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NONNIL-NOETHERIAN PAIRS OF THE FORM $(R,R[X])$ AND SOME RELATED RESULTS

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The rings considered in this paper are commutative with identity and are nonzero. An ideal of a ring is said to be nonnil if it contains an element that is not nilpotent. Let R be a ring. We say that R is nonnil-Noetherian (resp., nonnil-Laskerian) if each proper nonnil ideal of R is finitely generated (resp., each proper nonnil ideal of R admits a primary decomposition). Whenever R is a subring of a ring T, then it is assumed that R contains the identity element of T. Let $R \subseteq T$ be rings. We say that (R,T) is a nonnil-Noetherian pair (resp., nonnil-Laskerian pair) if each intermediate ring A between R and T is nonnil-Noetherian (resp., nonnil-Laskerian). This paper aims to characterize R such that $(R,R[X])$ is a nonnil-Noetherian pair (resp., nonnil-Laskerian pair), where $R[X]$ is the polynomial ring in one variable X over R. Also, this paper aims to characterize pairs of the form $(R,R[X])$ such that each intermediate ring A between R and $R[X]$ has a property that is related to being nonnil-Noetherian (resp., nonnil-Laskerian).

1. Introduction

The rings considered in this paper are commutative with identity, and they are nonzero. Throughout this paper, unless otherwise specified, we use R to denote

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a ring. The work carried out in this paper is motivated by the research work of Badawi [2] on Nonnil-Noetherian rings, the research work of Hizem and Benhissi [12] on Nonnil-Noetherian rings and by the study of several other researchers on similar concepts, such as nonnil-m-formally Noetherian rings and nonnil-Laskerian rings, see [3, 17].

The concept of a nonnil-Noetherian ring was introduced and investigated by Badawi [2]. We denote the nilradical of R by $Nil(R)$. An ideal I of R is said to be *nonnil* if $I \not\subseteq Nil(R)$. We use f.g. for finitely generated. We say that R is a *nonnil-Noetherian ring* if every proper nonnil ideal of R is f.g. [2]. We denote the set of all prime ideals of R by $Spec(R)$ and the set of all maximal ideals of R by $Max(R)$. If a set A is a subset of a set B and $A \neq B$, then we denote it by using $A \subset B$ or $B \supset A$. Recall that $Nil(R)$ is a *divided prime ideal* of R if $Nil(R) \in Spec(R)$ and for any $x \in R\backslash Nil(R)$, $Nil(R) \subset Rx$ [2]. With the assumption that $Nil(R)$ is a divided prime ideal of R, Badawi proved several interesting theorems on nonnil-Noetherian rings, see [2, Section 2]. Hizem and Benhissi [12] generalized some of the results from [2] to a nonnil-Noetherian ring without any assumption on the nilradical, and they investigated the ring of power series over a nonnil-Noetherian ring.

The research work of Badawi [2] inspired a lot of investigation by several researchers, replacing Noetherian with other interesting properties of rings. For a few of such research work that are relevant to this paper, one can refer to [3, 17].

Let $m \in \mathbb{N}$. The notion of m-formally Noetherian ring was introduced and investigated by Khalifa [14]. The ring R is said to be a *m-formally Noetherian ring* if for every increasing sequence $(I_n)_{n\geq 0}$ of ideals of R, the increasing sequence of ideals $(\sum_{i_1+\cdots+i_m=n} I_{i_1}\cdots I_{i_m})_{n\geq 0}$ stabilizes. Khalifa [14] proved that many properties of Noetherian rings hold for m-formally Noetherian rings, and he studied m-formally variants of several well-known theorems on Noetherian rings. Recall that the ring R is said to be *nonnil-m-formally Noetherian* if for any increasing sequence of nonnil ideals $(I_n)_{n\geq 0}$ of R, the increasing sequence of ideals $(\sum_{i_1+\cdots+i_m=n} I_{i_1}\cdots I_{i_m})_{n\geq 0}$ stabilizes [3]. Dabbabi and Maatallah investigated nonnil-m-formally variants of several well-known theorems on Noetherian rings, and they studied the transfer of nonnil-m-formally Noetherian property to trivial extension and amalgamation algebra along an ideal [3].

Recall that R is said to be a *Laskerian ring* (resp., *strongly Laskerian ring*) if each proper ideal of R admits a primary (resp., strong primary) decomposition [10]. Heinzer and Lantz [10] proved several inspiring and interesting theorems on Laskerian rings, and they showed that Laskerian rings possess some of the properties of Noetherian rings. Several researchers have investigated the properties of Laskerian rings, for example, the reader can refer to [6, 11, 18]. The concept of a nonnil-Laskerian ring was introduced and studied by Moulahi [17]. Recall that R is a *nonnil-Laskerian ring* if each proper nonnil ideal of R admits a primary decomposition [17]. We say that R is a *nonnil-strongly Laskerian ring* if each proper nonnil ideal of R admits a strong primary decomposition. Moulahi [17] proved that nonnil-Laskerian rings enjoy analogs of many properties of Laskerian rings, and he studied the nonnil-Laskerian property over the polynomial and power series rings.

The modules considered in this paper are unitary. Let M be a module over R. Recall that *M satisfies (accr)* (resp., *satisfies (accr*)*) if for every submodule N of M and every finitely generated (resp., principal) ideal B of R, the increasing sequence of submodules $(N :_M B) \subseteq (N :_M B^2) \subseteq (N :_M B^3) \subseteq \cdots$ terminates. We say that *R satisfies (accr)* (resp., *satisfies (accr*)*) if it satisfies (accr) (resp., (accr*)) as a module over itself [15, Definition 1]. Lu proved several interesting and inspiring theorems on modules and rings satisfying (accr), see [15, 16]. It is known that for an R-module M, the properties (accr) and (accr*) are equivalent [15, Theorem 1].

We say that *R satisfies nonnil-accr* (resp., *satisfies nonnil-accr**) if for every nonnil ideal I of R and every finitely generated (resp., principal) ideal B of R, the increasing sequence of ideals $(I :_R B) \subseteq (I :_R B^2) \subseteq (I :_R B^3) \subseteq \cdots$ is stationary. From the proof of [15, Theorem 1], it follows that the properties nonnil-accr and nonnil-accr* are equivalent.

Whenever T is an extension ring of R, it is assumed that R contains the identity element of T. Let T be an extension ring of R. We denote the set of all intermediate rings between R and T by $[R,T]$. We say that (R,T) is a *Noetherian pair* if each $A \in [R,T]$ is Noetherian. We use the abbreviation NP for Noetherian pair. For several interesting theorems on Noetherian pairs of rings, see [7, 22]. Let $m \in \mathbb{N}$. We say that (R,T) is a *nonnil-Noetherian pair* (resp., *nonnil-m-formally Noetherian pair*) if each $A \in [R,T]$ is nonnil-Noetherian (resp., nonnil-m-formally Noetherian). We use the abbreviation N-NP (resp., N-m-FNP) for nonnil-Noetherian pair (resp., nonnil-m-formally Noetherian pair). Let $R[X]$ denote the polynomial ring in one variable X over R. In Section 2 of this paper, we characterize R such that $(R,R[X])$ is an N-NP (resp., N-m-FNP). We say that (R,T) is a *Laskerian pair* (resp., *strongly Laskerian pair*) if each $A \in [R,T]$ is Laskerian (resp., strongly Laskerian). We use the abbreviation LP (resp., SLP) for Laskerian pair (resp., strongly Laskerian pair). For a discussion about some results on Laskerian pairs, one can refer to [20]. We say that (R,T) is a *nonnil-Laskerian pair* (resp., *nonnil-strongly Laskerian pair*) if each $A \in [R,T]$ is nonnil-Laskerian (resp., nonnil-strongly Laskerian). We use the abbreviation N-LP (resp., N-SLP) for nonnil-Laskerian pair (resp., nonnil-strongly Laskerian pair). In Section 3 of this paper, we characterize R such that $(R,R[X])$ is an N-

LP (resp., N-SLP). We say that (R,T) is an *accr pair* if A satisfies (accr) for each $A \in [R,T]$. For a discussion about some results on accr pairs, see [21]. We say that (R,T) is a *nonnil-accr pair* if A satisfies nonnil-accr for any $A \in [R,T]$. We use the abbreviation N-accr pair for nonnil-accr pair. In Section 4 of this paper, we characterize R such that $(R,R[X])$ is an N-accr pair.

2. When is $(R,R[X])$ an N-NP?

Unless otherwise specified, we use R to denote a ring. This section aims to characterize R such that $(R,R[X])$ is an N-NP, where N-NP stands for nonnil-Noetherian pair. We use the following lemma in the proof of the main result of this section.

Lemma 2.1. *Let R be nonnil-Noetherian but not Noetherian. Then, $Nil(R) \in Spec(R)$.*

Proof. Assume that R is nonnil-Noetherian but not Noetherian. Hence, there exists an ideal I of R with $I \subseteq Nil(R)$ such that I is not finitely generated. If $Nil(R) \notin Spec(R)$, then we can find $a,b \in R \backslash Nil(R)$ such that $ab \in Nil(R)$. Hence, there exists $n \in \mathbb{N}$ such that $(ab)^n = 0$. Observe that $a^n \in R \backslash Nil(R)$, $a^n \in I + Ra^n$, $b^n \in R \backslash Nil(R)$, and $b^n \in (I :_R a^n)$. Therefore, $I + Ra^n$ and $(I :_R a^n)$ are nonnil ideals of R. So, both $I + Ra^n$ and $(I :_R a^n)$ are f.g. ideals of R. Let $x_1,\ldots,x_t \in I$ and $r_1,\ldots,r_t \in R$ be such that $I + Ra^n = \sum_{i=1}^t R(x_i + r_i a^n)$. Let $y_1,\ldots,y_k \in (I :_R a^n)$ be such that $(I :_R a^n) = \sum_{j=1}^k Ry_j$. Then, it follows as in the proof of [13, Theorem 7] that $I = \sum_{i=1}^t Rx_i + \sum_{j=1}^k R(y_j a^n)$, a contradiction to the assumption that I is not finitely generated. Therefore, $Nil(R) \in Spec(R)$. □

We use the abbreviation m.c. subset for multiplicatively closed subset.

Lemma 2.2. *Let $R \subseteq T$ be rings. Let J be any proper ideal of T and let $I = J \cap R$. If (R,T) is an N-NP, then the following statements hold.*
(1) $(\frac{R}{I}, \frac{T}{J})$ is an N-NP.
(2) $(S^{-1}R, S^{-1}T)$ is an N-NP for any m.c. subset S of R.

Proof. (1) Let $B \in [\frac{R}{I}, \frac{T}{J}]$. Then, $B = \frac{A}{J \cap A}$ for some $A \in [R,T]$. Note that A is a nonnil-Noetherian ring, so B is nonnil-Noetherian by [12, Example 1.4]. Therefore, $(\frac{R}{I}, \frac{T}{J})$ is an N-NP.

(2) Let $W \in [S^1R, S^{-1}T]$. Then, $W = S^{-1}A$ for some $A \in [R,T]$. Since A is nonnil-Noetherian, it follows that $W = S^{-1}A$ is nonnil-Noetherian by [2, Theorem 2.7]. Therefore, $(S^{-1}R, S^{-1}T)$ is an N-NP. □

The Krull dimension of R is referred to as the dimension of R, and it is denoted by *dimR*. A ring A is said to be *reduced* if $Nil(A) = (0)$. We say that a ring A is *quasi-local* if A has only one maximal ideal. A Noetherian quasi-local ring is called a *local ring*.

Theorem 2.3. *The following statements are equivalent:*

(1) $(R,R[X])$ *is an N-NP.*

(2) *Either there exist* $n \in \mathbb{N}$ *and fields* $F_1,\dots,F_n$ *such that* $R \cong F_1 \times \cdots \times F_n$ *as rings or* $(R,\mathfrak{m})$ *is a local Artinian ring with* $\mathfrak{m} \neq (0)$.

Proof. (1) $\Rightarrow$ (2) Assume that the statement (1) holds. Then, $R[X]$ is nonnil-Noetherian. So, R is Noetherian by [12, Theorem 3.3 ((4) $\Rightarrow$ (2))]. Let $\mathfrak{p} \in Spec(R)$. Then, $\mathfrak{p}[X] \in Spec(R[X])$ and $\mathfrak{p}[X] \cap R = \mathfrak{p}$. Note that $(\frac{R}{\mathfrak{p}}, \frac{R[X]}{\mathfrak{p}[X]})$ is an N-NP by Lemma 2.2(1). Observe that each $A \in [\frac{R}{\mathfrak{p}}, \frac{R[X]}{\mathfrak{p}[X]}]$ is a nonnil-Noetherian integral domain. Hence, $(\frac{R}{\mathfrak{p}}, \frac{R[X]}{\mathfrak{p}[X]})$ is an NP. From $\frac{R[X]}{\mathfrak{p}[X]} \cong \frac{R}{\mathfrak{p}}[X]$ as rings, it follows that $(\frac{R}{\mathfrak{p}}, \frac{R}{\mathfrak{p}}[X])$ is an NP. Therefore, $\frac{R}{\mathfrak{p}}$ is a field by [22, Corollary 5]. Thus, any $\mathfrak{p} \in Spec(R)$ is maximal, so $dimR = 0$. As R is Noetherian and $dimR = 0$, it follows that R is Artinian by [1, Theorem 8.5].

If $(R,R[X])$ is an NP, then there exist $n \in \mathbb{N}$ and fields $F_1,\dots,F_n$ such that R is ring-isomorphic to $F_1 \times \cdots \times F_n$ by [8, Theorem 2.3 ((2) $\Rightarrow$ (3))]. If $(R,R[X])$ is not an NP, then there exists $A \in [R,R[X]]$ such that A is not Noetherian. As A is nonnil-Noetherian but not Noetherian, $Nil(A) \in Spec(A)$ by Lemma 2.1. As any reduced nonnil-Noetherian ring is Noetherian, it follows that $Nil(A) \neq (0)$. Let $f(X) \in Nil(A)\backslash\{0\}$. Let $f(X) = \sum_{i=0}^{k} r_i X^i$, where $r_i \in R$ for each $i \in \{0,\dots,k\}$ with $r_k \neq 0$. As $r_i \in Nil(R)$ for each $i \in \{0,\dots,k\}$ by [1, Exercise 2 (*ii*), p.11], it follows that $r_k \in Nil(R)\backslash\{0\}$. Therefore, $Nil(R) \neq (0)$. Since $Nil(R) = Nil(A) \cap R$, it follows that $Nil(R) \in Spec(R)$. Hence, $Nil(R) \in Max(R)$. Therefore, $Spec(R) = Max(R) = \{Nil(R)\}$. Let us denote $Nil(R)$ by $\mathfrak{m}$. Thus, if $(R,R[X])$ is an N-NP but not an NP, then $(R,\mathfrak{m})$ is a local Artinian ring with $\mathfrak{m}$ as its unique maximal ideal and $\mathfrak{m} \neq (0)$.

(2) $\Rightarrow$ (1) If there exist $n \in \mathbb{N}$ and fields $F_1,\dots,F_n$ such that $R \cong F_1 \times \cdots \times F_n$ as rings, then $(R,R[X])$ is an NP by [8, Theorem 2.3 ((3) $\Rightarrow$ (2))]. Assume that $(R,\mathfrak{m})$ is a local Artinian ring with $\mathfrak{m} \neq (0)$. Note that $\mathfrak{m} = Nil(R)$. Hence, $\mathfrak{m}$ is nilpotent by [1, Proposition 8.4]. Let $t \in \mathbb{N}\backslash\{1\}$ be such that $\mathfrak{m}^t = (0)$. Let $A \in [R,R[X]]$. Let $\overline{A}$ denote the integral closure of A in $R[X]$. Since $\mathfrak{m}[X]^t = (0)$, it follows that $R+\mathfrak{m}[X]$ is integral over R, so it is integral over A. Note that $\mathfrak{m}[X]$ is the only maximal ideal of $R+\mathfrak{m}[X]$. Either $\overline{A} = R+\mathfrak{m}[X]$ or $R+\mathfrak{m}[X] \subset \overline{A}$. If $\overline{A} = R+\mathfrak{m}[X]$, then $\mathfrak{m}[X] \cap A \in Max(A)$ by [1, Corollary 5.8] and $(\mathfrak{m}[X] \cap A)^t = (0)$. Hence, $\mathfrak{m}[X] \cap A$ is the unique maximal ideal of A. Therefore, A has no proper nonnil ideal. So, A is nonnil-Noetherian. If $\overline{A} \supset R+\mathfrak{m}[X]$, then A is Noetherian, see [20, Theorem 1.1 (Case 2 of its proof)]. Therefore, $(R,R[X])$ is an N-NP. As $\mathfrak{m}[X]$ is not a f.g. R-module, $\mathfrak{m}[X]$ is not a f.g. ideal of $R+\mathfrak{m}[X]$

by [7, Proposition 1.3]. Hence, $R+\mathfrak{m}[X]$ is not Noetherian, so $(R,R[X])$ is not an NP. $\square$

Let $m \in \mathbb{N}$. Note that R is a m-formally Noetherian ring if and only if I^m is f.g. for any ideal I of R by [14, Theorem 2.3 ((1) $\Leftrightarrow$ (10))]. For a pair of rings $R \subseteq T$, we say that (R,T) is a *m-formally Noetherian pair* if each $A \in [R,T]$ is a m-formally Noetherian ring. In Theorem 2.7, we characterize R such that $(R,R[X])$ is an m-FNP for some $m \in \mathbb{N}$, where m-FNP stands for m-formally Noetherian pair. We use the following lemmas in its proof.

Lemma 2.4. *For a pair of rings $R \subseteq T$, if (R,T) is an m-FNP for some $m \in \mathbb{N}$, then the following statements hold.*
(1) $(\frac{R}{J\cap R}, \frac{T}{J})$ *is an m-FNP for any proper ideal J of T.*
(2) $(S^{-1}R, S^{-1}T)$ *is an m-FNP for any m.c. subset S of R.*

Proof. It can be shown that the homomorphic image of a m-formally Noetherian ring is m-formally Noetherian, and if a ring A is m-formally Noetherian, then $S^{-1}A$ is m-formally Noetherian for any m.c. subset S of A by using [14, Theorem 2.3 ((1) $\Leftrightarrow$ (10))]. Hence, using arguments similar to those in the proof of Lemma 2.2, one can prove this lemma. $\square$

Lemma 2.5. *For a pair of rings $R \subseteq T$, if $|Max(R)| < \infty$, and if for each $\mathfrak{m} \in Max(R)$, $(R_{\mathfrak{m}}, T_{R\backslash\mathfrak{m}})$ is an m-FNP for some $m \in \mathbb{N}$, then (R,T) is an m-FNP.*

Proof. For each $\mathfrak{m} \in Max(R)$, $R\backslash\mathfrak{m}$ is a m.c. subset of R. By hypothesis, $|Max(R)| < \infty$. Let $Max(R) = \{\mathfrak{m}_i \mid i \in \{1,\ldots,n\}\}$. Let $A \in [R,T]$ and let I be any ideal of A. Let $i \in \{1,\ldots,n\}$. Note that $A_{R\backslash\mathfrak{m}_i} \in [R_{\mathfrak{m}_i}, T_{R\backslash\mathfrak{m}_i}]$. By hypothesis, $(R_{\mathfrak{m}_i}, T_{R\backslash\mathfrak{m}_i})$ is an m-FNP. As $I_{R\backslash\mathfrak{m}_i}$ is an ideal of $A_{R\backslash\mathfrak{m}_i}$, so $(I_{R\backslash\mathfrak{m}_i})^m$ is f.g. by [14, Theorem 2.3 ((1) $\Rightarrow$ (10))]. As $(I_{R\backslash\mathfrak{m}_i})^m = (I^m)_{R\backslash\mathfrak{m}_i}$, there exists a f.g. ideal F_i of A with $F_i \subseteq I^m$ such that $(I^m)_{R\backslash\mathfrak{m}_i} = (F_i)_{R\backslash\mathfrak{m}_i}$. We claim that $I^m = \sum_{i=1}^n F_i$. It is clear that $\sum_{i=1}^n F_i \subseteq I^m$. Let $a \in I^m$. Then, there exists $s_i \in R\backslash\mathfrak{m}_i$ such that $s_i a \in F_i$. Observe that $\sum_{i=1}^n Rs_i = R$. As $Rs_i a \subseteq F_i$ for each $i \in \{1,\ldots,n\}$, we get that $Ra = (\sum_{i=1}^n Rs_i)a = \sum_{i=1}^n Rs_i a \subseteq \sum_{i=1}^n F_i$. Hence, $a \in \sum_{i=1}^n F_i$. Thus, $I^m \subseteq \sum_{i=1}^n F_i$, so $I^m = \sum_{i=1}^n F_i$ is finitely generated. So, A is m-formally Noetherian. Hence, (R,T) is an m-FNP. $\square$

Lemma 2.6. *Let $(R,\mathfrak{m})$ be a local Artinian ring. Then $(R,R[X])$ is a k-FNP for some $k \in \mathbb{N}$.*

Proof. As $Nil(R) = \mathfrak{m}$, $\mathfrak{m}$ is nilpotent by [1, Proposition 8.4]. Let $k \in \mathbb{N}$ be such that $\mathfrak{m}^k = (0)$. Let $A \in [R,R[X]]$. We can now proceed as in the proof of (2) $\Rightarrow$ (1) of Theorem 2.3. Let $\overline{A}$ denote the integral closure of A in $R[X]$. Either $\overline{A} = R+\mathfrak{m}[X]$ or $\overline{A} \supset R+\mathfrak{m}[X]$. If $\overline{A} = R+\mathfrak{m}[X]$, then $\overline{A}$ is quasi-local with

$\mathfrak{m}[X]$ as its unique maximal ideal and $(\mathfrak{m}[X])^k = (0)$. Hence, A is quasi-local with $\mathfrak{m}[X] \cap A$ as its unique maximal ideal and $(\mathfrak{m}[X] \cap A)^k = (0)$. Therefore, for each proper ideal I of A, $I^k = (0)$ is f.g., so A is k-formally Noetherian. If $\overline{A} \supset R + \mathfrak{m}[X]$, then A is Noetherian, see [20, Theorem 1.1 (Case 2 of its proof)]. Hence, $(R,R[X])$ is a k-FNP. □

Theorem 2.7. *The following statements are equivalent:*

(1) $(R,R[X])$ *is an* m*-FNP for some* $m \in \mathbb{N}$.

(2) R *is Artinian.*

Proof. $(1) \Rightarrow (2)$ Assume that the statement (1) holds. Then, $R[X]$ is m-formally Noetherian for some $m \in \mathbb{N}$. Hence, R is Noetherian by [14, Corollary 2.12 ((2) $\Rightarrow$ (3))]. Let $\mathfrak{p} \in Spec(R)$. By Lemma 2.4(1), it follows as in the proof of (1) $\Rightarrow$ (2) of Theorem 2.3 that $(\frac{R}{\mathfrak{p}}, \frac{R}{\mathfrak{p}}[X])$ is an m-FNP. Let $A \in [\frac{R}{\mathfrak{p}}, \frac{R}{\mathfrak{p}}[X]]$. Then, A is an integral domain, and it is m-formally Noetherian. Therefore, A is Noetherian by [14, Corollary 2.7(3)]. Thus, $(\frac{R}{\mathfrak{p}}, \frac{R}{\mathfrak{p}}[X])$ is an NP. Hence, $\frac{R}{\mathfrak{p}}$ is a field by [22, Corollary 5], so $\mathfrak{p} \in Max(R)$. Therefore, $dim R = 0$. Hence, R is Artinian by [1, Theorem 8.5].

$(2) \Rightarrow (1)$ Assume that R is Artinian. Then, $Max(R)$ is finite by [1, Proposition 8.3]. Let $Max(R) = \{\mathfrak{m}_i \mid i \in \{1,\ldots,n\}\}$. Note that $R_{\mathfrak{m}_i}$ is a local Artinian ring with $\mathfrak{m}_i R_{\mathfrak{m}_i}$ as its unique maximal ideal. As $Nil(R_{\mathfrak{m}_i}) = \mathfrak{m}_i R_{\mathfrak{m}_i}$, $\mathfrak{m}_i R_{\mathfrak{m}_i}$ is nilpotent by [1, Proposition 8.4]. Let $k_i \in \mathbb{N}$ be such that $(\mathfrak{m}_i R_{\mathfrak{m}_i})^{k_i} = (0)$. If $k = max(k_1,\ldots,k_n)$, then $(\mathfrak{m}_i R_{\mathfrak{m}_i})^k = (0)$ for each $i \in \{1,\ldots,n\}$. Let $i \in \{1,\ldots,n\}$. Note that $(R[X])_{R\setminus\mathfrak{m}_i} = R_{\mathfrak{m}_i}[X]$. The proof of Lemma 2.6 shows that $(R_{\mathfrak{m}_i}, R_{\mathfrak{m}_i}[X] = (R[X])_{R\setminus\mathfrak{m}_i})$ is a k-FNP. Hence, $(R,R[X])$ is a k-FNP by Lemma 2.5. □

Let $m \in \mathbb{N}$. It is known that R is a nonnil-m-formally Noetherian ring if and only if I^m is f.g. for each nonnil ideal I of R by [3, Theorem 2.4 ((1) $\Leftrightarrow$ (10))]. In Theorem 2.10, we characterize R such that $(R,R[X])$ is an N-m-FNP, where N-m-FNP stands for nonnil-m-formally Noetherian pair. We use the following lemmas in its proof.

Lemma 2.8. *For a pair of rings* $R \subseteq T$, *if* (R,T) *is an N-*m*-FNP for some* $m \in \mathbb{N}$, *then the following statements hold.*

(1) $(\frac{R}{J \cap R}, \frac{T}{J})$ *is an N-*m*-FNP for any proper ideal* J *of* T.

(2) $(S^{-1}R, S^{-1}T)$ *is an N-*m*-FNP for any m.c. subset* S *of* R.

Proof. If A is a nonnil-m-formally Noetherian ring, then for any proper ideal I of A, $\frac{A}{I}$ is a nonnil-m-formally Noetherian ring by [3, Example 2.6], and $S^{-1}A$ is a nonnil-m-formally Noetherian ring for any m.c. subset S of A by [3, Corollary 2.33]. Hence, using arguments similar to those in the proof of Lemma 2.2, one can prove this lemma. □

An element $r \in R$ is said to be a *regular element* of R if r is not a zero-divisor of R.

Lemma 2.9. *If R is a nonnil-m-formally Noetherian ring, then for any regular element r of R with $Rr \neq R$, $\frac{R}{Rr}$ is Noetherian.*

Proof. As $\frac{R}{Nil(R)}$ is a nonnil-m-formally Noetherian ring by [3, Corollary 2.5], and $\frac{R}{Nil(R)}$ is a reduced ring, we get that it is an m-formally Noetherian ring. Hence, $\frac{R}{Nil(R)}$ is Noetherian by [14, Corollary 2.7(3)], so $\frac{R}{Nil(R)}$ satisfies a.c.c. on radical ideals. As $Nil(R) \subseteq \sqrt{I}$ for any ideal I of R, it follows that R satisfies a.c.c. on radical ideals. Hence, Rr has only a finite number of prime ideals of R minimal over it by [13, Theorem 88]. It follows from [1, Proposition 1.14] and [13, Theorem 10] that $\sqrt{Rr}$ is the intersection of prime ideals of R that are minimal over Rr. Let $\mathfrak{p} \in Spec(R)$. Then, $\frac{R}{\mathfrak{p}}$ is a nonnil-m-formally Noetherian ring, so it is an m-formally Noetherian ring. Since $\frac{R}{\mathfrak{p}}$ is an integral domain, it follows that $\frac{R}{\mathfrak{p}}$ is Noetherian by [14, Corollary 2.7(3)]. As Rr has only a finite number of prime ideals minimal over it, we get that $\frac{R}{\sqrt{Rr}}$ is Noetherian. Since $\sqrt{Rr}$ is a nonnil ideal of R, $(\sqrt{Rr})^m$ is f.g. by [3, Theorem 2.4 ((1) $\Rightarrow$ (10))]. Hence, it follows from [9, Lemma 1.12] that $\frac{R}{Rr}$ is Noetherian. $\square$

Theorem 2.10. *The following statements are equivalent:*

(1) $(R, R[X])$ *is an N-m-FNP for some* $m \in \mathbb{N}$.

(2) R *is Artinian.*

(3) $(R, R[X])$ *is an m-FNP for some* $m \in \mathbb{N}$.

Proof. (1) $\Rightarrow$ (2) Let $\mathfrak{p} \in Spec(R)$. Note that $(\frac{R}{\mathfrak{p}}, \frac{R}{\mathfrak{p}}[X])$ is an m-FNP. As each $A \in [\frac{R}{\mathfrak{p}}, \frac{R}{\mathfrak{p}}[X]]$ is an integral domain, it follows that $(\frac{R}{\mathfrak{p}}, \frac{R}{\mathfrak{p}}[X])$ is an NP by [14, Corollary 2.7(3)]. Therefore, $\frac{R}{\mathfrak{p}}$ is a field, so $\mathfrak{p} \in Max(R)$. Hence, $dimR = 0$. As $R[X]$ is a nonnil-m-formally Noetherian ring, X is a regular element of $R[X]$, it follows from Lemma 2.9 that $\frac{R[X]}{XR[X]}$ is Noetherian. Therefore, R is Noetherian, since $\frac{R[X]}{XR[X]} \cong R$ as rings. Hence, R is Artinian by [1, Theorem 8.5].

(2) $\Rightarrow$ (3) This follows from (2) $\Rightarrow$ (1) of Theorem 2.7.

(3) $\Rightarrow$ (1) This is clear. $\square$

3. When is $(R, R[X])$ an N-LP?

As in Section 2, unless otherwise specified, we use R to denote a ring. We use the abbreviation N-LP (resp., N-SLP) for nonnil-Laskerian pair (resp., nonnil-strongly Laskerian pair). This section aims to determine when $(R, R[X])$ is an N-LP (resp., N-SLP). First, we state and prove some lemmas that we use in the proof of the main result of this section.

Lemma 3.1. *A reduced ring is nonnil-Laskerian (resp., nonnil-strongly Laskerian) if and only if it is Laskerian (resp., strongly Laskerian).*

Proof. As any Laskerian (resp., strongly Laskerian) ring is nonnil-Laskerian (resp., nonnil-strongly Laskerian), the proof of this lemma is complete if we show that a reduced nonnil-Laskerian (resp., nonnil-strongly Laskerian) ring is Laskerian (resp., strongly Laskerian). Let R be a reduced nonnil-Laskerian (resp., nonnil-strongly Laskerian) ring. Hence, if I is any nonzero proper ideal of R, then I admits a primary (resp., strong primary) decomposition. If R is an integral domain, then it is clear that $(0) \in Spec(R)$, so it admits a primary (resp., strong primary) decomposition. If R is not an integral domain, then we can find a and b from $R\backslash\{0\}$ such that $ab = 0$. Since R is reduced, it follows from $ab = 0$ that $Ra \cap Rb = (0)$. As Ra and Rb admit primary (resp., strong primary) decomposition, we get that (0) admits a primary (resp., strong primary) decomposition. Therefore, R is Laskerian (resp., strongly Laskerian). □

If M is a module over R, then $R \times M$, the direct product of R-modules R and M can be made into a ring by defining multiplication as follows. For any $(r_1,m_1),(r_2,m_2) \in R \times M$, define $(r_1,m_1)(r_2,m_2) = (r_1r_2, r_1m_2 + r_2m_1)$. The ring obtained in this way is commutative with identity $(1,0)$, and it is called the ring constructed using *Nagata's principle of idealization*. We denote the ring thus obtained by $R(+)M$. In Example 3.3, we provide a non-reduced ring R such that R is nonnil-strongly Laskerian but not Laskerian. We use the following lemma in the proof of some examples in this paper.

Lemma 3.2. *Let $R = D(+)K$, where D is an integral domain, and K is its quotient field. If I is any nonnil ideal of R, then $(0)(+)K \subset I$.*

Proof. Note that $Nil(R) = (0)(+)K$. Thus, if I is any nonnil ideal of R, then $(d,\alpha) \in I$ for some $d \in D\backslash\{0\}$ and $\alpha \in K$. For any $\beta \in K$, $\beta = d(\frac{\beta}{d})$, so $(0,\beta) = (d,\alpha)(0,\frac{\beta}{d}) \in I$. Hence, $(0)(+)K \subseteq I$. As I is nonnil, $(0)(+)K \neq I$, so $(0)(+)K \subset I$. □

Example 3.3. $R = \mathbb{Z}(+)\mathbb{Q}$ is nonnil-strongly Laskerian but not Laskerian.

Proof. Note that $Nil(R) = (0)(+)\mathbb{Q}$. Let I be any proper nonnil ideal of R. As $\mathbb{Q}$ is the quotient field of $\mathbb{Z}$, it follows from Lemma 3.2 that $(0)(+)\mathbb{Q} \subset I$. Therefore, $I = J(+)\mathbb{Q}$ for some nonzero proper ideal J of $\mathbb{Z}$. Observe that $J = m\mathbb{Z}$ for some $m > 1$. Note that there exist distinct prime numbers $p_1,\ldots,p_k$ and positive integers $t_1,\ldots,t_k$ such that $m = \prod_{i=1}^{k} p_i^{t_i}$. Observe that $m\mathbb{Z} = \bigcap_{i=1}^{k} p_i^{t_i}\mathbb{Z}$ is the strong primary decomposition of $m\mathbb{Z}$ with $p_i^{t_i}\mathbb{Z}$ is a $p_i\mathbb{Z}$-primary ideal of $\mathbb{Z}$ for each $i \in \{1,\ldots,k\}$. Let $i \in \{1,\ldots,k\}$. It is not hard to verify that $\mathfrak{P}_i = p_i\mathbb{Z}(+)\mathbb{Q} \in Spec(R)$, and $\mathfrak{Q}_i = p_i^{t_i}\mathbb{Z}(+)\mathbb{Q}$ is a $\mathfrak{P}_i$-primary ideal of R, and

$\mathfrak{P}_i^{t_i} \subseteq \mathfrak{Q}_i$. Hence, $I = m\mathbb{Z}(+)\mathbb{Q} = \bigcap_{i=1}^k \mathfrak{Q}_i$ is a strong primary decomposition of I in R. Therefore, R is nonnil-strongly Laskerian. Since $\mathbb{Q}$ is a divisible and torsion-free $\mathbb{Z}$-module, and $\mathbb{Z}$ is not a field, it follows that $R = \mathbb{Z}(+)\mathbb{Q}$ is not Laskerian by [19, Lemma 2.5]. $\square$

Lemma 3.4. *The following statements hold.*
(1) *Let $\phi : R \to T$ be an onto homomorphism of rings. If R is nonnil-Laskerian (resp., nonnil-strongly Laskerian), then so is T.*
(2) *If R is nonnil-Laskerian (resp., nonnil-strongly Laskerian), then so is $S^{-1}R$ for any m.c. subset S of R.*

Proof. (1) Let J be any proper nonnil ideal of T. Let $I = \phi^{-1}(J)$. Note that I is a proper ideal of R, $Ker(\phi) \subseteq I$, and $J = \phi(I)$. Since $\phi(Nil(R)) \subseteq Nil(T)$ and $J \not\subseteq Nil(T)$, it follows that $I \not\subseteq Nil(R)$. As R is nonnil-Laskerian (resp., nonnil-strongly Laskerian) by assumption, there exist primary (resp., strongly primary) ideals $\mathfrak{q}_1, \ldots, \mathfrak{q}_k$ of R such that $I = \bigcap_{i=1}^k q_i$. As $Ker(\phi) \subseteq \mathfrak{q}_i$ for each $i \in \{1, \ldots, k\}$, it follows that $\phi(\mathfrak{q}_i)$ is a primary (resp., strongly primary) ideal of T for each $i \in \{1, \ldots, k\}$, and $J = \phi(I) = \bigcap_{i=1}^k \phi(\mathfrak{q}_i)$ is a primary (resp., strong primary) decomposition of J. Therefore, T is nonnil-Laskerian (resp., nonnil-strongly Laskerian).

(2) Let W be any proper nonnil ideal of $S^{-1}R$. Then, $W = S^{-1}I$ for some ideal I of R such that $I \cap S = \emptyset$ by [1, Proposition 3.11(ii)]. Since $Nil(S^{-1}R) = S^{-1}Nil(R)$ by [1, Corollary 3.12] and $W \not\subseteq Nil(S^{-1}R)$, it follows that $I \not\subseteq Nil(R)$. As R is nonnil-Laskerian (resp., nonnil-strongly Laskerian) by assumption, we can find primary (resp., strongly primary) ideals $q_1, \ldots, q_k$ of R such that $I = \bigcap_{i=1}^k \mathfrak{q}_i$. From $I \cap S = \emptyset$, it follows that $\mathfrak{q}_i \cap S = \emptyset$ for at least one $i \in \{1, \ldots, k\}$. Let $A = \{i \in \{1, \ldots, k\} \mid \mathfrak{q}_i \cap S = \emptyset\}$ and let $B = \{1, \ldots, k\} \setminus A$. If $i \in A$, then $S^{-1}\mathfrak{q}_i$ is a primary ideal of $S^{-1}R$ by [1, Proposition 4.8(ii)] and if $j \in B$, then $S^{-1}\mathfrak{q}_j = S^{-1}R$ by [1, Proposition 4.8(i)]. Let $i \in \{1, \ldots, k\}$. If $\mathfrak{q}_i$ contains a power of its radical, then $S^{-1}\mathfrak{q}_i$ will contain a power of its radical. Hence, $W = S^{-1}I = \bigcap_{i \in A} S^{-1}\mathfrak{q}_i$ is a primary (resp., strong primary) decomposition of W in $S^{-1}R$. Therefore, $S^{-1}R$ is nonnil-Laskerian (resp., nonnil-strongly Laskerian). $\square$

If I is any proper ideal of R, then the mapping $\phi : R \to \frac{R}{I}$ defined by $\phi(r) = r + I$ is an onto homomorphism of rings. Hence, if R is nonnil-Laskerian (resp., nonnil-strongly Laskerian), then so is $\frac{R}{I}$ by Lemma 3.4(1).

Lemma 3.5. *Let T be an extension ring of R. The following statements hold.*
(1) *If (R, T) is an N-LP (resp., N-SLP), then $(\frac{R}{J \cap R}, \frac{T}{J})$ is an N-LP (resp., N-SLP) for any proper ideal J of T.*
(2) *If (R, T) is an N-LP (resp., N-SLP), then $(S^{-1}R, S^{-1}T)$ is an N-LP (resp., N-SLP) for any m.c. subset S of R.*

Proof. Using arguments similar to those in the proof of Lemma 2.2, one can prove this lemma with the help of Lemma 3.4. □

Lemma 3.6. *If an ideal I of R admits a primary decomposition, then for any $r \in R$, the increasing sequence of ideals $(I :_R r) \subseteq (I ::_R r^2) \subseteq (I :_R r^3) \subseteq \cdots$ is stationary.*

Proof. For a proof of this lemma, see [15, Proposition 2]. □

Lemma 3.7. *If R is nonnil-Laskerian (resp., nonnil-strongly Laskerian) but not Laskerian (resp., not strongly Laskerian), then $Nil(R) \in Spec(R)$.*

Proof. Assume that R is nonnil-Laskerian (resp., nonnil-strongly Laskerian) but not Laskerian (resp., not strongly Laskerian). Then, there exists an ideal $I \subseteq Nil(R)$ such that I does not admit primary (resp., strong primary) decomposition. We claim that $Nil(R) \in Spec(R)$. Suppose that $Nil(R) \notin Spec(R)$. Then, we can find $a, b \in R \backslash Nil(R)$ such that $ab \in Nil(R)$. Hence, there exists $n \in \mathbb{N}$ such that $(ab)^n = 0$. As $a^n b^n = 0$, it follows that $b^n \in (I :_R a^n)$. Since $b^n \notin Nil(R)$, we get that $(I :_R a^n) \nsubseteq Nil(R)$. As $a^n \notin Nil(R)$, a^n cannot be in I. Hence, $(I :_R a^n) \neq R$, so $(I :_R a^n)$ is a proper nonnil ideal of R. Therefore, $(I :_R a^n)$ admits a primary (resp., strong primary) decomposition. It follows from Lemma 3.6 that the increasing sequence of ideals $((I :_R a^n) :_R a) \subseteq ((I :_R a^n) :_R a^2) \subseteq ((I :_R a^n) :_R a^3) \subseteq \cdots$ is stationary. Thus, there exists $j \in \mathbb{N}$ such that $((I :_R a^n) :_R a^t) = ((I :_R a^n) :_R a^j)$ for all $t \geq j$. Hence, $(I :_R a^{n+j}) = (I :_R a^{n+t})$ for all $t \geq j$, since $((I :_R r_1) :_R r_2) = (I :_R r_1 r_2)$ for any $r_1, r_2 \in R$. We claim that $I = (I + Ra^{n+j}) \cap (I :_R a^{n+j})$. It is clear that $I \subseteq (I + Ra^{n+j}) \cap (I :_R a^{n+j})$. Let $x \in (I + Ra^{n+j}) \cap (I :_R a^{n+j})$. Then, $x = y + ra^{n+j}$ for some $y \in I$ and $r \in R$. Hence, $xa^{n+j} = ya^{n+j} + ra^{2(n+j)}$. As $xa^{n+j}, y \in I$, we obtain that $ra^{2(n+j)} \in I$. So, $r \in (I :_R a^{2(n+j)}) = (I :_R a^{n+j})$. Thus, $ra^{n+j} \in I$, so $x = y + ra^{n+j} \in I$. Therefore, $(I + Ra^{n+j}) \cap (I :_R a^{n+j}) \subseteq I$. Hence, $I = (I + Ra^{n+j}) \cap (I :_R a^{n+j})$. As $a^{n+j}, b^{n+j} \in R \backslash Nil(R)$, $a^{n+j} \in I + Ra^{n+j}$, and $b^{n+j} \in (I :_R a^{n+j})$, we obtain that $I + Ra^{n+j}$ and $(I :_R a^{n+j})$ are nonnil ideals of R. Since $I \subseteq Nil(R)$ and a is not a unit in R, it follows that $I + Ra^{n+j} \neq R$. As $a^{n+j} \notin I$, we get that $(I :_R a^{n+j}) \neq R$. Thus, $I = (I + Ra^{n+j}) \cap (I :_R a^{n+j})$ is the intersection of two proper nonnil ideals of R. Since any proper nonnil ideal of R admits a primary (resp., strong primary) decomposition, we obtain that I admits a primary (resp., strong primary) decomposition, a contradiction to the assumption that I does not admit primary (resp., strong primary) decomposition. Hence, $Nil(R) \in Spec(R)$. □

We use the following proposition in the proof of Theorem 3.9.

Proposition 3.8. *If $Max(R) = \{\mathfrak{m}\}$ with $\mathfrak{m} \neq (0)$ and $dim R = 0$, then $(R, R[X])$ is an N-LP. Moreover, $(R, R[X])$ is an N-SLP if and only if $\mathfrak{m}^n = (0)$ for some $n \in \mathbb{N} \backslash \{1\}$.*

Proof. Assume that R is quasi-local with $\mathfrak{m}$ as its unique maximal ideal, $\mathfrak{m} \neq (0)$, and $dimR = 0$. Hence, $Spec(R) = \{\mathfrak{m}\}$, so $Nil(R) = \mathfrak{m}$ by [1, Proposition 1.8]. It follows from [1, Exercise 2(ii), p.11] that $Nil(R[X]) = Nil(R)[X]$. Thus, $Nil(R[X]) = \mathfrak{m}[X]$. Observe that $\frac{R[X]}{\mathfrak{m}[X]}$ is ring-isomorphic to $\frac{R}{\mathfrak{m}}[X]$. As $\frac{R}{\mathfrak{m}}$ is a field, we get that $\frac{R[X]}{\mathfrak{m}[X]}$ is a principal ideal domain (P.I.D.). Therefore, $Spec(R[X]) = \{\mathfrak{m}[X]\} \cup Max(R[X])$. As $dim\frac{R[X]}{\mathfrak{m}[X]} = 1$, we get that $dimR[X] = 1$. Let $A \in [R, R[X]]$. Let $\overline{A}$ denote the integral closure of A in $R[X]$. As each element of $\mathfrak{m}[X]$ is nilpotent, it follows that each element of $\mathfrak{m}[X]$ is integral over R. Therefore, $R + \mathfrak{m}[X]$ is integral over R, so integral over A. Hence, $R + \mathfrak{m}[X] \subseteq \overline{A}$. Either $\overline{A} = R + \mathfrak{m}[X]$ or $R + \mathfrak{m}[X] \subset \overline{A}$. If $\overline{A} = R + \mathfrak{m}[X]$, then $\overline{A}$ is quasi-local with $\mathfrak{m}[X]$ as its unique maximal ideal. Therefore, A is quasi-local with $\mathfrak{m}[X] \cap A$ as its unique maximal ideal. As each element of $\mathfrak{m}[X] \cap A$ is nilpotent, it follows that A does not have any proper nonnil ideal. Hence, A is trivially a nonnil-Laskerian (resp., nonnil-strongly Laskerian) ring. If $R + \mathfrak{m}[X] \subset \overline{A}$, then it follows as in the proof of [20, Theorem 1.1 (Case 2 of its proof)] that $R[X]$ is a finite integral extension of A. Since $dimR[X] = 1$, $dimA = 1$ by [5, 11.8]. Note that $Nil(A) = \mathfrak{m}[X] \cap A$ is the only minimal prime ideal of A. Thus, $Spec(A) = \{\mathfrak{m}[X] \cap A\} \cup Max(A)$. If $\mathfrak{p} \in Spec(A)$, then there exists $\mathfrak{q} \in Spec(R[X])$ such that $\mathfrak{q} \cap A = \mathfrak{p}$ by [1, Theorem 5.10]. Let I be any proper nonnil ideal of A. Then, $I \not\subseteq \mathfrak{m}[X] \cap A$. Thus, any prime ideal of A that contains I is necessarily maximal. Note that $IR[X] \not\subseteq \mathfrak{m}[X]$. Hence, $IR[X]$ is a proper nonnil ideal of $R[X]$. Hence, $\frac{IR[X]+\mathfrak{m}[X]}{\mathfrak{m}[X]}$ is a nonzero proper ideal of the P.I.D. $\frac{R[X]}{\mathfrak{m}[X]}$, so it can be contained in only a finite number of maximal ideals of $\frac{R[X]}{\mathfrak{m}[X]}$. Therefore, $IR[X]$ can be contained in only a finite number of maximal ideals of $R[X]$. Let $\mathfrak{M} \in Max(A)$ be such that $I \subseteq \mathfrak{M}$. Let $\mathfrak{P} \in Spec(R[X])$ be such that $\mathfrak{P} \cap A = \mathfrak{M}$. Note that $IR[X] \subseteq \mathfrak{P}$, so $\mathfrak{P} \in Max(R[X])$. Since $IR[X]$ is contained in only a finite number of maximal ideals of $R[X]$, we obtain that I can be contained in only a finite number of maximal ideals of A. Let $\{\mathfrak{M} \in Max(A) \mid \mathfrak{M} \supseteq I\} = \{\mathfrak{M}_i \mid i \in \{1, \ldots, t\}\}$. Let $i \in \{1, \ldots, t\}$. Let $f_i : A \to A_{\mathfrak{M}_i}$ be the usual homomorphism of rings given by $f_i(a) = \frac{a}{1}$. Observe that $Spec(A_{\mathfrak{m}_i}) = \{(\mathfrak{m}[X] \cap A)A_{\mathfrak{M}_i}, \mathfrak{M}_i A_{\mathfrak{M}_i}\}$. As $I \not\subseteq \mathfrak{m}[X] \cap A$, it follows that $\mathfrak{M}_i A_{\mathfrak{M}_i}$ is the only prime ideal of $A_{\mathfrak{M}_i}$ that contains $IA_{\mathfrak{M}_i}$. Hence, $\sqrt{IA_{\mathfrak{M}_i}} = \mathfrak{M}_i A_{\mathfrak{M}_i}$ by [1, Proposition 1.14]. Therefore, $IA_{\mathfrak{m}_i}$ is a $\mathfrak{M}_i A_{\mathfrak{M}_i}$-primary ideal of $A_{\mathfrak{M}_i}$ by [1, Proposition 4.2]. So, $f_i^{-1}(IA_{\mathfrak{M}_i})$ is a $\mathfrak{M}_i$-primary ideal of A. It is clear that $I = \bigcap_{i=1}^{t} f_i^{-1}(IA_{\mathfrak{M}_i})$ is a primary decomposition of I in A. Hence, we obtain that any proper nonnil ideal of A admits a primary decomposition, so $(R, R[X])$ is an N-LP.

Assume that $(R, R[X])$ is an N-SLP. So, $R[X]$ is nonnil-strongly Laskerian.

As $Nil(R[X]) = \mathfrak{m}[X]$ and $X \in \sqrt{XR[X]}$, we get that $\mathfrak{m}[X]+XR[X] \subseteq \sqrt{XR[X]}$. Since $\mathfrak{m}[X]+XR[X] \in Max(R[X])$, it follows that $\sqrt{XR[X]} = \mathfrak{m}[X]+XR[X]$. Hence, $XR[X]$ is a $\mathfrak{m}[X]+XR[X]$-primary ideal of $R[X]$ by [1, Proposition 4.2]. As $R[X]$ is nonnil-strongly Laskerian and $XR[X]$ is a nonnil ideal of $R[X]$, there exists $n \in \mathbb{N}$ such that $(\mathfrak{m}[X]+XR[X])^n \subseteq XR[X]$. Hence, $\mathfrak{m}^n \subseteq XR[X] \cap R = (0)$. Thus, $\mathfrak{m}^n = (0)$. Since $\mathfrak{m} \neq (0)$ by assumption, $n \in \mathbb{N}\backslash\{1\}$.

Conversely, assume that $\mathfrak{m}^n = (0)$ for some $n \in \mathbb{N}\backslash\{1\}$. It is shown in the first paragraph of this proof that $(R,R[X])$ is an N-LP. Let $A \in [R,R[X]]$. As $(R,R[X])$ is an N-LP, A is nonnil-Laskerian. We next show that any primary ideal $\mathfrak{q}$ of A with $\mathfrak{q} \not\subseteq Nil(A)$ is strongly primary. Let $\overline{A}$ denote the integral closure of A in $R[X]$. It is noted in the first paragraph of this proof that either $\overline{A} = R+\mathfrak{m}[X]$ or $\overline{A} \supset R+\mathfrak{m}[X]$. If $\overline{A} = R+\mathfrak{m}[X]$, then it is verified in the first paragraph of this proof that A has no proper nonnil ideal. Assume that $\overline{A} \supset R+\mathfrak{m}[X]$. Then, it is noted in the first paragraph of this proof that $R[X]$ is a finite integral extension of A and $dimR[X] = 1 = dimA$. Also, $Spec(A) = \{\mathfrak{m}[X] \cap A\} \cup Max(A)$. Observe that $Nil(A) = \mathfrak{m}[X] \cap A$. Let $\mathfrak{q}$ be a primary ideal of A such that $\mathfrak{q} \not\subseteq Nil(A)$. Then $\sqrt{q} = \mathfrak{P}$ for some $\mathfrak{P} \in Max(A)$. Since $R[X]$ is a finitely generated A-module, $\frac{R[X]}{\mathfrak{m}[X]}$ is a finitely generated $\frac{A}{\mathfrak{m}[X]\cap A}$-module. As $\frac{R[X]}{\mathfrak{m}[X]}$ is Noetherian, it follows from Eakin's Theorem [4, Theorem 2] that $\frac{A}{\mathfrak{m}[X]\cap A}$ is Noetherian. Hence, $\frac{\mathfrak{P}}{\mathfrak{m}[X]\cap A}$ is a f.g. ideal of $\frac{A}{\mathfrak{m}[X]\cap A}$. Therefore, there exist $f_1(X),\ldots,f_k(X) \in \mathfrak{P}\backslash(\mathfrak{m}[X] \cap A)$ such that $\mathfrak{P} = (\sum_{i=1}^{k} Af_i(X)) + \mathfrak{m}[X] \cap A$. From $\sqrt{q} = \mathfrak{P}$, it follows that there exists $t \in \mathbb{N}$ such that $(\sum_{i=1}^{k} Af_i(X))^t \subseteq \mathfrak{q}$. From $\mathfrak{m}^n = (0)$, it follows that $(\mathfrak{m}[X] \cap A)^n = (0) \subseteq \mathfrak{q}$. Therefore, $\mathfrak{P}^{n+t} = ((\sum_{i=1}^{k} Af_i(X)) + \mathfrak{m}[X] \cap A)^{t+n} \subseteq \mathfrak{q}$. Therefore, A is nonnil-strongly Laskerian, so $(R,R[X])$ is an N-SLP. □

In the following theorem, we characterize R such that $(R,R[X])$ is an N-LP.

Theorem 3.9. *The following statements are equivalent:*

(1) $(R,R[X])$ *is an N-LP.*

(2) *Either R is Artinian or $(R,\mathfrak{m})$ is quasi-local with $\mathfrak{m} \neq (0)$ and $dimR = 0$.*

Proof. (1) ⇒ (2) Assume that $(R,R[X])$ is an N-LP. If A is Laskerian for each $A \in [R,R[X]]$, then R is Artinian by [20, Theorem 1.1]. Suppose that A is not Laskerian for some $A \in [R,R[X]]$. Then, it follows from Lemma 3.1 that $Nil(A) \neq (0)$ and $Nil(A) \in Spec(A)$ by Lemma 3.7. Let $f(X) = \sum_{i=0}^{n} r_iX^i \in Nil(A)\backslash\{0\}$. Note that $r_i \in Nil(R)$ for each $i \in \{0,\ldots,n\}$ by [1, Exercise 2(ii), p.11]. As $r_i \neq 0$ for at least one $i \in \{0,\ldots,n\}$, it follows that $Nil(R) \neq (0)$. Since $Nil(R) = Nil(A) \cap R$, we obtain that $Nil(R) \in Spec(R)$. Therefore, $\frac{R}{Nil(R)}$ is an integral domain. Note that $Nil(R[X]) = Nil(R)[X]$ by [1, Exercise 2 (ii), p.11], and it is clear that $Nil(R)[X] \cap R = Nil(R)$. It follows from Lemma 3.5(1) that $(\frac{R}{Nil(R)},\frac{R[X]}{Nil(R)[X]})$ is an N-LP. Since $\frac{R[X]}{Nil(R)[X]}$ is ring isomorphic to $\frac{R}{Nil(R)}[X]$, we

get that $(\frac{R}{Nil(R)}, \frac{R}{Nil(R)}[X])$ is an N-LP. As each $A \in [\frac{R}{Nil(R)} \cdot \frac{R}{Nil(R)}[X]]$ is an integral domain, it follows from Lemma 3.1 that $(\frac{R}{Nil(R)}, \frac{R}{Nil(R)}[X])$ is an LP. Hence, $\frac{R}{Nil(R)}$ is a field by [20, Proposition 1,5]. Therefore, $Nil(R) \in Max(R)$. Let us denote $Nil(R)$ by $\mathfrak{m}$. It is clear that $(R, \mathfrak{m})$ is quasi-local and $\mathfrak{m} \neq (0)$. Since $Spec(R) = \{\mathfrak{m}\}$, it follows that $dimR = 0$.
$(2) \Rightarrow (1)$ If R is Artinian, then $(R, R[X])$ is an LP by [20, Theorem 1.1]. If $(R, \mathfrak{m})$ is quasi-local with $\mathfrak{m} \neq (0)$ and $dimR = 0$, then $(R, R[X])$ is an N-LP by Proposition 3.8. $\square$

Theorem 3.10. *The following statements are equivalent:*

(1) $(R, R[X])$ *is an N-SLP.*

(2) *Either R is Artinian or $(R, \mathfrak{m})$ is quasi-local with $\mathfrak{m} \neq (0)$ but $\mathfrak{m}^n = (0)$ for some $n \in \mathbb{N} \backslash \{1\}$.*

Proof. $(1) \Rightarrow (2)$ Assume that $(R, R[X])$ is an N-SLP. Therefore, $(R, R[X])$ is an N-LP. Therefore, either R is Artinian or $(R, \mathfrak{m})$ is quasi-local with $\mathfrak{m} \neq (0)$ and $dimR = 0$ by $(1) \Rightarrow (2)$ of Theorem 3.9. By Proposition 3.8, $\mathfrak{m}^n = (0)$ for some $n \in \mathbb{N} \backslash \{1\}$.

$(2) \Rightarrow (1)$ If R is Artinian then $(R, R[X])$ is an SLP by [20, Theorem 1.1]. If $(R, \mathfrak{m})$ is quasi-local with $\mathfrak{m} \neq (0)$ but $\mathfrak{m}^n = (0)$ for some $n \in \mathbb{N} \backslash \{1\}$, then it is clear that $dimR = 0$. Hence, we obtain from Proposition 3.8 that $(R, R[X])$ is an N-SLP. $\square$

The following example illustrates that an N-LP of the form $(R, R[X])$ can fail to be an N-SLP.

Example 3.11. Let $T = K[X_1, X_2, X_3, \ldots]$ be the polynomial ring in an infinite number of variables $\{X_i \mid i \in \mathbb{N}\}$ over a field K. Let I be the ideal of T given by $I = \sum_{n \in \mathbb{N}} TX_n^n$. If $R = \frac{T}{I}$, then $(R, R[X])$ is an N-LP but not an N-SLP, where $R[X]$ is the polynomial ring in one variable X over R.

Proof. It is clear that $Nil(R) = \sum_{n \in \mathbb{N}} R(X_n + I)$, $Nil(R) \in Max(R)$, and $Nil(R) \neq (0 + I)$. With $\mathfrak{m} = Nil(R)$, it follows that $(R, \mathfrak{m})$ is quasi-local with $\mathfrak{m} \neq (0 + I)$ and $dimR = 0$. Therefore, $(R, R[X])$ is an N-LP by Proposition 3.8. It is easy to show that $\mathfrak{m}^k \neq (0 + I)$ for all $k \in \mathbb{N}$. Hence, $(R, R[X])$ is not an N-SLP by Proposition 3.8. $\square$

4. When is $(R, R[X])$ an N-accr pair?

We use R to denote a ring. We use the abbreviation N-accr pair for nonnil-accr pair. This section aims to characterize R such that $(R, R[X])$ is an N-accr pair. First, we state and prove some preliminary results that we use in the proof of the main result of this section.

Remark 4.1. If R is nonnil-Laskerian, then any proper nonnil ideal of R admits a primary decomposition. Hence, for any proper nonnil ideal I of R and for any $r \in R$, the increasing sequence of ideals $(I :_R r) \subseteq (I :_R r^2) \subseteq (I :_R r^3) \subseteq \cdots$ is stationary by [15, Proposition 2]. Thus, R satisfies nonnil-accr*, so R satisfies nonnil-accr. Therefore, any nonnil-Laskerian ring satisfies nonnil-accr.

The following example illustrates that a ring satisfying nonnil-accr can fail to be nonnil-Laskerian.

Example 4.2. Let L be the field of algebraic numbers (that is, L is the algebraic closure of $\mathbb{Q}$) and let A be the ring of all algebraic integers. The ring $R = A(+)L$ satisfies nonnil-accr, but R is not nonnil-Laskerian.

Proof. Note that A is the integral closure of $\mathbb{Z}$ in L, and L is the quotient field of A. Since $dim\mathbb{Z} = 1$, $dimA = 1$ by [5, 11.8]. Hence, A satisfies (accr) by [15, Theorem 6]. Let J be any proper nonnil ideal of R. As L is the quotient field of A, it follows from Lemma 3.2 that $(0)(+)L \subset J$. Therefore, $J = I(+)L$ for some nonzero proper ideal I of A. Let $r = (b,\gamma)$ be any element of R. Note that for all $n \in \mathbb{N}$, $r^n = (b^n, \gamma_n)$ for some $\gamma_n \in L$ with $\gamma_1 = \gamma$. Let $n \geq 1$. We claim that $(J :_R r^n) = (I :_A b^n)(+)L$. As $(J :_R r^n)$ is a nonnil ideal of R, $(J :_R r^n) = I_n(+)L$ for some nonzero ideal I_n of A. Let $y \in A$. Note that $y \in I_n$ if and only if $(y,0) \in (J :_R r^n)$ if and only if $(y,0)r^n = (y,0)(b^n,\gamma_n) \in J = I(+)L$ if and only if $yb^n \in I$ if and only if $y \in (I :_A b^n)$. Therefore, $I_n = (I :_A b^n)$, so $(J :_R r^n) = (I :_A b^n)(+)L$. Since A satisfies accr, the increasing sequence of ideals $(I :_A b) \subseteq (I :_A b^2) \subseteq (I :_A b^3) \subseteq \cdots$ is stationary. Therefore, the increasing sequence of ideals $(J :_R r) \subseteq (J :_R r^2) \subseteq (J :_R r^3) \subseteq \cdots$ is stationary. Hence, R satisfies nonnil-accr*, so R satisfies nonnil-accr.

We next verify that R is not nonnil-Laskerian. Note that $Nil(R) = (0)(+)L$ is a prime ideal of R and $Spec(R) = \{(0)(+)L\} \cup \{\mathfrak{M}(+)L \mid \mathfrak{M} \in Max(A)\}$. Let $J = 2A(+)L$. It is clear that J is a proper nonnil ideal of R. It is known that any nonzero non-unit of A belongs to uncountably many maximal ideals of A, see [5, Proposition 42.8(i)]. Therefore, J has an uncountable number of prime ideals minimal over it. Hence, J cannot admit a primary decomposition by [1, Propositions 4.5 and 4.6]. Therefore, R is not nonnil-Laskerian. $\square$

If R satisfies (accr), then it is clear that R satisfies nonnil-accr. The following example illustrates that a ring satisfying nonnil-accr can fail to satisfy (accr).

Example 4.3. Let V be an infinite dimensional vector space over a field K. Let $T = K(+)V$ be the ring obtained by using Nagata's principle of idealization. Let $R = T[X]$ be the polynomial ring in one variable X over T. The ring R satisfies nonnil-accr, but R does not satisfy (accr).

Proof. Note that T is a quasi-local ring with $\mathfrak{m} = (0)(+)V$ as its unique maximal ideal and $\mathfrak{m}^2 = (0)(+)(0)$. Thus, $Spec(T) = \{\mathfrak{m}\}$. Therefore, $dimT = 0$. It follows from Proposition 3.8 that $(T, R = T[X])$ is an N-LP. Thus, $R = T[X]$ is nonnil-Laskerian, so R satisfies nonnil-accr by Remark 4.1. Since V is an infinite dimensional vector space over the field K, it follows that T is not Noetherian. Therefore, R does not satisfy (accr) by [16, Theorem 2]. $\square$

In the following proposition, we show that the properties (accr) and nonnil-accr are equivalent in the case of a reduced ring.

Proposition 4.4. *If R is reduced, then R satisfies (accr) if and only if R satisfies nonnil-accr.*

Proof. If a ring T (T can be non-reduced) satisfies (accr), then T satisfies nonnil-accr. Assume that R is reduced, and it satisfies nonnil-accr. As (0) is the only ideal that is not nonnil, for any nonzero ideal I of R and for any $r \in R$, the increasing sequence of ideals $(I :_R r) \subseteq (I :_R r^2) \subseteq (I :_R r^3) \subseteq \cdots$ is stationary. Thus, to prove R satisfies (accr), it is enough to show that the increasing sequence of ideals $((0) :_R r) \subseteq ((0) :_R r^2) \subseteq ((0) :_R r^3) \subseteq \cdots$ is stationary for any $r \in R$. If R is an integral domain, then $(0) \in Spec(R)$, so $((0) :_R r^n) = (0)$ for any $n \in \mathbb{N}$ if $r \neq 0$, and $((0) :_R r^n) = R$ for all $n \in \mathbb{N}$ if $r = 0$. If R is not an integral domain, then we can find a and b from $R \backslash \{0\}$ such that $ab = 0$. Since R is reduced, from $ab = 0$, it follows that $Ra \cap Rb = (0)$. As R satisfies nonnil-accr, for any $r \in R$, the increasing sequence of ideals $(Ra :_R r) \subseteq (Ra : r^2) \subseteq (Ra :_R r^3) \subseteq \cdots$ and the increasing sequence of ideals $(Rb :_R r) \subseteq (Rb :_R r^2) \subseteq (Rb :_R r^3) \subseteq \cdots$ are stationary. Hence, we can find positive integers k and t such that $(Ra :_R r^n) = (Ra :_R r^k)$ for all $n \geq k$ and $(Rb :_R r^n) = (Rb :_R r^t)$ for all $n \geq t$. Hence, $((0) :_R r^n) = ((0) :_R r^{k+t})$ for all $n \geq k+t$. Thus, for any $r \in R$, the increasing sequence of ideals $((0) :_R r) \subseteq ((0) :_R r^2) \subseteq ((0) :_R r^3) \subseteq \cdots$ is stationary. Therefore, R satisfies (accr*), so R satisfies (accr). $\square$

Proposition 4.5. *If R satisfies nonnil-accr, but it does not satisfy (accr), then $Nil(R) \in Spec(R)$.*

Proof. Assume that R satisfies nonnil-accr, but it does not satisfy (accr). Hence, there exists an ideal I of R with $I \subseteq Nil(R)$ and $r \in R$ such that the increasing sequence of ideals $(I :_R r) \subseteq (I :_R r^2) \subseteq (I :_R r^3) \subseteq \cdots$ is not stationary. Suppose that $Nil(R) \notin Spec(R)$. Then we can find a and b from $R \backslash Nil(R)$ such that $ab \in Nil(R)$. Let $n \in \mathbb{N}$ be such that $a^n b^n = 0$. Proceeding as in the proof of Lemma 3.7, it follows that $(I :_R a^n) \not\subseteq Nil(R)$. Since R satisfies nonnil-accr, the increasing sequence of ideals $((I :_R a^n) :_R a) \subseteq ((I :_R a^n) :_R a^2) \subseteq ((I :_R a^n) :_R a^3) \subseteq \cdots$ is stationary. Hence, it follows as in the proof of Lemma 3.7 that there exists $j \in \mathbb{N}$ such that $(I :_R a^{n+t}) = (I :_R a^{n+j})$ for all $t \geq j$. It can be shown as in the proof of Lemma 3.7 that $I+Ra^{n+j}, (I :_R a^{n+j})$ are nonnil ideals of R and $I=(I+Ra^{n+j})\cap(I :_R a^{n+j})$. Let us denote $I+Ra^{n+j}$ by I_1 and $(I :_R a^{n+j})$ by I_2. Since R satisfies nonnil-accr, the increasing sequence of ideals $(I_i :_R r) \subseteq (I_i :_R r^2) \subseteq (I_i :_R r^3) \subseteq \cdots$ is stationary for each $i \in \{1,2\}$. As $I = I_1 \cap I_2$, it follows that the increasing sequence of ideals $(I :_R r) \subseteq (I :_R r^2) \subseteq (I :_R r^3) \subseteq \cdots$ is stationary, a contradiction to the assumption that $(I :_R r) \subseteq (I :_R r^2) \subseteq (I :_R r^3) \subseteq \cdots$ is not stationary. Therefore, $Nil(R) \in Spec(R)$. $\square$

Lemma 4.6. *The following statements hold.*

(1) *If $\phi : R \to T$ is an onto homomorphism of rings, and if R satisfies nonnil-accr, then T satisfies nonnil-accr.*

(2) *If R satisfies nonnil-accr, then $S^{-1}R$ satisfies nonnil-accr for any m.c. subset S of R.*

Proof. This lemma can be proved using standard arguments, so we omit its proof. $\square$

If R satisfies nonnil-accr, then for any proper ideal I of R, $\frac{R}{I}$ satisfies nonnil-accr, since the mapping $\phi : R \to \frac{R}{I}$ given by $\phi(r) = r+I$ is an onto homomorphism of rings.

Lemma 4.7. *Let T be an extension ring of R. Then the following statements hold.*

(1) *If (R,T) is an N-accr pair, then $(\frac{R}{J\cap R}, \frac{T}{J})$ is an N-accr pair for any proper ideal J of T.*

(2) *If (R,T) is an N-accr pair, then $(S^{-1}R, S^{-1}T)$ is an N-accr pair for any m.c. subset S of R.*

Proof. Using arguments similar to those in the proof of Lemma 2.2, one can prove this lemma with the help of Lemma 4.6. $\square$

Theorem 4.8. *The following statements are equivalent:*

(1) *$(R,R[X])$ is an N-LP.*

(2) *$(R,R[X])$ is an N-accr pair.*

(3) *Either R is Artinian or $(R,\mathfrak{m})$ is a quasi-local ring with $\mathfrak{m} \neq (0)$ and $dim R = 0$.*

Proof. (1) $\Rightarrow$ (2) Assume that $(R,R[X])$ is an N-LP. It is noted in Remark 4.1 that any nonnil-Laskerian ring satisfies nonnil-accr. Therefore, we obtain that $(R,R[X])$ is an N-accr pair.

(2) $\Rightarrow$ (3) Assume that $(R,R[X])$ is an N-accr pair. If A satisfies (accr) for each $A \in [R,R[X]]$, then R is Artinian by [21, Theorem 1.1 ((1) $\Rightarrow$ (2))]. Assume that there exists $A \in [R,R[X]]$ such that A satisfies nonnil-accr, but A

does not satisfy (accr). It follows from Proposition 4.4 that $Nil(A) \neq (0)$ and $Nil(A) \in Spec(A)$ by Proposition 4.5, and it follows as in the proof of $(1) \Rightarrow (2)$ of Theorem 3.9 that $Nil(R) \neq (0)$ and $Nil(R) = Nil(A) \cap R \in Spec(R)$. As $Nil(R[X]) \cap R = Nil(R)$ and $(R, R[X])$ is an N-accr pair, $(\frac{R}{Nil(R)}, \frac{R[X]}{Nil(R[X])})$ is an N-accr pair. From $Nil(R) \in Spec(R)$, it follows that each $A \in [\frac{R}{Nil(R)}, \frac{R[X]}{Nil(R[X])}]$ is an integral domain, so A satisfies (accr) by Proposition 4.4. Since $\frac{R[X]}{Nil(R[X])}$ is ring-isomorphic to $\frac{R}{Nil(R)}[X]$, it follows that $(\frac{R}{Nil(R)}, \frac{R}{Nil(R)}[X])$ is an accr pair. Therefore, $\frac{R}{Nil(R)}$ is a field by [21, Proposition 1.3]. Hence, $Nil(R) \in Max(R)$. Let us denote $Nil(R)$ by $\mathfrak{m}$. Therefore, $(R, \mathfrak{m})$ is quasi-local with $\mathfrak{m} \neq (0)$ and $dimR = 0$.

$(3) \Rightarrow (1)$ If R is Artinian, then $(R, R[X])$ is an LP by [20, Theorem 1.1], so $(R, R[X])$ is an N-LP. If $(R, \mathfrak{m})$ is quasi-local with $\mathfrak{m} \neq (0)$ and $dimR = 0$, then $(R, R[X])$ is an N-LP by Proposition 3.8. $\square$

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