|_X \le c_6 \left\|A^\theta U_i\right\|_X \tag{50}$$

and that

$$\|A_1^\theta (u_1 - u_2)\|_{L^2(\Omega)} \le c_6 \left\|A^\theta (U_1 - U_2)\right\|_X . \tag{51}$$

Combining (50) and (51) with (49), we see that

$$\begin{aligned}
&\|F(U_1) - F(U_2)\|_X \\
&\le c_7 \left[\left\|A^\theta U_1\right\|_X \|U_1 - U_2\|_X + \left\|A^\theta (U_1 - U_2)\right\|_X \|U_2\|_X + \left\|A^\theta (U_1 - U_2)\right\|_X\right] \\
&\le c_7 \left(\|U_1\|_X + \|U_2\|_X + 1\right) \\
&\quad \cdot \left[\left\|A^\theta (U_1 - U_2)\right\|_X + \left(\left\|A^\theta U_1\right\|_X + \left\|A^\theta U_2\right\|_X\right) \|U_1 - U_2\|_X\right].
\end{aligned}$$

Therefore Lemma 2.1 asserts that (48) with $U_0 = (u_0, v_0)$ possesses a unique local solution

$$\begin{cases} u \in C^0([0, T_{u_0,v_0}]; L^2(\Omega)) \cap C^0((0, T_{u_0,v_0}]; H_{\mathrm{N}}^2(\Omega)) \cap C^1((0, T_{u_0,v_0}]; L^2(\Omega)), \\ v \in C^0([0, T_{u_0,v_0}]; H_{\mathrm{N}}^2(\Omega)) \cap C^0((0, T_{u_0,v_0}]; \mathcal{H}_{\mathrm{NN}}^4(\Omega)) \cap C^1((0, T_{u_0,v_0}]; H_{\mathrm{N}}^2(\Omega)), \end{cases}$$

where $T_{u_0,v_0} > 0$ is a positive constant. Also, the relation (46) entails

$$w \in C^0([0, T_{u_0,v_0}]; H_{\mathrm{N}}^2(\Omega)) \cap C^0((0, T_{u_0,v_0}]; \mathcal{H}_{\mathrm{NN}}^4(\Omega)).$$

Hence we obtain a unique local solution (u, v, w) of (1) with $\tau = 1$.

Step 2. We next show that

$$u(t), v(t), w(t) \ge 0 \quad \text{for all } t \in [0, T_{u_0,v_0}].$$

As in Step 2 in Section 3.1 we can verify that u, v and w are real-valued functions and $u(t), w(t) \ge 0$ for all $t \in [0, T_{u_0,v_0}]$. Thereafter, we deal with the function spaces $L^2(\Omega) = L^2(\Omega; \mathbb{R})$ and $H_{\mathrm{N}}^2(\Omega) = H_{\mathrm{N}}^2(\Omega; \mathbb{R})$ and prove that $v(t) \ge 0$ for $t \in (0, T_{u_0,v_0}]$. Let H be the cut-off function defined in (32) and set $\Psi(t) := \int_\Omega H(v(t))\,dx$ for $t \in [0, T_{u_0,v_0}]$. Then we see from the second equation in (1) with $\tau = 1$ that

$$\Psi'(t) = (H'(v), b\Delta v - \beta v + \alpha u)_{L^2(\Omega)} \tag{52}$$

for all $t \in (0, T_{u_0,v_0}]$. Here the Green formula implies that

$$\begin{aligned}
(H'(v), b\Delta v)_{L^2(\Omega)} &= -b \int_\Omega \nabla H'(v) \cdot \nabla v\,dx \\
&= -b \int_\Omega |\nabla H'(v)|^2\,dx \le 0.
\end{aligned} \tag{53}$$

Also, due to $H'(v)v \geq 0$, we have

$$(H'(v), -\beta v)_{L^2(\Omega)} = -\beta \int_\Omega H'(v)v\,dx \leq 0. \tag{54}$$

Similarly, since $H'(v) \leq 0$ and $u \geq 0$, we infer that

$$(H'(v), \alpha u)_{L^2(\Omega)} = \alpha \int_\Omega H'(v)u\,dx \leq 0. \tag{55}$$

Hence combining (53), (54) and (55) with (52) yields $\Psi'(t) \leq 0$ for $t \in (0, T_{u_0,v_0}]$, which means that $\Psi(t) \leq \Psi(0) = 0$ for $t \in [0, T_{u_0,v_0}]$. Consequently, $\Psi(t) \equiv 0$ holds, that is, $v(t) \geq 0$ for $t \in [0, T_{u_0,v_0}]$, which concludes the proof.

4.2. Global existence

Let $u_0 \in L^2(\Omega)$ and $v_0 \in H^2_{\mathrm{N}}(\Omega)$ be nonnegative functions satisfying $u_0, v_0 \neq 0$ and let $U = (u, v, w)$ be any local solution of the problem (1) with $\tau = 1$ on $[0, T_U]$ with $T_U > 0$ such that

$$\begin{cases} 0 \leq u \in C^0([0,T_U]; L^2(\Omega)) \cap C^0((0,T_U]; H^2_{\mathrm{N}}(\Omega)) \cap C^1((0,T_U]; L^2(\Omega)), \\ 0 \leq v \in C^0([0,T_U]; H^2_{\mathrm{N}}(\Omega)) \cap C^0((0,T_U]; \mathcal{H}^4_{\mathrm{NN}}(\Omega)) \cap C^1((0,T_U]; H^2_{\mathrm{N}}(\Omega)), \\ 0 \leq w \in C^0([0,T_U]; H^2_{\mathrm{N}}(\Omega)) \cap C^0((0,T_U]; \mathcal{H}^4_{\mathrm{NN}}(\Omega)). \end{cases}$$

We now show a priori estimates for u and v, which together with Lemma 2.2 enables us to obtain global existence in (1) with $\tau = 1$.

Proposition 4.1. *Let $\Omega \subset \mathbb{R}^n$ $(n \leq 3)$ be a bounded domain with C^3-boundary and let $\tau = 1$. Assume* (4)*. Then there is a constant $C > 0$ such that C is independent of T_U and*

$$\|u(t)\|^2_{L^2(\Omega)} + \|v(t)\|^2_{H^2(\Omega)} \leq C \quad \textit{for all } t \in [0, T_U].$$

Proof. Let $t_0 \in (0, T_U)$ and let $t \in (t_0, T_U)$. We note that Lemmas 3.2 and 3.3 hold also in the case $\tau = 1$. Multiplying the first equation in (1) by u and integrating it over Ω, we can compute as

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2\,dx = -a\int_\Omega |\nabla u|^2\,dx - \frac{\chi}{2}\int_\Omega u^2 \Delta v\,dx + \frac{\xi}{2}\int_\Omega u^2 \Delta w\,dx. \tag{56}$$

Applying the Hölder inequality to the second term on the right-hand side of (56), we see that

$$-\frac{\chi}{2}\int_\Omega u^2 \Delta v\,dx \leq \frac{\chi}{2}\|u(t)\|^2_{L^3(\Omega)} \|\Delta v(t)\|_{L^3(\Omega)}. \tag{57}$$

We next estimate the third term on the right-hand side of (56). By the third equation in (1), it follows that

$$\frac{\xi}{2}\int_\Omega u^2\Delta w\,dx = \frac{\xi}{2c}\int_\Omega u^2(\delta w-\gamma u)\,dx = \frac{\xi}{2c}\int_\Omega(\delta u^2 w-\gamma u^3)\,dx. \tag{58}$$

As in (42), taking $\varepsilon_1>0$ which will be fixed later, we have

$$\frac{\xi\delta}{2c}\int_\Omega u^2 w\,dx \le \frac{\xi\delta}{2c}\varepsilon_1\int_\Omega u^3\,dx + c_1(\varepsilon_1). \tag{59}$$

A combination of (58) and (59) yields

$$\frac{\xi\delta}{2c}\int_\Omega u^2\Delta w\,dx \le \frac{\xi\delta}{2c}\varepsilon_1\int_\Omega u^3\,dx + c_1(\varepsilon_1) - \frac{\xi\gamma}{2c}\int_\Omega u^3\,dx = \frac{\xi}{2c}(\delta\varepsilon_1-\gamma)\int_\Omega u^3\,dx + c_1(\varepsilon_1). \tag{60}$$

As a consequence of (56), (57) and (60), we deduce that

$$\frac{d}{dt}\|u(t)\|^2_{L^2(\Omega)} \le -2a\|\nabla u(t)\|^2_{L^2(\Omega)} + \left(\frac{\xi\delta}{c}\varepsilon_1 - \frac{\xi\gamma}{c}\right)\|u(t)\|^3_{L^3(\Omega)} + \chi\|u(t)\|^2_{L^3(\Omega)}\|\Delta v(t)\|_{L^3(\Omega)} + 2c_1(\varepsilon_1). \tag{61}$$

Here we have the inequality

$$-\frac{a}{4}\|\nabla u(t)\|^2_{L^2(\Omega)} \le -\|u(t)\|^2_{L^2(\Omega)} + c_2. \tag{62}$$

Indeed, we know from (44) that

$$c_3\|u(t)\|^{\frac{2}{\kappa}}_{L^2(\Omega)} - 1 \le \|\nabla u(t)\|^2_{L^2(\Omega)}$$

with $\kappa = \frac{1}{2}(\frac{1}{n}+\frac{1}{2})^{-1}\in(0,1)$. Since $\frac{2}{\kappa}>2$, the Young inequality implies

$$\frac{4}{a}\|u(t)\|^2_{L^2(\Omega)} \le c_3\|u(t)\|^{\frac{2}{\kappa}}_{L^2(\Omega)} + c_4 \le \|\nabla u(t)\|^2_{L^2(\Omega)} + c_4 + 1,$$

which entails (62). Adding (61) and (62) gives

$$\frac{d}{dt}\|u(t)\|^2_{L^2(\Omega)} \le -\frac{7}{4}a\|\nabla u(t)\|^2_{L^2(\Omega)} + \left(\frac{\xi\delta}{c}\varepsilon_1 - \frac{\xi\gamma}{c}\right)\|u(t)\|^3_{L^3(\Omega)} + \chi\|u(t)\|^2_{L^3(\Omega)}\|\Delta v(t)\|_{L^3(\Omega)} - \|u(t)\|^2_{L^2(\Omega)} + c_5(\varepsilon_1). \tag{63}$$

We next derive a differential inequality for $\|A_2 v(t)\|^2_{L^2(\Omega)}$, where A_2 is the realization of $-b\Delta+\beta I$. Multiplying the second equation in (1) with $\tau=1$ by $A_2^2 v$ and integrating it over Ω, we have

$$\int_\Omega v_t A_2^2 v\,dx = -\int_\Omega A_2 v A_2^2 v\,dx + \alpha\int_\Omega u A_2^2 v\,dx,$$

which along with the property that A_2 is a selfadjoint operator in $L^2(\Omega)$ warrants that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |A_2 v|^2\,dx + \int_\Omega |A^{\frac{3}{2}} v|^2\,dx = \alpha\int_\Omega A_2^{\frac{1}{2}} u A_2^{\frac{3}{2}} v\,dx.$$

Moreover, employing the Hölder inequality, applying Lemma 2.3 and using the Young inequality, we observe that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |A_2 v|^2\,dx + \int_\Omega |A_2^{\frac{3}{2}} v|^2\,dx &\le \alpha \|A_2^{\frac{1}{2}} u(t)\|_{L^2(\Omega)}\|A_2^{\frac{3}{2}} v(t)\|_{L^2(\Omega)}\\ &\le c_6\|u(t)\|_{H^1(\Omega)}\|A_2^{\frac{3}{2}} v(t)\|_{L^2(\Omega)}\\ &\le c_7\|u(t)\|^2_{H^1(\Omega)} + \frac{1}{2}\|A_2^{\frac{3}{2}} v(t)\|^2_{L^2(\Omega)},\end{aligned}$$

which means that

$$\frac{d}{dt}\int_\Omega |A_2 v|^2\,dx + \int_\Omega |A_2^{\frac{3}{2}} v|^2\,dx \le c_8\|u(t)\|^2_{H^1(\Omega)}. \tag{64}$$

By virtue of Lemma 2.3 with $\theta=\frac{1}{2}$, it follows that

$$\|A_2^{\frac{3}{2}} v(t)\|^2_{L^2(\Omega)} = \|A_2^{\frac{1}{2}}(A_2 v(t))\|^2_{L^2(\Omega)} \ge c_9\|A_2 v(t)\|^2_{H^1(\Omega)} \ge c_9\|A_2 v(t)\|^2_{L^2(\Omega)},$$

which in conjunction with (64) leads to

$$\begin{aligned}\frac{d}{dt}\|A_2 v(t)\|^2_{L^2(\Omega)} + c_9\|A_2 v(t)\|^2_{L^2(\Omega)} &\le c_8\|u(t)\|^2_{H^1(\Omega)}\\ &= c_8\|u(t)\|^2_{L^2(\Omega)} + c_8\|\nabla u(t)\|^2_{L^2(\Omega)}.\end{aligned} \tag{65}$$

Multiplying (63) by $\zeta>0$ which will be fixed later, we have

$$\begin{aligned}\zeta\frac{d}{dt}\|u(t)\|^2_{L^2(\Omega)} \le -\frac{7}{4}a\zeta\|\nabla u(t)\|^2_{L^2(\Omega)} + \left(\frac{\xi\delta}{c}\varepsilon_1 - \frac{\xi\gamma}{c}\right)\zeta\|u(t)\|^3_{L^3(\Omega)}\\ + \chi\zeta\|u(t)\|^2_{L^3(\Omega)}\|\Delta v(t)\|_{L^3(\Omega)} - \zeta\|u(t)\|^2_{L^2(\Omega)} + c_5(\varepsilon_1)\zeta,\end{aligned}$$

which along with (65) implies

$$\begin{aligned}&\frac{d}{dt}\left(\zeta\|u(t)\|^2_{L^2(\Omega)} + \|A_2 v(t)\|^2_{L^2(\Omega)}\right) + (\zeta - c_8)\|u(t)\|^2_{L^2(\Omega)} + c_9\|A_2 v(t)\|^2_{L^2(\Omega)}\\ &\le \left(c_8 - \frac{7}{4}a\zeta\right)\|\nabla u(t)\|^2_{L^2(\Omega)} + \left(\frac{\xi\delta}{c}\varepsilon_1 - \frac{\xi\gamma}{c}\right)\zeta\|u(t)\|^3_{L^3(\Omega)}\\ &\quad + \chi\zeta\|u(t)\|^2_{L^3(\Omega)}\|\Delta v(t)\|_{L^3(\Omega)} + c_5(\varepsilon_1)\zeta.\end{aligned} \tag{66}$$

Let $\zeta > c_8 + 1$ and take η such that

$$0 < \eta < \min\left\{c_9,\ 1 - \frac{c_8}{c_8+1},\ 3\beta\right\}. \tag{67}$$

Then we see that

$$0 < \eta < 1 - \frac{c_8}{c_8+1} < 1 - \frac{c_8}{\zeta},$$

that is,

$$\zeta\eta \le \zeta - c_8. \tag{68}$$

Set

$$\psi(t) := \zeta \|u(t)\|^2_{L^2(\Omega)} + \|A_2 v(t)\|^2_{L^2(\Omega)}.$$

Collecting (66)–(68), we obtain

$$\begin{aligned}
\frac{d\psi}{dt}(t) + \eta\psi(t) &= \frac{d\psi}{dt}(t) + \eta\left(\zeta\|u(t)\|^2_{L^2(\Omega)} + \|A_2 v(t)\|^2_{L^2(\Omega)}\right)\\
&\le \frac{d\psi}{dt}(t) + (\zeta - c_8)\|u(t)\|^2_{L^2(\Omega)} + c_9\|A_2 v(t)\|^2_{L^2(\Omega)}\\
&\le \left(c_8 - \frac{7}{4}a\zeta\right)\|\nabla u(t)\|^2_{L^2(\Omega)} + \left(\frac{\xi\delta}{c}\varepsilon_1 - \frac{\xi\gamma}{c}\right)\zeta\|u(t)\|^3_{L^3(\Omega)}\\
&\quad + \chi\zeta\|u(t)\|^2_{L^3(\Omega)}\|\Delta v(t)\|_{L^3(\Omega)} + c_5(\varepsilon_1)\zeta.
\end{aligned}$$

Then this differential inequality for ψ yields

$$\begin{aligned}
\psi(t) \le \psi(t_0)e^{-\eta(t-t_0)} &+ \left(c_8 - \frac{7}{4}a\zeta\right)\int_{t_0}^t e^{\eta(s-t)}\|\nabla u(s)\|^2_{L^2(\Omega)}\,ds\\
&+ \frac{\xi}{c}(\delta\varepsilon_1 - \gamma)\zeta\int_{t_0}^t e^{\eta(s-t)}\|u(s)\|^3_{L^3(\Omega)}\,ds\\
&+ \chi\zeta\int_{t_0}^t e^{\eta(s-t)}\|u(s)\|^2_{L^3(\Omega)}\|\Delta v(s)\|_{L^3(\Omega)}\,ds\\
&+ c_5(\varepsilon_1)\zeta\int_{t_0}^t e^{\eta(s-t)}\,ds.
\end{aligned} \tag{69}$$

In order to estimate the fourth term on the right-hand side of the inequality (69) we introduce two functions $\widetilde{u}$ and $\widetilde{v}$ by $\widetilde{u}(s) := e^{\frac{\eta(s-t)}{3}}u(s)$ and $\widetilde{v}(s) := e^{\frac{\eta(s-t)}{3}}v(s)$. Then we observe from the second equation in (1) with $\tau = 1$ that

$$\begin{aligned}
\widetilde{v}_s &= e^{\frac{\eta(s-t)}{3}}\left(v_s + \frac{\eta}{3}v\right)\\
&= e^{\frac{\eta(s-t)}{3}}\left(b\Delta v - \beta v + \alpha u + \frac{\eta}{3}v\right)\\
&= b\Delta\widetilde{v} - \left(\beta - \frac{\eta}{3}\right)\widetilde{v} + \alpha\widetilde{u}.
\end{aligned} \tag{70}$$

Here we note from (67) that $\beta - \frac{\eta}{3} > 0$. Moreover, since $v(t_0), \widetilde{v}(t_0) \in \mathcal{H}^4_{\mathrm{NN}}(\Omega)$, we know that $\Delta v(t_0), \Delta\widetilde{v}(t_0) \in H^2_{\mathrm{N}}(\Omega)$, in particular $\Delta v(t_0), \Delta\widetilde{v}(t_0) \in H^1(\Omega)$. By virtue of the assumption that Ω is of class C^3, it follows that $v(t_0), \widetilde{v}(t_0) \in H^3_{\mathrm{N}}(\Omega)$. Then the Sobolev embedding theorem implies $v(t_0), \widetilde{v}(t_0) \in W^{2,3}(\Omega)$. Therefore, applying Lemma 2.7 to (70) and relying on the fact $(x+y)^{\frac{1}{3}} \le x^{\frac{1}{3}} + y^{\frac{1}{3}}$ for all $x, y \ge 0$, we see that

$$\left(\int_{t_0}^t \|\Delta\widetilde{v}(s)\|^3_{L^3(\Omega)}\,ds\right)^{\frac{1}{3}} \le C_{\mathrm{MR}}^{\frac{1}{3}}\left[\|\Delta\widetilde{v}(t_0)\|_{L^3(\Omega)} + \frac{\alpha}{b}\left(\int_{t_0}^t \|\widetilde{u}(s)\|^3_{L^3(\Omega)}\,ds\right)^{\frac{1}{3}}\right]. \tag{71}$$

Now we deal with the fourth term on the right-hand side of the inequality (69). The definition of $\widetilde{v}$ and the Hölder inequality entail that

$$\begin{aligned}
&\int_{t_0}^t e^{\eta(s-t)}\|u(s)\|^2_{L^3(\Omega)}\|\Delta v(s)\|_{L^3(\Omega)}\,ds \\
&\le \int_{t_0}^t \|\widetilde{u}(s)\|^2_{L^3(\Omega)}\|\Delta\widetilde{v}(s)\|_{L^3(\Omega)}\,ds \\
&\le \left(\int_{t_0}^t \|\widetilde{u}(s)\|^3_{L^3(\Omega)}\,ds\right)^{\frac{2}{3}}\left(\int_{t_0}^t \|\Delta\widetilde{v}(s)\|^3_{L^3(\Omega)}\,ds\right)^{\frac{1}{3}}.
\end{aligned}$$

Combining (71) with this inequality yields

$$\begin{aligned}
&\int_{t_0}^t e^{\eta(s-t)}\|u(s)\|^2_{L^3(\Omega)}\|\Delta v(s)\|_{L^3(\Omega)}\,ds \\
&\le C_{\mathrm{MR}}^{\frac{1}{3}}\left[\|\Delta\widetilde{v}(t_0)\|_{L^3(\Omega)}\left(\int_{t_0}^t \|\widetilde{u}(s)\|^3_{L^3(\Omega)}\,ds\right)^{\frac{2}{3}}\right] + C_{\mathrm{MR}}^{\frac{1}{3}}\cdot\frac{\alpha}{b}\int_{t_0}^t \|\widetilde{u}(s)\|^3_{L^3(\Omega)}\,ds.
\end{aligned}$$

Taking $\varepsilon_2 > 0$ which will be fixed later and employing the Young inequality, we have

$$\begin{aligned}
&\int_{t_0}^t e^{\eta(s-t)}\|u(s)\|^2_{L^3(\Omega)}\|\Delta v(s)\|_{L^3(\Omega)}\,ds \\
&\le C_{\mathrm{MR}}^{\frac{1}{3}}\left(\varepsilon_2 + \frac{\alpha}{b}\right)\int_{t_0}^t \|\widetilde{u}(s)\|^3_{L^3(\Omega)}\,ds + c_{10}(\varepsilon_2).
\end{aligned} \tag{72}$$

Fixing ζ such that $\zeta > \max\{c_8 + 1, \frac{4c_8}{7a}\}$, we observe $c_8 - \frac{7}{4}a\zeta < 0$, which in conjunction with (69) and (72) ensures that

$$\begin{aligned}
\psi(t) &\le c_{11}(\varepsilon_1, \varepsilon_2) + \frac{\xi}{c}(\delta\varepsilon_1 - \gamma)\zeta\int_{t_0}^t \|\widetilde{u}(s)\|^3_{L^3(\Omega)}\,ds \\
&\quad + \chi C_{\mathrm{MR}}^{\frac{1}{3}}\left(\varepsilon_2 + \frac{\alpha}{b}\right)\zeta\int_{t_0}^t \|\widetilde{u}(s)\|^3_{L^3(\Omega)}\,ds \\
&= c_{11}(\varepsilon_1, \varepsilon_2) + \left(C_{\mathrm{MR}}^{\frac{1}{3}}\frac{\chi\alpha}{b} - \frac{\xi\gamma}{c} + \frac{\xi\delta}{c}\varepsilon_1 + \chi C_{\mathrm{MR}}^{\frac{1}{3}}\varepsilon_2\right)\zeta\int_{t_0}^t \|\widetilde{u}(s)\|^3_{L^3(\Omega)}\,ds.
\end{aligned}$$

Since we have assumed $C_{\mathrm{MR}}^{\frac{1}{3}}\frac{\chi\alpha}{b}-\frac{\xi\gamma}{c}<0$ in (4), fixing $\varepsilon_1,\varepsilon_2>0$ such that $C_{\mathrm{MR}}^{\frac{1}{3}}\frac{\chi\alpha}{b}-\frac{\xi\gamma}{c}+\frac{\xi\delta}{c}\varepsilon_1+\chi C_{\mathrm{MR}}^{\frac{1}{3}}\varepsilon_2<0$, we obtain the inequality $\psi(t)\le c_{11}$, so that

$$\zeta\|u(t)\|_{L^2(\Omega)}^2+\|A_2v(t)\|_{L^2(\Omega)}^2\le c_{11}\quad \text{for all } t\in[0,T_U].$$

Therefore standard elliptic regularity theory warrants that

$$\|u(t)\|_{L^2(\Omega)}^2+\|v(t)\|_{H^2(\Omega)}^2\le c_{12}\quad \text{for all } t\in[0,T_U].$$

This completes the proof of Proposition 4.1. □

Employing Proposition 4.1, we can prove Theorem 1.4.

Proof of Theorem 1.4. Thanks to Proposition 4.1 and Lemma 2.2 we immediately arrive at the conclusion of Theorem 1.4. □

REFERENCES

[1] G. Arumugam, J. Tyagi, *Keller–Segel chemotaxis models: a review*, Acta Appl. Math. 171 (2021), Paper No. 6, 82 pp.

[2] N. Bellomo, A. Bellouquid, Y. Tao, M. Winkler, *Toward a mathematical theory of Keller–Segel models of pattern formation in biological tissues*, Math. Models Methods Appl. Sci. 25 (2015), no. 9, 1663–1763.

[3] Y. Chiyo, *Large time behavior in a quasilinear parabolic–parabolic–elliptic attraction-repulsion chemotaxis system*, Arch. Math. (Brno) 59 (2023), no. 2, 163–171.

[4] Y. Chiyo, M. Marras, Y. Tanaka, T. Yokota, *Blow-up phenomena in a parabolic–elliptic–elliptic attraction-repulsion chemotaxis system with superlinear logistic degradation*, Nonlinear Anal. 212 (2021), Paper No. 112550, 14 pp.

[5] Y. Chiyo, T. Yokota, *Boundedness and finite-time blow-up in a quasilinear parabolic–elliptic–elliptic attraction-repulsion chemotaxis system*, Z. Angew. Math. Phys. 73 (2022), no. 2, Paper No. 61, 27 pp.

[6] A. Columbu, S. Frassu, G. Viglialoro, *Properties of given and detected unbounded solutions to a class of chemotaxis models*, Stud. Appl. Math. 151 (2023), no. 4, 1349–1379.

[7] A. Columbu, S. Frassu, G. Viglialoro, *Refined criteria toward boundedness in an attraction-repulsion chemotaxis system with nonlinear productions*, Appl. Anal. 103 (2024), no. 2, 415–431.

[8] A. Columbu, R. D. Fuentes, S. Frassu, *Uniform-in-time boundedness in a class of local and nonlocal nonlinear attraction-repulsion chemotaxis models with logistics*, Nonlinear Anal. Real World Appl. 79 (2024), Paper No. 104135, 14 pp.

[9] S. Frassu, R. Rodríguez Galván, G. Viglialoro, *Uniform in time L^∞-estimates for an attraction-repulsion chemotaxis model with double saturation*, Discrete Contin. Dyn. Syst. Ser. B 28 (2023), no. 3, 1886–1904.

[10] F. Heihoff, *Does strong repulsion lead to smooth solutions in a repulsion-attraction chemotaxis system even when starting with highly irregular initial data?*, Discrete Contin. Dyn. Syst. Ser. B 28 (2023), no. 10, 5216–5243.

[11] S. Ishida, T. Yokota, *Boundedness in a quasilinear fully parabolic Keller–Segel system via maximal Sobolev regularity*, Discrete Contin. Dyn. Syst. Ser. S 13 (2020), no. 2, 212–232.

[12] E. F. Keller, L. A. Segel, *Initiation of slime mold aggregation viewed as an instability*, J. Theoret. Biol. 26(3) (1970), 399–415.

[13] M. Luca, A. Chavez-Ross, L. Edelstein-Keshet and A. Mogilner, *Chemotactic signaling, microglia, and Alzheimer's disease senile plaques: Is there a connection?* Bull. Math. Biol. 65 (2003), 693–730.

[14] E. Nakaguchi, K. Noda, K. Osaki, K. Uemichi, *Global attractor for a two-dimensional chemotaxis system with linear degradation and indirectsignal production*, Jpn. J. Ind. Appl. Math. 37 (2020), no. 1, 49–80.

[15] E. Nakaguchi, K. Osaki, *Global existence of solutions to a parabolic–parabolic system for chemotaxis with weak degradation*, Nonlinear Anal. 74 (2011), no. 1, 286–297.

[16] E. Nakaguchi, K. Osaki, *Global solutions and exponential attractors of a parabolic–parabolic system for chemotaxis with subquadratic degradation*, Discrete Contin. Dyn. Syst. Ser. B 18 (2013), no. 10, 2627–2646.

[17] E. Nakaguchi, K. Osaki, *L_p-estimates of solutions to n-dimensional parabolic–parabolic system for chemotaxis with subquadratic degradation*, Funkcial. Ekvac. 59 (2016), no. 1, 51–66.

[18] E. Nakaguchi, K. Osaki, *Global existence of solutions to an n-dimensional parabolic–parabolic system for chemotaxis with logistic-type growth and superlinear production*, Osaka J. Math. 55 (2018), no. 1, 51–70.

[19] K. Noda, K. Osaki, *Global attractor and Lyapunov function for one-dimensional Deneubourg chemotaxis system*, Hiroshima Math. J. 49 (2019), no. 2, 251–271.

[20] K. Osaki, A. Yagi, *Finite dimensional attractor for one-dimensional Keller–Segel equations*, Funkcial. Ekvac. 44 (2001), no. 3, 441–469.

[21] K. Osaki, A. Yagi, *Global existence for a chemotaxis-growth system in $\mathbb{R}^2$*, Adv. Math. Sci. Appl. 12 (2002), no. 2, 587–606.

[22] A. Yagi, *Abstract Parabolic Evolution Equations and Their Applications*, Springer-Verlag, Berlin, 2010.

Y. CHIYO
Department of Mathematics
Tokyo University of Science
e-mail: ycnewssz@gmail.com

T. SAGA
Department of Mathematics
Tokyo University of Science
e-mail: taisho.saga@gmail.com

T. YOKOTA
Department of Mathematics
Tokyo University of Science
e-mail: yokota@rs.tus.ac.jp