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MOMENTS AND (p,q)-INTEGRAL REPRESENTATIONS OF (p,q)-CLASSICAL FORMS

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In [6], the authors introduce a new technical method for studying the $D_{p,q}$-classical orthogonal polynomials, where $D_{p,q}$ is the (p,q)-difference operator, using an algebraic approach. In this paper, we investigate $D_{p,q}$-classical orthogonal polynomials associated with the (p,q)-difference operator $D_{p,q}$. Using Pearson-type functional equations and a suitable decomposition of $D_{p,q}$, we simplify the description of the canonical $D_{p,q}$-classical forms and clarify their relation with the q-classical framework. We also study the moments and modified moments of the corresponding forms, derive recurrence relations, and establish several explicit (p,q)-integral representations for canonical cases.

1. Introduction and preliminary results

The motivation behind the present work stems from recent developments in the theory of orthogonal polynomials associated with generalized difference operators, particularly within the framework of (p,q)-calculus. The study of $D_{p,q}$-classical orthogonal polynomials may be viewed as a natural extension of both the classical and q-classical theories, preserving several structural properties while introducing new analytical and algebraic phenomena related to the two-parameter deformation.

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One of the first systematic investigations of this subject was carried out by Mohammad Masjed-Jamei, Farzad Soleyman, Iván Area and Juan J. Nieto in [4], where the notion of (p,q)-classical orthogonal polynomials was introduced and several characterization theorems were established. Their work generalized Hahn's classical characterization theory to the (p,q)-setting and provided a foundational framework for the study of orthogonal polynomials related to the (p,q)-difference operator.

Subsequently, Mabrouk Sghaier, Mohamed Zaatra and Mohamed Mechri in [6] developed an algebraic approach to the study of $D_{p,q}$-classical orthogonal polynomials. Their results provided Rodrigues-type formulas, recurrence relations, canonical classifications, and several structural properties of the associated polynomial sequences. In particular, they obtained explicit functional characterizations of $D_{p,q}$-classical forms and clarified the role played by the $D_{p,q}$-operator in the corresponding orthogonality theory. However, although the characterization and algebraic aspects of $D_{p,q}$-classical forms have been extensively investigated in [4, 6], the corresponding moment problem and the construction of explicit (p,q)-integral representations remain largely unexplored. In particular, the relationship between the functional equations satisfied by the $D_{p,q}$-classical forms and their associated moments has not yet been systematically developed. Since moment representations and integral representations constitute fundamental tools in the analysis of orthogonal polynomials, providing information about orthogonality measures, generating functions, and asymptotic properties, it becomes natural and important to investigate these aspects in the (p,q)-framework.

The main purpose of the present paper is therefore to complement and extend the characterization results established in [4, 6] by developing a systematic study of moments and (p,q)-integral representations associated with $D_{p,q}$-classical forms. More precisely, we derive explicit recurrence relations for the moments, establish modified moment representations, and construct suitable (p,q)-integral representations for several canonical families of $D_{p,q}$-classical orthogonal polynomials. These results provide a deeper analytical understanding of the corresponding forms and reveal new connections between their functional equations, orthogonality properties, and integral representations.

The structure of this paper is organized as follows: The first section introduces preliminary concepts and foundational material necessary for understanding the subsequent discussions. The second section is devoted to the study of $D_{p,q}$-classical orthogonal polynomials. Special attention is given to the functional equations of Pearson type and to their connection with the corresponding q-classical theory.

In the third section first of all, we address the technical aspects related to moments and (p,q)-integral representations, exploring their mathematical underpinnings and implications. We discuss how these representations can be utilized to derive important identities and relationships among the polynomials. Next, we present the moments of the $D_{p,q}$-classical forms and, where feasible, we derive their representations as consequences of their functional equations.

Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and let $\mathcal{P}'$ be its dual. We denote by $\langle u,f\rangle$ the action of $u\in\mathcal{P}'$ on $f\in\mathcal{P}$. For $n\geq 0$, $(u)_n:=\langle u,x^n\rangle$ are the moments of u. In particular, a form u is called symmetric if all of its moments of odd order are zero [1]. For any form u, any polynomial g, any $a\in\mathbb{C}\backslash\{0\}$ and any $b\in\mathbb{C}$, let $Du=u'$, gu, $h_a u$, δ_b and $x^{-1}u$ be the forms defined by duality

$$\langle u',f\rangle=-\langle u,f'\rangle\ ,\ \langle gu,f\rangle=\langle u,gf\rangle\ ,\ \langle h_a u,f\rangle=\langle u,h_a f\rangle\ ,$$

$$\langle\delta_b,f\rangle=f(b)\ ,\ \langle x^{-1}u,f\rangle=\langle u,\theta_0 f\rangle\ ,\ f\in\mathcal{P},$$

where $(h_a f)(x)=f(ax)$ and $\big(\theta_0 f\big)(x)=\dfrac{f(x)-f(0)}{x}$.

Let $\{S_n\}_{n\geq 0}$ be a sequence of monic polynomials with $\deg(S_n)=n$, $n\geq 0$, and let $\{u_n\}_{n\geq 0}$ be its dual sequence, $u_n\in\mathcal{P}'$, defined by

$$<u_n,S_m>:=\delta_{n,m}\ ,n\ ,\ m\geq 0\ .$$

The form u is called regular if there exists a polynomial sequence $\{S_n\}_{n\geq 0}$ such that

$$<u,S_mS_n>=r_n\delta_{n,m}\ ,\quad n\ ,m\geq 0\ ;\ r_n\neq 0\ ,\ n\geq 0\ .$$

The sequence $\{S_n\}_{n\geq 0}$ is said to be orthogonal (MOPS, in short) with respect to u. Necessarily, $u=\lambda u_0,\lambda\neq 0$. In this case, we have

$$u_n=r_n^{-1}S_nu_0\ ,\ n\geq 0\ .$$

According to Favard's Theorem, a MOPS is characterized by the following three-term recurrence relation (TTRR, in short) [1]

$$\begin{aligned}&S_0(x)=1\ ,\qquad S_1(x)=x-\beta_0\ ,\\&S_{n+2}(x)=(x-\beta_{n+1})S_{n+1}(x)-\gamma_{n+1}S_n(x)\ ,\quad n\geq 0\ ,\end{aligned}$$

with $(\beta_n,\gamma_{n+1})\in\mathbb{C}\times(\mathbb{C}-\{0\})$, $n\geq 0$.

Let us introduce the (p,q)-difference operator [9]

$$(D_{p,q}f)(x)=\frac{(h_pf)(x)-(h_qf)(x)}{(p-q)x}\ ,\qquad f\in\mathcal{P}\ ,\quad 0<q<p\leq 1\ or\ 1\leq p<q\ .$$

Remark 1.1. In the literature, the limit operator when $p \longrightarrow 1$, is known as Jackson operator [2].

From the definition, we obtain

$$D_{p,q} = H_{\frac{q}{p}} \circ h_p \ .$$

On account of the least equation we have,

$$t_{D_{p,q}} = h_p \circ t_{H_{\frac{q}{p}}} = \frac{1}{p-q}(h_p - h_q)x^{-1} \ ,$$

where ${}^tD_{p,q}$ denotes the transposed of $D_{p,q}$ .We can define $D_{p,q}$ from $\mathcal{P}'$ to $\mathcal{P}'$ by $D_{p,q} := -{}^tD_{p,q}$ so that

$$\langle D_{p,q}u, f\rangle = -\langle u, D_{p,q}f\rangle, \quad f \in \mathcal{P} \ , \quad u \in \mathcal{P}' \ .$$

In particular, this yields

$$(D_{p,q}u)_n = -[n]_{p,q}(u)_{n-1} \ , \quad n \geq 0 \ ,$$

where

$$(u)_{-1} = 0, \quad [n]_{p,q} := \frac{p^n - q^n}{p-q} \ .$$

For $a \in \mathbb{C} - \{0\}$ and f, $g \in \mathcal{P}$, we have the following results [6, 9]

$$D_{p,q} \circ h_a = a h_a \circ D_{p,q} \ \ in\, \mathcal{P} \ , \tag{1}$$

$$D_{p,q} \circ h_a = a^{-1} h_a \circ D_{p,q} \ \ in\, \mathcal{P}' \ , \tag{2}$$

$$D_{p,q}(fg)(x) = f(px)D_{p,q}g(x) + g(qx)D_{p,q}f(x) \ . \tag{3}$$

Let us recall some results [6].

Definition 1.2. A form u_0 is called $D_{p,q}$-classical when it is regular and satisfies the functional equation

$$D_{p,q}\big(h_{p^{-1}}(\Phi u_0)\big) + \Psi u_0 = 0 \ , \tag{4}$$

with

$$\Psi'(0) - \frac{p^{1-n}}{2}[n]_{p,q}\Phi''(0) \neq 0 \ , \ n \geq 0 \ .$$

2. The canonical cases

We agree that the decomposition

$$D_{p,q} = H_{\frac{q}{p}} \circ h_{,}$$

acting on the polynomial space, provides a substantial simplification in the description of the canonical $D_{p,q}$-classical forms. Indeed, since

$$\langle D_{p,q}(h_{p^{-1}}u), f\rangle = \langle u, H_{\frac{q}{p}}(f)\rangle, \quad f \in \mathcal{P}.$$

The study of $D_{p,q}$-classical forms can be reduced to Pearson-type equations involving the operator $H_{\frac{q}{p}}$. This approach not only simplifies the analysis of the canonical cases, but also makes their connection with the corresponding q-classical framework more transparent (see [3]). Furthermore, the recurrence coefficients of the three-term recurrence relation follow directly from Theorem 5.1 in [4]. The canonical situations are therefore reduced to the following three cases:

$$\Phi(x) = 1, \quad \Phi(x) = x - c, \quad \Phi(x) = (x-c)(x-d).$$

For this reason, the detailed classifications, explicit recurrence coefficients, and canonical examples are no longer developed in the main text; instead, they are collected and summarized in tables.

Table1: $D_{p,q}$-classical forms

Pearson equation	The coefficients of the TTRR
$D_{p,q}\left((h_{p^{-1}}u_0)\right)+(p+q)xu_0=0.$	$\beta_n=0\ ,\ n\geq 0\ ,$ $\gamma_{n+1}=\frac{q^n}{(p+q)p^{2n}}[n+1]_{p,q}\ ,\ n\geq 0\ .$
$D_{p,q}\left(h_{p^{-1}}(\tilde{u}_0)\right)+pb^{-1}(q-p)^{-1}\{x-(b+1)\}\tilde{u}_0=0.$	$\tilde{\beta}_n=(1+b)\frac{q^n}{p^n}\ ,\ n\geq 0\ ,$ $\tilde{\gamma}_{n+1}=-b\frac{q^n}{p^{2n+1}}(p^{n+1}-q^{n+1})\ ,\ n\geq 0\ .$
$D_{p,q}\Big(h_{p^{-1}}((x-1)\tilde{u}_0)\Big)$ $-p^3\left(\widehat{a}\widehat{b}(q-p)\right)^{-1}\Big(q^{-2}x+p^{-2}\widehat{a}\widehat{b}-q^{-1}p^{-1}(\widehat{a}+\widehat{b})\Big)\tilde{u}_0=0\ .$	$\tilde{\beta}_n=\frac{q^{n+1}}{p^{2n+2}}\{p^{n+1}(\widehat{a}\widehat{b}+\widehat{a}+\widehat{b})-q^n(p+q)\widehat{a}\widehat{b}\}\ ,\ n\geq 0\ ,$ $\tilde{\gamma}_{n+1}=\frac{q^{n+2}}{p^{4n+5}}\widehat{a}\widehat{b}(q^{n+1}-p^{n+1})(p^{n+1}$ $-q^{n+1}\widehat{a})(p^{n+1}-q^{n+1}\widehat{b})\ ,\ n\geq 0\ .$
$D_{p,q}\left(h_{p^{-1}}(x\tilde{u}_0)\right)-p\left(\widehat{b}(q-p)\right)^{-1}(q^{-1}px+\widehat{b}-1)\tilde{u}_0=0\ .$	$\tilde{\beta}_n=\frac{q^n}{p^{2n+1}}\{p^n(p\widehat{b}+q)-q^n(p+q)\widehat{b}\}\ ,\ n\geq 0\ ,$ $\tilde{\gamma}_{n+1}=\frac{q^{2n+2}}{p^{4n+3}}\widehat{b}(q^{n+1}-p^{n+1})(q^n\widehat{b}-p^n)\ ,\ n\geq 0\ .$
$D_{p,q}\left(h_{p^{-1}}(x\tilde{u}_0)\right)-p(q-p)^{-1}(x+1)\tilde{u}_0=0\ .$	$\tilde{\beta}_n=\frac{q^{n-1}}{p^{2n}}\{p^{n+1}-(p+q)q^n\}\ ,\ n\geq 0\ ,$ $\tilde{\gamma}_{n+1}=\frac{q^{3n}}{p^{4n+1}}(q^{n+1}-p^{n+1})\ ,\ n\geq 0\ .$
$D_{p,q}\left(h_{p^{-1}}(x\tilde{u}_0)\right)-p\left(q\widehat{a}(q-p)\right)^{-1}(px+q\widehat{a}-p)\tilde{u}_0=0\ .$	$\tilde{\beta}_n=\frac{q^n}{p^{2n+1}}\{p^{n+1}(\widehat{a}+1)-q^n(p+q)\widehat{a}\}\ ,\ n\geq 0\ ,$ $\tilde{\gamma}_{n+1}=\frac{q^{2n+1}}{p^{4n+3}}\widehat{a}(q^{n+1}-p^{n+1})(q^{n+1}\widehat{a}-p^{n+1})\ ,\ n\geq 0\ .$

Table2: $D_{p,q}$-classical forms

Pearson equation	The coefficients of the TTRR
$D_{p,q}\big(h_{p^{-1}}(x^2\tilde{u}_0)\big)$ $-p\big(q(q-p)b\big)^{-1}\big((p+qb)x-p\big)\tilde{u}_0=0$.	$\tilde{\beta}_n=q^np^n\frac{p^{2n}+p^{n+1}q^{n-1}b+p^nq^nb-q^{2n}b}{(p^{2n-1}+q^{2n-1}b)(p^{2n+1}+q^{2n+1}b)}$, $n\geq 0$, $\tilde{\gamma}_{n+1}=\frac{bp^{3n+2}q^{3n+1}(p^{n+1}-q^{n+1})(p^n+q^nb)}{(p^{2n}+q^{2n}b)(p^{2n+1}+q^{2n+1}b)^2(p^{2n+3}+q^{2n+3}b)}$, $n\geq 0$.
$D_{p,q}\big(h_{p^{-1}}(x^2\tilde{u}_0)\big)-p(q-p)^{-1}(x-q^{-\frac{3}{2}}p^{\frac{3}{2}})\tilde{u}_0=0$.	$\tilde{\beta}_n=q^{-n-\frac{3}{2}}p^{n+\frac{1}{2}}\{p^nq^{-n}(p+q)-q\}$, $n\geq 0$, $\tilde{\gamma}_{n+1}=p^{3n+3}q^{-4n-4}(p^{n+1}-q^{n+1})$, $n\geq 0$.
$D_{p,q}\big(h_{p^{-1}}(x(x-q^{-1}\widehat{b}^{-1}p)\tilde{u}_0\big)$ $+\big(\widehat{a}\widehat{b}(q-p)q^2\big)^{-1}\Big(p(p^2-\widehat{a}\widehat{b}q^2)x+p^2(q\widehat{a}-p)\Big)\tilde{u}_0=0$.	$\tilde{\beta}_n=q^np^{2n+1}\frac{(p^{n+1}+\widehat{a}\widehat{b}q^{n+1})(\widehat{a}+1)-\widehat{a}(1+\widehat{b})(p+q)q^n}{(p^{2n}-\widehat{a}\widehat{b}q^{2n})(p^{2n+2}-\widehat{a}\widehat{b}q^{2n+2})}$, $n\geq 0$, $\tilde{\gamma}_{n+1}=\widehat{a}q^{2n+1}p^{2n+3}(p^{n+1}-q^{n+1})$ $\times\frac{(p^{n+1}-\widehat{a}\widehat{b}q^{n+1})(p^{n+1}-\widehat{a}q^{n+1})(p^{n+1}-\widehat{b}q^{n+1})}{(p^{2n+1}-\widehat{a}\widehat{b}q^{2n+1})(p^{2n+2}-\widehat{a}\widehat{b}q^{2n+2})^2(p^{2n+3}-\widehat{a}\widehat{b}q^{2n+3})}$, $n\geq 0$
$D_{p,q}\big(h_{p^{-1}}(x(x-pq^{-1}\mu^{-1})\tilde{u}_0\big)$ $-p\big((q-p)q\mu\big)^{-1}\big((\mu q-p)x-p\big)\tilde{u}_0=0$.	$\tilde{\beta}_n=q^{n-1}p^{n+1}\frac{p^{2n}+\mu q^{2n}-(p+q)q^np^{n-1}}{(p^{2n-1}-\mu q^{2n-1})(p^{2n+1}-a\mu q^{2n+1})}$, $n\geq 0$, $\tilde{\gamma}_{n+1}=-p^{3n+3}q^{3n}\frac{(p^{n+1}-q^{n+1})(p^n-\mu q^n)}{(p^{2n}-\mu q^{2n})(p^{2n+1}-\mu q^{2n+1})^2(p^{2n+2}-\mu q^{2n+2})}$, $n\geq 0$.
$D_{p,q}\big(h_{p^{-1}}(x(x+wq^{-\frac{3}{2}}p^{\frac{3}{2}})\tilde{u}_0)\big)$ $-p(q-p)^{-1}\{x+(w-1)q^{-\frac{3}{2}}p^{\frac{3}{2}}\}\tilde{u}_0=0$.	$\tilde{\beta}_n=q^{-n-\frac{3}{2}}p^{n+\frac{1}{2}}\{(p+q)p^nq^{-n}-q-pw\}$, $n\geq 0$, $\tilde{\gamma}_{n+1}=p^{2n+3}q^{-4n-4}(q^{n+1}-p^{n+1})(wq^n-p^n)$, $n\geq 0$.
$D_{p,q}\big(h_{p^{-1}}((x-\widehat{b}^{-1}\widehat{c})(x-1)\tilde{u}_0)\big)$ $-p\big(\widehat{a}\widehat{b}q^2(q-p)\big)^{-1}\{(\widehat{a}\widehat{b}q^2-p^2)x-pq\widehat{a}\widehat{b}\widehat{Q}\}\tilde{u}_0=0$, $\widehat{M}=1+\widehat{b}^{-1}+\widehat{b}^{-1}\widehat{c}(1+\widehat{a}^{-1})$, $\widehat{N}=1+\widehat{a}+\widehat{c}(1+\widehat{b}^{-1})$, $\widehat{Q}=\frac{q}{p}(1+\widehat{b}^{-1}\widehat{c})-\widehat{b}^{-1}(1+\widehat{a}^{-1}\widehat{c})$.	$\tilde{\beta}_n=q^{n+1}p^n\widehat{a}\widehat{b}\frac{\widehat{M}(p^{2n+1}+\widehat{a}\widehat{b}q^{2n+1})-\widehat{N}(p+q)p^nq^n}{(p^{2n}-\widehat{a}\widehat{b}q^{2n})(p^{2n+2}-\widehat{a}\widehat{b}q^{2n+2})}$, $n\geq 0$, $\tilde{\gamma}_{n+1}=-\widehat{a}\widehat{c}q^{n+2}p^n(q^{n+1}-p^{n+1})(p^{n+1}-\widehat{a}\widehat{b}q^{n+1})$ $\times\frac{(p^{n+1}-\widehat{a}q^{n+1})(p^{n+1}-\widehat{b}q^{n+1})(p^{n+1}-\widehat{c}q^{n+1})(p^{n+1}-\widehat{a}\widehat{b}\widehat{c}^{-1}q^{n+1})}{(p^{2n+1}-\widehat{a}\widehat{b}q^{2n+1})(p^{2n+2}-\widehat{a}\widehat{b}q^{2n+2})^2(p^{2n+3}-\widehat{a}\widehat{b}q^{2n+3})}$, $n\geq 0$.
$D_{p,q}\big(h_{p^{-1}}((x+\widehat{a}q^{-\frac{1}{2}}p^{\frac{1}{2}})(x+\widehat{b}q^{-\frac{1}{2}}p^{\frac{1}{2}})\tilde{u}_0)\big)$ $-p(q-p)^{-1}\big(x+q^{-\frac{1}{2}}p^{\frac{1}{2}}(\widehat{a}+\widehat{b})-p^{\frac{3}{2}}q^{-\frac{3}{2}}\big)\tilde{u}_0=0$.	$\tilde{\beta}_n=q^{-n-\frac{3}{2}}p^{n+\frac{1}{2}}\{(p+q)p^nq^{-n}-q(\widehat{a}+\widehat{b}+1)\}$, $n\geq 0$, $\tilde{\gamma}_{n+1}=q^{-4n-4}p^{n+1}(p^{n+1}-q^{n+1})(p^{n+1}-\widehat{a}q^{n+1})(p^{n+1}-\widehat{b}q^{n+1})$, $n\geq 0$.
$D_{p,q}\big(h_{p^{-1}}((x-c)(x-d))\tilde{u}_0)\big)$ $-p(q-p)^{-1}(x-c-d)\tilde{u}_0=0$.	$\tilde{\beta}_n=q^{-n}p^n(c+d)$, $\tilde{\gamma}_{n+1}=q^{-2n-1}p^ncd(p^{n+1}-q^{n+1})$, $n\geq 0$.

3. Moments and (p,q)-Integral Representations

3.1. General framework: moments and integral representations

Proposition 3.1. *Let $\{\Phi_n(x.,a,p,q)\}_{n\geq 0}$ be the polynomial sequence defined by*

$$\Phi_0(x,a,p,q):=1\ ,\quad \Phi_n(x,a,p,q):=\prod_{k=1}^{n}(p^{k-1}x-q^{k-1}a)\ ,\ n\geq 1\ ,\ a\in\mathbb{C}\ .$$

Then, for any form $u \in \mathcal{P}'$, *its moments satisfy*

$$(u)_n = \sum_{\nu=0}^{n} \left[\begin{array}{c} n \\ \nu \end{array} \right]_{p,q} a^{\nu} (u)_{n-\nu}^{\Phi(a,p,q)} \ , \quad n \geq 0 \ , \tag{5}$$

where

$$\left[\begin{array}{c} n \\ k \end{array} \right]_{p,q} := \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!}, \quad 0 \leq k \leq n, \quad n \geq 0,$$

and $(u)_n^{\Phi(a,p,q)} :=< u, \Phi_n(.,a,p,q) >$, $n \geq 0$, *the so-called modified moments with respect to* $\{\Phi_n\}_{n\geq 0}$,

Proof. The polynomial sequence $\{\Phi_n(x.,a,p,q)\}_{n\geq 0}$ satisfies

$$\Phi_{n+1}(x,a,p,q) = (p^n x - q^n a)\Phi_n(x,a,p,q) \ , \ n \geq 0 \ , \tag{6}$$

$$(D_{p,q}\Phi_{n+1})(x,a,p,q) = [n+1]_{p,q}\Phi_n(x,a,p,q) \ , \ n \geq 0 \ . \tag{7}$$

The relation (7) shows that the polynomial sequence $\{\Phi_{n+1}(x,a,p,q)\}_{n\geq 0}$ is a $D_{p,q}$-Appell polynomial sequence (see [8]). Its dual sequence $\{w_n(a,p,q)\}_{n\geq 0}$ fulfils

$$w_0(a,p,q) = \delta_a, \qquad w_{n+1}(a,p,q) = -\frac{\left(h_{p^{-1}} \circ D_{p,q}\right) w_n(a,p,q)}{[n+1]_{p,q}} \ , \quad n \geq 0 \ .$$

Consequently, we get from (2)

$$w_n(a,p,q) = \frac{(-1)^n p^{-\frac{n(n-1)}{2}}}{[n]_{p,q}!} \left(h_{p^{-1}} \circ D^n_{p,q}\right) \delta_a \ , \quad n \geq 0 \ .$$

In particular, the sequence $\{\Phi_n(x,0,p,q) = x^n\}_{n\geq 0}$ is $D_{p,q}$-Appell and we have

$$w_n(0,p,q) = \frac{(-1)^n}{[n]_{p,q}!} (h_{p^{-1}} \circ D^n_{p,q}) \delta = \frac{(-1)^n}{n!} D^n \delta \ , \quad n \geq 0 \ .$$

Thus, for any $u \in \mathcal{P}'$, we have

$$u = \sum_{n\geq 0} < u, \Phi_n(.,a,p,q) > \frac{(-1)^n}{[n]_{p,q}!} \left(h_{p^{-n}} \circ D^n_{p,q}\right) \delta_a \ .$$

From (6), by induction we can easily prove

$$x^n = \sum_{\nu=0}^{n} \left[\begin{array}{c} n \\ \nu \end{array} \right]_{p,q} a^{\nu} \Phi_{n-\nu}(x,a,p,q) \ , \quad n \geq 0 \ .$$

Therefore, we can deduce (5) □

Proposition 3.2. *Let u_0 be a $D_{p,q}$-classical form satisfying the functional equation* (4)*. Then, the modified moments $(u)_n^{\Phi(a,p,q)}$ satisfy the recurrence system*

$$\Psi'(0)(u_0)_1^{\Phi(a,p,q)} + a\Psi'(0) + \Psi(0) = 0\ , \tag{8}$$

$$\begin{aligned}
&p^{-n-1}\{\frac{p^{-n}}{2}\Phi''(0)[n+1]_{p,q} - \Psi'(0)\}(u_0)^{\Phi}_{n+2}(a,p,q)\\
&+\{p^{-n}[\frac{1}{2}p^{-n-1}q^n(p+q)a\Phi''(0)+\Phi'(0)][n+1]_{p,q}\\
&-p^{-n-1}q^{n+1}a\Psi'(0)-\Psi(0)\}(u_0)_{n+1}^{\Phi(a,p,q)}+\{\Phi(0)+p^{-n}q^n a\Phi'(0)\\
&+\frac{1}{2}p^{-2n}q^{2n}a^2\Phi''(0)\}[n+1]_{p,q}(u_0)_n^{\Phi(a,p,q)} = 0\ ,\ n \geq 0\ .
\end{aligned} \tag{9}$$

Proof. Taking into account

$$\Phi(x) = \frac{1}{2}\Phi''(0)x^2 + \Phi'(0)x + \Phi(0)\ ,\quad \Psi(x) = \Psi'(0)x + \Psi(0)\ ,$$

$$p^n x\Phi_n(x,a,p,q) = \Phi_{n+1}(x,a,p,q) + q^n a\Phi_n(x,a,p,q)\ ,\quad n \geq 0\ ,$$

$$\begin{aligned}
p^{2n+1}x^2\Phi_n(x,a,p,q) &= \Phi_{n+2}(x,a,p,q) + q^n(p+q)a\Phi_{n+1}(x,a,p,q)\\
&\quad + q^{2n}pa^2\Phi_n(x,a,p,q)\ ,\quad n \geq 0\ ,
\end{aligned}$$

and shut that

$$\langle D_{p,q}\big(h_{p^{-1}}(\Phi u_0)\big) + \Psi u_0, \Phi_n(.,a,p,q)\rangle = 0\ ,\ n \geq 0\ ,$$

we can deduce (8)-(9). □

Remark 3.3. When $a = 0$, we have $(u_0)_n^{\Phi(0,p,q)} = p^{\frac{n(n-1)}{2}}(u_0)_n$ and the system (8)-(9) becomes

$$\Psi'(0)(u_0)_1 + \Psi(0) = 0\ , \tag{10}$$

$$\begin{aligned}
\{\frac{1}{2}\Phi''(0)[n+1]_{p,q} - p^n\Psi'(0)\}(u_0)_{n+2} + \{\Phi'(0)[n+1]_{p,q} - p^n\Psi(0)\}(u_0)_{n+1}&\\
+\Phi(0)[n+1]_{p,q}(u_0)_n = 0\ ,\ n \geq 0\ .&
\end{aligned} \tag{11}$$

Proposition 3.4. *Assume that there exists a function $U_{p,q}(x)$ and a contour $\mathcal{C}$ such that*

$$\langle u_0, f(x)\rangle = \int_{\mathcal{C}} f(x)U_{p,q}(x)d_{p,q}x\ ,\quad (u_0)_0 = \int_{\mathcal{C}} U_{p,q}(x)d_{p,q}x = 1. \tag{12}$$

Then, u_0 satisfies the functional equation (4) *if and only if the weight function $U_{p,q}$ verifies*

$$D_{p,q}\big(h_{q^{-1}}(\Phi(x)U_{p,q}(x))\big)+p^{-1}\Psi(x)U_{p,q}(x)=0\ , \tag{13}$$

together with the boundary condition

$$h_{q^{-1}}\big(\Phi U_{p,q}\big)(x)\big(h_{p^{-1}}f\big)(x)|_{\mathcal{C}}=0\ , \tag{14}$$

and $U_{p,q}$ satisfies the functional equation

$$\big(D_{p,q}U_{p,q}\big)(x)=-\frac{\big(D_{p,q}\Phi\big)(x)+p^{-1}q\Psi(qx)}{\Phi(px)}U_{p,q}(qx)\ . \tag{15}$$

Proof. Using (4), (12) and taking into account of (1), we get

$$\int_{\mathcal{C}}\Psi(x)U_{p,q}(x)f(x)d_{p,q}x-p\int_{\mathcal{C}}\Phi(x)U_{p,q}(x)D_{p,q}\big((h_{p^{-1}}f)(x)\big)d_{p,q}x=0\ . \tag{16}$$

Or from (3), we have

$$\begin{aligned}\Phi(x)U_{p,q}(x)D_{p,q}\big((h_{p^{-1}}f)(x)\big)&=D_{p,q}\big(h_{q^{-1}}(U_{p,q}\Phi)(h_{p^{-1}}f)\big)\\&\quad-fD_{p,q}\big(h_{q^{-1}}(\Phi U_{p,q})\big)\ .\end{aligned} \tag{17}$$

Then, (16) reduces to

$$\begin{aligned}&\int_{\mathcal{C}}\{D_{p,q}\Big(h_{q^{-1}}(\Phi U_{p,q})\Big)+p^{-1}\Psi U_{p,q}\}(x)f(x)d_{p,q}x\\&\qquad\qquad-h_{q^{-1}}\big(\Phi U_{p,q}\big)(x)(h_{p^{-1}}f)(x)|_{\mathcal{C}}=0\ ,\end{aligned}$$

since $\int D_{p,q}(f)d_{p,q}x=f(x)$ (see [9, 10]).

In order that (12) yields the representation of a solution of equation (4), the conditions (13) and (14) must hold.

From (13) and (3), we have

$$\big(D_{pq^{-1},1}U_{p,q}\big)(x)=-\frac{\big(D_{pq^{-1},1}\Phi\big)(x)+p^{-1}q\Psi(x)}{\Phi(pq^{-1}x)}U_{p,q}(x)\ . \tag{18}$$

With x changed into qx, we obtain from (18) the desired result (15).

Conversely, assume that $U_{p,q}$ satisfies (13) and (14). After multiplying (13) by $f\in\mathcal{P}$ and integrating over $\mathcal{C}$, we get

$$\int_{\mathcal{C}}\Big(D_{p,q}\big(h_{q^{-1}}(\Phi(x)U_{p,q}(x))\big)+p^{-1}\Psi(x)U_{p,q}(x)\Big)d_{p,q}x=0\ .$$

Taking into account (17) and the boundary condition, we obtain for all $f\in\mathcal{P}$

$$\langle D_{p,q}\big(h_{q^{-1}}(\Phi u_0)\big)+\Psi u_0,f\rangle=0\ .$$

Therefore, u_0 satisfies (4). □

Now, let us recall the following standard notations [7, 9, 11]

$$\left(a\ominus b\right)_{p,q}^{n}:=\begin{cases} 1\,, & n=0\,,\\ \prod_{\nu=0}^{n-1}(ap^{\nu}-bq^{\nu})\,, & n\geq 1\,,\end{cases}$$

$$\left(a\oplus b\right)_{p,q}^{n}:=\begin{cases} 1\,, & n=0\,,\\ \prod_{\nu=0}^{n-1}(ap^{\nu}+bq^{\nu})\,, & n\geq 1\,,\end{cases}$$

$$\left(a\ominus b\right)_{p,q}^{+\infty}=\prod_{k=0}^{+\infty}\left(ap^{k}-bq^{k}\right)\,,$$

$$\left(a\oplus b\right)_{p,q}^{+\infty}=\prod_{k=0}^{+\infty}\left(ap^{k}+bq^{k}\right)\,,$$

$$\int_{0}^{a}f(x)d_{p,q}x=\begin{cases} (q-p)a\sum_{k=0}^{+\infty}\frac{p^{k}}{q^{k+1}}f(\frac{p^{k}}{q^{k+1}}a)\,, & \frac{p}{q}<1\,,\\ (p-q)a\sum_{k=0}^{+\infty}\frac{q^{k}}{p^{k+1}}f(\frac{q^{k}}{p^{k+1}}a)\,, & \frac{p}{q}>1\,,\end{cases}$$

$$\int_{a}^{b}f(x)d_{p,q}x=\int_{0}^{b}f(x)d_{p,q}x-\int_{0}^{a}f(x)d_{p,q}x\,,$$

$$\int_{0}^{+\infty}f(x)d_{p,q}x=\begin{cases} (p-q)\{\sum_{i=0}^{+\infty}\frac{q^{i}}{p^{i+1}}f(\frac{q^{i}}{p^{i+1}}) & \\ +\sum_{i=1}^{+\infty}\frac{q^{-i}}{p^{-i+1}}f(\frac{q^{-i}}{p^{-i+1}})\}\,, & \frac{p}{q}>1\,,\\ (q-p)\{\sum_{i=0}^{+\infty}\frac{p^{i}}{q^{i+1}}f(\frac{p^{i}}{q^{i+1}}) & \\ +\sum_{i=1}^{+\infty}\frac{p^{-i}}{q^{-i+1}}f(\frac{p^{-i}}{q^{-i+1}})\}\,, & \frac{q}{p}>1\,,\end{cases}$$

$$\int_{a}^{\infty}f(x)d_{p,q}x=\begin{cases} (p-q)a\sum_{k=0}^{\infty}\frac{q^{-k}}{p^{-k-1}}f\left(\frac{q^{-k}}{p^{-k-1}}a\right)\,, & \frac{p}{q}>1\,,\\ (q-p)a\sum_{k=0}^{\infty}\frac{p^{-k}}{q^{-k-1}}f\left(\frac{p^{-k}}{q^{-k-1}}a\right)\,, & \frac{q}{p}>1\,.\end{cases}$$

3.2. Canonical cases: explicit moments and integral representations

Before we consider the above fourteen canonical functional equations, and in each case provide an integral and moments representation of the corresponding linear functional, let us recall the following basic result.

Lemma 3.5. *Consider the form $D_{p,q}(y)(x) = A(x)y(qx)$. Then, we find*

$$y(x) = \begin{cases} K\displaystyle\prod_{i=0}^{+\infty}\Big\{1+(1-\frac{q}{p})(\frac{q}{p})^i xA(\frac{q^i}{p^{i+1}}x)\Big\}\ , & 0<q<p\leq 1\ , \\ K\displaystyle\prod_{i=0}^{+\infty}\Big\{1+(\frac{p}{q}-1)(\frac{p}{q})^i xA(\frac{p^i}{q^{i+1}}x)\Big\}^{-1}\ , & 1\leq p<q\ . \end{cases} \tag{19}$$

Proof. For $0 < \frac{q}{p} < 1$ see [5]. Now, from the definition of $D_{p,q}$, we have

$$y(px) = \{1+(p-q)xA(x)\}y(qx)\ .$$

By replacing qx by x in the above equation, we get

$$y(\frac{p}{q}x) = \{1+(\frac{p}{q}-1)xA(\frac{x}{q})\}y(x)\ . \tag{20}$$

From (20), we obtain by using the recurrence relation

$$y((\frac{p}{q})^n x) = \prod_{i=0}^{n-1}\Big\{1+(\frac{p}{q}-1)(\frac{p}{q})^i xA(\frac{p^i}{q^{i+1}}x)\Big\}y(x)\ , \quad n\geq 1\ .$$

If $n \to +\infty$ with $0 < \frac{p}{q} < 1$, then we can deduce the desired result. ☐

Lemma 3.6. *We have for $x\xi \neq 0$, $p>0$, $q>0$ and $p\neq q$*

$$\Big(D_{p,q}\exp\Big(-\frac{\ln^2(|\xi|)}{2\ln(\frac{q}{p})}\Big)\Big)(x) = \frac{p^{\frac{1}{2}}q^{\frac{1}{2}}|x|-1}{(p-q)x}\exp\Big(-\frac{\ln^2(q|x|)}{2\ln(\frac{q}{p})}\Big)\ , \tag{21}$$

$$\Big(D_{p,q}\exp\Big(-\frac{\ln^2(|\xi|)}{2\ln(\frac{p}{q})}\Big)\Big)(x) = \frac{p^{-\frac{1}{2}}q^{-\frac{1}{2}}-|x|}{(p-q)x|x|}\exp\Big(-\frac{\ln^2(q|x|)}{2\ln(\frac{p}{q})}\Big)\ , \tag{22}$$

$$\begin{aligned}&\Big(D_{p,q}\exp\Big(-\tfrac{\ln^2(|\xi|)}{2\ln(\frac{q}{p})}\Big)\big(1\ominus a\xi\big)_{1,\frac{p}{q}}^{+\infty}\Big)(x)\\ &\quad=\tfrac{(pq)^{\frac{1}{2}}|x|+aqx-1}{(p-q)x(1-aqx)}\exp\Big(-\tfrac{\ln^2(q|x|)}{2\ln(\frac{q}{p})}\Big)\big(1\ominus aqx\big)_{1,\frac{p}{q}}^{+\infty}\ ,\end{aligned} \tag{23}$$

$$\begin{aligned}&\left(D_{p,q}\exp\Big(-\tfrac{\ln^2(|\xi|)}{2\ln(\frac{p}{q})}\Big)\big(1\ominus a\xi\big)^{+\infty}_{1,\frac{q}{p}}\right)(x)\\ &\quad=\tfrac{(pq)^{-\frac{1}{2}}(1-apx)-|x|}{(p-q)x|x|}\exp\Big(-\tfrac{\ln^2(q|x|)}{2\ln(\frac{p}{q})}\Big)\big(1\ominus aqx\big)^{+\infty}_{1,\frac{q}{p}}\ ,\end{aligned}\tag{24}$$

$$\begin{aligned}&\left(D_{p,q}\exp\Big(-\tfrac{\ln^2(|\xi|)}{2\ln(\frac{p}{q})}\Big)\big(1\ominus a\xi\big)^{+\infty}_{1,\frac{q}{p}}\big(1\ominus b\xi\big)^{+\infty}_{1,\frac{q}{p}}\right)(x)\\ &=\tfrac{(pq)^{-\frac{1}{2}}(1-apx)(1-bpx)-|x|}{(p-q)x|x|}\exp\Big(-\tfrac{\ln^2(q|x|)}{2\ln(\frac{p}{q})}\Big)\big(1\ominus aqx\big)^{+\infty}_{1,\frac{q}{p}}\big(1\ominus bqx\big)^{+\infty}_{1,\frac{q}{p}}\ .\end{aligned}\tag{25}$$

Proof. Taking into account the definition of the operator $D_{p,q}$ and such that $p|x|=\frac{p}{q}(q|x|)$, we can deduce the desired results (21) and (22). On the other hand, thanks to (3), (21), (22) and taking account of

$$\Big(D_{p,q}(1\ominus a\xi)^{+\infty}_{1,\frac{p}{q}}\Big)(x)=\frac{aq}{(p-q)(1-aqx)}(1\ominus a\xi)^{+\infty}_{1,\frac{p}{q}}(qx)\ ,\tag{26}$$

$$\Big(D_{p,q}(1\ominus a\xi)^{+\infty}_{1,\frac{q}{p}}\Big)(x)=-\frac{ap}{p-q}(1\ominus a\xi)^{+\infty}_{1,\frac{q}{p}}(qx)\ ,\tag{27}$$

we get (23) and (24). Finally, (25) follows from (3), (24) and (27). □

Now, we are able to calculate the moments and to give an integral representation of any canonical form satisfying either (8)-(9) or (10)-(11).

(A_1) $\Phi(x)=1$.

- u_0 satisfies (4) with

$$\Phi(x)=1,\quad \Psi(x)=(p+q)x.\tag{28}$$

From (28), the system (10)-(11) becomes

$$(u_0)_1=0\ ,\quad (u_0)_{n+2}-\frac{[n+1]_{p,q}}{(p+q)p^n}(u_0)_n=0\ ,\ n\geq 0\ .$$

Consequently,

$$(u_0)_{2n}=\frac{[2n]_{p,q}![2n+2]_{p,q}}{(p+q)^n p^{n(n-1)}\prod_{\nu=0}^{n}[2\nu+2]_{p,q}}\ ,\quad (u_0)_{2n+1}=0\ ,\ n\geq 0\ .$$

Using (28) and (15), we get

$$\big(D_{p,q}U_{p,q}\big)(x)=-p^{-1}q^2(p+q)xU_{p,q}(qx)\ .$$

Whence, where $1 \le p < q$ we get from (19)

$$U_{p,q}(x) = \frac{K}{\big(1 \oplus (q^2-p^2)p^{-1}x^2\big)^{+\infty}_{1,\frac{p^2}{q^2}}} \ , \ x \in \ R \ , \tag{29}$$

where

$$K = \frac{1}{2}\bigg(\int_0^{+\infty} \frac{d_{p,q}x}{\big(1 \oplus (q^2-p^2)p^{-1}x^2\big)^{+\infty}_{1,\frac{p^2}{q^2}}}\bigg)^{-1} \ .$$

For any $n \ge 0$, we have

$$\begin{aligned} |x^n U_{p,q}(q^{-1}x)| &\le \frac{K|x|^n}{\big(1 \oplus (q^2-p^2)p^{-1}q^{-2}x^2\big)^{n}_{1,\frac{p^2}{q^2}}} \\ &\le K\Big(\frac{(\frac{q}{p})^n}{q^{-2}p^{-1}(q^2-p^2)}\Big)^{n+1}\frac{1}{|x|^{n+2}} \ . \end{aligned}$$

Thus, $U_{p,q}$ given by (29) is an admissible solution.
When $0 < q < p \le 1$, we obtain from (28) and (15)

$$U_{p,q}(x) = \begin{cases} K\big(1 \ominus (p^2-q^2)p^{-2}q^2x^2\big)^{+\infty}_{1,\frac{q^2}{p^2}} \ , \quad |x| \le \frac{p}{\sqrt{(p^2-q^2)}} \ , \\ 0 \ , \quad |x| > \frac{p}{\sqrt{(p^2-q^2)}} \ , \end{cases}$$

where

$$K = \frac{1}{2}\bigg(\int_0^{p\left((p^2-q^2)\right)^{-\frac{1}{2}}} \big(1 \ominus (p^2-q^2)p^{-2}q^2x^2\big)^{+\infty}_{1,\frac{q^2}{p^2}} d_{p,q}x\bigg)^{-1} \ .$$

- u_0 verifies (4) with

$$\tilde{\Phi}(x) = 1, \qquad \tilde{\Psi}(x) = p\big(b(q-p)\big)^{-1}(x-(b+1)). \tag{30}$$

Then, the system (8)-(9) becomes

$$\begin{aligned} (\tilde{u}_0)_1^{\Phi(a,p,q)} &= b+1-a \ , \\ (\tilde{u}_0)_{n+2}^{\Phi(a,p,q)} &= ((b+1)p^{n+1}-aq^{n+1})(\tilde{u}_0)_{n+1}^{\Phi(a,p,q)} \\ &\quad -bp^n(p^{n+1}-q^{n+1})(\tilde{u}_0)_n^{\Phi(a,p,q)} \ , \ n \ge 0 \ . \end{aligned} \tag{31}$$

With $a \in \{1,b\}$, we can obtain the following first-order recurrence relation (for $n \geq 0$,)

$$\begin{aligned}
&(\tilde{u}_0)_1^{\Phi(1,p,q)} = b,\\
&(\tilde{u}_0)_{n+2}^{\Phi(1,p,q)} - bp^{n+1}(\tilde{u}_0)_{n+1}^{\Phi(1,p,q)} = (p^{n+1}-q^{n+1})\{(\tilde{u}_0)_{n+1}^{\Phi(1,p,q)}\\
&\qquad\qquad - bp^n(\tilde{u}_0)_n^{\Phi(1,p,q)}\}\ ,\\
&(\tilde{u}_0)_1^{\Phi(b,p,q)} = 1\ ,\\
&(\tilde{u}_0)_{n+2}^{\Phi(b,p,q)} - p^{n+1}(\tilde{u}_0)_{n+1}^{\Phi(b,p,q)} = b(p^{n+1}-q^{n+1})\{(\tilde{u}_0)_{n+1}^{\Phi(b,p,q)}\\
&\qquad\qquad - p^n(\tilde{u}_0)_n^{\Phi(b,p,q)}\}\ .
\end{aligned}$$

This yields

$$(\tilde{u}_0)_n^{\Phi(a,p,q)} = \begin{cases} p^{\frac{n(n-1)}{2}} b^n\ , & a=1\ , \\ p^{\frac{n(n-1)}{2}}\ , & a=b\ , \end{cases} \quad n \geq 0\ . \tag{32}$$

Finally, from (15) and (30), we obtain

$$\big(D_{p,q}U_{p,q}\big)(x) = -q\big(b(q-p)\big)^{-1}(qx-b-1)U_{p,q}(qx)\ .$$

Whence, if $b<0$ and $0<q<p\leq 1$, then we get from (19)

$$U_{p,q}(x) = \begin{cases} 0\ , \quad x<pb\ , \\ K\big(1\ominus \frac{q}{p}b^{-1}x\big)_{1,\frac{q}{p}}^{+\infty}\big(1\ominus\frac{q}{p}x\big)_{1,\frac{q}{p}}^{+\infty}\ , \quad pb\leq x\leq p\ , \\ 0\ , \quad x>p\ , \end{cases}$$

where

$$K = \left(\int_{pb}^{p}\big(1\ominus\frac{q}{p}b^{-1}x\big)_{1,\frac{q}{p}}^{+\infty}\big(1\ominus\frac{q}{p}x\big)_{1,\frac{q}{p}}^{+\infty}d_{p,q}x\right)^{-1}\ .$$

For $1\leq p<q$, the problem is open.

Remark 3.7. (i) When $a=0$ and $b=-1$ we have from (31)

$$(\tilde{u}_0)_1 = 0\ , \quad (\tilde{u}_0)_{n+2} = p^n(p^{n+1}-q^{n+1})(\tilde{u}_0)_n\ ,\ n\geq 0\ .$$

Consequently,

$$(\tilde{u}_0)_{2n} = p^{n(n-1)}\big(p\ominus q\big)_{p^2,q^2}^{n}\ , \qquad (\tilde{u}_0)_{2n+1} = 0\ ,\ n\geq 0\ .$$

(ii) By virtue of (5) and (32), we have

$$(\tilde{u}_0)_n = \sum_{\nu=0}^{n} \begin{bmatrix} n \\ \nu \end{bmatrix}_{p,q} p^{-\nu(n-\nu)} b^{\nu} \ , \quad n \geq 0 \ .$$

(A_2) $\Phi(x) = x - c$, $c \in \mathbb{C}$.

$(A_{2.1})$ $c \neq 0$.

u_0 fulfils (4) with

$$\begin{aligned} &\tilde{\Phi}(x) = x - 1 \ , \\ &\tilde{\Psi}(x) = -p^3 \big(\widehat{a}\widehat{b}(q-p)\big)^{-1} \{q^{-2}x + p^{-2}\widehat{a}\widehat{b} - q^{-1}p^{-1}(\widehat{a}+\widehat{b})\} \ . \end{aligned} \tag{33}$$

Then, the system (8)-(9) becomes

$$\begin{aligned} (\tilde{u}_0)_1^{\Phi(a,p,q)} &= qp^{-1}(\widehat{a}+\widehat{b}) - a - q^2 p^{-2}\widehat{a}\widehat{b} \ , \\ (\tilde{u}_0)_{n+2}^{\Phi(a,p,q)} &= \{(p^n q(\widehat{a}+\widehat{b}) - (a + p^{-2}q^2\widehat{a}\widehat{b})q^{n+1}\}(\tilde{u}_0)_{n+1}^{\Phi(a,p,q)} \\ &\quad - \widehat{a}\widehat{b}q^2 p^{-2}(p^n - aq^n)(p^{n+1} - q^{n+1})(\tilde{u}_0)_n^{\Phi(a,p,q)} \ , \ n \geq 0 \ . \end{aligned}$$

When we choose $a = qp^{-1}\widehat{a}$, we obtain for $n \geq 0$

$$\begin{aligned} (\tilde{u}_0)_1^{\Phi(qp^{-1}\widehat{a},p,q)} &= qp^{-1}\widehat{b}(1 - qp^{-1}\widehat{a}) \ , \\ (\tilde{u}_0)_{n+2}^{\Phi(qp^{-1}\widehat{a},p,q)} &- qp^{-1}\widehat{b}(p^{n+1} - q^{n+2}\widehat{a})(\tilde{u}_0)_{n+1}^{\Phi(qp^{-1}\widehat{a},p,q)} \\ &= qp^{-1}\widehat{a}(p^{n+1} - q^{n+1})\{(\tilde{u}_0)_{n+1}^{\Phi(qp^{-1}\widehat{a},p,q)} \\ &\quad - qp^{-1}\widehat{b}(p^n - p^{-1}\widehat{a}q^{n+1})(\tilde{u}_0)_n^{\Phi(qp^{-1}\widehat{a},p,q)}\} \ . \end{aligned}$$

Hence,

$$(\tilde{u}_0)_n^{\Phi(qp^{-1}\widehat{a},p,q)} = (qp^{-1}\widehat{b})^n \big(1 \ominus q\widehat{a}\big)_{p,q}^n \ , \qquad n \geq 0 \ .$$

Likewise, when $a = qp^{-1}\widehat{b}$, we get

$$(\tilde{u}_0)_n^{\Phi(qp^{-1}\widehat{b},p,q)} = (qp^{-1}\widehat{a})^n \big(1 \ominus q\widehat{b}\big)_{p,q}^n \ , \quad n \geq 0 \ .$$

Now, from (33) and (15), we have

$$\begin{aligned} &\big(D_{p,q}U_{p,q}\big)(x) \\ &= -\frac{1 - qp^2\big(\widehat{a}\widehat{b}(q-p)\big)^{-1}[q^{-1}x + p^{-2}\widehat{a}\widehat{b} - q^{-1}p^{-1}(\widehat{a}+\widehat{b})]}{px-1} U_{p,q}(qx) \ . \end{aligned}$$

Moreover, when $\widehat{a}<0,\ 0<\widehat{b}\leq 1$ and $0<q<p\leq 1$, we get

$$U_{p,q}(x)=\begin{cases} 0\,, \quad x<p\widehat{a}\,, \\ K\frac{(1\ominus\widehat{b}^{-1}x)^{+\infty}_{p,q}(1\ominus\widehat{a}^{-1}x)^{+\infty}_{1,\frac{q}{p}}}{(1\ominus x)^{+\infty}_{p,q}}\,, \quad p\widehat{a}\leq x\leq p\widehat{b}\,, \\ 0\,, \quad x>p\widehat{b}\,, \end{cases}$$

where

$$K=\left(\int_{p\widehat{a}}^{p\widehat{b}}\frac{(1\ominus\widehat{b}^{-1}x)^{+\infty}_{p,q}(1\ominus\widehat{a}^{-1}x)^{+\infty}_{1,\frac{q}{p}}}{(1\ominus x)^{+\infty}_{p,q}}d_{p,q}x\right)^{-1}\,.$$

When $0<\widehat{a}\leq 1, \widehat{b}<0$, by exchanging $\widehat{a}$ and $\widehat{b}$, we get

$$U_{p,q}(x)=\begin{cases} 0\,, \quad x<p\widehat{b}\,, \\ K\frac{(1\ominus\widehat{b}^{-1}x)^{+\infty}_{p,q}(1\ominus\widehat{a}^{-1}x)^{+\infty}_{1,\frac{q}{p}}}{(1\ominus x)_{p,q}}\,, \quad p\widehat{b}\leq x\leq p\widehat{a}\,, \\ 0\,, \quad x>p\widehat{a}\,, \end{cases}$$

where

$$K=\left(\int_{p\widehat{b}}^{p\widehat{a}}\frac{(1\ominus\widehat{b}^{-1}x)^{+\infty}_{p,q}(1\ominus\widehat{a}^{-1}x)^{+\infty}_{1,\frac{q}{p}}}{(1\ominus x)^{+\infty}_{p,q}}d_{p,q}x\right)^{-1}\,.$$

When $1\leq p<q$, the problem is open.

$(A_{2.2})$ $c=0$.

$(A_{2.2.1})$ u_0 satisfies (4) with

$$\tilde{\Phi}(x)=x, \qquad \tilde{\Psi}(x)=-p\big(\widehat{b}(q-p)\big)^{-1}(q^{-1}px+\widehat{b}-1)\,. \quad (34)$$

Since $\tilde{\Phi}(0)=0$, (10)-(11) becomes

$$(\tilde{u}_0)_{n+1}=qp^{-n-1}(p^n-\widehat{b}q^n)(\tilde{u}_0)_n\,, \qquad n\geq 0\,.$$

Therefore,

$$(\tilde{u}_0)_n=q^np^{-\frac{n(n+1)}{2}}\big(1\ominus\widehat{b}\big)^n_{p,q}\,, \qquad n\geq 0\,.$$

From (34) and (15), we obtain

$$\big(D_{p,q}U_{p,q}\big)(x)=-\frac{1-pq(\widehat{b}(q-p))^{-1}[px+\widehat{b}-1]}{px}U_{p,q}(qx)\,. \quad (35)$$

If $\widehat{b} = (\frac{q}{p})^{\tau}$, then (35) becomes

$$\big(D_{p,q}\xi^{1-\tau}U_{p,q}\big)(x) = p(q-p)^{-1}\big(\xi^{1-\tau}U_{p,q}\big)(qx)\ .$$

Then, for $0 < q < p \leq 1$ and $0 < b < 1$, we get

$$U_{p,q}(x) = \begin{cases} 0\ , \quad x \leq 0\ , \\ Kx^{\tau-1}(1\ominus x)^{+\infty}_{1,\frac{q}{p}}\ , \quad 0 < x < p\ , \quad \tau > 0\ , \\ 0\ , \quad x \geq p\ , \end{cases}$$

where

$$K = \left(\int_0^p x^{\tau-1}(1\ominus x)^{+\infty}_{1,\frac{q}{p}} d_{p,q}x\right)^{-1}\ .$$

When $1 \leq p < q$ and $b > 1$, we have

$$U_{p,q}(x) = \begin{cases} 0\ , \quad x \geq 0\ , \\ K\frac{|x|^{\tau-1}}{(1\oplus\frac{p}{q}|x|)^{+\infty}_{1,\frac{p}{q}}}\ , \quad x < 0\ , \quad \tau > 0\ , \end{cases}$$

where

$$K = \left(\int_{-\infty}^0 \frac{|x|^{\tau-1}}{(1\oplus\frac{p}{q}|x|)^{+\infty}_{1,\frac{p}{q}}} d_{p,q}x\right)^{-1}\ .$$

$(A_{2.2.2})$ u_0 verifies (4) with

$$\tilde{\Phi}(x) = x, \qquad \tilde{\Psi}(x) = -p(q-p)^{-1}(x+1)\ . \tag{36}$$

System (10)-(11) becomes

$$(\tilde{u}_0)_{n+1} = -(\frac{q}{p})^n(\tilde{u}_0)_n\ ,\ n \geq 0\ .$$

This yields

$$(\tilde{u}_0)_n = (-1)^n(\frac{q}{p})^{\frac{n(n-1)}{2}}\ , \qquad n \geq 0\ .$$

From (36) and (15), we can deduce

$$\big(D_{p,q}U_{p,q}\big)(x) = -\frac{(p-q)^{-1}(q^2x+p)}{px}U_{p,q}(qx)\ .$$

Equivalently,

$$\big(D_{p,q}\xi^{\frac{3}{2}}U_{p,q}\big)(x) = -\frac{p^{\frac{1}{2}}q^{\frac{1}{2}}x+1}{(p-q)x}\big(\xi^{\frac{3}{2}}U_{p,q}\big)(qx)\ . \tag{37}$$

Then, from (21) and (37), we have for $1 \leq p < q$

$$U_{p,q}(x) = \begin{cases} 0\,, \quad x \geq 0 \\ K|x|^{-\frac{3}{2}} \exp\Big(- \frac{\ln^2|x|}{2\ln(\frac{q}{p})}\Big)\,, \quad x < 0\,, \end{cases}$$

with

$$K = \Big(\int_0^{+\infty} x^{-\frac{3}{2}} \exp\Big(- \frac{\ln^2(x)}{2\ln(\frac{q}{p})}\Big) d_{p,q}x \Big)^{-1}\,.$$

The problem remains open to give integral representation for $0 < q < p \leq 1$.

$(A_{2.2.3})$ u_0 fulfils (4) with

$$\tilde{\Phi}(x) = x, \qquad \tilde{\Psi}(x) = -p(q\widehat{a}(q-p))^{-1}(px + q\widehat{a} - p)\,. \quad (38)$$

System (10)-(11) becomes

$$(\tilde{u}_0)_{n+1} = p^{-n-1}(p^{n+1} - \widehat{a}q^{n+1})(\tilde{u}_0)_n\,,\ n \geq 0\,.$$

This yields

$$(\tilde{u}_0)_n = p^{-\frac{n(n+1)}{2}} \big(p \ominus \widehat{a}q\big)^n_{p,q}\,, \qquad n \geq 0\,.$$

Further, from (38) and (15), we can deduce

$$\big(D_{p,q}U_{p,q}\big)(x) = -\frac{1 - \big(\widehat{a}(q-p)\big)^{-1}(qpx + q\widehat{a} - 1)}{px} U_{p,q}(qx)\,. \quad (39)$$

If $\widehat{a} = (\frac{q}{p})^\tau$, then (39) reduces to

$$\big(D_{p,q}\xi^{-\tau}U_{p,q}\big)(x) = -q(p-q)^{-1}\big(\xi^{-\tau}U_{p,q}\big)(qx)\,.$$

Then, for $0 < q < p \leq 1$ and $0 < \widehat{a} < 1$, we have

$$U_{p,q}(x) = \begin{cases} 0\,, \quad x \leq 0 \\ Kx^\tau(1 \ominus qp^{-1}x)^{+\infty}_{1,\frac{q}{p}}\,, \quad 0 < x < p\,, \quad \tau > -1\,, \end{cases}$$

with

$$K = \Big(\int_0^{p} x^\tau(1 \ominus qp^{-1}x)^{+\infty}_{1,\frac{q}{p}} d_{p,q}x \Big)^{-1}\,.$$

When $1 \leq p < q$ and $\widehat{a} > 1$, we have

$$U_{p,q}(x) = \begin{cases} 0\ , \quad x \geq 0\ , \\ K\frac{|x|^{\tau}}{(1 \oplus \frac{p}{q}|x|)_{1,\frac{p}{q}}^{+\infty}}\ , \quad x < 0\ , \quad \tau > -1\ , \end{cases}$$

with

$$K = \left(\int_0^{+\infty} \frac{x^{\tau}}{(1 \oplus \frac{p}{q}x)_{1,\frac{p}{q}}^{+\infty}} d_{p,q}x \right)^{-1} .$$

(B_1) $\Phi(x) = (x-c)(x-d)$, $cd = 0$.

$(B_{1.1})$ $c = 0, d = 0$.

$(B_{1.1.1})$ u_0 satisfies (4) with

$$\widehat{\Phi}(x) = x^2, \qquad \widehat{\Psi}(x) = -p(qb(p-q))^{-1}[(p+qb)x-p]\ . \tag{40}$$

The system (10)-(11) becomes

$$(\tilde{u}_0)_{n+1} = \frac{p^{n+1}}{p^{n+1}+bq^{n+1}}(\tilde{u}_0)_n\ ,\ n \geq 0\ .$$

Consequently,

$$(\tilde{u}_0)_n = \frac{p^{\frac{n(n+1)}{2}}}{\big(p \oplus bq\big)_{p,q}^{n}}\ , \quad n \geq 0\ , \quad b \neq -(\frac{p}{q})^m\ , \quad m \geq 1\ .$$

From (40) and (15), we obtain

$$\big(D_{p,q}U_{p,q}\big)(x) = -\frac{[b(p-q)]^{-1}}{px^2}[(bp+q)x-1]U_{p,q}(qx)\ . \tag{41}$$

If $b = (\frac{q}{p})^{\tau}$, then we can deduce from (41)

$$\big(D_{p,q}\xi^{\frac{1}{2}-\tau}U_{p,q}\big)(x) = \\ \frac{(p-q)^{-1}}{x^2}\{(pq)^{-\frac{1}{2}}(1-qx)-x\}\big(\xi^{\frac{1}{2}-\tau}U_{p,q}\big)(qx)\ .$$

Then, from (24), we get for $0 < q < p \leq 1$ and $\tau \in \mathbb{R}$

$$U_{p,q}(x) = \begin{cases} Kx^{\tau-\frac{1}{2}}\exp\Big(-\frac{\ln^2(x)}{2\ln(\frac{p}{q})}\Big)\big(1 \ominus qp^{-1}x\big)_{1,\frac{q}{p}}^{+\infty}\ , \\ x > 0\ , \\ 0 \quad x \leq 0\ , \end{cases}$$

where

$$K = \Big(\int_0^{+\infty} x^{\tau-\frac{1}{2}} \exp\Big(-\frac{\ln^2(x)}{2\ln(\frac{p}{q})}\Big)\big(1 \ominus qp^{-1}x\big)^{+\infty}_{1,\frac{q}{p}} d_{p,q}x\Big)^{-1} .$$

The problem remains open to give integral representation for $q > p \geq 1$.

$(B_{1.1.2})$ u_0 verifies (4) with

$$\widehat{\Phi}(x) = x^2, \qquad \widehat{\Psi}(x) = -p(q-p)^{-1}(x - q^{-\frac{3}{2}} p^{\frac{3}{2}}) . \tag{42}$$

System (10)-(10) becomes

$$(\widehat{u}_0)_{n+1} = (\frac{q}{p})^{-(n+3/2)}(\widehat{u}_0)_n,\ n \geq 0.$$

Hence,

$$(\widehat{u}_0)_n = (\frac{q}{p})^{-\frac{1}{2}n(n+2)}\ ,\ n \geq 0\ .$$

From (42) and (15), we get

$$\big(D_{p,q}U_{p,q}\big)(x) = -\frac{x - (qp)^{-\frac{1}{2}}}{(p-q)x^2}\ .$$

Then, from (22) we obtain for $0 < q < p \leq 1$

$$U_{p,q}(x) = \begin{cases} K\exp\Big(-\frac{\ln^2(x)}{2\ln(\frac{p}{q})}\Big)\ , & x > 0\ , \\ 0 \quad x \leq 0\ , \end{cases}$$

where

$$K = \Big(\int_0^{+\infty} \exp\Big(-\frac{\ln^2(x)}{2\ln(\frac{p}{q})}\Big) d_{p,q}x\Big)^{-1} .$$

When $1 \leq p < q$, the problem remains open to give integral representation.

$(B_{1.2})$ $c \neq 0, d = 0.$

$(B_{1.2.1})$ u_0 fulfils (4) with

$$\begin{aligned} &\widehat{\Phi}(x) = x(x - p(\widehat{b}q)^{-1}), \\ &\widehat{\Psi}(x) = (\widehat{a}\widehat{b}q^2(q-p))^{-1}\{p(p^2 - \widehat{a}\widehat{b}q^2)x + p^2(q\widehat{a} - p)\}\ . \end{aligned} \tag{43}$$

(10)-(11) becomes

$$(\widehat{u}_0)_{n+1} = \frac{1 - \widehat{a}(\frac{q}{p})^{n+1}}{1 - \widehat{a}\widehat{b}(\frac{q}{p})^{n+2}}(\widehat{u}_0)_n\ ,\ n \geq 0\ .$$

This yields

$$(\widehat{u}_0)_n = \frac{p^n\big(p\ominus\widehat{a}q\big)^n_{p,q}}{\big(p^2\ominus\widehat{a}\widehat{b}q^2\big)^n_{p,q}}\ , \quad n\geq 0\ .$$

Further, from (15) and (43) we get

$$\begin{aligned}\big(D_{p,q}U_{p,q}\big)(x) = -\frac{[\widehat{a}\widehat{b}q(q-p)]^{-1}}{x[x-(q\widehat{b})^{-1}]}&\{q(1-\widehat{a}\widehat{b})x\\ &+(\widehat{a}-1)\}U_{p,q}(qx)\ .\end{aligned}\tag{44}$$

If $\widehat{a}=(\frac{q}{p})^{\tau-1}$, then we obtain from (44)

$$\big(D_{p,q}\xi^{1-\tau}U_{p,q}\big)(x) = \frac{(\widehat{b}-1)[\widehat{b}(p-q)]^{-1}}{[(q\widehat{b})^{-1}-x]}\big(\xi^{1-\tau}U_{p,q}\big)(qx)\ .$$

For $0<q<p\leq 1$ and $\widehat{b}\in]-\infty,0[\bigcup]0,1]$, we have

$$U_{p,q}(x) = \begin{cases} 0\ , \quad x\leq 0\ ,\\ Kx^{\tau-1}\frac{\big(p\ominus qx\big)^{+\infty}_{p,q}}{\big(p\ominus q\widehat{b}x\big)^{+\infty}_{p,q}}\ , \quad 0<x<p\ , \quad \tau>0\ ,\\ 0 \quad x\geq p\ ,\end{cases}$$

where

$$K=\left(\int_0^p x^{\tau-1}\frac{\big(p\ominus qx\big)^{+\infty}_{p,q}}{\big(p\ominus q\widehat{b}x\big)^{+\infty}_{p,q}}d_{p,q}x\right)^{-1}\ .$$

Next, for $q>p\geq 1$ two cases arise.
When $\widehat{b}\geq 1$, we have

$$U_{p,q}(x) = \begin{cases} 0\ , \quad x\leq 0\ ,\\ Kx^{\tau-1}\frac{\big(1\ominus\widehat{b}x\big)^{+\infty}_{q,p}}{\big(1\ominus x\big)^{+\infty}_{q,p}}\ , \quad 0<x<p\widehat{b}^{-1}\ , \quad \tau>0\ ,\\ 0\ , \quad x\geq p\widehat{b}^{-1}\ ,\end{cases}$$

with

$$K=\left(\int_0^{p\widehat{b}^{-1}} x^{\tau-1}\frac{\big(1\ominus\widehat{b}x\big)^{+\infty}_{q,p}}{\big(1\ominus x\big)^{+\infty}_{q,p}}d_{p,q}\right)^{-1}\ .$$

When $\widehat{b} < 0$, we have

$$U_{p,q}(x) = \begin{cases} 0\,, \quad x \leq p\widehat{b}^{-1}\,, \\ K \mid x \mid^{\tau-1} \dfrac{\left(1 \ominus \widehat{b}x\right)^{+\infty}_{q,p}}{\left(1 \ominus x\right)^{+\infty}_{q,p}}\,, \quad \widehat{b}^{-1} < x < 0\,, \quad \tau > 0\,, \\ 0\,, \quad x \geq 0\,, \end{cases}$$

with

$$K = \left(\int_0^{p|\widehat{b}|^{-1}} x^{\tau-1} \frac{\left(1 \oplus \widehat{b}x\right)^{+\infty}_{q,p}}{\left(1 \oplus x\right)^{+\infty}_{q,p}} d_{p,q}x \right)^{-1} .$$

$(B_{1.2.2})$ u_0 satisfies (4) with

$$\begin{aligned} \widehat{\Phi}(x) &= x(x - pq^{-1}\mu^{-1})\,, \\ \widehat{\Psi}(x) &= -p(\mu q(q-p))^{-1}\{(\mu q - p)x - p\}\,. \end{aligned} \tag{45}$$

System (10)-(11) gives

$$(\widehat{u}_0)_{n+1} = -\frac{pq^{n+1}}{p^{n+2} - \mu q^{n+2}} (\widehat{u}_0)_n\,,\ n \geq 0\,. \tag{46}$$

Hence,

$$\begin{aligned} &(\widehat{u}_0)_n = (-1)^n \frac{p^n q^{\frac{n(n+1)}{2}}}{\left(p^2 \ominus \mu q^2\right)^n_{p,q}}\,, \quad n \geq 0\,, \\ &\mu \neq (\tfrac{p}{q})^m\,, \quad m \geq 2\,. \end{aligned}$$

From (15) and (45) we have

$$\begin{aligned} \left(D_{p,q}U_{p,q}\right)(x) = -\frac{[\mu(q-p)]^{-1}}{px[x - (q\mu)^{-1}]} &\{(q - p\mu)x \\ &+ pq^{-1}\}U_{p,q}(qx)\,. \end{aligned}$$

Therefore, with $\mu := (\frac{q}{p})^{\tau}$, we get

$$\left(D_{p,q}\xi^{1-\tau}U_{p,q}\right)(x) = \frac{[q\mu(p-q)]^{-1}}{x[x - (q\mu)^{-1}]} \left(\xi^{1-\tau}U_{p,q}\right)(qx)\,.$$

If $0 < q < p \leq 1$ and $0 < \mu < 1$, then we obtain

$$U_{p,q}(x) = \begin{cases} 0\,, \quad x \leq 0\,, \\ Kx^{\tau-1}\left(1 \ominus \mu^{-1}x^{-1}\right)^{+\infty}_{1,\frac{p}{q}}\,, \\ 0 < x < p\mu^{-1}\,, \quad \tau > 0\,, \\ 0\,, \quad x \geq p\mu^{-1}\,, \end{cases}$$

with

$$K^{-1} = \int_0^{p\mu^{-1}} x^{\tau-1} \big(1 \ominus \mu^{-1}x^{-1}\big)_{1,\frac{p}{q}}^{+\infty} d_{p,q}x \ .$$

Now, (46) is equivalent to

$$\big(D_{p,q}\xi^{\frac{3}{2}}U_{p,q}\big)(x) = \frac{(q\mu-(pq)^{\frac{1}{2}})x-1}{(p-q)x(1-q\mu x)}\big(\xi^{\frac{3}{2}}U_{p,q}\big)(qx) \ .$$

Therefore, for $q > p \geq 1$ and $\mu < 0$, we have from (23)

$$U_{p,q}(x) = \begin{cases} 0 \ , \quad x \leq p\mu^{-1} \ , \\ K|x|^{-\frac{3}{2}} \big(1 \ominus \mu x\big)_{1,\frac{p}{q}}^{+\infty} \exp\Big(-\frac{\ln^2|x|}{2\ln(\frac{q}{p})}\Big) \ , \\ p\mu^{-1} < x < 0 \ , \quad \tau > 0 \ , \\ 0 \ , \quad x \geq 0 \ , \end{cases}$$

with

$$K^{-1} = \int_0^{-p\mu^{-1}} x^{-\frac{3}{2}} \big(1 \oplus \mu x\big)_{1,\frac{p}{q}}^{+\infty} \exp\Big(-\frac{\ln^2(x)}{2\ln(\frac{q}{p})}\Big) d_{p,q}x \ .$$

$(B_{1.2.3})$ u_0 verifies (4) with

$$\begin{aligned} \widehat{\Phi}(x) &= x(x+wq^{-\frac{3}{2}}p^{\frac{3}{2}}) \ , \\ \widehat{\Psi}(x) &= -p(q-p)^{-1}\{x+(w-1)q^{-\frac{3}{2}}p^{\frac{3}{2}}\} \ . \end{aligned} \tag{47}$$

System (10)-(11) becomes

$$(\widehat{u}_0)_{n+1} = \frac{q^{n+\frac{3}{2}}}{p^{2n+\frac{3}{2}}}(p^n - wq^n)(\widehat{u}_0)_n \ , \ n \geq 0 \ .$$

Hence,

$$(\widehat{u}_0)_n = \frac{q^{\frac{n(n+2)}{2}}}{p^{\frac{n(2n+1)}{2}}}\big(1 \ominus w\big)_{p,q}^{n} \ , \qquad n \geq 0 \ .$$

Moreover, from (15) and (47) we get

$$\big(D_{p,q}U_{p,q}\big)(x) = -\frac{(p-q)^{-1}}{x(x+wq^{-\frac{3}{2}}p^{\frac{1}{2}}}\{x+wq^{-\frac{3}{2}}p^{\frac{1}{2}} \\ -q^{-\frac{1}{2}}p^{-\frac{1}{2}}\}U_{p,q}(qx) \ .$$

For $w := (\frac{q}{p})^{\tau}$, we obtain

$$\big(D_{p,q}\xi^{1-\tau}U_{p,q}\big)(x) = \frac{(q-p)^{-1}}{(x + wq^{-\frac{3}{2}}p^{\frac{1}{2}}}\big(\xi^{1-\tau}U_{p,q}\big)(qx)\ .$$

If $0 < q < p \leq 1$, then we get

$$U_{p,q}(x) = \begin{cases} 0\ , \quad x \leq 0\ , \\ K\frac{|x|^{\tau-1}}{\big(1 \oplus w^{-1}p^{-\frac{3}{2}}q^{\frac{3}{2}}x\big)_{1,\frac{q}{p}}^{+\infty}}\ , \quad x > 0\ , \quad \tau > 0\ , \end{cases}$$

where

$$K^{-1} = \int_0^{+\infty} \frac{x^{\tau-1}}{\big(1 \oplus w^{-1}p^{-\frac{3}{2}}q^{\frac{3}{2}}x\big)_{1,\frac{q}{p}}^{+\infty}} d_{p,q}x\ .$$

When $1 \leq p < q$ and $w > 1$, we have

$$U_{p,q}(x) = \begin{cases} 0\ , \quad x \leq -wq^{-\frac{1}{2}}p^{\frac{3}{2}}\ , \\ Kx^{\tau-1}\big(1 \oplus w^{-1}p^{-\frac{1}{2}}q^{-\frac{1}{2}}x\big)_{1,\frac{p}{q}}^{+\infty}\ , \\ -wq^{-\frac{1}{2}}p^{\frac{3}{2}} < x < 0\ , \quad \tau > 0\ , \\ 0\ , \quad x \geq 0\ , \end{cases}$$

with

$$K^{-1} = \int_0^{wq^{-\frac{1}{2}}p^{\frac{3}{2}}} x^{\tau-1}\big(1 \ominus w^{-1}p^{-\frac{1}{2}}q^{-\frac{1}{2}}x\big)_{1,\frac{p}{q}}^{+\infty} d_{p,q}x\ .$$

(B_2) $cd \neq 0$.

In this case, taking into account

$$\begin{aligned} \Psi(0) &= \Psi(a) - a\Psi^{'}(0)\ , \\ \Phi^{'}(0) &= \Phi^{'}(a) - 2a\ , \\ \Phi(0) &= a^2 + \Phi(a) - a\Phi^{'}(a)\ , \end{aligned}$$

the system (8)-(9) reduces to

$$\begin{aligned} \Lambda_{n+1}(a,p,q) - ap^{n-1(p^{n+1}-q^{n+1})}\Lambda_n(a,p,q) & \\ = \Lambda_0(a,p,q)(u_0)_{n+1}^{\Phi(a,p,q)} \quad n \geq 0\ , & \end{aligned} \tag{48}$$

$$\Lambda_0(a,p,q) = \Psi(a) + p\big(a(q-p)\big)^{-1}\Phi(a)\ , \quad a \neq 0\ , \tag{49}$$

where for $n \geq 0$,

$$\begin{aligned}\Lambda_n(a,p,q) := {} & p^{1-2n}\{[n]_{p,q} - p^{n-1}\Psi'(0)\}(u_0)_{n+1}^{\Phi(a,p,q)} \\ & + p^{1-n}\{a(q-p)p^{-2n}[n]_{p,q}^2 + p^{-n}[n]_{p,q}\Phi'(a) \\ & + \big(a(q-p)\big)^{-1}\Phi(a)\}(u_0)_n^{\Phi(a,p,q)} \ .\end{aligned} \tag{50}$$

$(B_{2.1})$ u_0 fulfils (4) with

$$\begin{aligned}\widehat{\Phi}(x) &= (x - \widehat{b}^{-1}\widehat{c})(x-1) \ , \\ \widehat{\Psi}(x) &= -p\big(\widehat{a}\widehat{b}q^2(q-p)\big)^{-1}\{(\widehat{a}\widehat{b}q^2 - p^2)x + pq(\widehat{a}+\widehat{c}) \\ & \quad - q^2\widehat{a}(\widehat{b}+\widehat{c})\} \ .\end{aligned} \tag{51}$$

Moreover, from (15) and (51) we obtain

$$\begin{aligned}\big(D_{p,q}U_{p,q}\big)(x) = -p\frac{\big(\widehat{a}\widehat{b}(q-p)\big)^{-1}}{(px-\widehat{b}^{-1}\widehat{c})(px-1)}&\{p(1-\widehat{a}\widehat{b})x + \widehat{a}(\widehat{b}+\widehat{c}) \\ & - (\widehat{a}+\widehat{c})\}U_{p,q}(qx) \ .\end{aligned}$$

If $0 < q < p \leq 1$, then we have

$$U_{p,q}(x) = \begin{cases} 0 \ , \quad x \leq q\widehat{c} \ , \\ K\dfrac{\big(1\ominus\widehat{a}^{-1}x\big)_{p,q}^{+\infty}\big(1\ominus\widehat{c}^{-1}x\big)_{p,q}^{+\infty}}{\big(1\ominus\widehat{c}^{-1}\widehat{b}x\big)_{p,q}^{+\infty}\big(1\ominus x\big)_{p,q}^{+\infty}} \ , \quad q\widehat{c} < x < q\widehat{a} \ , \\ 0 \ , \quad x \geq q\widehat{a} \ , \end{cases}$$

where

$$K^{-1} = \int_{q\widehat{c}}^{q\widehat{a}} \frac{\big(1\ominus\widehat{a}^{-1}x\big)_{p,q}^{+\infty}\big(1\ominus\widehat{c}^{-1}x\big)_{p,q}^{+\infty}}{\big(1\ominus\widehat{c}^{-1}\widehat{b}x\big)_{p,q}^{+\infty}\big(1\ominus x\big)_{p,q}^{+\infty}} d_{p,q}x \ ,$$

with $0 < \widehat{a} < 1$, $0 < \widehat{b} < 1$ and $\widehat{c} < 0$.
Or

$$U_{p,q}(x) = \begin{cases} 0 \ , \quad x \leq q\widehat{a} \ , \\ K\dfrac{\big(1\ominus\widehat{a}^{-1}x\big)_{p,q}^{+\infty}\big(1\ominus\widehat{c}^{-1}x\big)_{p,q}^{+\infty}}{\big(1\ominus\widehat{c}^{-1}\widehat{b}x\big)_{p,q}^{+\infty}\big(1\ominus x\big)_{p,q}^{+\infty}} \ , \quad q\widehat{a} < x < q\widehat{c} \ , \\ 0 \ , \quad x \geq q\widehat{c} \ , \end{cases}$$

where

$$K^{-1} = \int_{q\widehat{a}}^{q\widehat{c}} \frac{\big(1\ominus\widehat{a}^{-1}x\big)_{p,q}^{+\infty}\big(1\ominus\widehat{c}^{-1}x\big)_{p,q}^{+\infty}}{\big(1\ominus\widehat{c}^{-1}\widehat{b}x\big)_{p,q}^{+\infty}\big(1\ominus x\big)_{p,q}^{+\infty}} d_{p,q}x \ ,$$

with $0 < \widehat{c} < 1$, $0 < \widehat{b} < 1$ and $\widehat{a} < 0$.
When $1 \leq p < q$, we obtain

$$U_{p,q}(x) = \begin{cases} 0\,, \quad x \leq p^{-1}q^2\widehat{b}\widehat{c}^{-1}\,, \\ K\dfrac{\left(q\ominus p\widehat{b}^{-1}\widehat{c}x\right)_{q,p}^{+\infty}\left(q\ominus px\right)_{q,p}^{+\infty}}{\left(q\ominus p\widehat{a}^{-1}x\right)_{q,p}^{+\infty}\left(q\ominus p\widehat{c}^{-1}x\right)_{q,p}^{+\infty}}\,, \\ p^{-1}q^2\widehat{b}\widehat{c}^{-1} < x < p^{-1}q^2\,, \\ 0\,, \quad x \geq p^{-1}q^2\,, \end{cases}$$

where

$$K^{-1} = \int_{p^{-1}q^2\widehat{b}\widehat{c}^{-1}}^{p^{-1}q^2} \frac{\left(q\ominus p\widehat{b}^{-1}\widehat{c}x\right)_{q,p}^{+\infty}\left(q\ominus px\right)_{q,p}^{+\infty}}{\left(q\ominus p\widehat{a}^{-1}x\right)_{q,p}^{+\infty}\left(q\ominus p\widehat{c}^{-1}x\right)_{q,p}^{+\infty}} d_{p,q}x\,,$$

with $\widehat{a} > 1$, $\widehat{c} > 1$, $\widehat{b} < 0$ or $\widehat{a} > 1$, $\widehat{b} > 1$, $\widehat{c} < 0$.
Now, from (49) and (51) we have

$$\Lambda_0(\widehat{a}qp^{-1}, p, q) = \Lambda_0(\widehat{c}qp^{-1}, p, q) = 0.$$

Then, from (48) we get

$$\Lambda_n(a, p, q) = 0\,, \quad n \geq 0\,, \quad a \in \{\widehat{a}qp^{-1}, \widehat{c}qp^{-1}\}\,.$$

Therefore, from (50) we can deduce

$$\begin{array}{l} (\tilde{u}_0)_{n+1}^{\Phi(\widehat{a}qp^{-1},p,q)} = \\ \widehat{c}qp^{-n-1}\frac{(p^{n+1}-\widehat{a}q^{n+1})(p^{n+1}-\widehat{a}\widehat{b}\widehat{c}^{-1}q^{n+1})}{p^{n+2}-\widehat{a}\widehat{b}q^{n+2}}(\tilde{u}_0)_n^{\Phi(\widehat{a}qp^{-1},p,q)}\,, \\ (\tilde{u}_0)_{n+1}^{\Phi(\widehat{c}qp^{-1},p,q)} = \\ \widehat{a}qp^{-n-1}\frac{(p^{n+1}-\widehat{b}q^{n+1})(p^{n+1}-\widehat{c}q^{n+1})}{p^{n+2}-\widehat{a}\widehat{b}q^{n+2}}(\tilde{u}_0)_n^{\Phi(\widehat{c}qp^{-1},p,q)}\,, \end{array}$$

with

$$\begin{array}{l} \widehat{a}, \widehat{b}, \widehat{c} \neq (pq^{-1})^{m+1}, \\ \widehat{a}\widehat{b} \neq (pq^{-1})^{m+2}, \\ \widehat{a}\widehat{b}\widehat{c}^{-1} \neq (pq^{-1})^{m+2}, m \geq 0. \end{array}$$

Here, we have respectively (for $n \geq 0$),

$$(\tilde{u}_0)_n^{\Phi(\widehat{a}qp^{-1},p,q)} = \widehat{c}q^n p^{-\frac{n(n+1)}{2}} \frac{\left(p\ominus\widehat{a}q\right)_{p,q}^n\left(p\ominus\widehat{a}\widehat{b}\widehat{c}^{-1}q\right)_{p,q}^n}{\left(p^2\ominus\widehat{a}\widehat{b}q^2\right)_{p,q}^n}\,, \tag{52}$$

$$(\tilde{u}_0)_n^{\Phi(\widehat{c}qp^{-1},p,q)} = \widehat{a}q^n p^{-\frac{n(n+1)}{2}} \frac{\left(p\ominus\widehat{b}q\right)_{p,q}^n\left(p\ominus\widehat{c}q\right)_{p,q}^n}{\left(p^2\ominus\widehat{a}\widehat{b}q^2\right)_{p,q}^n}\,. \tag{53}$$

$(B_{2.2})$ u_0 satisfies (4) with

$$\begin{aligned} \widehat{\Phi}(x) &= (x+\widehat{a}^{-1}q^{-\frac{1}{2}}p^{\frac{1}{2}})(x+\widehat{b}^{-1}q^{-\frac{1}{2}}p^{\frac{1}{2}}) \ , \\ \widehat{\Psi}(x) &= -p(q-p)^{-1}\{xq^{-\frac{1}{2}}p^{\frac{1}{2}}(\widehat{a}+\widehat{b})-q^{-\frac{3}{2}}p^{\frac{3}{2}}\} \ , \end{aligned} \tag{54}$$

Moreover, from (15) and (54), we get

$$\big(D_{p,q}U_{p,q}\big)(x) = -\frac{(p-q)^{-1}[p^2x+q^{-\frac{1}{2}}p^{\frac{3}{2}}(\widehat{a}+\widehat{b}-1)]}{(px+\widehat{a}q^{-\frac{1}{2}}p^{\frac{1}{2}})(px+\widehat{b}q^{-\frac{1}{2}}p^{\frac{1}{2}})}U_{p,q}(qx) \ .$$

If $0<q<p\leq 1$, $0<\widehat{a}<1$ and $0<\widehat{b}<1$, then we obtain

$$U_{p,q}(x) = \begin{cases} 0 \ , \quad x \leq -\widehat{ab}p^{\frac{1}{2}}q^{\frac{1}{2}} \ , \\ K\dfrac{\left(1\oplus(\widehat{ab})^{-1}q^{\frac{1}{2}}q^{-\frac{1}{2}}x\right)_{p,q}^{+\infty}}{\left(1\oplus\widehat{a}^{-1}q^{\frac{1}{2}}p^{-\frac{1}{2}}x\right)_{p,q}^{+\infty}\left(q^{-\frac{1}{2}}p^{\frac{1}{2}}\oplus\widehat{b}^{-1}x\right)_{1,\frac{q}{p}}^{+\infty}} \ , \\ x > -\widehat{ab}p^{\frac{1}{2}}q^{\frac{1}{2}} \ , \end{cases}$$

where

$$K^{-1} = \int_{-\widehat{ab}p^{\frac{1}{2}}q^{\frac{1}{2}}}^{+\infty} \frac{\left(1\oplus(\widehat{ab})^{-1}q^{\frac{1}{2}}q^{-\frac{1}{2}}x\right)_{p,q}^{+\infty}}{\left(1\oplus\widehat{a}^{-1}q^{\frac{1}{2}}p^{-\frac{1}{2}}x\right)_{p,q}^{+\infty}\left(q^{-\frac{1}{2}}p^{\frac{1}{2}}\oplus\widehat{b}^{-1}x\right)_{1,\frac{q}{p}}^{+\infty}}d_{p,q}x \ .$$

When $1\leq p<q$, $\widehat{a}\geq 1$ and $\widehat{b}<0$ we have

$$U_{p,q}(x) = \begin{cases} 0 \ , \quad x \leq -\widehat{a}q^{\frac{1}{2}}p^{-\frac{1}{2}} \ , \\ K\dfrac{\left(\widehat{a^{-1}}q^{-\frac{1}{2}}p^{\frac{1}{2}}x\oplus 1\right)_{q,p}^{+\infty}\left(1\oplus q^{-\frac{1}{2}}p^{\frac{1}{2}}\widehat{b}^{-1}x\right)_{1,\frac{p}{q}}^{+\infty}}{\left(1\oplus(\widehat{ab})^{-1}q^{-\frac{1}{2}}p^{\frac{1}{2}}x\right)_{q,p}^{+\infty}} \ , \\ -\widehat{a}q^{\frac{1}{2}}p^{-\frac{1}{2}} < x < -\widehat{b}q^{\frac{1}{2}}p^{-\frac{1}{2}} \ , \\ 0 \ , \quad x \geq -\widehat{b}q^{\frac{1}{2}}p^{-\frac{1}{2}} \ , \end{cases} \tag{55}$$

with

$$K^{-1} = \int_{-\widehat{a}q^{\frac{1}{2}}p^{-\frac{1}{2}}}^{-\widehat{b}q^{\frac{1}{2}}p^{-\frac{1}{2}}} \frac{\left(\widehat{a^{-1}}q^{-\frac{1}{2}}p^{\frac{1}{2}}x\oplus 1\right)_{q,p}^{+\infty}\left(1\oplus q^{-\frac{1}{2}}p^{\frac{1}{2}}\widehat{b}^{-1}x\right)_{1,\frac{p}{q}}^{+\infty}}{\left(1\oplus(\widehat{ab})^{-1}q^{\frac{1}{2}}p^{-\frac{1}{2}}x\right)_{q,p}^{+\infty}}d_{p,q}x \ . \tag{56}$$

When $1\leq p<q$, $\widehat{b}\geq 1$ and $\widehat{a}<0$ it is sufficient to permute $\widehat{a}$ and $\widehat{b}$ in (55)-(56).

Now, proceeding as in (52)-(53), we can easily prove that

$$(\tilde{u}_0)_n^{\Phi(-\widehat{ab}q^{\frac{1}{2}}q^{-\frac{1}{2}},p,q)} = q^{-\frac{n(n+2)}{2}}p^{-\frac{n^2}{2}}\left(p\ominus\widehat{a}q\right)_{p,q}^{n}\left(p\ominus\widehat{b}q\right)_{p,q}^{n} \ , \quad n\geq 0 \ .$$

$(B_{2.3})$ u_0 verifies (4) with

$$\Phi(x) = x^2 - 1\ , \quad \Psi(x) = p(p-q)^{-1}x\ . \tag{57}$$

Therefore, the system (10)-(11) yields

$$(u_0)_{2n+1} = 0\ , \quad (u_0)_{2n} = -q^{-n^2}\big((p^2 \ominus q^2)^n_{p^2,q^2}\ , \quad n \geq 0\ .$$

Now, from (57) and (15) we have

$$\big(D_{p,q}U_{p,q}\big)(x) = -\frac{(p-q)^{-1}p^2x}{p^2x^2-1}U_{p,q}(qx)\ .$$

Here, for $1 \leq p < q$, we get from (19)

$$U_{p,q}(x) = \begin{cases} 0\ , & x \leq -q\ , \\ \big(1 \ominus p^2q^{-2}x^2\big)^{+\infty}_{1,\frac{p^2}{q^2}}\ , & -q < x < q\ , \\ 0\ , & x \geq q\ , \end{cases}$$

with

$$K^{-1} = 2\int_0^q \big(1 \ominus p^2q^{-2}x^2\big)^{+\infty}_{1,\frac{p^2}{q^2}} d_{p,q}x\ .$$

When $0 < q < p \leq 1$, the problem is open.

4. Concluding remarks

In this paper, we studied the family of $D_{p,q}$-classical orthogonal polynomials and the related regular forms associated with the (p,q)-difference operator $D_{p,q}$. Using Pearson-type functional equations and the decomposition of $D_{p,q}$, we derived a simplified characterization of the canonical cases and established their connection with the corresponding q-classical setting. We also determined the associated recurrence coefficients and presented a classification of the canonical cases. Furthermore, moment relations were derived, leading to corresponding (p,q)-integral representations of the underlying linear functionals. These results contribute to a deeper understanding of the structure and properties of $D_{p,q}$-classical polynomials.

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