LE MATEMATICHE
Vol. LXXXI (2026) – Issue I, pp. 87–108
doi: 10.4418/2026.81.1.5

ON THE COMPLEMENTS OF UNION OF OPEN BALLS AND RELATED SET CLASSES

M. LONGINETTI - P. MANSELLI - A. VENTURI

1. Introduction

Looking for classes of sets more general than the convex sets, one has to look for properties of the convex sets maintained and for properties of the convex sets generalized.

Here a family of sets is studied: the sets that are the complements of a non empty union of open balls of radius R, generalizing the property of the closed convex sets as complements of union of open half spaces. As the family of convex sets, this family of sets is closed with respect to the intersection (see Proposition 3.5 and Remark 4.3).

This family has been introduced by Perkal [10], used in Walther [16], in Cuevas, Fraiman, Pateiro-López [1] and in [6]; the family of these sets is called R-bodies there.

Here further properties of the R-bodies are provided, also in comparison with other classes of sets.

In §3 the R-bodies are introduced; some of their properties, proved in [10] and in [6], are recalled. Other formulas for the R-hulloid of a body E, the minimal R-body which contains E, denoted by $co_R(E)$, are made explicit in Theorem 3.6 and Corollary 5.2.

Received on March 4, 2025

AMS 2010 Subject Classification: Primary: 52A01; Secondary: 52A30
Keywords: generalized convexity, convex hull, rolling bodies, reach positive sets, support cones

In §3.1 it has been introduced the following definition: an open ball B of radius R is R-supporting the body A at a, when $a \in \partial B$ and B is in the complement of A. The set $\mathcal{N}_R(A,a)$ of the unit vectors v, normal to a R supporting ball B at $a \in \partial A$ is introduced, generalizing the property of supporting half space for convex sets. The properties of the intersection of A with the boundary of a R-supporting ball at a boundary point of $co_R(A)$, are studied in Theorem 3.9.

Let us call A a R-supported body if A is a body and $\mathcal{N}_R(A,a)$ is non empty for all $a \in \partial A$. The family of R-supported bodies contains the family of R-bodies. In [1] these sets were called R-rolling sets; in [1, Proposition 2] the authors introduced a body with a R-supporting unit vectors at every boundary point which is not a R-body, see Remark 3.8 for more examples.

In §4 the R-bodies and the R-supported bodies have been matched with the sets of reach greater or equal than R and with bodies E having balls rolling freely inside E and in the complement of E, [16].

In §5 the family of R-cones is introduced. An R-cone with vertex x is the R-body obtained by intersection of the complements of a given family of open balls of radius R, having x on their boundaries. If $R \to \infty$ the R-cones have limit convex cones. In Corollary 5.3 a representation of a R-body A by R- cones is given.

In §5.1 the relations between $\mathcal{N}_R(A,a)$, the tangent set $\mathrm{Tan}(A,a)$ and the normal cone $\mathrm{Nor}(A,a)$ are investigated, see Theorems 5.4 and 5.5.

A characterization of the family of sets of reach greater or equal than R is obtained in Theorems 5.6, 5.7 through the property of convexity of $\mathcal{N}_R(A,a)$.

2. Definitions and preliminaries

As in [13], a non empty closed subset of the Euclidean space $\mathbb{R}^d$, $d \geq 1$, will be called a *body*.

Let $K \subset \mathbb{R}^d$; $int(K)$ will be the interior of K, ∂K the boundary of K, $cl(K)$ or $\overline{K}$ the closure of K, $K^c = \mathbb{R}^d \setminus K$. For every set $K \subset \mathbb{R}^d$, $co(K)$ is the convex hull of K. The elements of $\mathbb{R}^d$ are called vectors, the zero vector of $\mathbb{R}^d$ is denoted by o. Let $B(z,\rho) = \{x \in \mathbb{R}^d : |x-z| < \rho\}, S^{d-1} = \partial B(o,1)$ and let $D(z,\rho) = cl(B(z,\rho))$. The notations $B_\rho(x), D_\rho(x)$ will also be used respectively for open, closed balls of radius ρ centered at x. The usual scalar product between vectors $u,v \in \mathbb{R}^d$ will be denoted by $\langle u,v \rangle$. The closed segment with end points $x_1,x_2 \in \mathbb{R}^d$ is denoted by $[x_1,x_2]$. The Minkowski sum $E+F$ of two sets E and F is given by $\{x+y : x \in E, y \in F\}$. A cone C is a subset of $\mathbb{R}^d$ with the following property: when $x \in C$, then $\forall \lambda > 0$ $\lambda x \in C$. A closed cone C contains its vertex o. The set $C \cap (-C)$ is the apex set of a closed cone C. C is a closed pointed cone if $C \cap (-C) = \{o\}$.

Let A be a body. Let $q \in A$; the *tangent cone* of A at q is defined in [4] as:

$$\mathrm{Tan}\,(A,q) = \{v \in \mathbb{R}^d : \forall \varepsilon > 0\,, \exists x \in A \cap B_\varepsilon(q)\ , \exists r > 0\,\text{s.t.}\ |r(x-q) - v| < \varepsilon\}.$$

Let us recall that if $\mathrm{Tan}\,(A,q) \neq \{0\}$ then

$$S^{d-1} \cap \mathrm{Tan}\,(A,q) = \bigcap_{\varepsilon>0} cl(\{\frac{x-q}{|x-q|} : x \in A \cap B(q,\varepsilon), x \neq q\}).$$

The *normal cone* at q to A is the non empty closed convex cone, given by:

$$\mathrm{Nor}\,(A,q) = \{u \in \mathbb{R}^d : \langle u,v\rangle \leq 0 \quad \forall v \in \mathrm{Tan}\,(A,q)\}. \tag{1}$$

The *dual* cone of a cone K is

$$K^\star = \{y \in \mathbb{R}^d : \langle y,x\rangle \geq 0 \quad \forall x \in K\}.$$

Thus

$$\mathrm{Nor}\,(A,q) = -\{\mathrm{Tan}\,(A,q)\}^\star. \tag{2}$$

Let A be a body of $\mathbb{R}^d$ and $R > 0$. Let

$$A_R = \{x \in \mathbb{R}^d : \text{dist}\ (x,A) < R\} = \cup_{a\in A} B(a,R), \tag{3}$$

$$A'_R = (A_R)^c = \{x \in \mathbb{R}^d : \text{dist}\ (x,A) \geq R\} = \cap_{a\in A}(B(a,R))^c. \tag{4}$$

Definition 1. *([3]) Let A be a body. Let*

$$Unp(A) = \{x \in \mathbb{R}^d : \textit{there exists a unique point } \xi_A(x) \in A \textit{ nearest to } x\}.$$

If $A \subset \mathbb{R}^d$, $a \in A$, then reach(A,a) is the supremum of all numbers ρ such that for every $x \in B(a,\rho)$ there exists a unique point $b \in A$ satisfying: $|b-x| =$ dist (x,A):

$$reach(A,a) := \sup\{\rho > 0 : B(a,\rho) \subset Unp(A)\};$$

and:

$$reach(A) := \inf\{reach(A,a) : a \in A\}.$$

Definition 2. *For $a \in A$, let*

$$Q^{(a)} := \{v \in \mathbb{R}^d : \text{dist}\ (a+v,A) = |v|\}.$$

Let us recall the following facts:

Proposition 2.1. [3, Theorem 4.8,(2), (7) and (12)]. Let A be a body, $a \in A$, then

i) $Q^{(a)} \subset \mathrm{Nor}(A,a)$.

ii) If $x \notin A, x \in Unp(A), a = \xi_A(x) \in \partial A, reach(A,a) = \rho > 0$, then

$$A \subset (B(a+\rho\frac{x-a}{|x-a|},\rho))^c.$$

iii) If $reach(A,a) > r > 0$, then

$$\mathrm{Nor}(A,a) = \{\lambda v : \lambda \geq 0, |v| = r, \xi_A(a+v) = a\};$$

$\mathrm{Tan}(A,a)$ is the dual cone of $-\mathrm{Nor}(A,a)$.

Definition 3. *Let $b_1, b_2 \in \mathbb{R}^d, |b_1 - b_2| < 2R$ and let $\mathfrak{h}(b_1,b_2)$ be the intersection of all closed balls of radius R containing b_1, b_2.*

Proposition 2.2. ([2, Theorem 3.8], [12, Lemma 3]) The body A has reach $\geq R$ if and only if for every $b_1, b_2 \in A, 0 < |b_1 - b_2| < 2R$ the set $A \cap \mathfrak{h}(b_1,b_2)$ is connected.

Remark 2.3. *The R-hull of a set E was introduced in [2, Definition 4.1] as the minimal set $\hat{E}$ of reach $\geq R$ containing E. Therefore, if $reach(A) \geq R$, then A coincides with its R-hull. The R-hull of a set E may not exist, see [2, Example 2].*

Proposition 2.4. [2, Theorem 4.4 and Theorem 4.6] Let $A \subset \mathbb{R}^d$.

i) If $reach(A'_R) \geq R$ then A admits R-hull $\hat{A}$ and

$$\hat{A} = (A'_R)'_R.$$

ii) If A admits R-hull then $reach(A'_R) \geq R$.

Let us also recall the following result, see also [11, Lemma 4.3] :

Proposition 2.5. [2, Theorem 3.10] Let A be a closed set such that $reach(A) \geq R > 0$. If D is a closed set such that $A \cap D \neq \emptyset$ and for every $a, b \in D$ with $|a-b| < 2R$:

$$\mathfrak{h}(a,b) \subset D$$

holds, then $reach(A \cap D) \geq R$.

3. *R*-bodies

Next definitions have been introduced in [10] and in [6].

Let R be a fixed positive real number. From now on B will denote a general open ball of radius R, $B(x)$ will be the open ball of center $x \in \mathbb{R}^d$ and radius R and $D = cl(B), D(x) = cl(B(x))$.

Definition 4. *Let A be a body, A will be called a R-body if $\forall y \in A^c$, there exists an open ball B in $\mathbb{R}^d$, satisfying: $y \in B \subset A^c$. This is equivalent to say:*

$$A^c = \cup\{B : B \cap A = \emptyset\};$$

that is:

$$A = \cap\{B^c : B \cap A = \emptyset\}.$$

Notice that $\mathbb{R}^d$ is a R-body, since there are no points y in its complement.

Definition 5. *Let $E \subset \mathbb{R}^d$ be a body with the property that there exists $B \subset E^c$. The body:*

$$co_R(E) := \cap\{B^c : B \cap E = \emptyset\}$$

*will be called the **R-hulloid** of E, see [6]. If there are no balls $B \subset E^c$ then $co_R(E) = \mathbb{R}^d$.*

Remark 3.1. *In [10] the sets defined in Definition 4 are called $2R$ convex sets and the sets defined in Definition 5 are called $2R$ convex hulls. On the other hand Valentine [15, pp. 99-101] and Fenchel [5, p.42] use the name R-convex sets for convex sets with special properties depending on R. To avoid misunderstandings we decided in [6] to call R-bodies and R-hulloids the sets defined in Definition 4 and in Definition 5 respectively.*

Remark 3.2. *The R-hulloid of a bounded set always exists. Let us notice that $co_R(E)$ is a R-body (by definition) and $E \subset co_R(E)$. Moreover A is a R-body if and only if $A = co_R(A)$.*

Clearly every convex body E is a R-body (for all positive R) and its convex hull $co(E) = E$ coincides with its R-hulloid.

Proposition 3.3. [6, Theorem 3.11 and 3.10] For $d \geq 2$, every closed non empty subset of a affine linear proper subset of $\mathbb{R}^d$ is a R-body; closed subsets of the boundary of a ball of radius greater or equal than R are R-bodies too.

Remark 3.4. *Perkal in [10] proved that, if A is a body and $int(A) \neq \emptyset$, then:*

$$cl\Big(\bigcup_{r>R} co_r(A)\Big) = co(A). \tag{5}$$

Walther ([16]) claimed that if A is a body and $int(co(A)) \neq \emptyset$, then (5) *holds.*

If $int(co(A)) = \emptyset$, equality (5) may not be true: let A be a not connected body which ia a subset of an affine linear proper subset of $\mathbb{R}^d$. Then, by Proposition 3.3, for all positive r, the r-hulloids $co_r(A) = A$ and $A \neq co(A)$; thus (5) does not hold.

Proposition 3.5. Let E be a non empty set. The following facts have been proved in [10].

- **a** $\quad co_R(E) = (E'_R)'_R$;
- **b** Let $A^{(\alpha)}$ be R-bodies, $\alpha \in \mathcal{I}$ index set , then $\cap_{\alpha\in\mathcal{I}} A^{(\alpha)}$ is an R-body;
- **c** $co_R(E) \subset co(E)$ for all $R > 0$;
- **d** $0 < R_1 \leq R_2 \Rightarrow co_{R_1}(E) \subseteq co_{R_2}(E)$.

It is easy to prove also the following:

- **e** $\partial E \subset \partial co_R(E)$;
- **f** $int(E) \subset int(co_R(E))$ for all $R > 0$.

In [6, Theorem 3.4] the following fact was proved:

$$co_R(E) = E_R \cap \Big(\partial(E_R)\Big)'_R. \tag{6}$$

Moreover:

Theorem 3.6. Let E be a body. Then:

$$(\partial E_R)'_R = co_R(E) \bigcup E'_{2R}. \tag{7}$$

Proof. Let us prove (7). This is equivalent to prove that:

$$(\partial E_R)_R = (co_R(E))^c \bigcap E_{2R}. \tag{8}$$

By (3) with $A = \partial E_R$, it follows that:

$$(\partial E_R)_R = \cup\{B(x) : x \in \partial(E_R)\}.$$

Then, as $\partial E_R = \{x : \text{dist}\ (x,E) = R\}$:

$$(\partial E_R)_R = \bigcup\{B(x) : \text{dist}\ (x,E) = R\}. \tag{9}$$

Therefore:

$$(\partial E_R)'_R = \bigcap\{B^c(x) : \text{dist}\ (x,E) = R\}. \tag{10}$$

From (9), it holds:

$$(\partial E_R)_R = \tag{11}$$
$$\Big(\cup\{B(x) : \text{dist } (x,E) \geq R\}\Big) \bigcap \Big(\cup\{B(x) : \text{dist } (x,E) \leq R\}\Big).$$

Since the following Minkowski sums hold:

$$E_{2R} = E + B(o,2R) = (E + B(o,R)) + B(o,R) = E_R + B(o,R),$$

then:

$$E_{2R} = \cup\{B(x) : \text{dist } (x,E) \leq R\}. \tag{12}$$

By Definition 5:

$$(co_R(E))^c = \cup\{B(x) : \text{dist } (x,E) \geq R\}. \tag{13}$$

(8) follows from (11) and last two equalities. □

3.1. *R*-supporting balls and *R*-supported bodies

Definition 6. *Let A be a body. Let $a \in \partial A$. Let $v \in S^{d-1}$. We say that the ball $B(a+Rv)$ is R-*supporting *A at a if:*

$$A \subset (B(a+Rv))^c;$$

v will be called a unit vector R-supporting A at a.

In other words, $v \in S^{d-1}$ is a unit vector R-supporting A at a if and only if for all $b \in A$

$$\langle v, a-b \rangle \geq -\frac{|a-b|^2}{2R}. \tag{14}$$

In [8, §2.2] a closed ball $D_R = cl(B)$ of radius R is defined an outer support (closed) ball of A if $D_R \cap A \neq \emptyset$ and $D_R \cap int(A) = \emptyset$. The closure of a R-supporting ball B is an outer closed support ball D_R and conversely.

Golubyatnikov and Rovensky [8] have defined the class $\mathcal{K}_2^{1/R}$ of bodies A, with non empty interior, satisfying the following property:

$$\forall x \in A^c \text{ there exists a closed ball } \; D_R \ni x : D_R \cap int(A) = \emptyset. \tag{15}$$

The class of R-bodies, is strictly contained in the class $\mathcal{K}_2^{1/R}$, see [6, Theorem 6.1].

Definition 7. *Let A be a body. If $a \in \partial A$, let us denote:*

$$\mathcal{N}_R(A,a) :\equiv \{v \in S^{d-1} : A \subset (B(a+Rv))^c\} \tag{16}$$

the set of unit vectors R-supporting A at a.

Cuevas and others in [1, Proposition 2] proved that for a R-body A and for $a \in \partial A$ the set of R-supporting unit vectors $\mathcal{N}_R(A,a)$ is not empty; they also call a body A a R-rolling set, if for every $a \in \partial A$ there is a R-supporting ball B of A; therefore $\mathcal{N}_R(A,a)$ is non empty. Since the R-rolling set name is used in [16] with another meaning (see §4), here a different name has been used.

Definition 8. *A body A is called a R-supported body if for every point $a \in \partial A$ the set $\mathcal{N}_R(A,a)$ of the directions R-supporting A is non empty.*

From [6, Theorem 3.6] a more general result follows:

Proposition 3.7. If E is a body and $A = co_R(E), a \in \partial A$ then there exists a ball $B \subset A^c$, with $a \in \partial B$; moreover $\partial B \cap \partial E \neq \emptyset$.

Remark 3.8. *Proposition 3.7 implies that, if A is a R-body, then for all $a \in \partial A$ the set of R-supporting unit vectors $\mathcal{N}_R(A,a)$ is non empty. Then A is a R-supported body, The converse of this fact it is not true, see Proposition 2 in [1]. Other examples: let V be the set of the vertices of an equilateral triangle T with circumradius less than R. At each $v \in V = \partial V$, $\mathcal{N}_R(V,v)$ is non empty; V is not a R-body since the center of T lies in V^c but does not belong to an open disk of radius R, avoiding V. In [6, formula (16), Theorem 5.7] it has been defined a body E with non empty interior, with a R-supporting ball at every point of its boundary which is not a R-body.*

The following facts hold:

a) the family of R-bodies is a proper subset of the family of R-supported bodies;

b) every closed subset of the boundary of a R-body is a R-supported body;

c) the intersection of two R-supported bodies is a R-supported body;

d) if a R-body A is not connected, then the closed connected components of A are R-supported bodies.

Lemma 3.1. *Let E be a R-supported body and $A = co_R(E)$, then:*

i) $\partial E \subset \partial A$;

ii) $int(E) \subset int(A)$;

iii) $\mathcal{N}_R(E,a) = \mathcal{N}_R(A,a)\ \forall a \in \partial E$;

iv) $co_R(E) = E_R \cap \{(B(x))^c : x = a + R\theta, a \in \partial E, \theta \in \mathcal{N}_R(E,a)\}$.

Proof. Let $a \in \partial E$. Let $\theta \in \mathcal{N}_R(E,a)$, then

$$a \in \partial B(a+R\theta) \quad \text{and} \quad B(a+R\theta) \cap E = \emptyset.$$

By definition of $A = co_R(E)$, $B(a+R\theta) \subset A^c$. Therefore $a \in \partial A$ and $\theta \in \mathcal{N}_R(A,a)$. Then, i) is proved and $\mathcal{N}_R(E,a) \subseteq \mathcal{N}_R(A,a)$. Since $E \subset A = co_R(E)$, then ii) is obvious. To prove iii), let $a \in \partial E$; then for $\theta \in \mathcal{N}_R(A,a)$ it holds $B(a+R\theta) \cap A = \emptyset$. Since $E \subset A$ then $B(a+R\theta) \cap E = \emptyset$. Then, $\theta \in \mathcal{N}_R(E,a)$ and equality in iii) is proved.

From (6), (10) and

$$\partial E_R = \{x : dist(x,E) = R\} = \{a+R\theta : a \in \partial E, \theta \in \mathcal{N}_R(E,a)\},$$

iv) follows. □

3.2. R-supporting balls to the R-hulloid of a body

Definition 9. *Let S be a sphere in $\mathbb{R}^d$ of radius $\rho > 0$ centered at the origin; there is a one-to-one map between closed cones in $\mathbb{R}^d$ and closed subsets of S. For every closed cone K of $\mathbb{R}^d$ let $\mathcal{K} = K \cap S$; let $\mathcal{K}$ be a closed subset of S, let $K = \{\lambda v : v \in \mathcal{K}, \lambda \geq 0\}$ the related cone in $\mathbb{R}^d$.*

For $\mathcal{K} \subset S$ let us define the spherical convex hull

$$co^S_{sph}(\mathcal{K}) :\equiv co(K) \cap S.$$

A closed convex cone C is pointed if:

$$ap(C) := C \cap (-C) = \{o\}.$$

Last definitions have similar extensions for a sphere S centered at every point $c \in \mathbb{R}^d$, not necessarily at the origin o and for cones with vertex $c \neq o$.

Let us introduce the following notations in this section:

Let $H := \{x \in \mathbb{R}^d : \langle x, v\rangle = 0\}$, with $v \in S^{d-1}$,

$$H^+ := \{x \in \mathbb{R}^d : \langle x,v\rangle > 0\}, H^- := \{x \in \mathbb{R}^d : \langle x,v\rangle < 0\};$$

$$\mathbf{H}^+ = \{x \in \mathbb{R}^d ; \langle x,v\rangle \geq 0\}, \mathbf{H}^- := \{x \in \mathbb{R}^d : \langle x,v\rangle \leq 0\}.$$

As in [13, §1.3], let us define A and B strictly separated by H if

$$A \subset H^+, \quad B \subset H^-$$

or conversely.

From now on let E be a body, $a \in \partial co_R(E)$, $a \notin E$, $B(o,R)$ an open ball R-supporting $co_R(E)$ at a. With no restrictions it can be assumed that B is centered at o.

By Proposition 3.7, $\partial B \cap E \neq \emptyset$. Let $S = \partial B$.

Lemma 3.2. *Let E be a body, $a \in \partial co_R(E)$. Let $B(o,R)$ be an open ball R-supporting $co_R(E)$ at a. Let $S = \partial B(o,R)$ and let F be the cone related to $\mathfrak{F} = E \cap S$. Assume that $C = co(F)$ is pointed and $a \notin C$. Then:*

i) there exists $v \in S^{d-1}$ and $H := \{x \in \mathbb{R}^d : \langle x, v\rangle = 0\}$ so that H strictly separates $\{a\}$ from $\mathfrak{F}$, that is:

$$\{a\} \subset H^+, \quad \mathfrak{F} \subset H^-;$$

ii) let $t > 0, B_t = \{x : |x - tv| < R\}$. If t is sufficiently small, then

$$a \in B_t, \quad B_t \cap E = \emptyset. \tag{17}$$

Proof. Let us consider $C = co(F) = co(\mathfrak{F} \cup \{o\})$ and let $A = co(\{a, o\})$, then $ap(A) = \{o\}, ap(C) = \{o\}$. Since $a \notin C$, then $A \cap C = \{o\}$. A result of Klee [9, Theorem 2.7], in $\mathbb{R}^d$, see also [14, Theorem 4.2], states that for two closed convex cones A and C satisfying the above conditions there exists a hyperplane $H := \{x \in \mathbb{R}^d : \langle x, v\rangle = 0\}$, which *sharply* separates A and C, that is:

$$A \setminus ap(A) \subset H^+, \quad C \setminus ap(C) \subset H^-.$$

So i) is proved.

Let us prove now ii).

Let $\cos\phi = \langle \frac{a}{|a|}, v\rangle > 0$ since $a \in H^+$; if $0 < t < 2R\cos\phi$ then

$$|a - tv|^2 = R^2 + t^2|v|^2 - 2Rt\cos\phi < R^2,$$

so $a \in B_t$.

Let us show that $B_t \cap E = \emptyset$. The open ball $B_t = (B_t \cap B) \cup (B_t \setminus B)$ is union of two non empty sets, for $0 < t$ small enough. Since $B \cap E = \emptyset$ by assumption, then $(B_t \cap B) \cap E = \emptyset$. Moreover $\overline{B_t} \setminus B \subset H^+$ for $t > 0$ and for $t \to 0^+$ the set $\overline{B_t} \setminus B \to S \cap \mathbf{H}^+$. By i) $\mathfrak{F} \subset H^-$, thus $S \cap \mathbf{H}^+$ and E are disjoint closed sets; then for $t > 0$ small enough, $B_t \setminus B$ and E are disjoint sets too. Then B_t and E are disjoint sets for $t > 0$ small enough. □

Theorem 3.9. Let E be a body in $\mathbb{R}^d$. Let $a \in \partial co_R(E) \setminus E$. Then, there exists a ball B, R-supporting $co_R(E)$ at a. Let $S = \partial B$ (with no restrictions it can be assumed that B is centered at o). Then

i) $\partial E \cap S$ contains at least two points.

Let $\mathcal{F} = \partial E \cap S$ and F be the related cone and $C = co(F)$, then

ii) if C is a pointed cone, then there exist distinct points $x_1,\dots,x_s \in \partial E \cap S$, $2 \leq s \leq d$, such that

$$a \in co^S_{sph}(\{x_1,\dots,x_s\}). \tag{18}$$

Proof. By Proposition 3.7, if $a \in \partial co_R(E) \setminus E$, there exists at least a support ball $B \subset (co_R(E))^c \subset E^c$ such that $a \in \partial B = S$ and $\mathfrak{F} = \partial E \cap S$ is a non empty closed set. Let us prove i). By contradiction, let $\mathfrak{F} = \{z\}$. Since $a \notin E$, $a \neq z$, then by Lemma 3.2, if $t > 0$ is sufficiently small, $a \in B_t \subset E^c$, impossible as $a \in co_R(E)$.

Let us prove ii). By assumption C is a pointed cone. If $a \notin C$ by ii) of Lemma 3.2 there exists $t > 0$, sufficiently small, such that $B_t \ni a$, $B_t \cap E = \emptyset$; impossible since $a \in co_R(E)$. Thus $a \in C$. By [5, Theorem 7 pag. 12] a is a linear combinations of s vectors of F, $s \leq d$, then (18) follows. □

Remark 3.10. *Under the notations of Lemma 3.2, in case C is not a pointed cone, let $r = dim(ap(C))$. Then [5, Theorem 7] (for convex cones) proves that there exist $(r+1)$ distinct points $x_1,\dots,x_{r+1} \in \partial E \cap S$, with $1 \leq r \leq d$, such that*

$$\bigcup_{\lambda_1,\lambda_2,\dots,\lambda_{r+1} \geq 0} \{\sum_{i=1}^{r+1} \lambda_i x_i\} = C \cap (-C). \tag{19}$$

Let us notice that (19), with $r = 1$, implies that there are two opposite points x_1, x_2 on the spherical surface S. When $r = 2$ there are three points x_1, x_2, x_3 such that the circumradius of $co(\{x_1,x_2,x_3\})$ is equal to R.

4. R-supported bodies vs sets of reach $\geq R$ and R-rolling sets

Lemma 4.1. *Let A be a body, $a \in \partial A$. Let $\mathcal{N}(A,a) = \mathrm{Nor}\,(A,a) \cap S^{d-1}$, then*

$$\mathcal{N}_R(A,a) \subseteq \mathcal{N}(A,a). \tag{20}$$

Proof. If $\mathcal{N}_R(A,a) \neq \emptyset$, let $B(a+Rv)$ be R-supporting A at a, then $A \subset (B(a+Rv))^c$. Let

$$w = \lim_{z_n \in A, z_n \to a} \frac{z_n - a}{|z_n - a|} \in \mathrm{Tan}\,(A,a);$$

by (14), with $b = z_n$ the inequalities

$$\langle v, \frac{z_n - a}{|z_n - a|} \rangle \leq \frac{|z_n - a|}{2R}$$

hold, $\forall n \in \mathbb{N}$. Thus the inequality $\langle v, w \rangle \leq 0$ holds. Then $v \in \mathrm{Nor}\,(A,a)$. □

Theorem 4.1. Let A be a body, $a \in \partial A$. If $reach(A,a) \geq R$, then

$$\mathcal{N}(A,a) = \mathcal{N}_R(A,a).$$

Proof. Let $w \in \operatorname{Nor}(A,a)$, $0 < r < reach(A,a)$; then by iii) of Proposition 2.1,

$$\xi_A(a+rw) = a.$$

Therefore, by ii) of Proposition 2.1, $B_r(a+rw)$ is r-supporting A at a; then by (14) for all $z \in A \setminus \{a\}$ the inequalities

$$\langle w, \frac{z-a}{|z-a|} \rangle \leq \frac{|z-a|}{2r} \tag{21}$$

hold. By a limit argument for $r \to R^-$ then (21) holds for $r = R$ and the unit vector w is R-supporting A at a, so $w \in \mathcal{N}_R(A,a)$. This proves that

$$\mathcal{N}(A,a) \subseteq \mathcal{N}_R(A,a).$$

The opposite inclusion follows from (20). □

Remark 4.2. *It was noticed in [2, Corollary 4.7] and proved in [1, Proposition 1] that, when the R-hull exists, it coincides with the R-hulloid. If A has reach greater or equal than R, then (see Remark 2.3) A has R-hull, which coincides with A and with its R-hulloid, and A is a R-body; then the class of R-bodies contains the class of sets of reach greater or equal than R. The family of the sets of reach greater or equal than R is not closed with respect to the intersection.*

Remark 4.3. *Let $\mathcal{R}$ be the family of the sets of reach R. The family of the R-bodies is the minimal family containing $\mathcal{R}$ and closed with respect to the intersection.*

Proof. Let $\mathcal{F}$ be a family closed with respect to the intersection, such that $\mathcal{F} \supset \mathcal{R}$. Let A be a R-body, then $A = \cap\{B^c : B \cap A = \emptyset\}$. As $reach(B^c) = R$ for every $B^c \supset A$, then A is intersection of sets of reach R, which are in the family. Then $A \in \mathcal{F}$. □

Next theorem gives necessary and sufficient conditions for a R-supported body to have reach greater or equal than R.

Theorem 4.4. Let $d \geq 2$, A be a body in $\mathbb{R}^d$. Then the following properties are equivalent:

i) A is a R-supported body and

$$\operatorname{Nor}(A,a) \cap S^{d-1} = \mathcal{N}_R(A,a) \quad \forall a \in \partial A; \tag{22}$$

ii) $reach(A) \geq R$.

Proof. If $reach(A) \geq R$, from Theorem 4.1 equality holds in (20); from Remark 4.2, A is a R-body, then it is a R-supported body. Then ii) implies i).

By contradiction let us assume that i) holds and $reach(A) < R$. Then there exists $a \in \partial A$, $x \in B(a) \cap A^c$, $x \notin Unp(A)$. Thus there exists $a_1 \in \partial A$, $a_1 \neq a$ satisfying:

$$\text{dist}\,(x,A) = |x-a| = |x-a_1| = R_1 < R.$$

Let $v = x - a$, then

$$\text{dist}\,(a+v,A) = \text{dist}\,(x,A) = |x-a| = |v|,$$

so $v \in Q^{(a)}$, see Definition 2. By i) of Proposition 2.1, $v \in \text{Nor}\,(A,a)$. Then $\theta = v/|v|$ is a unit vector R-supporting A at a; that is $\theta \in \mathcal{N}_R(A,a)$, thus $A \subset (B(a+R\theta))^c$. Since $x - a = R_1\theta$, $R_1 < R$ and θ is the inner unit normal to $B(a+R\theta)$ and to $B(a+R_1\theta)$ at a, then $B(a+R_1\theta,R_1) \subset B(a+R\theta)$. Then $a_1 \in A \cap B(a+R\theta)$, so $\theta \notin \mathcal{N}_R(A,a)$. Contradiction. □

Example 1. The following sets are examples of R-bodies where strict inclusion holds in (22); then, they have reach less than R:

a) $A = \{a,b\}$, with $|a-b| < R$;

b) $H \setminus B(o,r)$, with $r < R$, H plane in R^3, through o (see Proposition 3.3);

c) in [1, fig.1(a)] there is a R-body with non empty interior and strict inequality holds in (22).

Lemma 4.2. *Let A and $\overline{A^c}$ be R-supported bodies. If:*

$$a_0 \in \partial A \cap \partial \overline{A^c}, \tag{23}$$

then there exist two open balls $B(y_0), B(y_1)$ satisfying:

i) $B(y_0) \subset A^c, B(y_1) \subset (\overline{A^c})^c$;

ii) $B(y_0) \cap B(y_1) = \emptyset, \{a_0\} = \partial B(y_0) \cap \partial B(y_1)$.

Proof. For every $A \subset \mathbb{R}^d$ it holds $\mathbb{R}^d = int(A) \cup \partial A \cup A^c$, $\overline{A^c} = \partial A \cup A^c$, and

$$int(A) = (\overline{A^c})^c. \tag{24}$$

As $a_0 \in \partial A$ and A is a R-supported body, there exists an open ball $B(y_0) \subset A^c$, where $a_0 \in \partial B(y_0)$. By assumptions (23), $a_0 \in \partial \overline{A^c}$ too and $\overline{A^c}$ is a R-supported body, then there exists an open ball $B(y_1) \subset (\overline{A^c})^c$, $a_0 \in \partial B(y_1)$ and i) is proved.

Moreover by (24) the two open balls $B(y_0), B(y_1)$ also satisfy ii).□

Theorem 4.5. Let A and $\overline{A^c}$ be R-supported bodies and

$$\partial A = \partial \overline{A^c} \tag{25}$$

then, $reach(A) \geq R$.

Proof. Let us assume, by contradiction that $reach(A) < R$. Then there exists $a_0 \in \partial A$, so that $reach(A, a_0) < R$. Then $\exists\, r_1 < R, \exists\, x_1 \in A^c$ and $a_1 \in \partial A \setminus \{a_0\}$ satisfying:

$$\begin{aligned} r_1 = \text{dist}\,(x_1, A) = |x_1 - a_0| = |x_1 - a_1| < R, \\ B(x_1, r_1) \subset A^c. \end{aligned} \tag{26}$$

By assumption (25), $a_0 \in \partial \overline{A^c}$ and by Lemma 4.2 there exist two open balls $B(y_0), B(y_1)$ satisfying: i), ii). Since $a_0 \in \partial B(x_1, r_1)$ and inclusion (26) holds then, $B(x_1, r_1) \subset B(y_0, R)$ and

$$a_1 \in \partial B(x_1, r_1) \setminus \{a_0\} \subset B(y_0, R) \subset A^c.$$

So $a_1 \in A^c$, in contradiction with $a_1 \in \partial A$. □

Remark 4.6. *Let A be a body, $X = A^c$; as $\partial \overline{X} \subset \partial X$, then:*

$$\partial \overline{A^c} \subseteq \partial A^c = \partial A.$$

Last inclusion can be strict, see also next example.

Example 2. If (25) does not hold, then it can be $reach(A) < R$. As example let $|x_0 - x_1| > 3R, |x_1 - x_2| < R/4$ and let

$$A = D(x_0) \cup \{x_1\} \cup \{x_2\}.$$

Then A is a R-body and $\overline{A^c}$ is a R-body too. But $reach(A) = |x_1 - x_2|/2 < R$.

Remark 4.7. *Walther ([16]) considered the class of non empty, path connected, compact sets A, with the following property: a ball of radius R rolls freely in A and in $\overline{A^c}$. These path connected compact sets A are such that A and $\overline{A^c}$ are R-bodies ([16, Theorem 1]).*

Walther proved several other properties of the boundary of these sets A:

Proposition 4.8. [16, formula (33)] Let A and $\overline{A^c}$ be R-bodies and A path connected compact set, then (25) holds.

From last proposition and Theorem 4.5, for the Walther's R-rolling sets a property, not stated in [16], can be proved.

Theorem 4.9. Let A and $\overline{A^c}$ be R-bodies and let A be a path connected compact set, then

$$reach(A) \geq R.$$

5. *R*-cones

Let us introduce a class of *R*-bodies, that will be called *R*-cones. As the *R*-bodies are a generalization of convex sets, the *R*-cones are a generalization of convex cones.

Definition 10. *Let $\mathcal{K}$ be a body in S^{d-1}. A R*-cone *with vertex o (more simply a R-cone) is the R-body:*

$$C_{\mathcal{K}} := \bigcap_{v\in\mathcal{K}} (B(Rv,R))^c. \tag{27}$$

Let $x \in \mathbb{R}^d$, a R-cone with vertex x is the R-body:

$$C^x_{\mathcal{K}} := x + C_{\mathcal{K}} = \bigcap_{v\in\mathcal{K}} (B(x+Rv,R))^c. \tag{28}$$

Let us notice that $x \in \partial C^x_{\mathcal{K}}$ and the unit vectors in $\mathcal{K}$ are *R*-supporting $C^x_{\mathcal{K}}$ at its vertex *x*, that is

$$\mathcal{K} \subset \mathcal{N}_R(C^x_{\mathcal{K}}, x). \tag{29}$$

Moreover

$$\mathcal{K}_1 \subset \mathcal{K}_2 \Rightarrow C^x_{\mathcal{K}_1} \supset C^x_{\mathcal{K}_2}. \tag{30}$$

Theorem 5.1. Let *E* be a *R*-supported body. Then

$$\bigcap_{a\in\partial E} C^a_{\mathcal{N}_R(E,a)} = (\partial E_R)'_R. \tag{31}$$

Proof. By definition:

$$C^a_{\mathcal{N}_R(E,a)} = \bigcap_{v\in\mathcal{N}_R(E,a)} (B(a+Rv))^c = \bigcap_{(B(a+Rv))^c\supset E} B(a+Rv))^c. \tag{32}$$

Let us notice that $a \in \partial E, v \in \mathcal{N}_R(E,a)$ if and only if:

$$x = a + Rv \quad \text{satisfies} \quad \text{dist }(x,E) = R.$$

Thus

$$\bigcap_{a\in\partial E} C^a_{\mathcal{N}_R(E,a)} = \bigcap_{x:\ \text{dist }(x,E)=R} (B(x))^c. \tag{33}$$

From (10) of Theorem 3.6 the formula (31) follows. □

From (31) and (6) we have:

Corollary 5.2. Let E be a *R*-supported body. Then:

$$co_R(E) = E_R \bigcap \Big(\cap_{a\in\partial E} C^a_{\mathcal{N}_R(E,a)} \Big). \tag{34}$$

Corollary 5.3. Let A be a R-supported body. Then A is a R-body if and only if:

$$A = A_R \bigcap \Big(\cap_{a\in\partial A} C^a_{\mathcal{N}_R(A,a)} \Big). \tag{35}$$

Proof. If A is a R-body then $A = co_R(A)$ and (35) follows from Corollary 5.2 with $E = A$. Conversely if (35) holds for a R-supported body A, then by (31) and (6), with A in place of E, it follows that $A = co_R(A)$. □

5.1. *R*-cones, Tangent cones and Normal cones

For simplicity, let us denote $\mathcal{N}_R(C_\mathcal{K}) = \mathcal{N}_R(C_\mathcal{K}, o)$, $N(C_\mathcal{K}) = \mathrm{Nor}\,(C_\mathcal{K}, o)$ and $\mathrm{Tan}\,(C_\mathcal{K}) = \mathrm{Tan}\,(C_\mathcal{K}, o)$.

Lemma 5.1. *Let A be a body, $a \in \partial A$ then*

$$\mathrm{Tan}\,(A,a) \subset C_{\mathcal{N}_R(A,a)}. \tag{36}$$

Proof. Since for every $v \in \mathcal{N}_R(A,a)$, the inclusion $A \subset (B(a+Rv))^c$ holds, then

$$\mathrm{Tan}\,(A,a) \subset \mathrm{Tan}\,((B(a+Rv))^c, a).$$

Moreover

$$a + \mathrm{Tan}\,((B(a+Rv))^c, a) \subset (B(a+Rv))^c.$$

By previous inclusions:

$$a + \mathrm{Tan}\,(A,a) \subset \cap_{v\in\mathcal{N}_R(A,a)} (B(a+Rv))^c$$

and by (28):

$$\cap_{v\in\mathcal{N}_R(A,a)} (B(a+Rv))^c = a + C_{\mathcal{N}_R(A,a)}.$$

Therefore:

$$a + \mathrm{Tan}\,(A,a) \subset a + C_{\mathcal{N}_R(A,a)}$$

and (36) is proved. □

Lemma 5.2. *Let $\mathcal{K}$ be a body in S^{d-1} and let $C_\mathcal{K}$ be its related R-cone with vertex o. If $B(Rv)$ is a R-supporting ball to $C_\mathcal{K}$ at o, then $v \in \mathcal{K}$.*

Proof. In our notations, the claim of the lemma is:

$$\mathcal{N}_R(C_\mathcal{K}) = \mathcal{K}. \tag{37}$$

By definition:

$$\mathcal{K} \subseteq \mathcal{N}_R(C_\mathcal{K}).$$

Let $v \in \mathcal{N}_R(C_{\mathcal{K}})$, then:

$$B(Rv) \subset \bigcup_{w\in\mathcal{K}} B(Rw).$$

Let $\{w_n\}$ a sequence of points in $B(Rv)$ with limit $2Rv$. There exists a sequence of unit vectors $\{v_n\}$ such that $w_n \in B(Rv_n)$. Up to a subsequence $v_n \to \overline{v} \in \mathcal{K}$ and $|w_n - 2Rv_n| < R$. Then

$$|2Rv - R\overline{v}| = \lim_n |w_n - Rv_n| \leq R.$$

Thus $|2v-\overline{v}|^2 \leq 1$; then: $4+1-4\langle v,\overline{v}\rangle \leq 1$ and $\langle v,\overline{v}\rangle \geq 1$. Therefore, since $v,\overline{v}$ are unit vectors, equality holds in Schwartz's inequality:

$$\langle v,\overline{v}\rangle = |v||\overline{v}|.$$

Then, $v = \overline{v} \in \mathcal{K}$. So (37) is proved. □

Theorem 5.4. Let $\mathcal{K}$ be a body in S^{d-1} and K its related cone in $\mathbb{R}^d$. Let $\mathcal{N} = \mathrm{Nor}\,(C_{\mathcal{K}}) \cap S^{d-1}$. Then

a) $\mathrm{Tan}\,(C_{\mathcal{K}}) = -K^\star$;

b) $\mathcal{N} = co_{sph}(\mathcal{K})$;

c) $\mathrm{Tan}\,(C_{\mathcal{K}}) \setminus \{o\} \subset int(C_{\mathcal{K}})$.

Proof. Let $w \in \mathcal{K}$ then

$$C_{\mathcal{K}} \subset C_{\{w\}}$$

and

$$\mathrm{Tan}\,(C_{\mathcal{K}}) \subset \mathrm{Tan}\,(C_{\{w\}}) = -\{w\}^\star. \tag{38}$$

From (38) it follows

$$\mathrm{Tan}\,(C_{\mathcal{K}}) \subset -\bigcap_{w\in\mathcal{K}} \{w\}^\star = -K^\star. \tag{39}$$

Let $\theta \in K^\star$, then $\forall v \in \mathcal{K}$, $\langle \theta, v\rangle \geq 0$. This implies that $\forall v \in \mathcal{K}$ the ball $B(Rv)$ is R-supporting the half line $\{-\lambda\theta, \lambda \geq 0\}$ at o . Then

$$\{-\lambda\theta, \lambda \geq 0\} \subset C_{\mathcal{K}}, \quad \forall \theta \in K^\star \tag{40}$$

and therefore

$$-\theta \in \mathrm{Tan}\,(C_{\mathcal{K}}).$$

Then

$$-K^\star \subset \mathrm{Tan}\,(C_{\mathcal{K}}).$$

From (39) a) is proved.

Moreover from a) it follows that $\mathrm{Tan}\,(C_{\mathcal{K}}) = -(co(K))^{\star}$; (2) and the bipolar theorem imply that

$$N(C_{\mathcal{K}}) = -(\mathrm{Tan}\,(C_{\mathcal{K}}))^{\star} = ((co(K))^{\star})^{\star} = co(K).$$

b) follows.

Let us prove now c). First let us prove that

$$\mathrm{Tan}\,(C_{\mathcal{K}}) \subset C_{\mathcal{K}}.$$

This inclusion follows from (37) and (36), with $A = C_{\mathcal{K}}$, $a = o$. If o is an isolated point of $C_{\mathcal{K}}$, then $\mathrm{Tan}\,(C_{\mathcal{K}}) = \{o\}$ and c) is trivial. In case o is not an isolated point of $C_{\mathcal{K}}$, let $y \neq 0, y \in \mathrm{Tan}\,(C_{\mathcal{K}})$. By previous inclusion $y \in C_{\mathcal{K}}$. Let us prove that

$$\mathrm{dist}\,(y, (C_{\mathcal{K}})^c) = \mathrm{dist}\,(y, \cup_{v \in \mathcal{K}}(B(Rv, R))) > 0.$$

By contradiction: if $\mathrm{dist}\,(y, \cup_{v \in \mathcal{K}}(B(Rv, R))) = 0$, since $\mathcal{K}$ is compact, then there exists $u \in \mathcal{K}$, such that $y \in \partial B(Ru, R)$. Since $y \neq o$ and $\partial B(Ru, R)$ strictly convex: $\langle y, u \rangle > 0$. Since by a): $y \in \mathrm{Tan}\,(C_{\mathcal{K}}) = -K^*$, then $\langle y, u \rangle \leq 0$, contradiction. □

Theorem 5.5. Let $\mathcal{K}$ be a body contained in an hemisphere of S^{d-1} and let $\mathcal{N} = \mathrm{Nor}\,(C_{\mathcal{K}}, o) \cap S^{d-1}$. Then the following properties are equivalent:

a) $\mathcal{K}$ is spherically convex ;

b) $C_{\mathcal{K}}$ is a set of reach greater or equal than R;

c) $C_{\mathcal{K}} = C_{\mathcal{N}}$.

Proof. By (3) and (27) :

$$(R\mathcal{K})'_R = C_{\mathcal{K}} \tag{41}$$

holds for every set $\mathcal{K} \subset S^{d-1}$.

Let us assume that a) holds.

If $\mathcal{K}$ is spherically convex on a hemisphere of S^{d-1}, it follows that $R\mathcal{K}$ is convex on a hemisphere of a ball of radius R. Then for every $a, b \in R\mathcal{K}, |a-b| < 2R$, the set $R\mathcal{K} \cap \mathfrak{h}(a, b)$ is connected (see Definition 3). Then by Proposition 2.2 $reach(R\mathcal{K}) \geq R$.

Thus $R\mathcal{K}$ has R-hull and $R\mathcal{K} = co_R(R\mathcal{K})$. By ii) of Proposition 2.4

$$reach((R\mathcal{K})'_R) \geq R.$$

This fact and equality (41) imply that $C_{\mathcal{K}}$ is a set of reach greater or equal than R; b) is proved.

Let us assume that b) holds.

From Theorem 4.1 any direction v which lies in the normal cone at o of $C_{\mathcal{K}}$ is R-supporting it at o; then $\forall v \in \mathcal{N}$:

$$(B(Rv))^c \supset C_{\mathcal{K}}.$$

Then $C_{\mathcal{K}} \subset C_{\mathcal{N}}$. The opposite inclusion follows from (20) and c) is proved.

Then by (37) it follows that $\mathcal{K} = \mathcal{N}$. Therefore since by definition $\mathcal{N}$ is spherically convex then $\mathcal{K}$ is spherically convex too and a) follows.

Let assume that c) holds.

By (37) the set $\mathcal{K}$ is the set of the R-supporting unit vectors of $C_{\mathcal{K}}$ at the origin o; similarly $\mathcal{N}$ is the set of the R-supporting unit vectors of $C_{\mathcal{N}}$ at o, then by c) $\mathcal{K} = \mathcal{N}$; by b) of Theorem 5.4 a) follows. □

Next example shows that a R-supported body A, with $\mathcal{N}_R(A,p)$ spherically convex for every $P \in \partial A$, may not be a R-body, so it has reach less than R.

Example 3. Let P be a convex polygon contained in a circle of radius R, $A = \partial P$. Then for every $p \in \partial A = A$, $\mathcal{N}_R(A,p)$ is non empty and convex: if p is inside a side of P, $\mathcal{N}_R(\partial A,p)$ is a single vector normal at p to the sides of P; if p is a corner of P, then $\mathcal{N}_R(\partial A,p)$ is spherically convex since it is an arc in a semicircle. The body A is not a R-body so does not have reach greater or equal than R.

Theorem 5.6. Let $d \geq 2$, A be a body in $\mathbb{R}^d$. If $reach(A) \geq R$ then A is a R-body and for all $a \in \partial A$ the set of unit vectors R-supporting A at a is spherically convex.

Proof. If A has reach greater or equal than R, then by Remark 4.2 A is a R-body. Theorem 4.4 and Lemma 4.1 prove the equality:

$$\mathcal{N}(A,a) = \mathcal{N}_R(A,a)$$

for all $a \in \partial A$. Then, $\mathcal{N}_R(A,a)$ is convex for all $a \in \partial A$. □

For the family of planar R-bodies the converse statement holds.

Theorem 5.7. Let $d = 2$, let A be a planar R-body. If, for all $a \in \partial A$, the set $\mathcal{N}_R(A,a)$ is a spherically convex set, then A has reach greater or equal than R.

Proof. Let us assume, by contradiction, that $reach(A) < R$. By Proposition 2.2, there exist $b_1, b_2 \in A$, $|b_1 - b_2| < 2R$, so that $A \cap \mathfrak{h}(b_1,b_2)$ is not connected. Then there exist $a_1, a_2 \in A \cap \mathfrak{h}(b_1,b_2)$, so that $A \cap \mathfrak{h}(a_1,a_2) = \{a_1,a_2\}$.

Let

$$\mathfrak{h}(a_1,a_2) = cl(B(x_1)) \cap cl(B(x_2))$$

and let

$$H(a_1,a_2) = B(x_1) \cup B(x_2). \tag{42}$$

As A is a R-body, by [6, Theorem 4.5 and lemma 4.1]

$$\mathfrak{h}(a_1,a_2) \setminus \{a_1,a_2\} \subset A^c \tag{43}$$

implies

$$H(a_1,a_2) \subset A^c.$$

From (42) it follows

$$B(x_1) \cup B(x_2) \subset A^c.$$

Since $a_i \in A \cap \partial B(x_i)$, $i = 1,2$, then each ball $B(x_i)$ is R-supporting A both in a_1 and at a_2. Then, in particular,

$$v_i = \frac{x_i - a_1}{|x_i - a_1|}, \quad i = 1,2$$

are R-supporting vectors of A at a_1. As, by assumption, $\mathcal{N}_R(A,a_1)$ is spherically convex, then it contains all unit vectors connecting v_1 with v_2; then

$$u = \frac{(a_2 - a_1)}{|a_2 - a_1|} \in \mathcal{N}_R(A,a_1).$$

Since $B(a_1 + Ru) \ni a_2$, there is a contradiction. □

6. Open questions

Let us point out some open questions:

a) Is Theorem 5.7 true for $d > 2$? Let us notice that it is true for a R-cone, Theorem 5.5.

b) Let $E \subset \mathbb{R}^d$, $d > 2$, be a connected body, contained in an open ball of radius R, then is $co_R(E)$ connected? For $d = 2$ the statement is true, see [6, Theorem 4.8] with a detailed proof in arXiv:2210.04276, Appendix.

c) If E is the set V of the vertices of a simplex in $\mathbb{R}^d$, the boundary of the R-hulloid $co_R(V)$ has properties which follow from Theorem 3.9. Is it possible to describe completely the shape of $co_R(V)$?

   In two dimensions this description is made in [6, Theorem 4.2]. In [7] a complete characterization of the shape of $co_R(V)$ is given for any well centered tetrahedron and for any triangular pyramid.

Funding

This work has been partially supported by INDAM-GNAMPA(2023).

REFERENCES

[1] A. Cuevas, R. Fraiman, B. Pateiro-López, *On statistical properties of sets fulfilling rolling-type conditions*, Adv. App. Probab. (S.G.S.A.) 2012; 44: 311-329.

[2] A. Colesanti, P. Manselli, *Geometric and Isoperimetric Properties of sets of Positive Reach in* $\mathbb{E}^d$, Atti Semin Mat Fis Univ Modena Reggio Emilia. 2010; 57: 97-113.

[3] H. Federer *Curvature measures*. Trans. Amer. Math. Soc. 1959; 93: 418-491.

[4] H. Federer, *Geometric Measure Theory*, Springer 1969.

[5] W. Fenchel, *Convex Cones, Sets and Functions*, Princeton Univ. Press, 1953.

[6] M. Longinetti, P. Manselli, A. Venturi, *On the complements of union of open balls of fixed radius in the Euclidean space*, Le Matematiche 2023; 1: 3-22.

[7] M. Longinetti, S. Naldi, A. Venturi. *R-hulloid of the vertices of a tetrahedron*. https://arxiv.org/abs/2406.00658, submitted.

[8] V. Golubyatnikov, V. Rovenski, *Determination of sets with positive reach by their projection type images*, Inv. Ill-Posed Problems 2011; 19: 407–428.

[9] V.L. Klee, *Separation properties of convex cones*, Proc. Amer. Math. Soc. 6 (1955) 313–318.

[10] J. Perkal, *Sur les ensembles* ε*-convexes*, Colloq. Math 1956 Fasc 1, 1-10.

[11] J. Ratay, *Determination of spherical area measures by means of dilations volumes*, Math Nachr. 2002; 235: 143-162.

[12] J. Ratay, *On estimation of the Euler number by projections of thin slabs*, Adv Appl Prob. 2004; 36: 715-724.

[13] R. Schneider, *Convex bodies: the Brunn-Minkowski theory*, Cambridge University Press, Cambridge, 1993.

[14] V. Soltan, *Polarity and separation of cones*, Linear Algebra and its Applications 2018; 538: 212-224.

[15] F. A. Valentine, *Convex sets*, Mc Graw-Hill, 1964.

[16] G. Walther, *On a Generalization of Blaschke's Rolling Theorem and the Smoothing of Surfaces*, Math. Meth. Appl. Sci. 1999; 22: 301-316.

M. LONGINETTI
Dipartimento DIMAI,
Università di Firenze
e-mail: marco.longinetti@unifi.it

P. MANSELLI
e-mail: paolo.manselli@unifi.it

A. VENTURI
e-mail: adriana.venturi@unifi.it