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ANALYTIC BRANCHES AND HYPERSURFACE SECTIONS

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Dedicated to Silvio Greco in occasion of his 60-th birthday.

We study the behaviour of analytic branches of a projective variety with respect to hypersurface sections and we give conditions under which their number and their orders are preserved.

Introduction.

Let $X \subseteq \mathbb{P}^n_k$ be an algebraic variety over an algebraically closed field k of any characteristic. Starting point of this note was a question, posed in [10], if there is a Seidenberg type theorem for analytic branches; more precisely when is it true that the number of branches of X at a singular point P equals the number of branches at P of a "general" hypersurface section of X ?

We approach this problem from different points of view and by different methods.

In Section 1 we study the case of a closed point P belonging to a singular subvariety $Y \subseteq X$, with $\dim Y \geq 1$; using standard local algebra, we give some sufficient conditions under which the number of branches at P and their orders are preserved by hypersurface sections; then we show by examples that this fact does not hold in general, but that such conditions are always verified when Y is $1-$codimensional and $\operatorname{char} k = 0$; in these particular hypotheses and if

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moreover X is a hypersurface we prove that our results follow also by Zariski's equisingularity theory and they are classically well known in $\mathbb{P}^3$ for multiple curves on surfaces (see [7]).

In Section 2 we discuss the case of isolated points. When $\operatorname{char} k = 0$ we use local Bertini type theorems, especially a result of Flenner (see [9]) about normality, and we get in particular sufficient conditions under which the branches of X at P are well behaved with respect to the general element of a linear system on X. We observe also that, in any characteristic, when $\dim X = 2$, something quite different may happen.

Section 3 deals with analytic branches of an algebraic variety X and linear systems on X from a global point of view and in any characteristic. Main tool of our work is a Bertini theorem for geometrically unibranch schemes, due to Zhang (see [17]). We obtain in particular a global statement on the good behaviour of branches of a projective variety under hyperplane sections.

0. We recall some generalities about the branches of a scheme at its points. For more details see [10], [2].

Definition 0.1. Let X be a k-scheme, $\overline{X}$ be the normalization of X_{red} and $\nu : \overline{X} \to X$ be the canonical morphism; let $x \in X$. A *(geometric) branch* of X at x is a (geometric) point of the $k(x)$-scheme $\nu^{-1}(x)$. The number of branches of X at x is denoted by $b(x)$ (or $b_X(x)$ when necessary). If A is a local ring, the (geometric) branches of A are the (geometric) branches of $\operatorname{spec}(A)$ at its closed point.

If A is a ring, we denote by $\min(A)$ the set of minimal primes of A and by $c(A)$ their number; $\max(A)$ shall denote the set of maximal ideals of A.

Lemma 0.2. *Let ${}^hA$ be the henselization of the local ring A; there is a canonical bijection between the set of branches of* $\operatorname{spec}(A)$ *and* $\min({}^hA)$.

Definition 0.3. Let A be a local reduced ring and $p \in \min({}^hA)$, we call *order of the branch* corresponding to p the multiplicity $e({}^hA/p)$. The branch is said to be *linear* if ${}^hA/p$ is a regular ring.

Definition 0.4. A local ring A is said to be *unibranch* (UB) if $b(A) = 1$. A scheme X is said to be UB at x if $O_{X,x}$ is such.

1. In this section we examine the behaviour of branches at closed points of an algebraic variety $X \subseteq \mathbb{P}^n_k$, over an algebraically closed field k, with respect to sections with an hypersurface F. We show at first some sufficient condition under which the equality $b_X(P) = b_{X\cap F}(P)$ holds, for a point P belonging to a reduced irreducible subvariety $Y \subseteq X$ of dimension ≥ 1, when k is a field of any characteristic. Secondly we show that, when Y is a 1-codimensional subvariety of an hypersurface X and $\operatorname{char} k = 0$, our results are a consequence of the classical theory of equisingularity.

Theorem 1.1. *Let (A, m, k) be a local reduced ring. Suppose that:*

1) *$\overline{A}$ is regular;*
2) *the conductor of A has only one associated prime p;*
3) *A/p is regular and $(\overline{A}/p\overline{A})_{red}$ is étale over A/p.*

Let $f \in A$ be an element such that $\overline{f} := f \bmod p$ in A/p belongs to a regular system of parameters. Then there is a canonical map

$$\{branches\ of\ A\} \longleftrightarrow \{branches\ of\ A/fA\}$$

which is one to one and preserves the orders of branches.

Proof.

(a) Since f is not in p, it follows by (2) that the conductor is not contained in any minimal associated prime q to fA; hence $(A/fA)_q = (\overline{A}/f\overline{A})_{\underline{q}}$, for any such q ; therefore the integral closures of $(A/fA)_{red}$ and $(\overline{A}/f\overline{A})_{red}$ coincide.

(b) Put $B = A/p$ and $C = (\overline{A}/p\overline{A})_{red} = \overline{A}/I$; let $m_1, \cdots, m_n$ be the maximal ideals of C. Consider, for each $i = 1, \cdots, n$, the following diagram:

$$\begin{array}{ccc} B & \xrightarrow{h} & C_{m_i} \\ \downarrow & & \downarrow \\ B/\overline{f}B & \xrightarrow{h'} & C_{m_i} \otimes_B B/\overline{f}B \end{array}$$

Observe that $C_{m_i} \otimes_B B/\overline{f}B = C_{m_i}/\overline{f}C_{m_i}$ and that, by hypothesis (3), $B/\overline{f}B$ is regular and h is étale; h' too is étale, therefore $C_{m_i}/\overline{f}C_{m_i}$ is regular; ([8] IV_2 6.5.2, IV_4 17.6.1).

(c) For each $m \in \max(\overline{A})$, put $R = \overline{A}_m$ and $R' = C_{mC}$. By hypothesis (1) and (3), R and R' are regular. Now $R'/\overline{f}R' = R/(f, t_1, \cdots, t_s)$ is by (b) a regular local ring, hence f belongs to a regular system of parameters of R, therefore R/fR is a regular ring.

(d) By hypothesis $f \in m \subseteq rad(\overline{A})$ and by (c) $\overline{A}/f\overline{A}$ is a regular ring, hence we obtain the bijective map $\max(\overline{A}) \longleftrightarrow \max(\overline{A}/f\overline{A}) \longleftrightarrow \max(\overline{\overline{A}/f\overline{A}})$ and, by (a), the correspondence of the thesis.

Let now A and f be as above; let ${}^hA$ be the henselization of A and let $q \in \min({}^hA)$ be the minimal prime corresponding to a fixed branch of A. Put $D = ({}^hA)/q$ and $D' = D/fD$; since f is in particular a superficial element with respect to the maximal ideal of D, by [16] p. 287, we have that $e(D) = e(D')$ and the thesis follows.

Definition 1.2. Let $X \subseteq \mathbb{P}^n_k$ be a reduced projective variety and let $Y \subseteq X$ be a reduced and irreducible subvariety; let P be a closed point of Y; we say that a hypersurface H is *transversal* to Y at P if H does not contain Y nor it is tangent to it at P.

Remark 1.3. In 1.1 hypothesis (3) is necessary. In fact, consider for example, in the affine space k^3, the surface V having equation $XZ^2 - Y^2 = 0$; let $A = k[U^2, UV, V]$ be the coordinate ring of V and let $p = (UV, V)$ be the ideal which corresponds to the double line $Y = Z = 0$; at the origin O the homomorphism $k[U^2] \to k[U]$ is ramified and there is only one branch of order 2; moreover not all the sections which are transversal to the double line have, at O, only one branch of order 2: for example the sections of V with planes having equation $aX + bY = 0$ $(a \neq 0, b \neq 0)$, are cubic curves, which are reducible in a parabola and a line tangent to each other at O.

Theorem 1.4. *Let X be a reduced projective variety over an algebraically closed field k, let $\nu : \overline{X} \to X$ be the normalization of X and let $Y \subseteq X$ be a reduced and irreducible singular subvariety such that:*

(a) $Y \not\subseteq \nu(Sing(\overline{X}))$;

(b) *ν induces a morphism $\nu^{-1}(Y)_{red} \to Y$, which is étale over a non empty open $U \subseteq Y$.*

Then there exists a non empty open $V \subseteq Y$ such that, for any closed point $P \in V$ and for each hypersurface H transversal to Y at P, we have a map

$$\{branches\ of\ X\ at\ P\} \longleftrightarrow \{branches\ of\ X \cap H\ at\ P\}$$

which is one to one and order preserving.

Proof. Since $\overline{X}$ is normal and ν is a closed morphism, the closed subset $\nu(Sing(\overline{X})$ has codimension ≥ 2; consider then the open non empty subset $W = X - \nu(Sing(\overline{X}))$: we may restrict W in such a way that, for each closed point $P \in W$, Y is the unique reduced irreducible singular subvariety which contains P; moreover we may suppose that P is regular for Y. By hypothesis $\nu^{-1}(Y)_{red} \to Y$ is étale over a non empty open $U \subseteq Y$. Put now $V = U \cap W$ and $A = \mathcal{O}_{X,P}$ for each closed point $P \in V$ and let p be the prime ideal of A corresponding to Y; put moreover $B = A/p$ and let f be an element of the maximal ideal of A such that $\overline{f} = f \ mod \ p$ belongs to a regular system of parameters. A and f verify the hypothesis of 1.1 and the thesis follows.

Remark 1.5.
(1) When $\operatorname{codim}(Y) = 1$, hypothesis (a) is always verified, being $\operatorname{codim}(\nu(Sing(\overline{X})) \geq 2$.
(2) If $\operatorname{char} k = 0$, there always exists an open non empty set $U \subseteq Y$ such that $\nu^{-1}(U)_{red} \to U$ is étale. If $\operatorname{char} k > 0$, such a U may be empty and there are cases of subvarieties not well behaved with respect to transversal sections, as the following example shows.

Example 1.6. Assume that k is an algebraically closed field of characteristic $p > 0$. Put $A = k[X, Y, Z]/(Y^p + XZ^p) = k[x, y, z] = k[U^p, UV, V]$ and $W = \operatorname{spec}(A)$. The non normal locus of W is the line $L : y = z = 0$ of multiplicity p on W, which corresponds to the ideal $p = (UV, V)$. Let $Q = (a, 0, 0)$, with $a \neq 0$, be a closed point of L. Observe that $\overline{A} = k[U, V]$, $A/p = k[U^p]$, $\overline{A}/p\overline{A} = k[U]$; then the tangent cone at the point Q is $\operatorname{spec}(k[X, Y, Z]/(Y + bZ)^p)$, where $b^p = a$ and, at Q, there is only one branch of order p. Consider now the surface $F : (a-X)(1+Y+bZ+YZ^p)+Y(Y+bZ)^{p-1} = 0$; F is L-transversal at Q and the corresponding section of W has at Q one linear branch and one branch of order $p-1$, having the same tangent of multiplicity p. We have in fact:

$$(V, F) = (Y^p + XZ^p, (Y+bZ)^p(1+Y+bZ+YZ^p)+YZ^p(Y+bZ)^{p-1}) =$$

$$= (Y^p + XZ^p, (Y+bZ)^{p-1}((Y+bZ)(1+Y+bZ+YZ^p)+YZ^p))$$

Consider now the same example with $\operatorname{char} k \neq p$: in this case we obtain an L-transversal section of W, which has, at Q, p linear branches corresponding to p distinct tangents; hence the situation is similar to the case of $\operatorname{char} k = 0$.

Assume now that the ground field k is algebraically closed of characteristic 0; let X be a hypersurface of $\mathbb{P}^{n+1}_k$ and let $Y \subseteq X$ be an irreducible singular subvariety of codimension 1. In these hypothesis Theorem 1.4 is a consequence of the classical theory of equisingularity developed by Zariski in the early 1960's (see [13]).

Let $P \in Y$ be a simple point of Y , put $A = \hat{\mathcal{O}}_{X,P}$ and let p be the prime ideal of Y in A.

Definition 1.7. ([15] 3.1) The elements $y_1, \cdots, y_{n-1}$ of the maximal ideal m of A are said *Y-transversal parameters* if their images in A/p form a regular system of parameters.

Set $A_{(y)} = A/(Ay_1 + \cdots + Ay_{n-1})$ and call $\text{spec}(A_{(y)})$ an *Y-transversal section* of X at P; observe that $\dim(A_{(y)}) = 1$ (see[15] 3.5), hence $\text{spec}(A_{(y)})$ is a plane algebroid curve if it has no multiple components. Let Q be the generic point of Y; then $\text{spec}(\widehat{A_p})$ is the only $Y-$transversal section of X at Q, since in this case $\dim(\widehat{A_p}) = 1$ and hence the empty set is the only set of Y-transversal parameters; moreover $\text{spec}(\widehat{A_p})$ is a plane algebroid curve defined over the field $k(p)$.

Definition 1.8. ([15] 4.1) X is said to be *equisingular* at P along Y if there exists a Y-transversal section $\text{spec}(A_{(y)})$ of X at P, such that $\text{spec}(A_{(y)})$ is a curve and such that $\text{spec}(\widehat{A_{(y)}})$ and $\text{spec}(\widehat{A_p})$ have equivalent singularities at P and at Q respectively.

Here equivalence of singularity of plane curves is intended in the sense defined by Zariski (see [13]), by comparing the successive locally quadratic transformations which resolve the singularities of the two curves at P and at Q respectively.

Proposition 1.9. *If X is equisingular at P along Y, then all Y-transversal sections of X at P are curves with equivalent singularities at P.*

Proof. See [15] 5.3.

It can be shown in numerous ways (see[14]) that given a hypersurface X, whose singular locus Y is non singular and of codimension 1, the points of Y where X is equisingular form a dense open subset of Y.

Corollary 1.10. *Let X and Y be as above, then there exists a non empty open $V \subseteq Y$ such that:*

(a) *for each closed point $P \in V$, there is a one to one correspondence $\{branches\ of\ X\ at\ P\} \longleftrightarrow \{branches\ of\ any\ Y-transversal\ section\ of\ X\ at\ P\} \longleftrightarrow \{\ geometric\ branches\ of\ X\ at\ the\ generic\ point\ of\ Y\ \}$*

(b) *Theorem 1.4 holds for each closed point $P \in V$.*

Proof.

(a) Let $y_1, \cdots, y_{n-1}$ be a set of Y-transversal parameters of A. By hypothesis A is a Cohen-Macaulay ring; moreover $\dim(A) = n$ and $\dim(A/(Ay_1 + \cdots + Ay_{n-1})) = 1$; hence $y_1, \cdots, y_{n-1}$ form a regular sequence in A. Since A is a local ring, $\overline{y}_2, \cdots, \overline{y}_{n-1}$ is a regular sequence in A/Ay_1; in the same way $\overline{y}_3, \cdots, \overline{y}_{n-1}$ is a regular sequence in $A/(Ay_1 + Ay_2)$ and so on. As X is equisingular along Y at x, the ring $A/(Ay_1, \cdots, Ay_{n-1})$ is reduced; by [10] 6.5 it follows that $A/(Ay_1, \cdots, Ay_{n-i})$ is reduced, for each $i = 2, \cdots, n$ and that

$$c(A) \leq c(A/Ay_1) \leq \cdots \leq c(A/(Ay_1, \cdots, Ay_{n-1}))$$

On the other hand, as a consequence of equisingularity, we have $c(A/(Ay_1, \cdots, Ay_{n-1})) = c(\widehat{A_p})$ and, shrinking if necessary the open set V, we have , by [2] 3.2, $c(A) = c(\widehat{A_p})$, hence the conclusion follows by [2] 3.3(i).

(b) By the proof of (a), we have in particular that $c(A) = c(A/Ay_1)$.

Remark 1.11.

(1) 1.10 generalizes a classical result about "general points" of a multiple curve on a surface (see e.g. [7] vol.II, pg. 640).

(2) Statement 1.10 does not hold in general in positive characteristic, since it is based on a concept of equivalence of plane curve singularities which works only in the zero characteristic case, because of pathologies due to inseparability (see for example [1]).

2. Let P be a closed point of an algebraic variety $X \subseteq \mathbb{P}^n_k$: we know (see example 1.6) that the equality $b_X(P) = b_{X\cap F}(P)$ does not hold in general. In this section we examine various situations from this point of view ; we discuss in particular the case of isolated points.

Proposition 2.1. *Let $P \in X$ be any closed point, assume that k is an algebraically closed field of any characteristic and let F be any hypersurface through P and not tangent to any branch of X at P; then if the branches of X at P are linear, $X \cap F$ has at P the same number of branches and they are all linear.*

Proof. Put $A = {}^h\mathcal{O}_{X,P}$, let q be a minimal prime of A. Put $B = A/q$, let m be the maximal ideal of B and let $f \in m$ be an element corresponding to F. For such an f put $B' = B/fB$; by hypothesis f is superficial with respect to m and $e(B) = 1$, hence $e(B) = e(B') = 1$; therefore B' is regular, in particular it is the ring of a linear branch. Put now $A' = A/fA$; we have: $e(A) = \Sigma_q e(A/q)$ and $e(A') = \Sigma_{q'} e(A'/q')$, where q (resp. q') ranges in the set of minimal primes of A (resp. A') (see [2] 2.7); since f is superficial, we have $e(A) = e(A')$ and hence $b(A) = b(A')$.

Lemma 2.2. *Let k be a field of characteristic* 0*, (A, m) a local excellent k-algebra, B a finite normal A-algebra and let $x_1, \cdots, x_n$ be elements of m; for $\lambda = (\lambda_1, \cdots, \lambda_n) \in k^n$, put $x_\lambda = \Sigma\lambda_i x_i$; suppose moreover $m = rad(x_1, \cdots, x_n)$. Then:*

(i) *there is a non empty open set $U \subseteq k^n$ such that, if $\lambda \in U$,* $\mathrm{spec}(B/x_\lambda B) - V(mB/x_\lambda B)$ *is normal;*

(ii) *if moreover $depth(B_M) \geq 3$, for each $M \in$* $\max(B)$ *and $\lambda \in U$, then $B \otimes_A A/x_\lambda A$ is normal.*

Proof. See[9] 3.4 and [5] 1.4, 1.8.

Theorem 2.3. *Let $X \subseteq \mathbb{P}^n_k$ be a reduced irreducible variety over an algebraically closed field k of characteristic zero, such that* $\dim(X) \geq 3$ *and let $\nu : \overline{X} \to X$ be the normalization morphism. Let $P \in X$ be an isolated singularity and assume that $depth(\mathcal{O}_{\overline{X},Q}) \geq 3$ for each $Q \in \nu^{-1}(P)$. Let S be a linear system on X, which has P as base point and which separates the tangent vectors at P (see [12] 7.3, p. 152); then, for a general element F of S, there is a map:*

$$\{branches\ of\ X\ at\ P\} \leftrightarrow \{branches\ of\ F\ at\ P\}$$

which is one to one and order preserving.

Proof. If P is an isolated normal singularity the thesis follows immediately by 2.2(ii). If P is an isolated non-normal singularity put $A := \mathcal{O}_{X,P}$ and let m be the maximal ideal of A; since P is an isolated singularity, the conductor of A in $\overline{A}$ is m-primary; moreover for a regular element $f \in m$, the conductor is not contained in any minimal associated prime to fA and we deduce (as in the proof of 1.1, part (a)), that the integral closures of $(A/fA)_{red}$ and of $(\overline{A}/f\overline{A})_{red}$ coincide. Let now $m = (x_1, \cdots, x_n)$; for $\lambda \in k^n$, put $x_\lambda = \Sigma\lambda_i x_i$; then by 2.2 there exists a non empty open set U such that for $\lambda \in U$, $\overline{A}/x_\lambda\overline{A}$ is normal, therefore we have the conclusion by the correspondence

$$\max(\overline{A}) \leftrightarrow \max(\overline{A}/x_\lambda\overline{A})$$

Nothing we can say if $\dim(X) = 2$ using the above methods; in this hypothesis, however we can observe that something quite different might happen: we present in particular a situation at which the equality $b_X(P) = b_{X\cap F}(P)$ does never hold.

Proposition 2.4. *Assume that k is an algebraically closed field of any characteristic. Let $X \subseteq \mathbb{P}^n_k$ be a 2-dimensional integral cone and let P be its vertex. Then, for any general hypersurface F of a sufficiently large degree containing P, $X \cap F$ has at P exactly $e(\mathcal{O}_{X,P})$ linear branches.*

Proof. Put $A = \mathcal{O}_{X,P}$ and let $f \in \mathcal{O}_{X,P}$ be an element corresponding to an hypersurface F passing through P; let $B = A/fA$. Then A/fA is a 1-dimensional ring and, since X is a cone, we have $\mathrm{Gr}(B) = A/\tilde{f}A$, where $\tilde{f}$ is the initial form of f in A. By Bertini's Theorem (see e. g. [11] 6.11 (2) with $d = 2$) and by an argument similar to [4] Lemma 5, it follows that for any general hypersurface F of a sufficiently large degree passing through P, $X \cap F \backslash P$ is a (geometrically) reduced scheme; this means in particular that $\mathrm{Proj}\,\mathrm{Gr}(B)$ is reduced. As it is well known, the closed points of $\mathrm{Proj}\,\mathrm{Gr}(B)$ are in 1-1 correspondence with the tangents to $X \cap F$ at P, hence, by [3] 2.3 and 2.4 (i), it follows that the branches of B are all linear and their number is $e(B)$; moreover $e(B) = e(A)$, since f is superficial.

Corollary 2.5. *Assume that $P \in X$ is an isolated singular point of the surface X and that the projectivized tangent cone to X at P is integral, then for any general hypersurface F of sufficiently large degree passing through P, $X \cap F$ has at P $e(\mathcal{O}_{X,P})$ linear branches.*

Proof. Put $A = \mathcal{O}_{X,P}$; it is sufficient to apply the proof of 2.4 to the "cone" $\mathrm{spec}(\mathrm{Gr}(A))$, since $\mathrm{spec}\,(\mathrm{Gr}(A/fA))$ is the tangent cone to $X \cap F$ at P.

3. Using Bertini type theorems, we can study from a global point of view the behaviour of branches of an algebraic variety X with respect to the general element of a linear system on X in any characteristic, in particular we deduce a result on the branches of a projective variety under hyperplane sections.

Theorem 3.1. *Let X be a scheme of finite type over an algebraically closed field k of any characteristic and let $f : X \longrightarrow \mathbb{P}^n_k$ be a morphism. Then there exists a non empty open subset $U \subseteq (\mathbb{P}^n_k)^\vee$ such that, for each hyperplane $H \in U$ and for each closed point $x \in f^{-1}(H)$, the branches of X at x are in one to one correspondence with the branches of $f^{-1}(H)$ at x.*

Proof. Let $\mathbb{P} = \mathbb{P}^n_k$ and let Z be the reduced subscheme of $\mathbb{P} \times \mathbb{P}^\vee$ whose set of closed points is $\{(x, H) \in \mathbb{P} \times \mathbb{P}^\vee | x \in H\}$. Consider the commutative diagram

$$\begin{tikzcd}
\overline{X} \arrow[r, "\nu"] & X \arrow[r, "f"] & \mathbb{P} & \\
\overline{X} \times_{\mathbb{P}} Z \arrow[r] \arrow[u] & X \times_{\mathbb{P}} Z \arrow[r] \arrow[u] & Z \arrow[r] \arrow[u] & \mathbb{P}^\vee
\end{tikzcd}$$

where ν is the normalization morphism. Clearly, for each hyperplane $H \subseteq \mathbb{P}$ corresponding to an element s of $\mathbb{P}^\vee$, we have, by the commutativity of the diagram above: $f^{-1}(H) \cong X_s \cong X \times_{\mathbb{P}} Z_s$ and $\overline{X}_s \cong \overline{X} \times_{\mathbb{P}} Z_s$. By the theorem of generic flatness and by [6] Lemma 4, there exists a non empty open $U \subseteq \mathbb{P}$ such that, for each $s \in U$, the induced morphism $\nu_{|\overline{X}_s} : \overline{X}_s \to X_s$ is birational. Moreover the fiber of $\nu_{|\overline{X}_s}$ over a point $x \in X_s$ can be identified with the fiber of ν over $x \in X$.

Now, by [17] Theorem 1.2, we may assume that, for each $s \in U$, $\overline{X}_s$ is geometrically UB, or equivalently (see[8]I 3.5) that the normalization morphism $\overline{(\overline{X}_s)_{red}} \to \overline{X}_s$ is a universal homeomorphism. This completes the proof.

Corollary 3.2. *Let V be a scheme of finite type over an algebraically closed field k; let S be a finite dimensional linear system on V and let $V \longrightarrow \mathbb{P}^n_k$ be the rational map corresponding to S. Let D be a general element* [1] *of S, considered as a subscheme of V. Then at each closed point $x \in D$, but perhaps at the base points of S, the branches of X at x are in one to one correspondence with the branches of D at x.*

Proof. Let X be the complementary of the base locus of S. Then X is open and we may apply Theorem 3.1 to the morphism $X \to \mathbb{P}^n_k$ induced by S.

Corollary 3.3. *Let $X \subseteq \mathbb{P}^n_k$ be a projective reduced variety over an algebraically closed field k of any characteristic. Then the number of branches of X at its closed points is preserved by the generic hyperplane section.*

Remark 3.4.

(1) If $\mathrm{char}\, k = 0$ the last step of the proof of Theorem 3.1 follows also by [6] Theorem 1 and the following Corollary 1 with $\mathcal{P} = Normal$: in this hypothesis, indeed, after shrinking the open set U if necessary, $\overline{X}_s$ is normal, for each $s \in U$.

[1] General element means as usual element of a suitable dense open subset of the projective space parameterizing S.

(2) If char $k = 0$ Corollary 3.3 is a consequence of the local statements of the first section. Let indeed H be a generic hyperplane; by [9] 5.2 we have that $H \cap Nor(X) \subseteq Nor(H \cap X)$, moreover H does not contain any component of the non normal locus of X, nor it is tangent to any of them; let Y be a singular subvariety of X and let $V \subseteq X$ be an open set as in 1.3, then $V \cap H$ is non empty ([5] 3.4) and the conclusion follows.

Note added in proofs. recently G. Castaldo and G. Ilardi gave analogous results about hypersurrface sections in the case of ordinary multiplr subvarieties of codimension one and char $k = 0$ (see Communications in Algebra, 29 - 7 (2001), pp. 2923–2933).

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