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VECTOR BUNDLES ON REDUCIBLE CURVES

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Dedicated to Silvio Greco in occasion of his 60-th birthday.

Let $X = X_1 \cup \cdots \cup X_s$ be a reduced and connected union of irreducible projective curves $X_1, \dots, X_s$. Fix integers r, k, d, δ with $1 \leq k < r$. Which vector bundles E on X of rank r and degree d are an extension of a vector bundle G of rank $r - k$ and degree $d - \delta$ by a vector bundle H of rank k and degree δ ? Namely, which vector bundles E on X of rank r and degree d have subbundles H of rank k and degree δ ? What happens if E is "general" in a suitable sense ? Here we solve this problem under certain assumptions on E and X and relate this question to the computation of the Lange invariant $s_k(E)$ of E giving the maximal degree of the subsheaves of E with constant rank k.

Introduction.

In this paper we study some problems concerning vector bundles and torsion-free sheaves on a reducible connected projective curve X. Almost all our results concern *multistable vector bundles*, i.e. the vector bundles E on X such that for every irreducible component Y of X the vector bundle $E_{|Y}$ is stable. We denote by $M(Y; r, d)$ the moduli scheme of stable locally free sheaves (vector bundles) on Y with rank r and degree d. Quite often we will assume that the vector bundle E on X is "general" in a suitable sense.

Now we state our main results. The definitions used are described in 1.

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Theorem 1. *Let X be a reduced and connected projective curve and $X_1, \dots, X_s$ its irreducible components. Assume that the normalization of every X_i has genus at least* 2.

Fix integers r and k with $r > k \geq 1$ and $(a_1, \dots, a_s)$, $(b_1, \dots, b_s) \in \mathbb{Z}^s$ such that $\frac{a_i}{k} < \frac{b_i}{r-k}$ for every i. Then there exist multistable vector bundles H, E, G with $rank(H) = k$, $rank(E) = r$, $rank(G) = r - k$, $multideg(H) = (a_1, \dots, a_s)$, $multideg(E) = (a_1 + b_1, \dots, a_s + b_s)$, $multideg(G) = (b_1, \dots, b_s)$ (see Definition 1.5.3) and fitting in an exact sequence

$$0 \to H \to E \to G \to 0 \tag{1}$$

Furthermore, if every singular point of X lies on at most two irreducible components, we may take as H (resp. G) a general multistable vector bundle on X (see Definition 1.11) with rank k and multidegree $(a_1, \dots, a_s)$ (resp. rank $(r-k)$ and multidegree $(b_1, \dots, b_s)$) .

Moreover, for any such H and G the middle term E of a general extension as (1) *is a multistable vector bundle.*

Theorem 2. *Let Y be an integral projective curve whose normalization has genus at least* 2. *Set $g := p_a(Y)$. Fix integers r, d with $r \geq 2$ and let a, b, u, v be the unique integers such that $a + b = u + v = d$ and*

$$(r-1)(g-1) \leq b-(r-1)a \leq (r-1)g, \quad (r-1)(g-1) \leq (r-1)v-u \leq rg.$$

Then there is a non-empty Zariski open set W of $M(Y; r, d)$ with the following properties:

- **(a)** *For every $E \in W$ there is $H \in Pic^a(Y)$ and $G \in M(Y; r-1, b)$ such that E fits in an exact sequence* (1).
- **(b)** *For every $E \in W$ there is $H \in M(Y; r-1, u)$ and $G \in Pic^v(Y)$ such that E fits in an exact sequence* (1).
- **(c)** *For a general $H \in Pic^a(Y)$ and a general $G \in M(Y; r-1, b)$ the middle term of a general extension* (1) *is an element of W.*
- **(d)** *For a general $H \in M(Y; r-1, u)$ and a general $G \in Pic^v(Y)$ the middle term of a general extension* (1) *is an element of W.*
- **(e)** *For every $E \in W$ there is no integer $a' > a$ such that E is an extension of a rank $(r-1)$ vector bundle of degree $(d-a')$ by a line bundle of degree a'.*
- **(f)** *For every $E \in W$ there is no integer $u' > u$ such that E is an extension of a line bundle of degree $(d-u')$ by a rank $(r-1)$ vector bundle of degree u'.*

Teorem 3. *Let X be a nodal union of two smooth curves X_1 and X_2 meeting (quasi-transversally) at a point. Assume $g_1 := p_a(X_1) \geq 2$ and $g_2 := p_a(X_2) \geq 2$. Fix integers r, k, d_1 and d_2 with $r > k \geq 1$.*

Let E be a general multistable vector bundle on X (see Definition 1.11) with rank r and multidegree (d_1, d_2). Let e_i, ϵ_i, $i = 1, 2$, be the only integers with $e_i, \epsilon_i \in \{0, 1, \dots, r-1\}$ and $e_i + k(r-k)(g_i - 1) \equiv kd_i$ modulo r, $\epsilon_i + k(r-k)g_i \equiv kd_i$ modulo r.

Then the Lange invariant $s_k(E)$ of E (see Definition 1.6) satisfies the following conditions:

$$s_k(E) \geq e_1 + e_2 + k(r-k)(g_1 + g_2 - 2),$$
$$s_k(E) \leq k(r-k)(g_1+g_2-1) + min\,\{e_1+\epsilon_2, \epsilon_1+e_2, e_1+e_2+min\,\{k^2, (r-k)^2\}\}.$$

Teorem 4. *Let X be a connected curve of compact type such that all of its irreducible components have genus at least two. Order the irreducible components $X_1, \dots, X_s$ of X so that $X_{[i-1]} := X_1 \cup \dots \cup X_{i-1}$ is connected and card $(X_{[i-1]} \cap X_i) = 1$ for every integer i with $2 \leq i \leq s$. Set $g_i := p_a(X_i)$.*

Fix a multidegree $(d_1, \dots, d_s)$ associated to this ordering and integers $r, k, a_1, \dots, a_s$ such that $r > k \geq 1$, $kd_1 - ra_1 \geq k(r-k)(g_1 - 1)$ and $kd_j - ra_j \geq k(r-k)g_j$ for every $2 \leq j \leq s$.

Then a general multistable vector bundle E on X (see Definition 1.11) with multidegree $(d_1, \dots, d_s)$ fits in an exact sequence (1) with H and G multistable vector bundles respectively of rank k and $r-k$ and multidegree $(a_1, \dots, a_s)$ and $(d_1 - a_1, \dots, d_s - a_s)$. Furthermore we may assume that H and G are general.

The last assertion of the above Theorem means that the general extension of a general G and a general H is multistable and that the family of all middle terms of general extension (1) with G multistable rank $(r-k)$ vector bundle with multidegree $(d_1 - a_1, \dots, d_s - a_s)$ and H multistable rank k vector bundle with multidegree $(a_1, \dots, a_s)$ covers a dense subset of the set of all multistable vector bundles on X with rank r and multidegree $(d_1, \dots, d_s)$.

We work over an algebraically closed field $\mathbf{k}$ of characteristic 0.

1. Preliminaries.

Definition 1.1. Let R be a ring and M be a R-module. The *torsion module of* M is $Tors(M) := \{m \in M \,/\, am = 0$ for some non-0-divisor $a \in R\}$. M is called *torsion-free* if and only if $Tors(M) = (0)$.

If X is a scheme and F is a coherent sheaf on X, then F is called *torsion-free* if F_x is a torsion-free $\mathcal{O}_{X,x}$-module for every $x \in X$.

Remark 1.2. A subsheaf of a torsion-free sheaf is torsion-free.

If X is a smooth scheme, then a sheaf F on X is torsion-free if and only if it is locally free (i.e. it is a vector bundle).

If X is a nodal curve (i.e. for every singular point P of X the completion of $\mathcal{O}_{X,P}$ is isomorphic to $\mathbf{k}[[x, y]]/(xy)$), then a sheaf F on X is torsion-free if and only if it is a sheaf of depth 1, as defined in [17] p. 146 Definition 1 and Lemma 2.

Let Y be an integral projective curve. Recall that for any rank r torsion-free sheaf F on Y its degree $deg(F)$ is defined by the relation $deg(F) = \chi(F) + r(p_a(Y) - 1)$.

We recall the following definition.

Definition 1.3. Let F be a torsion-free sheaf on an integral projective curve. Denote by $\mu(F) := \frac{deg(F)}{rank(F)}$ the *slope* of F. We say that F is *stable* (resp. *semistable*) if for every subsheaf A of F we have $\mu(A) < \mu(F)$ (resp. $\mu(A) \leq \mu(F)$) or equivalently for every quotient sheaf Q of F we have $\mu(Q) > \mu(F)$ (resp. $\mu(Q) \geq \mu(F)$).

Assume $p_a(Y) \geq 2$. For all integers r, d with $d > 0$ let $M(Y; r, d)$ be the moduli scheme of stable locally free sheaves E on Y with rank r and degree d. The scheme $M(Y; r, d)$ is a Zariski open subset of the moduli scheme of all torsion-free stable sheaves on Y with rank r and degree d. The scheme $M(Y; r, d)$ is a non-empty smooth irreducible algebraic variety with dimension $r^2(p_a(Y) - 1) + 1$ (see [15] remark at p. 167).

Definition 1.4. Let Y be an integral projective curve and F be a rank r torsion-free sheaf on Y. Fix an integer k with $1 \leq k < r$ and consider a rank k subsheaf A of F of maximal degree. The integer $s_k(F) := k(deg(F)) - r(deg(A))$ is called a *Lange invariant of* F. The name is due to its introduction and study in [12].

Note that a rank r torsion-free sheaf F on an integral projective curve Y is stable (resp. semistable) if and only if $s_k(F) > 0$ (resp. $s_k(F) \geq 0$) for every $1 \leq k < r$.

From now on X denotes a reduced and connected projective curve with irreducible components $X_1, \dots, X_s$.

Definition 1.5.

1. Let F be a torsion-free sheaf on X. The sheaf $F_{|X_i}/Tors(F_{|X_i})$ is a torsion-free sheaf on the irreducible component X_i, let r_i be its rank . We call the ordered s-ple $(r_1, \dots, r_s)$ the *multirank of* F. If $r_i = r$ for every i, we say that F has a constant rank r.

2. If F is a torsion-free sheaf on X with constant rank r, then its *degree* $deg(F)$ is defined by the formula $deg(F) = \chi(F) + r(\chi(\mathcal{O}_X))$.
3. If E is a vector bundle on X, then it has a constant rank r, in this case we will call the ordered s-ple of integers $multideg(E) := (deg(E_{|X_1}), \dots, deg(E_{|X_s}))$ the *multidegree of* E. We have $deg(E) = \sum_{i=1}^{s} deg(E_{|X_i})$.
4. A vector bundle E on X is called *multistable* (resp. *multisemistable*) if for every irreducible component X_i of X the vector bundle $E_{|X_i}$ is stable (resp. semistable).
5. Let F be a torsion-free sheaf on X with constant rank r such that it is a flat limit of a flat family $\{F_\lambda\}_{\lambda \in T}$ of locally free sheaves with T integral and quasi-projective. The multidegree of F_λ for a general $\lambda \in T$ is well-defined and we will say that such multidegree is the *multidegree* of F with respect to the partial smoothing $\{F_\lambda\}_{\lambda \in T}$.

Definition 1.3 of stable (resp. semistable) sheaf and Definition 1.4 of Lange invariant can be extended to reducible reduced and connected curves X, by considering subsheaves of constant rank.

Definition 1.6. Let F a torsion-free sheaf on X with constant rank r.

1. Fix a positive integer $k < r$.The *pure Lange invariant of order k of F* is the integer $s_k(F) := k(deg(F)) - r(deg(A))$, where A is a subsheaf of F with constant rank k and A has maximal degree among all such subsheaves. Since we require that A has constant rank, it easy to check that $s_k(F)$ is a well-defined integer.
2. We say that F is *stable* (resp. *semistable*) if $s_k(F) > 0$ (resp. $s_k(F) \geq 0$ for every $1 \leq k < r$).

Remark 1.7. The above definitions were used for example by Ellingsrud, Hirschowitz in [7] and Heine, Kurke in [9].

The properness of the $Quot$-scheme implies that the function $s_k(F)$ is lower semicontinuos in flat families of torsion-free sheaves on X with constant rank r and in flat families of curves.

We point out that a multistable (resp. multisemistable) vector bundle E on X is stable (resp. semistable) in the sense of the above definition (see [2] Lemma 1.1).

A different definition of stability is given by Seshadri in [17]. In 4 we will give some remarks on the differences between the two definitions.

Definition 1.8. A reduced and connected projective curve X is said to be *of compact type* if the scheme $Pic^0(X)$ is compact. Recall that the group scheme $Pic^0(X)$ is compact if and only if each irreducible component of X is smooth,

X has only ordinary nodes as singularities and the graph associated to the irreducible components of X is a tree. The last assertion means that we may order the irreducible components of X (say $X_1, \dots, X_s$) in such a way that for $2 \leq i \leq s$, $X_{[i-1]} := X_1 \cup \cdots X_{i-1}$ is connected and $card\,(X_{[i-1]} \cap X_i) = 1$.

Definition 1.9. Let X be a reduced and connected projective curve.

1. We say that X is *quasi-nodal* if every $P \in Sing(X)$ such that P lies on at least two irreducible components of X is an ordinary node of X.
2. We say that X is *of quasi-compact type* if X is quasi-nodal and the graph associated to the irreducible components of X is a tree.

Note that if X is a curve of quasi-compact type with irreducible components $X_1, \dots X_s$, then $p_a(X) = \sum_{i-1}^{s} p_a(X_i)$.

Remark-Definition 1.10. Let X be a reduced and connected projective curve with irreducible components $X_1, \dots, X_s$ such that every singular point of X is contained in at most two irreducible components. Consider the natural morphism $\pi : X_1 \sqcup \cdots \sqcup X_s \to X$ (the symbol "$\sqcup$" denotes the disjoint union). The morphism π induces the following exact sequence

$$0 \to \mathcal{O}_X \to \pi_*(\mathcal{O}_{X_1} \oplus \cdots \oplus \mathcal{O}_{X_s}) \to \mathcal{O}_S \to 0 \tag{2}$$

where S is the 0-dimensional scheme given by the singularities of π. Hence S is supported on the points $P \in X$ lying on two different irreducible components of X.

A vector bundle E on X with rank r and multidegree $(d_1, \dots, d_s)$ gives the following exact sequence

$$0 \to E \to E_{|X_1} \oplus \cdots \oplus E_{|X_s} \xrightarrow{\beta} E_{|S} \to 0 \tag{3}$$

where $E_{|S} \cong \mathcal{O}_S^{\oplus r}$ and, for $1 \leq i \leq s$, $E_{|X_i}$ is a vector bundle on X_i with rank r and degree d_i.

Now we describe the morphism β. Let Z be a connected component of the 0-dimensional scheme S given by the intersections of the irreducible components of X. Assume that $P := Z_{red}$ lies on X_1 and X_2.

The 0-dimensional algebra $\mathcal{O}_Z$ is local and it has $\mathcal{O}_P = \mathbf{k}$ as residue field. Since Z is 0-dimensional, we have $(E_{|X_1})_{|Z} \cong (E_{|X_2})_{|Z} \cong E_{|Z} \cong \mathcal{O}_Z^{\oplus r}$. The morphism β is given, locally at P by an isomorphism of $(E_{|X_1})_{|Z}$ with $(E_{|X_2})_{|Z}$ which will be called the *gluing datum of* $E_{|X_1}$ and $E_{|X_2}$ at P.

Notice that $Hom(\mathcal{O}_Z^{\oplus r}, \mathcal{O}_Z^{\oplus r})$ is a matrix algebra over $\mathcal{O}_Z$ and a matrix gives a gluing datum if and only if it is invertible. In particular, seeing $\mathcal{O}_Z$ as a finite dimensional vector space over **k** and hence as an irreducible variety, the set of all gluing data is parametrized by an irreducible variety denoted $Gl(r, \mathcal{O}_Z)$.

On the other hand, for any choice of rank r and degree d_i vector bundles E_i on X_i, $i = 1, \dots, s$, and of gluing data at each common point of two irreducible components of X (i.e. for every surjective morphism $E_1 \oplus \cdots \oplus E_s \to \mathcal{O}_S^{\oplus r}$) there is vector bundle E on X with $E_{|X_i} = E_i$.

Definition 1.11. Let X be a reduced and connected projective curve with irreducible components $X_1, \dots, X_s$ such that every singular points of X lies on at most two irreducible components.

1. A vector bundle E on X with rank r and multidegree $(d_1, \dots, d_s)$ is called *general multistable* vector bundle if the restriction $E_{|X_i}$ is a general element (i.e. an element of an open set) of the moduli space $M(X; r, d_i)$, for $1 \leq i \leq s$ and the gluing data are general.
2. We will say that an irreducible flat family $\{E_\alpha\}_{\alpha \in T}$ of vector bundles on X with rank r and multidegree $(d_1, \dots, d_s)$ is a *general family* of vector bundles on X with that rank and multidegree if for a general α the vector bundle E_α is general multistable and for every point P lying on two irreducible components of X the gluing data of E_α at P are general.

2. Proof of Theorem 1.

We will prove simultaneously Theorem 1 and the following result which considers also curves of geometric genus 1.

Proposition 2.1. *Let X be a reduced and connected projective curve and $X_1, \dots, X_s$ its irreducible components. Assume that the normalization of every X_i has genus at least* 1.

Fix integers r and k with $r > k \geq 1$ and multidegrees $(a_1, \dots, a_s)$ and $(b_1, \dots, b_s)$ such that $\frac{a_i}{k} < \frac{b_i}{r-k}$ for every i. Then there exist multisemistable vector bundles H, E, G with $rank(H) = k$, $rank(E) = r$, $rank(G) = r - k$, $multideg(H) = (a_1, \dots, a_s)$, $multideg(E) = (a_1 + b_1, \dots, a_s + b_s)$, $multideg(G) = (b_1, \dots, b_s)$ and fitting in an exact sequence (1) *: $0 \to H \to E \to G \to 0$.*

We may find H, G and E such that for every irreducible component X_i with normalization of genus at least 2 *the restrictions $H_{|X_i}, E_{|X_i}$ and $G_{|X_i}$ are stable and furthermore the bundles $H_{|X_i}, G_{|X_i}$ are general in their moduli scheme.*

We may find H, G and E such that for every irreducible component X_i which is smooth of genus 1 the bundles $H_{|X_i}$, $E_{|X_i}$ and $G_{|X_i}$ are polystable (i.e. they are direct sum of stable vector bundles with the same slope).

We may find H, G and E such that for every irreducible component X_i which is singular and with normalization of genus 1 the bundles $H_{|X_i}$ and $G_{|X_i}$ are stable and general in their moduli scheme and the bundle $E_{|X_i}$ is polystable.

Remark 2.2. Let Y be an integral projective curve and $\pi : Z \to Y$ its normalization. For every vector bundle E on Y we have $\pi_*\pi^*(E) \cong E \otimes \pi_*(\mathcal{O}_Z)$ and there is a natural map $f : H^1(Y, E) \to H^1(Y, \pi_*\pi^*(E))$. Since $\pi_*(\mathcal{O}_Z)/\mathcal{O}_Y$ is supported by a finite set, we have $H^1(Y, \pi_*\pi^*(E)/E) = (0)$. Hence f is surjective. Moreover we have $H^1(Y, \pi_*\pi^*(E)) = H^1(Z, \pi^*(E))$, since π is finite.

Take vector bundles A, B on Y and apply these observations to the vector bundle $Hom(B, A)$. We obtain that every extension of $\pi^*(B)$ by $\pi^*(A)$ is a vector bundle isomorphic to a vector bundle $\pi^*(U)$ with U extension of B by A.

Lemma 2.3. *Let X be a reduced and connected projective curve, H and G be vector bundles on X and C be the union of some irreducible components of X. Then for every exact sequence*

$$0 \to H_{|C} \to M \to G_{|C} \to 0 \tag{4}$$

of vector bundles on C there is an extension

$$0 \to H \to N \to G \to 0 \tag{5}$$

of vector bundles on X such that the sequence (4) is the restriction of the sequence (5) to C.

Proof. Let $\mathcal{J}$ be the ideal sheaf of C in X. Since $dim(X) = 1$, $h^2(X, \mathcal{J} \otimes Hom(G, H)) = 0$ holds. Hence the natural restriction map $H^1(X, Hom(G, H)) \to H^1(C, Hom(G_{|C}, H_{|C}))$ is surjective and we conclude.

Lemma 2.4. *Let Y be an integral projective curve and $\pi : Z \to Y$ its normalization. For every vector bundle F on Z there is a vector bundle E on Y with $F \cong \pi^*(E)$.*

Furthermore, if $\pi^*(E)$ is stable (resp. semistable), then E is stable (resp. semistable).

Proof. The result is true and well-known if $rank(F) = 1$ because $Pic(Y)$ is an extension of $Pic(Z)$ by an affine commutative connected group. Since for every integer $r \geq 2$ a rank r vector bundle on Z is obtained making $r-1$ times an extension by a line bundle, we obtain an exact sequence of vector bundles $0 \to L \to F \to G \to 0$, with $rank(L) = 1$, $rank(G) = r-1$. By the inductive hypothesis there exist vector bundles L' and G' on Y with $L = \pi^*(L')$ and $G = \pi^*(G')$. So by Remark 2.2 there exists an extension E of G' by L' with $F = \pi^*(E)$.

We have $deg(\pi^*(A)) = deg(A)$ for every vector bundle A on Y. Assume E non-stable (resp. non-semistable) and let Q be a quotient sheaf of E with $\mu(Q) \leq \mu(E)$ (resp. $\mu(Q) < \mu(E)$) and $rank(Q) < rank(E)$.

Set $B := \pi^*(Q)/Tors(\pi^*(Q))$. We have $deg(B) \leq deg(Q)$ (see [5] Prop. 3.2.4 part (2) or, in the rank 1 case [6] Lemma 1). There is a morphism $f : \pi^*(E) \to B$ which is surjective outside finitely many points. Hence $rank(Im(f)) = rank(B) < rank(E)$ and $deg(Im(f)) \leq deg(B)$. So $\pi^*(E)$ is not stable (resp. not semistable).

Proof of Theorem 1 and Proposition 2.1. It is sufficient to prove the first part of Theorem 1 and Proposition 2.1. First we consider Theorem 1. By [16] Thm. 0.1 and Thm. 0.2, the result is true if X is smooth. Now assume $s = 1$. Since the normalization of X has genus at least 2, the result is true by the smooth case just quoted and by above Remark 2.2 and Lemma 2.4.

Let $s \geq 2$. For general multistable H and G, the restrictions $H_{|X_i}$ and $G_{|X_i}$ are general stable bundles on X_i and by the case $s = 1$ we obtain as general extension of $G_{|X_i}$ by $H_{|X_i}$ a stable bundle E_i. The restriction map $H^1(X, Hom(G, H)) \to H^1(X_i, Hom(G_{|X_i}, H_{|X_i}))$ is surjective, for $i = 1, \dots, s$ (see Lemma 2.3), then we obtain as general extension of G by H a multistable vector bundle E.

Proposition 2.1 is done in the same way, just quoting [3] 1 for the particular case of smooth elliptic curves.

3. Proofs of Theorems 2, 3 and 4.

We need the following well-known result.

Lemma 3.1. *Let Y be an integral projective curve with $g := p_a(Y) \geq 2$. For all integers r and d with $r > 0$ and a general $E \in M(Y; r, d)$, we have $h^0(Y, E) = max\{0, d + r(1-g)\}$ and $h^1(Y, E) = max\{0, r(g-1) - d\}$.*

Proof. The result is obvious and well-known for $r = 1$. Assume $r \geq 2$. For $d \geq r(g-1)$, take a general $L \in Pic^{g-1}(Y)$ and a general $M \in$

$Pic^{d-(r-1)(g-1)}(Y)$, then set $F := L^{\oplus(r-1)} \oplus M$. By the case $r = 1$ we have $h^1(Y, F)$ =0 and by Riemann-Roch $h^0(Y, F) = d + r(1 - g)$.

The proofs of [11] Prop. 2.1 and Cor. 2.2 work verbatim when Y is a singular irreducible projective curve with arithmetic genus $g \geq 2$.

Hence every vector bundle on Y is the flat limit of a flat family of stable vector bundles on X. Thus by semicontinuity we conclude for $d \geq r(g - 1)$.

The case $d < r(g - 1)$ is similar and left to the reader.

Proof of Theorem 2. We will consider parts (b), (d) and (f), the other parts requiring only sraightforward modifications.

By Theorem 1, for Y integral, we know that for general $H \in M(Y; r-1, u)$ and a general $G \in Pic^v(Y)$ the general extension of G by H is stable. In this way varying G, H and the extension class we obtain an irreducible constructible subset T of $M(Y; r, d)$.

We want to check that T is dense in $M(Y; r, d)$, i.e. that $dim(T) = r^2(g - 1) + 1$. Since for every $H \in M(Y; r - 1, u)$ and every $G \in Pic^v(Y)$ we have $h^0(Y, Hom(G, H)) = 0$ by definition of stability, we obtain

$$h^1(Y, Hom(G, H)) = (r-1)(g-1+\mu(G)-\mu(H)) = (r-1)(g-1)+(r-1)v+u.$$

Since $dim(Pic^v(Y)) = g$ and $dim(M(Y; r - 1, u)) = (r - 1)^2(g - 1) + 1$, to conclude that $dim(T)$ has the expected value it is sufficient to prove that for general H and G and every extension (1) the family of all $H' \in M(Y; r - 1, u)$ which are contained in E and for which the inclusion $H' \to E$ is near to the inclusion of H in E given by (1) has the expected dimension $(r - 1)(g - 1) + (r - 1)v - u = \chi(H^\vee \otimes G)$.

By the theory of the $Quot$-scheme it is sufficient to prove that $h^1(Y, H^\vee \otimes G) = 0$ for general H and G. This is true by Lemma 3.1 applied to the integer $r' := r - 1$. Hence we have proved parts (b) and (c).

Now we will prove part (f). Assume the existence of an integer $u' > u$ for which part (f) fails. The proofs of [11] Prop. 2.1 and Cor. 2.2 work verbatim for a singular irreducible curve and show that any rank $r - 1$ vector bundle on Y is the flat limit of a flat family of stable vector bundles. Hence the isomorphic classes of the possible quotient bundles depends on at most $(r - 1)^2(g - 1) + 1$ parameters, while the possible line subbundles depend on at most g parameters.

Fix any extension (1) with E stable. Since E is simple, we have $h^0(Y, Hom(G, H)) = 0$. Hence $h^1(Y, Hom(G, H)) = rank(G)rank(H)(g - 1 + \mu(G) - \mu(H))$.

Since $u' > u$, we obtain that the set of all isomorphism classes of middle terms of such extension cannot cover an open subset of $M(Y; r, d)$.

We cannot expect exactly the same type of results for reducible curves and to show the differences we will analyze the case of curves of compact or quasi-compact type with two irreducible components.

Lemma 3.2. *Let Y be a smooth projective curve with genus $g \geq 2$ and E be a rank r degree d stable vector bundle on Y. Let e be the only integer such that $0 \leq e \leq r-1$ and $e + k(r-k)(g-1) \equiv kd$ modulo r.*

If E is general, the Lange invariant of E is $s_k(E) = e + k(r-k)(g-1)$.

Moreover each irreducible component $\mathcal{T}$ of the algebraic set of all maximal degree rank k subbundles of E has dimension e.

Proof. The first assertion follows by [16] Thm. 0.2 and [13] Cor. 3.13 and Rem.3.14, or [11] 4. Now let H be a general member of $\mathcal{T}$, we have $deg((E/H) \otimes H^{\vee}) = k(deg(E/H)) - (r-k)(deg(H)) = k(deg(E)) - r(deg(H)) = s_k(E) = e + k(r-k)(g-1)$.

Furthermore $h^1(Y, (E/H) \otimes H^{\vee}) = 0$, because $deg((E/H) \otimes H^{\vee}) \geq k(r-k)(g-1)$ and we may assume that the pair $(H, E/H)$ is general in the product of the two moduli schemes (see [11] 4, [12] and [4] for an explanation of way this follows from [16] Thm. 0.1). Thus by Riemann-Roch $h^0(Y, (E/H) \otimes H^{\vee}) = e$ and the last assertion follows from the theory of $Quot$-schemes.

Lemma 3.3. *Let X be the union of two smooth curves X_1, X_2 meeting quasi-transversally at a point P. Denote by $\pi : \tilde{X} := X_1 \sqcup X_2 \to X$ the normalization of X. Consider a rank r vector bundle E on X and the following exact sequences induced by X and E:*

$$0 \to \mathcal{O}_X \to \pi_*(\mathcal{O}_{X_1} \oplus \mathcal{O}_{X_2}) \to \mathcal{O}_{\{P\}} \cong \mathbf{k} \to 0 \tag{6}$$

$$0 \to E \to E_{|X_1} \oplus E_{|X_2} \stackrel{\beta}{\to} E_{|P} \cong \mathbf{k}^{\oplus r} \to 0 \tag{7}$$

Let $1 \leq k < r$. A saturated rank k subsheaf H of E is given by a pair (H_1, H_2), where H_i is a rank k subbundle on X_i, $i = 1, 2$, holding the following exact sequence

$$0 \to H \to H_1 \oplus H_2 \to Im(\beta_{|(H_1 \oplus H_2)}) \to 0 \tag{8}$$

where $H_i = H_{|X_i}/Tors(H_{|X_i})$ and $Im(\beta_{|(H_1 \oplus H_2)})$ is the vector subspace of $E_{|P} \cong \mathbf{k}^{\oplus r}$ generated by $H_{1|P}$ and $H_{2|P}$.

Moreover we have the following assertions:

1. *The completion at P of the sheaf H is isomorphic to $R^{\oplus a} \oplus \mathfrak{m}^{\oplus(k-a)}$, where R is the completion of the local ring $\mathcal{O}_{X,P}$, $\mathfrak{m}$ is the maximal ideal of R and $a := dim(H_{1|P} \cap H_{2|P})$. The fiber at P of the restriction $H_{|X_i}$ of H to X_i, $i = 1, 2$, has as completion the module $R^{\oplus a} \oplus R_i^{\oplus(k-a)} \oplus \mathbf{k}^{\oplus(k-a)}$, where R_i is the completion of $\mathcal{O}_{X_i}$ at P. Thus $\mathbf{k}^{\oplus(k-a)}$ gives the torsion of $H_{|X_i}$.*
2. *$deg(H) = deg(H_1) + deg(H_2) - (k - a)$.*

Proof. Every rank k subsheaf H of E is torsion-free, i.e. of depth 1 as in [17] parts VII and VIII. Then the first assertion follows by [17] p. 165 and Prop. 3 at p. 166. Note also that as R-module $\mathfrak{m}$ is isomorphic to $R_1 \oplus R_2$ (see [17] p. 165).

For the second assertion, note that $Im(\beta_{|(H_1 \oplus H_2)})$ is a vector space of dimension $b = 2k - a$, because it is the vector subspace of $E_{|P} \cong \mathbf{k}^{\oplus r}$ generated by the k-dimensional vector spaces $H_{1|P}$ and $H_{2|P}$, meeting at a vector space with dimension a.

Moreover we have $\chi(H) = \chi(H_1) + \chi(H_2) - (2k - a) = deg(H_1) - k(g_1 - 1) + deg(H_2) - k(g_2 - 1) - (2k - a) = deg(H_1) + deg(H_2) - (k - a) - k(g_1 + g_2 - 1)$.

Proposition 3.4. *Let X be the union of two smooth curves X_1, X_2 meeting quasi-transversally at a point P. Then for a vector bundle E on X we have*

$$s_k(E_{|X_1}) + s_k(E_{|X_2}) \le s_k(E) \le s_k(E_{|X_1}) + s_k(E_{|X_2}) + r(min\,\{k, r - k\}).$$

Proof. Let H_i be a maximal rank k subsheaf of $E_{|X_i}$, $i = 1, 2$. By Lemma 3.3, every subsheaf N of E satisfies $deg(N) \le deg(H_1) + deg(H_2)$ and we obtain the first inequality.

Since two k-dimensional vector subspaces V_1 and V_2 of $E_{|P} \cong \mathbf{k}^r$ satisfy $dim(V_1 \cap V_2) \ge max\{0, 2k - r\}$, for the subsheaf H of E defined by H_1 and H_2 we have $deg(H) \ge deg(H_1) + deg(H_2) - (k - max\{0, 2k - r\}) = deg(H_1) + deg(H_2) - min\{k, r - k\}$ (see Lemma 3.3). For a maximal rank k subsheaf N of E we have $deg(N) \ge deg(H)$ and so we conclude.

Proof of Theorem 3. Note that (see Lemma 3.2) a maximal rank k subbundle of E_i has degree $\frac{1}{r}(kd_i - e_i - k(r - k)(g_i - 1))$, $i = 1, 2$. Then by Proposition 3.4 a maximal rank k subsheaf N of E satisfies $deg(N) \le \frac{1}{r}(k(d_1 + d_2) - e_1 - e_2 - k(r - k)(g_1 + g_2 - 2))$ and $deg(N) \ge \frac{1}{r}(k(d_1 + d_2) - e_1 - e_2 - k(r - k)(g_1 + g_2 - 2)) - min\{k, r - k\}$.

Since E_2 is a general stable vector bundle, it has rank k subbundles H_2' with degree $a_2 := \frac{1}{r}(kd_2 - \epsilon_2 - k(r - k)(g_2))$ (see [16] Thm. 0.3).

We have $deg((E_2/H_2') \otimes H_2'^{\vee}) = k(d_2 - a_2) - (r - k)a_2 = kd_2 - ra_2 \geq k(r - g_2)g_2$. Hence $\chi((E_2/H_2') \otimes H_2'^{\vee}) \geq k(r - k)$.

Moreover the full statement of [16] Thm. 0.3 implies that the set of all fibers over P of all possible rank k degree a_2 subbundles H_2' of the fixed bundle E_2 covers an open dense subset of the Grassmannian of the k-dimensional vector subspaces of $E_{2|P} \cong \mathbf{k}^r$.

Hence we may find at least one such H_2' which may be glue to the maximal rank k subbundle H_1 of E_1 obtaining a locally free subbundle H' of E with the quotient E/H' locally free (see Remark 3.6) and $deg(H') = \frac{1}{r}(k(d_1 + d_2) - e_1 - \epsilon_2 - k(r - k)(g_1 + g_2 - 1))$. Hence Theorem 3 follows.

Remark 3.5. Use the notation of Lemma 3.3. A general multistable rank r vector bundle E on X with multidegree (d_1, d_2) is given by a triple (E_1, E_2, β), with E_i a general element of $M(X_i; r, d_i)$, $i = 1, 2$ and $\beta : E_1 \oplus E_2 \to \mathcal{O}_P^{\oplus r}$ a general morphism, i.e. β is a general isomorphism from $E_{1|P} \cong \mathbf{k}^r$ to $E_{2|P} \cong \mathbf{k}^r$.

For $i = 1, 2$, consider a component irreducible $\mathcal{T}_i$ of the algebraic set of all maximal degree rank k subbundles of $E_{|X_i}$ and the morphism $\varphi_i : \mathcal{T}_i \to G(k, E_{|P})$ associating to $H_i \in \mathcal{T}_i$ the fiber $H_{i|P}$, here $G(k, E_{|P})$ denotes the Grassmannian of the k-dimensional vector subspaces of the vector space $E_{|P} \cong \mathbf{k}^r$. Let $\phi := \varphi_1 \times \varphi_2$ and U be the open set given by the pairs (V_1, V_2) of k-dimensional vector subspaces of $E_{|P}$ such that $dim(V_1 \cap V_2) = max\{0, 2k - r\}$.

We can prove that the open subset $\phi^{-1}(U)$ of $\mathcal{T}_1 \times \mathcal{T}_2$ is non-empty.

Indeed let H_i be an element of $\mathcal{T}_i$, $i = 1, 2$, since the above morphism β is general, we have that $(\beta(H_{1|P}), H_{2|P})$ is an element of the open U.

Thus for the subsheaf H of E defined by H_1 and H_2 we have $deg(H) = deg(H_1) + deg(H_2) - (k - max\{0, 2k - r\}) = \frac{1}{r}(kd_1 - e_1 - k(r - k)(g_1 - 1) + kd_2 - e_2 - k(r - k)(g_2 - 1)) - min\{k, r - k\} = \frac{1}{r}(k(d_1 + d_2) - e_1 - e_2 - k(r - k)(g_1 + g_2 - 2)) - min\{k, r - k\}$.

A problem to compute $s_k(E)$ for general E is also to consider saturated rank k subsheaves H of E such that for at least one index $i \in \{1, 2\}$ the saturation of $H_{|X_i}/Tors(H_{|X_i})$ is not a maximal degree rank k subbundle of $E_{|X_i}$.

Remark 3.6. We use the notation of the proof of Theorem 3. Let N_i a rank k maximal subbundle of $E_i := E_{|X_i}$, $i = 1, 2$. The subsheaf N of E defined by N_1 and N_2 is a subbundle, i.e. the quotient E/N is locally free, if and only if we have $dim(N_{1|P} \cap N_{2|P}) = k$.

We can choose gluing data of E_1 and E_2 (the morphism β) to have the above condition. In this case we have $s_k(E) = e_1 + e_2 + k(r - k)(g_1 + g_2 - 2)$.

Proof of Theorem 4. The case $s = 1$ is [16] Thm. 0.2. Now let $s \geq 2$, assume that the result is true for the curve $X_{[s-1]} := X_1 \cup \cdots \cup X_{s-1}$. Since $kd_s - ra_s \geq k(r-k)g_s$, by [16] Thm. 0.3, a general $E_s \in M(X_s; r, d_s)$ is an extension of some $G_s \in M(X_s; r-k, d_s - a_s)$ by some $H_s \in M(X_s; k, a_s)$ and we may assume that the pair (H_s, G_s) is general.

Let us consider the vector bundles H', E', G' as in the statement of Theorem 4 for the curve $X_{[s-1]}$. Set $\{p\} := X_{[s-1]} \cap X_s$. By assumption P is a smooth point of $X_{[s-1]}$ and X_s.

As in the proof of Theorem 3, by using the full statement of [16] Thm. 0.3, we obtain that the set of all fibers over P of all possible rank k degree d_s subbundles H_s of the fixed bundle E_s covers an open dense subset of the Grassmannian of the k-dimensional vector subspaces of $E_{s|P} \cong \mathbf{k}^r$.

Hence we may find at least one such H_s which may be glue to H' obtaining a locally free subbundle H of E with the quotient $G := E/H$ locally free (see Remark 3.6).

Theorem 3.7. *Let X be a reduced and connected smoothable projective curve. Let E be a torsion-free sheaf on X with constant rank r and such that E is a flat limit of a flat family of locally free sheaves on a flat family of curves.*

Then for every integer k with $1 \leq k < r$ we have $s_k(E) \leq k(r-k)p_a(X)$.

Proof. Let $\mathcal{X} := \{X_\lambda\}_{\lambda \in \Delta}$ be a flat family of curves, with X_λ smooth for general $\lambda \in \Delta$. Let $\mathcal{E} := \{E_\lambda\}_{\lambda \in \Delta}$ be a flat family of sheaves with E_λ sheaf on X_λ for every $\lambda \in \Delta$ and $X_o = X$, $E_o = E$ for $o \in \Delta$. Restricting if necessary Δ to a smaller neighborhood of o, we assume E_λ locally free for every $\lambda \neq o$.

By [1] Thm. 0.1, for every $\lambda \neq o$ we have $s_k(E_\lambda) \leq k(r-k)p_a(X_\lambda)$. Hence we may apply the properness of the relative *Quot*-scheme to obtain the semicontinuity for the function $s_k(E)$. Hence we have $s_k(E) = s_k(E_o) \leq s_k(E_\lambda)$ for general λ and we conclude.

4. Further remarks.

The first two results of this section (i.e. Propositions 4.1 and 4.2) concern the extension of vector bundles from some component of a reduced curve X to all X. In particular Proposition 4.2 may be used to define a notion of "general" without taking a polarization of X. The other examples of this section (i.e. Example 4.3 and Proposition 4.5) point out to a difference between the notion of stability given in Definition 1.6 (according to [7], [9]) and that one given in [17], even in the rank 1 case. Furthermore Example 4.4 point out another phenomenon which may arise for non-locally free sheaves.

Proposition 4.1. *Let X be a reduced and connected projective curve and D be the union of some of the irreducible components of X.*

(a) *Let F be a vector bundle on D. Then there exists a vector bundle E on X with $E_{|D} \cong F$.*

(b) *Let $\{F_t\}_{t \in T}$ be a flat family of vector bundles on D parametrized by an integral quasi-projective variety and $s \in T$. Then there exist an open finite map $\alpha : U \to T$ with $s \in \alpha(U)$ and a flat family $\{E_\lambda\}_{\lambda \in U}$ of vector bundles on X parametrized by U such that for every $\lambda \in U$ we have $E_{\lambda|D} \cong F_{\alpha(\lambda)}$. If $\{F_t\}_{t \in T}$ is induced by a vector bundle on $D \times T$, then we may take $U = T$ and $\alpha = id_T$.*

Proof. Now we prove the assertion (a). Let C be the closure in X of $X \setminus D$. We may assume $C \neq \emptyset$, otherwise $D = X$. Take a finite open covering $\{U_j\}_{j \in J}$ of X such that the open covering $\{U_j \cap D\}_{j \in J}$ of D is a trivializing covering for F, moreover for every $j \in J$ either $U_j \cap D = \emptyset$ or U_j contains a unique point of $Y \cap D$ and such that for every $P \in Y \cap D$ there is a unique index $j_P \in J$ with $P \in U_{j_P}$. Set $J' := \{j \in J : U_j \cap D \neq \emptyset\}$.

The transition functions of the vector bundle F with respect to the covering $\{U_j \cap D\}_{j \in J}$ of D define a vector bundle G on the open subset $W := \bigcup_{j \in J'} U_j$ of X. We have $G_{|D} \cong F$.

Take the open covering of X formed by $X \setminus D$ and by the open sets U_j, $j \in J'$.

By construction $X \setminus D$ does not intersect $U_i \cap U_j$ for every $i, j \in J'$, $i \neq j$. Hence we may use arbitrary transition functions on $(X \setminus D) \cap U_j$, $j \in J'$, to define together with G a vector bundle E on X with $E_{|D} \cong F$.

For the assertion (b), note that up to a quasi-finite extension of T, near s we may find an open covering $\{U_j\}_{j \in J}$ of X as in the proof of part (a) such that $\{U_j\}_{j \in J}$ is a trivializing open covering for F_t for all t "near" to s in the finite flat topology. Hence the first assertion of part (b) follows.

By Lefschetz principle we reduce to the case in which $\mathbf{k}$ is the complex number field. By GAGA it is sufficient to prove the same assertion for complex analytic vector bundles. Now the existence of such good "simultaneously trivializing " open covering is obvious because every complex analytic vector bundle on one-dimensional reduced complex space without compact positive dimensional components (i.e. Stein) is trivial; indeed the proof in the case of a smooth Stein Riemann Surface given in [8] Thm. 30 (i.e. reduction to the rank 1 case and use of the exponential sequence) works verbatim in the general case.

Proposition 4.2. *Let X be a reduced and connected projective curve and D be the union of some of the irreducible components of X. Let E be a vector bundle on X. Set $F := E_{|D}$. Then the natural restriction map $\tau : H^1(X, End(E)) \to$*

$H^1(D, End(F))$ is surjective and the natural restriction map ρ from the germ $\Delta(E)$ at E of the local deformation space of E to the germ $\Delta(F)$ at F of the local deformation space of F is surjective.

Proof. Let $\mathcal{J}$ be the ideal sheaf of D in X. Since $dim(X) = 1$, we have $h^2(X, \mathcal{J} \otimes End(E)) = 0$. Hence τ is surjective.

Note that $H^1(X, End(E))$ (resp. $H^1(D, End(F))$ is the tangent space at the point parametrizing E (resp. F) of the germ $\Delta(E)$ (resp. $\Delta(F)$). Since $dim(X) = dim(D) = 1$, we have $H^2(X, End(E)) = H^2(D, End(F)) = (0)$. Hence both deformation functors are unobstructed, i.e. the germs $\Delta(E)$ and $\Delta(F)$ are smooth. Since τ is the differential of ρ, the map ρ from the germ $\Delta(E)$ to the germ $\Delta(F)$ is a submersion at E, as wanted.

Recall that a stable sheaf according Seshadri ([17] p. 153) is simple ([17] part c) of Prop. 10 and Prop. 12 at p. 154). Now we give examples of rank 1 torsion-free sheaves, that are stable according Definition 1.6 but non-simple, so they are not stable according [17].

Example 4.3. We give examples of reduced connected curves X with rank 1 torsion-free sheaves L on X such that $h^0(X, End(L)) \geq 2$.

Let X be a quasi-nodal projective curve with at least two irreducible components. Fix one of the irreducible components, D, of X. Let L be a rank 1 torsion-free sheaf on X such that L is not locally free at every point of D wich is common with another irreducible component of X.

Call S the union of these singular points of X, i.e. $S := (Sing(X) \cap D) \setminus Sing(D)$. Fix $\lambda, \mu \in \mathbf{k} \setminus \{0\}$. Multiply a germ of a section of L over $D \setminus S$ by λ and over $X \setminus D$ by μ. The case $a = 0$, $b = c = 1$ of [17] Prop. 7 at p. 171 shows that this automorphism of $L_{|X \setminus S}$ extends to an automorphism of L over each point of S. Hence $h^0(X, End(L)) \geq 2$.

Note that if X is the union of two smooth irreducible curves X_1, X_2 meeting quasi-transversally at a point P, the sheaf $L = \mathcal{O}_X(-P)$ satisfies the above conditions.

Definition 4.4. Let X be a reduced curve and F a torsion-free sheaf on X. We define $Sing(F) := \{P \in X : F \text{ is not locally free at } P\}$.

Obviously, we have $Sing(F) \subseteq Sing(X)$ for any reduced curve X and any torsion-free sheaf F on X.

Proposition 4.5. *Let X be a quasi-nodal projective curve. Let L be a rank 1 torsion-free sheaf on X. Set $S := \{P \in Sing(L) : P$ belongs to two irreducible components of $X\}$.*

Let $f : Z \to X$ be the partial normalization of X which normalizes exactly the points of S. If c is the number of connected components of Z, then $h^0(X, End(L)) = c$.

Proof. The proof of the above example gives easily the inequality $h^0(X, End(L)) \geq c$. Hence it is sufficient to prove the opposite inequality. Set $M := f^*(L)/Tors(f^*(L))$. Since L is torsion-free, we have an injective map $H^0(X, End(L)) \to H^0(Z, End(M))$. Hence it is sufficient to prove that for every connected component Y of Z the sheaf $M_{|Y}$ is simple.

By construction $M_{|Y}$ is locally free at every point of Y common to two irreducible components of Y. Hence for every irreducible component W of Y the sheaf $M_{|W}$ is torsion-free. Hence $M_{|W}$ is simple ([5] Lemma 3.5.1 part 1). Fix $P \in Y_{reg}$ and call W the irreducible component of Y containing P.

In order to obtain a contradiction, we assume $M_{|Y}$ not simple. Hence, just using that the fiber of M at P is one-dimensional, we obtain the existence of $\alpha \in H^0(Y, End(M_{|Y}))$, $\alpha \neq 0$, with α vanishing at P.

Since $M_{|W}$ is simple, we have $\alpha_{|W} \equiv 0$. Since $M_{|Y}$ is locally free at every point of Y common to two irreducible components of Y, we obtain for the same reason that α vanishes identically on every irreducible component of Y intersecting W. And so on. Since Y is connected, in a finite number of steps we obtain a contradiction.

If we do not assume the condition “ $\frac{a_1}{k} < \frac{b_i}{r-k}$ for every i ” (resp. “ $\frac{a_1}{k} \leq \frac{b_i}{r-k}$ for every i ”) in the statement of Theorem 1 (resp. Proposition 2.1) the situation may be completely different as shown by the following example, which explains why we study mainly multisemistable and multistable vector bundles.

Example 4.6. Let X be a quasi-nodal projective curve with s irreducible components, say $X_1, \ldots, X_s$. Assume $s \geq 2$. Set $S := X_1 \cap (X_2 \cup \cdots \cup X_s)$ and $\sigma := card(S)$. Set $g_i := p_a(X_i) \geq 1$. Fix integers a_i and b_i for $1 \leq i \leq s$, with the condition $b_1 \leq a_1 - \sigma - 2g_1 + 1$. Fix $L, M \in Pic(X)$ with $deg(L_{|X_i}) = a_i$ and $deg(M_{|X_i}) = b_i$ for every i.

Let E be any extension of M by L.

(a) Since $deg(L_{|X_1}) \geq deg(M_{|X_1}) + 2g_1 - 1$, the restriction of this extension to X_1 splits, i.e. $E_{|X_1} \cong (L_{|X_1} \oplus M_{|X_1})$. So E is not multisemistable.

(b) By Riemann-Roch we have $h^0(X_1, Hom(M_{|X_1}, L_{|X_1})(-S)) \neq 0$ and hence there exists $u \in H^0(X_1, End(E_{|X_1}))$ with $u_{|S} = 0$. Thus we may extend u to $v \in H^0(X, End(E))$ with $v_{|X_1} = u$ and $v_{|X_i} = 0$ for every $i \geq 2$. Hence E is not simple. Thus by [17] part c) of Prop. 10 p. 154, E

is not stable in the sense of [17] pp. 153-154 for any choice of polarization on X.

(c) If $deg(L) = \sum_{i=1}^{s} a_i < \sum_{i=1}^{s} b_i = deg(M)$, the above example shows also that, without any restiction, the so-called Lange conjecture (see [16]) is false for reducible curves, at least for certain polarizations, in the sense of [17] p. 153, even in rank two and for vector bundles.

Furthermore this example shows that Theorem 1 does not hold for stable sheaves in the sense of [17].

The main difference between the notion of stability considered in [17] pp. 153-154 and in [7], [9] is not that in [17] a choice of a polarization is added, but that in [7] and [9] the sheaf has constant rank and one have to check the slope inequality only for proper subsheaves with constant rank. With the definition of [17] pp. 153-154, one has to check the stability or semistability of a sheaf of constant rank (or even of a vector bundle) by considering the slope inequality for all subsheaves, even of non-constant rank. As shown by Example 4.3 and Proposition 4.5 this is essential to obtain that stability implies simpleness ([17] part c) of Prop. 10 and Prop. 12 at p. 154).

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