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ON S-ASYMPTOTICALLY (ω, Q)-PERIODIC MILD SOLUTIONS FOR SOME NEUTRAL FRACTIONAL STOCHASTIC EVOLUTION EQUATIONS

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Recent results concerning (ω, Q)-periodic functions are extended to recurrent stochastic processes in the square mean sense, termed S-asymp -totic (ω, Q)-periodic processes, where Q denotes a linear isomorphism from a Hilbert space to itself. These quasi-periodic stochastic processes covers S-asymptotic ω-(anti-)periodic processes, Bloch and (ω, c)-perio -dic stochastic processes in Hilbert spaces. We prove the completeness, convolution and superposition theorems for the S-asymptotic (ω, Q)-perio -dic process in abstract spaces. We also consider some existence results for S-asymptotically (ω, Q)-periodic mild solutions to stochastically forced fractional evolution equations under some different conditions. Somme examples are given to illustrate the existence results.

1. Introduction

In the last decades, many scholars have great interest in studying the properties of periodic solutions of stochastic and deterministic evolution equations due to their importance for both pure and applied mathematics. Many real-world phenomena do not satisfy conditions of strict periodicity, which are often tough to fulfil. Researchers have developed some generalized quasi-periodic functions,

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such as almost periodic functions, asymptotic periodic functions, asymptotic almost periodic functions, pseudo almost periodic functions and S-asymptotic periodic functions and so on, to better investigate and represent these periodic behaviours and their mathematical models. These type of functions are not exactly periodic, but posses some periodic characteristics. They are helpful for modelling complex systems that have fluctuations or perturbations. For more details on these subjects, see the monographs [16, 21, 22] and references therein.

Particularly, the class of (ω, c)-periodic functions was introduced and formally examined by E. Alvarez, A. Gómez and M. Pinto [11] in 2018 and rapidly explored by many others such as Abadias et al. [8], Mophou and N'Guérékata [9], Li et al. [19]. Assume that $(\mathbb{E}, \|\cdot\|)$ is a complex Banach space, $J = [0, \infty)$ or $J = \mathbb{R}$ and let $\omega > 0$ and c a non-zero complex number. A continuous function $\vartheta : J \to \mathbb{E}$ is said to be (ω, c)–periodic if $\vartheta(t + \omega) = c\vartheta(t)$. By Floquet's theorem, linear system involving periodic coefficients generate (ω, c)–periodic solutions. Some mathematicians have also examined how small changes can affect (ω, c)-periodic functions in abstract spaces. For example, Alvarez, Castillo and Pinto [12, 13] defined the concepts of (ω, c)-asymptotically periodic functions and (ω, c)-pseudo periodic functions in abstract spaces and applied them to the abstract Cauchy problem of first order and the Lasota-Wazewska model with unbounded and ergodic production of red cells. Recently, the concept of pseudo S-asymptotically (ω, c)-periodic was introduced by Chang et al. [14], which extends the S-asymptotically ω-periodic functions. A study on fundamental properties and applications of S-asymptotically ω-periodic functions can be found in [15, 16, 21]. Recent progress in stochastic models, have broadened the purview of periodicity analysis. For instance, authors of [3] explored S-asymptotically (ω, c)-type periodicity and its applications to stochastic evolution equations, whereas [25] studies the Stepanov-like pseudo S-asymptotically (ω, c)-periodic solutions of a class of stochastic integro-differential equations. For more developments on quasi-periodicity of stochastic systems, we refer to [28, 29].

Very recently, M. Feĉkan, K. Liu and J. Wang [23] generalized the notions Q-affine-periodicity [24] and (ω, c)-periodicity [11] to (ω, Q)-periodicity, where Q is linear isomorphism on a Banach space X (precisely a continuous function $\vartheta : J \to \mathbb{E}$ is referred as (ω, Q)–periodic if $\vartheta(t + \omega) = Q\vartheta(t)$). The conceptual necessity of replacing the scalar factor $c \in \mathbb{C}$ by a linear isomorphism Q is already visible at the deterministic level. In the foundational work of Zhang, Yang and Li [24], a system $\vartheta' = f(t, \vartheta)$ is called Q-affine-periodic if there exist $T > 0$ and $Q \in GL(n)$ such that

$$f(t+T, \vartheta) \;=\; Q f\big(t, Q^{-1}\vartheta\big), \qquad \forall\,(t, \vartheta) \in \mathbb{R} \times \mathbb{R}^n.$$

This single structural condition simultaneously encodes three qualitatively distinct types of behaviour, depending on the choice of Q. When $Q = \mathrm{id}$, the system is reduced to classical T-periodicity; when $Q = -\mathrm{id}$, it captures anti-periodicity; and when $Q \in SO(n)$, it describes rotating periodicity: solutions that rotate in the state space as time advances by one period T. None of these last two cases can be subsumed under an (ω, c)-periodic framework with a mere scalar c as demonstrated in [24]. As a concrete illustration, consider the simple equation

$$\vartheta' + 2\vartheta = e^{-t}.$$

For any $\tau > 0$, setting $Q = e^{-\tau}$ gives $f(t+\tau, \vartheta) = e^{-\tau} f(t, e^{\tau}\vartheta)$ with $f(t, \vartheta) = -2\vartheta + e^{-t}$, and all solutions satisfy the dissipativity condition $|e^{\tau m}\vartheta(t+m\tau)| \leq e^{-2t}|c| + 1$, which is uniformly bounded. By the main theorem of [24], this system therefore admits a Q-affine-periodic solution, which is precisely $\vartheta(t) = e^{-t}$, a solution that is distinct from any classically periodic solution. Moreover, they proved that affine-dissipativity, ultimate boundedness of the rescaled orbit $Q^{-m}\vartheta(t+mT)$, is sufficient for the existence of a Q-affine-periodic solution, constituting a far-reaching generalisation of Yoshizawa's classical theorem for dissipative periodic systems. Beyond scalar rescaling, phenomena that require a linear isomorphism Q arise in several concrete settings. In multidimensional neural networks, synaptic coupling between nodes introduces a linear mixing of state variables between successive periods that no scalar c can encode; see [27] for (ω, Q)-periodic solutions applied to Hopfield-type models. In stochastic dynamical systems subject to affine-periodic forcing, both the drift and diffusion coefficients satisfy $f(t+T, \vartheta) = Qf(t, Q^{-1}\vartheta)$ and $g(t+T, \vartheta) = Qg(t, Q^{-1}\vartheta)$ for an invertible matrix Q, so that when $Q \in SO(n)$ the model describes rigid-body rotation under stochastic perturbation [32, 33]. In rotating and precessing mechanical systems, $Q \in SO(n)$ is the rotation matrix by the precession angle accumulated over one period, a setting extensively studied in [24]. Finally, in interconnected biological or economic compartment models, the coupling matrix between compartments induces a non-scalar transformation of the state vector across successive periods [30, 31]. In each of these settings, replacing the scalar c by the operator Q is not a formal abstraction but a modelling necessity: the structural evolution of the state between successive periods is linear but non-scalar.

The stochastic and fractional analogue of this picture is the subject of the present paper. More precisely, classical S-asymptotic periodicity is insufficient for the systems described above for the same reason that (ω, c)-periodicity is insufficient: the asymptotic periodicity of the orbit is governed, in the limit, not by a scalar rescaling but by a linear transformation Q. This motivates the introduction of S-asymptotically (ω, Q)-periodic stochastic processes in the square-mean sense, as the natural stochastic-fractional counterpart of the affine-periodic framework of [24]. Specifically, taking $Q = \mathrm{id}$ recovers the classical square-mean S-asymptotically ω-periodic processes, while $Q = -\mathrm{id}$ yields the square-mean S-asymptotically ω-anti-periodic ones. Setting $Q = cI$ for $c \in \mathbb{C} \setminus \{0\}$ recovers the S-asymptotically (ω, c)-periodic stochastic processes of [3], and for general $Q \in GL(H)$ one captures Bloch-type and rotating-periodic stochastic behaviour not accessible by any of the above special cases. The aim of this work is to develop this stochastic analogue of (ω, Q)-periodicity by introducing the concept of S-asymptotically (ω, Q)-periodic processes in the square-mean sense, and to establish some fundamental properties of such stochastic processes, including completeness, convolution invariance, and a superposition principle. As an application, we develop existence and uniqueness theorems for S-asymptotically (ω, Q)-periodic process solutions to

$$\begin{cases} d\mathcal{N}(t, v_t) = \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} A\mathcal{N}(s, v_s)dsdt + f(t, v_t)dt + g(t, v_t)d\mathbb{W}(t), \, t \geq 0 \\ v(t) = \psi(t), t \in [-q, 0] \end{cases} \tag{1}$$

where $\alpha \in (1,2)$, $\mathcal{N}(t, \varphi) = \varphi(0) + h(t, \varphi)$, with $A : D(A) \subset \mathbb{L}^2(\Omega, \mathbb{H}) \to \mathbb{L}^2(\Omega, \mathbb{H})$ a linear densely defined operator of sectorial type on a real separable Hilbert space $\mathbb{H}$, v_t be defined by $v_t(\ell) = v(t+\ell)$ pour $\ell \in [-q, 0]$. $f : \mathbb{R}^+ \times \mathcal{B}_0 \to \mathbb{L}^2(\Omega, \mathbb{H})$ and $g : \mathbb{R}^+ \times \mathcal{B}_0 \to \mathbb{L}^2(\Omega, \mathbb{H})$ are continuous functions subject to some additional conditions where $\mathbb{L}^2(\Omega, \mathbb{H}$ and the phase space $\mathcal{B}_0$ are appropriate functions spaces which will be appointed later. $\mathbb{W}(t)$ is a one-sided and standard one-dimensional Brownian motion on a separable Hilbert space $\mathbb{K}$. Specifically, under global and local Lipschitz condition on non-linear terms, we show two existence and uniqueness theorems (see Theorem 4.2 and Theorem 4.3). Finally, two examples are also provided to illustrate our theoretical outcomes.

2. Preliminaries and notations

Let $A : D(A) \subseteq \mathbb{H} \to \mathbb{H}$ be a closed, linear and sectorial operator of type μ and angle θ which means that there exist $0 < \theta < \pi/2$, $M > 0$ and $\mu \in \mathbb{R}$ such that the resolvent $\rho(A)$ of A exists outside the sector

$$\mu + S_\theta = \{\mu + \lambda : \lambda \in \mathbb{C}, |arg(-\lambda)| < \theta\} \text{ and } \|(\lambda - A)^{-1}\| \leq \frac{M}{|\lambda - \mu|}, \lambda \notin \mu + S_\theta.$$

The operator A is the generator of a solution operator if there exist $\mu \in \mathbb{R}$ and a strongly continuous function $\mathcal{J}_\alpha : \mathbb{R}^+ \to \mathcal{B}(\mathbb{H})$ such that

$$\{\lambda^\alpha : Re(\lambda) > \mu\} \subset \rho(A) \text{ and } \lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}x = \int_0^\infty e^{-\lambda t}\mathcal{J}_\alpha(t)dt, \mathrm{Re}(\lambda) > \mu, x \in \mathbb{H}.$$

$\mathcal{J}_\alpha(\cdot)$ is called the solution operator generated by A. From [2], it follows that if A is sectorial of type μ with $1<\theta<\pi(1-\frac{\alpha}{2})$, then A is the generator of a solution operator given by

$$\mathcal{J}_\alpha(t)=\frac{1}{2\pi i}\int_\gamma e^{\lambda t}\lambda^{\alpha-1}(\lambda^\alpha-A)^{-1}d\lambda,$$

where γ is a is a suitable path lying outside the sector $\mu+S_\theta$. According to [1] if A is a sectorial operator of type $\mu<0$, for some $M>0$ and $0<\theta<\pi(1-\frac{\alpha}{2})$, there is $C>0$ such that

$$\|\mathcal{J}_\alpha(t)\|\leq\frac{CM}{1+|\mu|t^\alpha},\quad t\geq 0.$$

By using the same setting as in [3], we suppose that $(\Omega,\mathcal{F},\mathbb{P})$ represents a probability space, $\mathbb{H}$ is separable Hilbert space, and $\mathbb{K}$ indicates a real separable Hilbert space. For convenience, the same notations $\|\cdot\|$ and $(\cdot,\cdot)$ are applied to denote the norms and the inner products in $\mathbb{H}$ and $\mathbb{K}$. We denote by $\mathcal{L}(\mathbb{K},\mathbb{H})$ the Banach space of all bounded linear operators from $\mathbb{K}$ to $\mathbb{H}$ endowed with the topology defined by the operator norm, and $\mathbb{L}^2(\Omega,\mathbb{H})$ stands for the collection of all strongly-measurable, square-integrable $\mathbb{H}$-valued random variables, which is a Hilbert space endowed with the norm

$$\|v\|_{\mathbb{L}^2}=(\mathsf{E}\|v\|^2)^{1/2},\, u\in\mathbb{L}^2(\Omega,\mathbb{H})$$

where $\mathsf{E}(\cdot)$ is the expectation defined by $\mathsf{E}\|v\|^2=\int_\Omega\|v\|^2d\mathbb{P}$. In addition, let

$$\mathbb{L}_0^2(\Omega,\mathbb{H})=\{v\in L^2(\Omega,\mathbb{H})\mid v\text{ is }\mathcal{F}_0\text{ measurable}\}$$

and

$$\mathcal{B}_0=C\left([-q,0],L_0^2(\Omega,\mathbb{H})\right)$$

the Banach space of all bounded continuous stochastic process from $[-q,0]$ into $L_0^2(\Omega,\mathbb{H})$ equipped with the norm $\|v\|_{\mathcal{B}_0}^2=\sup_{-q\leq r\leq 0}\mathsf{E}\|v(r)\|^2$.

Definition 2.1. A stochastic process $v:\mathbb{R}^+\to L^2(\Omega,\mathbb{H})$ is said to be:

(i) bounded if there exists a constant $M>0$ such that $\|v\|_\infty=\sup_{t\in\mathbb{R}^+}\|v(t)\|<M$.

(ii) continuous if $\lim_{t\to s}E\|v(t)-v(s)\|^2=0\quad\text{for all}\quad s\in\mathbb{R}^+$.

We denote $\mathcal{BC}(\mathbb{R}^+, L^2(\Omega,\mathbb{H}))$ resp. $\mathcal{C}(\mathbb{R}^+, L^2(\Omega,\mathbb{H}))$ the Banach space of all bounded and continuous (resp. continuous) stochastic processes v from $\mathbb{R}^+$ into $L^2(\Omega,\mathbb{H})$ with the norm $\|v\|_\infty = \Big(\sup_{t\in\mathbb{R}^+} E\|v(t)\|^2\Big)^{1/2}$. The space $\mathcal{BC}(\mathbb{R}^+ \times \mathcal{B}_0, L^2(\Omega,\mathbb{H}))$ is the collection of all functions $h : \mathbb{R}\times\mathcal{B}_0 \to, L^2(\Omega,\mathbb{H})$ such that $h(\cdot,\phi) \in \mathcal{BC}(\mathbb{R}^+, L^2(\Omega,\mathbb{H}))$ uniformly for each $\phi \in \mathcal{B}_0$.

In the sequel, the set $\mathcal{C}(\mathbb{R}^+ \times L^2(\Omega,\mathbb{H}), L^2(\Omega,\mathbb{H}))$ represents the collection of all continuous function $g : \mathbb{R}^+ \times L^2(\Omega,\mathbb{H}) \to L^2(\Omega,\mathbb{H})$.

3. Square-mean S-asymptotically (ω, Q)-periodic process

We commence this section with the definition of square mean S–asymptotically (ω,Q)-periodic processes and basic facts about the space of these processes.

Definition 3.1. Let $Q : \mathbb{H} \to \mathbb{H}$ a linear isomorphism. A stochastic process $v \in \mathcal{BC}(\mathbb{R}^+, \mathbb{L}^2(\Omega,\mathbb{H}))$ is called square mean S-asymptotically (ω,Q)-periodic if for given $\omega > 0$

$$\lim_{t\to+\infty} E\|v(t+\omega) - Qv(t)\|^2 = 0.$$

The collection of square mean S-asymptotically (ω,Q)-periodic stochastic process $v : \mathbb{R}^+ \to L^2(\Omega,\mathbb{H})$ is then denoted by $\mathcal{SAP}_{\omega,Q}(\mathbb{R}^+, L^2(\Omega,\mathbb{H}))$.

Definition 3.2. Let $Q : \mathbb{H} \to \mathbb{H}$ a linear isomorphism, $q \geq 0$ and $\omega > 0$. A stochastic process $v : [-q,+\infty) \to \mathbb{L}^2(\Omega,\mathbb{H})$ is referred to be square mean S-asymptotically (ω,Q)-periodic if $v|_{[-q,0]} \in C\big([-q,0], L^2(\Omega,\mathbb{H})\big)$ and $v^+ = v|_{[0,+\infty)} \in \mathcal{SAP}_{\omega,Q}(\mathbb{R}^+, L^2(\Omega,\mathbb{H}))$.

The collection of theses stochastic process will be denoted by $\mathcal{SAP}_{\omega,Q}([-q,+\infty), L^2(\Omega,\mathbb{H}))$. As a starting point we establish the following two lemmas.

Lemma 3.3. Let $v_1, v_2 \in \mathcal{SAP}_{\omega,Q}(\mathbb{R}^+, L^2(\Omega,\mathbb{H}))$, $\alpha \in \mathbb{C}$ and $b \in \mathbb{R}^+$. Then:

(i) $v_1 + \alpha v_2 \in \mathcal{SAP}_{\omega,Q}(\mathbb{R}^+, L^2(\Omega,\mathbb{H}))$.

(ii) $\big(\mathcal{SAP}_{\omega,Q}(\mathbb{R}^+, L^2(\Omega,\mathbb{H})), \|\cdot\|_\infty\big)$ is a Banach space, where $\|v\|_\infty^2 = \sup_{t\in\mathbb{R}^+}\big\{E\|v(t)\|^2\big\}$, $v \in \mathcal{SAP}_{\omega,Q}(\mathbb{R}^+, L^2(\Omega,\mathbb{H}))$.

Proof. (i) Let $\alpha \in \mathbb{C}$ and $\varepsilon > 0$. From definition 3.1, there exists a constant $T_\varepsilon > 0$ such that for each $t \geq T_\varepsilon$, we have:

$$E\|v_1(t+\omega) - Qv_1(t)\|^2 < \frac{\varepsilon}{4}, \quad E\|v_2(t+\omega) - Qv_2(t)\|^2 < \frac{\varepsilon}{4(|\alpha|^2+1)}$$

and

$$\begin{aligned} &E\|(v_1+\alpha v_2)(t+\omega)-Q(v_1+\alpha v_2)(t)\|^2 \\ &\qquad\le 2E\|v_1(t+\omega)-Qv_1(t)\|^2+2|\alpha|^2E\|v_2(t+\omega)-Qv_2(t)\|^2 \\ &\qquad\le \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon. \end{aligned}$$

Thus $v_1+\alpha v_2\in\mathcal{SAP}_{\omega,Q}(\mathbb{R}^+,L^2(\Omega,\mathbb{H}))$.

(ii) Let $\{v_n\}_{n\in\mathbb{N}}\subset\mathcal{SAP}_{\omega,Q}(\mathbb{R}^+,L^2(\Omega,\mathbb{H}))$ such that $v_n\to X$ as $n\to+\infty$. Then for any $\varepsilon>0$, there exist $N_\varepsilon>0$ and $T_\varepsilon>0$ such that

$$\|v_n-X\|_\infty^2<\frac{\varepsilon}{(9|Q|^2_{\mathcal{L}(\mathbb{H})}+1)},\quad \text{for}\quad n\ge N_\varepsilon$$

and

$$E\|v_n(t+\omega)-Qv_n(t)\|^2<\varepsilon,\quad\text{for}\quad t\ge T_\varepsilon.$$

For $t\ge T_\varepsilon$, we have

$$\begin{aligned} &E\|X(t+\omega)-QX(t)\|^2 \\ &=E\|X(t+\omega)-v_n(t+\omega)+v_n(t+\omega)-Qv_n(t)+Qv_n(t)+QX(t)\|^2 \\ &\le 3E\|X(t+\omega)-v_n(t+\omega)\|^2+3E\|v_n(t+\omega)-Qv_n(t)\|^2 \\ &\qquad+3E\|Qv_n(t)-QX(t)\|^2 \\ &\le 3\|X(t+\omega)-v_n(t+\omega)\|_\infty^2+3\|v_n(t+\omega)-Qv_n(t)\|_\infty^2 \\ &\qquad+3\|Q\|^2_{\mathcal{L}(\mathbb{H})}\|v_n(t)-X(t)\|_\infty^2 \\ &\le\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon. \end{aligned}$$

This implies that the space $\mathcal{SAP}_{\omega,Q}(\mathbb{R}^+,L^2(\Omega,\mathbb{H}))$ is a closed sub-space of $\mathcal{BC}(\mathbb{R}^+,L^2(\Omega,\mathbb{H}))$. Thus it is a Banach space endowed with the norm $\|\cdot\|_\infty$.

□

Lemma 3.4. Let $X:[-q,+\infty)\to L^2(\Omega,\mathbb{H})$ be a continuous stochastic process with $X_0\in\mathcal{B}_0$ and $X^+=X_{|[0,+\infty)}\in\mathcal{SAP}_{\omega,Q}(\mathbb{R}^+,L^2(\Omega,\mathbb{H}))$. Then, the stochastic process $X_{(\cdot)}:\mathbb{R}^+\to\mathcal{B}_0$ (expressed by $X_t(r)=X(t+r)$, $r\in[-q,0]$) belongs to $\mathcal{SAP}_{\omega,Q}(\mathbb{R}^+,\mathcal{B}_0)$

Proof. First, let's show that $t \mapsto X_t \in \mathcal{BC}(\mathbb{R}^+, \mathcal{B}_0)$. For $t \geq 0$, since $X_t \in \mathcal{B}_0$ is continuous on $[-q,0]$ which is compact, there exists $r^* \in [-q,0]$ such that

$$\begin{aligned}\|X_t\|_{\mathcal{B}_0}^2 = \sup_{-q\leq r\leq 0} E\|X(t+r)\|^2 &= E\|X(t+r^*)\|^2 \\ &\leq \sup_{r\in[-q,0]} E\|X_0(r)\|^2 + \sup_{t\in\mathbb{R}^+} E\|X^+(t)\|^2 \Big\} \\ &< \infty.\end{aligned}$$

For $s \in \mathbb{R}^+$, we have

$$\begin{aligned}\lim_{t\to s}\|X_t - X_s\|_{\mathcal{B}_0}^2 &= \lim_{t\to s}\left(\sup_{-q\leq r\leq 0} E\|X(t+r) - X(s+r)\|^2\right) \\ &= \lim_{t\to s} E\|X(t+r^*) - X(s+r^*)\|^2 = 0.\end{aligned}$$

Moreover, for $t > -r^*$, we have

$$\begin{aligned}\|X_{t+\omega} - QX_t\|_{\mathcal{B}_0}^2 &= \sup_{-q\leq r\leq 0} E\|X(t+\omega+r) - QX(t+r)\|^2 \\ &= E\|X(t+r^*+\omega) - QX(t+r^*)\|^2 \\ &= E\|X^+(t+r^*+\omega) - QX^+(t+r^*)\|^2 \to 0 \quad \text{as} \quad t\to\infty\end{aligned}$$

This means that $\lim_{t\to+\infty}\left\|X_{t+\omega} - QX_t\right\|_{\mathcal{B}_0}^2 = 0$. Thus $X_{(\cdot)} \in \mathcal{SAP}_{\omega,Q}(\mathbb{R}^+, \mathcal{B}_0)$. □

For the convenience, we introduce the following space

$$\mathcal{SAP}_{\omega,Q}(\mathbb{R}^+ \times \mathcal{B}_0, \mathbb{X}) = \left\{\begin{array}{l} h \in \mathcal{BC}(\mathbb{R}^+ \times \mathcal{B}_0, \mathbb{X}) \text{ such that} \\ \lim_{t\to+\infty} E\|h(t+\omega, b) - Qh(t, Q^{-1}b)\|^2 = 0 \\ \text{for all } b \in \mathcal{B}_0. \end{array}\right\}$$

where $\mathbb{X}$ is a Banach space.

Now, we state some compositions and convolution theorems, under the following conditions: Let $h \in \mathcal{BC}(\mathbb{R}^+ \times \mathcal{B}_0, L^2(\Omega, \mathbb{H}))$ and consider the following conditions:

(A1) There exist $L > 0$ such that, for all $\varphi, \psi \in \mathcal{B}_0$, $t \in \mathbb{R}^+$,

$$\|h(t,\varphi) - h(t,\psi)\|^2 \leq L\|\varphi - \psi\|_{\mathcal{B}_0}^2.$$

(A2) For any $\varepsilon > 0$ and any bounded subset C in $\mathbb{L}^2(\Omega, \mathbb{H})$, there exists constants $T_{\varepsilon,C}$ and $\delta_{\varepsilon,C} > 0$ such that

$$E\|h(t,\varphi_1) - h(t,\varphi_2)\|^2 \leq \varepsilon$$

for all $\varphi_1, \varphi_2 \in \mathcal{B}_0$ with $E\|\varphi_1 - \varphi_2\|^2 \leq \delta_{\varepsilon,C}$ and $t \geq T_{\varepsilon,C}$.

(**A3**) There exists a function $L : \mathbb{R}^+ \to \mathbb{R}^+$ such that for each $\lambda > 0$ and for all $\varphi_1, \varphi_2 \in \mathcal{B}_0$

$$E\|h(t,\varphi_1) - h(t,\varphi_2)\|^2 \le L(\lambda)\|\varphi_1 - \varphi_2\|^2_{\mathcal{B}_0},$$

with $\|\varphi_1\|^2_{\mathcal{B}_0} \le \lambda$, $\|\varphi_2\|^2_{\mathcal{B}_0} \le \lambda$ uniformly for all $t \in \mathbb{R}^+$.

Theorem 3.5. Let $h \in \mathcal{SAP}_{\omega,Q}(\mathbb{R}^+ \times \mathcal{B}_0, L^2(\Omega,\mathbb{H}))$ verifying the assumption (**A1**). Then for every $\varphi(\cdot) \in \mathcal{SAP}_{\omega,Q}(\mathbb{R}^+,\mathcal{B}_0)$, $t \mapsto z(t) = h(t,\varphi(t)) \in \mathcal{SAP}_{\omega,Q}(\mathbb{R}^+, L^2(\Omega,\mathbb{H}))$.

Proof. Since $\varphi \in \mathcal{SAP}_{\omega,Q}(\mathbb{R}^+,\mathcal{B}_0)$, then $\lim_{t\to+\infty} E\|Q^{-1}\varphi(t+\omega) - Q\varphi(t))\|^2 = 0$

$$\begin{aligned}
&E\|h(t+\omega,\varphi(t+\omega)) - Qh(t,\varphi(t))\|^2\\
&\quad\le 2E\|h(t+\omega,\varphi(t+\omega)) - Q\varphi(t,Q^{-1}\varphi(t+\omega))\|^2\\
&\qquad + 2E\|Qh(t,Q^{-1}\varphi(t+\omega) - Qh(t,\varphi(t))\|^2\\
&\quad\le 2E\|h(t+\omega,\varphi(t+\omega)) - Qf(t,Q^{-1}\varphi(t+\omega))\|^2\\
&\qquad + 2L\|Q\|^2_{\mathcal{L}(\mathbb{H})}E\|Q^{-1}\varphi(t+\omega) - \varphi(t)|^2 \underset{t\to+\infty}{\longrightarrow} 0
\end{aligned}$$

i.e $h(\cdot,\varphi(\cdot)) \in \mathcal{SAP}_{\omega,Q}(\mathbb{R}, L^2(\Omega,\mathbb{H}))$. □

Theorem 3.6. Let $h \in \mathcal{SAP}_{\omega,Q}(\mathbb{R}^+ \times \mathcal{B}_0, L^2(\Omega,\mathbb{H}))$ verifying the assumption (**A2**). Then for every $\varphi(\cdot) \in \mathcal{SAP}_{\omega,Q}(\mathbb{R}^+,\mathcal{B}_0)$, $h(\cdot,\varphi(\cdot)) \in \mathcal{SAP}_{\omega,Q}\big(\mathbb{R}^+, \mathbb{L}^2(\Omega,\mathbb{H})\big)$.

Proof. Let $\varepsilon > 0$. For $\varphi \in \mathcal{SAP}_{\omega,Q}\big(\mathbb{R}^+,\mathcal{B}_0\big)$ and $h \in \mathcal{SAP}_{\omega,Q}(\mathbb{R}^+ \times \mathcal{B}_0, L^2(\Omega,\mathbb{H}))$, there exists $T_\varepsilon > 0$ such that for $t \ge T_\varepsilon$,

$$E\|h(t+\omega,\varphi(t+\omega)) - Qh(t,Q^{-1}\varphi(t+\omega))\|^2 < \frac{\varepsilon}{4} \quad \text{and} \|\varphi(t+\omega) - Q\varphi(t)\|^2_{\mathcal{B}_0} \le \varepsilon.$$

Using condition (**A2**), there exists $\delta_{\varepsilon,C} := \varepsilon$ and $T_{\varepsilon,C} := T_\varepsilon$ such that

$$E\|h(t,Q^{-1}\varphi(t+\omega)) - h(t,\varphi(t))\|^2 < \frac{\varepsilon}{4\|Q\|^2_{\mathcal{L}(\mathbb{H})}}$$

anytime $\|\varphi(t+\omega) - Q\varphi(t)\|^2_{\mathcal{B}_0} \le \varepsilon$ and $t \ge T_\varepsilon$.
We have

$$\begin{aligned}
&E\|h(t+\omega,\varphi(t+\omega)) - Qh(t,\varphi(t))\|^2\\
&\quad\le 2E\|h(t+\omega,\varphi(t+\omega)) - Qh(t,Q^{-1}\varphi(t+\omega))\|^2\\
&\qquad + 2E\|Qh(t,Q^{-1}\varphi(t+\omega)) - Qh(t,\varphi(t))\|^2\\
&\quad\le 2E\|h(t+\omega,\varphi(t+\omega)) - Qh(t,Q^{-1}\varphi(t+\omega))\|^2\\
&\qquad + 2\|Q\|^2_{\mathcal{L}(\mathbb{H})}E\|h(t,Q^{-1}\varphi(t+\omega) - h(t,\varphi(t)))\|^2\\
&\quad< \varepsilon \quad \text{for} \quad t \ge T_\varepsilon.
\end{aligned}$$

Hence $h(\cdot,\varphi(\cdot)) \in \mathcal{SAP}_{\omega,Q}\big(\mathbb{R}^+, \mathbb{L}^2(\Omega,\mathbb{H})\big)$ □

Since (**A2**) $\Rightarrow$ (**A3**), we get the following result.

Corollary 3.7. Let $h \in \mathcal{SAP}_{\omega,Q}(\mathbb{R}^+ \times \mathcal{B}_0, L^2(\Omega,\mathbb{H}))$ verifying the assumption **(A3)**. Then for each $\varphi(\cdot) \in \mathcal{SAP}_{\omega,Q}(\mathbb{R}^+,\mathcal{B}_0)$, $t \mapsto h(t,\varphi(t)) \in \mathcal{SAP}_{\omega,Q}(\mathbb{R}^+,\mathbb{L}^2(\Omega,\mathbb{H}))$.

Theorem 3.8. *Let $(\mathcal{J}_\alpha(t))_{t\geq 0}$ be an integrable solution operator on $\mathbb{H}$ such that*

$$\|\mathcal{J}_\alpha(t)\| \leq \frac{K}{1+|\mu|t^\alpha} \text{ for all } t \geq 0 \text{ with } K>0,$$

$f \in \mathcal{SAP}_{\omega,Q}(\mathbb{R}^+,L^2(\Omega,\mathbb{H}))$ and $g \in \mathcal{SAP}_{\omega,Q}(\mathbb{R}_+,L^2(\Omega,\mathcal{L}(\mathbb{K},\mathbb{H}))$, then

$$z(t) := \int_0^t \mathcal{J}_\alpha(t-s)f(s)ds + \int_0^t \mathcal{J}_\alpha(t-s)g(s)dW(s) \in \mathcal{SAP}_{\omega,Q}(\mathbb{R}^+,L^2(\Omega,\mathbb{H})).$$

Proof. We subdivide the proof in two claims.
Claim1. $z = z_f + z_g \in \mathcal{BC}(\mathbb{R}^+,L^2(\Omega,\mathbb{H}))$ where $z_f(t) = \int_0^t \mathcal{J}_\alpha(t-s)f(s)ds$ and $z_g(t) = \int_0^t \mathcal{J}_\alpha(t-s)g(s)dW(s)$, $t \geq 0$. By using Cauchy-Schwartz inequality and Itô's isometry property, we obtain,

$$\begin{aligned}
&E\|z(t)\|^2 \\
&\leq 2(E\|z_f(t)\|^2 + E\|z_g(t)\|^2) \\
&\leq 2E\Big(\int_0^t \|\mathcal{J}_\alpha(t-s)\|\|f(s)\|ds\Big)^2 + 2\int_0^t \|\mathcal{J}_\alpha(t-s)\|^2 E\|g(s)\|^2 ds \\
&\leq 2\int_0^t \frac{K}{1+|\mu|(t-s)^\alpha}ds \int_0^t \frac{K}{1+|\mu|(t-s)^\alpha}E\|f(s)\|^2 ds \\
&\qquad + 2(K)^2\|g\|_\infty^2 \int_0^t \frac{1}{(1+|\mu|(t-s)^\alpha)^2}ds \\
&\leq 2(K)^2\|f\|_\infty^2 \Big(\int_0^t \frac{K}{1+|\mu|(t-s)^\alpha}ds\Big)^2 + \frac{(K)^2}{2}\|g\|_\infty^2 \int_0^t \frac{1}{1+|\mu|^2(t-s)^{2\alpha}}ds \\
&\leq 2(K)^2\|f\|_\infty^2 \Big(\int_0^{|\mu|t^\alpha} \frac{z^{\frac{1}{\alpha}-1}}{\alpha|\mu|^{\frac{1}{\alpha}}(1+z)}dz\Big)^2 + (K)^2\|g\|_\infty^2 \int_0^{|\mu|^2t^{2\alpha}} \frac{z^{\frac{1}{2\alpha}-1}}{\alpha|\mu|^{\frac{1}{\alpha}}(1+z)}dz \\
&\leq 2(K)^2\|f\|_\infty^2 \int_0^\infty \frac{z^{\frac{1}{\alpha}-1}}{\alpha|\mu|^{\frac{1}{\alpha}}(1+z)}dz + 2(K)^2\|g\|_\infty^2 \int_0^\infty \frac{z^{\frac{1}{2\alpha}-1}}{\alpha|\mu|^{\frac{1}{\alpha}}(1+z)}dz \\
&\leq 2\frac{(K)^2|\mu|^{-1/\alpha}}{\alpha}\Gamma\big(\frac{1}{\alpha}\big)\Gamma\big(1-\frac{1}{\alpha}\big)\|f\|_\infty^2 + 2\frac{(K)^2|\mu|^{-1/\alpha}}{\alpha}\|g\|_\infty^2\Gamma\big(\frac{1}{2\alpha}\big)\Gamma\big(1-\frac{1}{2\alpha}\big) \\
&\leq 2\left[\frac{(K)^2|\mu^{-1/\alpha}\pi}{\alpha(\sin\frac{\pi}{\alpha})}\|f\|_\infty^2 + \frac{(K)^2|\mu|^{-1/\alpha}\pi}{\alpha(\sin\frac{\pi}{2\alpha})}\|g\|_\infty^2\right] < \infty
\end{aligned}$$

Let $t,t_0\in\mathbb{R}^+$ such that $t>t_0$, we have

$$
\begin{aligned}
&E\|z_f(t)-z_f(t_0)\|^2\\
&=E\Big\|\int_0^t \mathcal{J}_\alpha(t-s)f(s)ds-\int_0^{t_0}\mathcal{J}_\alpha(t_0-s)f(s)ds\Big\|^2\\
&\le E\Big\|\int_0^t \mathcal{J}_\alpha(s)f(t-s)ds-\int_0^{t_0}\mathcal{J}_\alpha(s)f(t_0-s)ds\Big\|^2\\
&\le E\Big\|\int_{t_0}^t \mathcal{J}_\alpha(s)\Big(f(t-s)ds-f(t_0-s)\Big)ds\Big\|^2\\
&\le (K)^2\int_{t_0}^t\frac{1}{1+|\mu|(t-s)^\alpha}ds\int_{t_0}^t\Big(\frac{1}{1+|\mu|(t-s)^\alpha}E\|f(t-s)ds-f(t_0-s)\|^2\Big)ds\\
&\le 4(K)^2\|f\|_\infty^2(t-t_0)^2\to 0 \text{ as } t\to t_0.
\end{aligned}
$$

and

$$
\begin{aligned}
E\|z_g(t)-z_g(t_0)\|^2&=E\Big\|\int_0^t \mathcal{J}_\alpha(t-s)g(s)dW(s)-\int_0^{t_0}\mathcal{J}_\alpha(t_0-s)g(s)dW(s)\Big\|^2\\
&=E\Big\|\int_0^t\mathcal{J}_\alpha(s)g(t-s)dW(t-s)-\int_0^{t_0}\mathcal{J}_\alpha(s)f(t_0-s)dW(t_0-s)\Big\|^2\\
&\le E\Big\|\int_{t_0}^t \mathcal{J}_\alpha(s)\Big(g(t-s)-g(t_0-s)\Big)d\widetilde{W}(s)\Big\|^2\\
&\le 4\int_{t_0}^t \|\mathcal{J}_\alpha(t-s)\|^2E\|z_g(s)\|^2ds\\
&\le 4(K)^2\|g\|_\infty^2(t-t_0)\to 0 \text{ as } t\to t_0.
\end{aligned}
$$

where, for $\xi\ge s\ge 0$, $\widetilde{W}(s)=W(\xi)-W(\xi-s)$ define a Brownian motion and has the same distribution as W. Therefore, Ito's isometry property yield that

$$
\begin{aligned}
E\|z_g(t)-z_g(t_0)\|^2&\le E\Big\|\int_{t_0}^t \mathcal{J}_\alpha(s)\Big(g(t-s)-g(t_0-s)\Big)d\widetilde{W}(s)\Big\|^2\\
&\le 4\int_{t_0}^t \|\mathcal{J}_\alpha(t-s)\|^2E\|z_g(s)\|^2ds\\
&\le 4(K)^2\|g\|_\infty^2(t-t_0)\to 0 \text{ as } t\to t_0.
\end{aligned}
$$

Hence

$$
E\|z(t)-z(t_0)\|^2\le 2E\|z_f(t)-z_f(t_0)\|^2+2E\|z_g(t)-z_g(t_0)\|^2\to 0 \text{ as } t\to t_0^+.
$$

Similarly, we get $E\|z(t)-z(t_0)\|^2\to 0$ as $t\to t_0^-$.

Thus, $z=z_f+z_g\in\mathcal{BC}(\mathbb{R}^+,L^2(\Omega,\mathbb{H}))$.

Claim 2. $\lim_{t\to+\infty} E\|z(t+\omega)-Qz(t)\|^2 = 0.$

We show that $\lim_{t\to\infty} E\|z_f(t+\omega)-Qz_f(t)\|^2 = 0$. We have

$$\begin{aligned}
&E\|z_f(t+\omega)-Qz_f(t)\|^2\\
&= E\Big\|\int_0^{t+\omega}\mathcal{J}_\alpha(t-s)f(s)ds - Q\int_0^t \mathcal{J}_\alpha(t+\omega-s)h(s)ds\Big\|^2\\
&\leq 2E\Big\|\int_0^{\omega}\mathcal{J}_\alpha(t+\omega-s)f(s)ds\Big\|^2 + 2E\Big\|\int_0^t \mathcal{J}_\alpha(t-s)\big(f(s+\omega)-Qf(s)\big)ds\Big\|^2\\
&\leq 2E\Big(\int_0^{\omega}\|\mathcal{J}_\alpha(t+\omega-s)\|\|f(s)\|ds\Big)^2 + 2E\Big(\int_0^t \|\mathcal{J}_\alpha(t-s)\|\|f(s+\omega)-Qf(s)\|ds\Big)^2\\
&\leq 2(K)^2\int_0^{\omega}\frac{1}{1+|\mu|(t+\omega-s)^\alpha}ds\int_0^{\omega}\frac{1}{1+|\mu|(t+\omega-s)^\alpha}E\|f(s)\|^2ds\\
&\quad + 2(K)^2\int_0^{t}\frac{1}{1+|\mu|(t-s)^\alpha}ds\int_0^{t}\frac{1}{1+|\mu|(t-s)^\alpha}E\|f(s+\omega)-Qf(s)\|^2ds\\
&:= 2A(t)+2B(t).
\end{aligned}$$

Next, we show that $\lim_{t\to\infty} A(t) = \lim_{t\to\infty} B(t) = 0$.

- For the first term $A(t)$ on the right hand side, we get

$$\begin{aligned}
A(t) &= (K)^2\int_0^{\omega}\frac{1}{1+|\mu|(t+\omega-s)^\alpha}ds\int_0^{\omega}\frac{1}{1+|\mu|(t+\omega-s)^\alpha}E\|f(s)\|^2ds\\
&\leq (K)^2\|f\|_\infty^2\Big(\int_0^{\omega}\frac{1}{1+|\mu|(t+\omega-s)^\alpha}ds\Big)^2\\
&\leq (K)^2\|f\|_\infty^2\Big(\int_0^{\omega}\frac{1}{(1+|\mu|t^\alpha)}ds\Big)^2 = \frac{(K)^2\omega^2}{(1+|\mu|t^\alpha)^2}\|f\|_\infty^2\\
&\leq \frac{(K)^2\omega^2}{(1+|\mu|^2t^{2\alpha})}\|f\|_\infty^2 \to 0 \text{ as } t\to\infty.
\end{aligned}$$

- For the second term $B(t)$ on the right hand side. Let $\varepsilon > 0$. Since $f \in \mathcal{SAP}_{\omega,Q}(\mathbb{R}_+, L^2(\Omega,\mathbb{H}))$, there exists $T_\varepsilon > 0$ such that,
$E\|f(t+\omega)-Qf(t)\|^2 < \dfrac{\varepsilon|\mu|^{\frac{1}{2\alpha}}\alpha^2\sin^2(\frac{\pi}{\alpha})}{(K)^2\pi^2}$, for $t \geq T_\varepsilon$. We have

$$\begin{aligned}
B(t) =& (K)^2\int_0^{t}\frac{ds}{1+|\mu|(t-s)^\alpha}\int_0^{t}\frac{1}{1+|\mu|(t-s)^\alpha}E\|f(s+\omega)-Qf(s)\|^2ds\\
=& (K)^2\int_0^{T_\varepsilon}\frac{ds}{1+|\mu|(t-s)^\alpha}ds\int_0^{T_\varepsilon}\frac{1}{1+|\mu|(t-s)^\alpha}E\|f(s+\omega)-Qf(s)\|^2ds\\
&+ (K)^2\int_{T_\varepsilon}^{t}\frac{ds}{1+|\mu|(t-s)^\alpha}ds\int_{T_\varepsilon}^{t}\frac{1}{1+|\mu|(t-s)^\alpha}E\|f(s+\omega)-Qf(s)\|^2ds.
\end{aligned}$$

Note that

$$\int_0^{T_\varepsilon} \frac{1}{1+|\mu|(t-s)^\alpha} ds \int_0^{T_\varepsilon} \frac{1}{1+|\mu|(t-s)^\alpha} E\|f(s+\omega)-Qf(s)\|^2 ds$$
$$\leq \frac{T_\varepsilon}{(1+|\mu|^2(t-T_\varepsilon)^{2\alpha})} \int_0^{T_\varepsilon} 2(1+\|Q\|^2_{\mathcal{L}(\mathbb{H})})\|f\|^2_\infty ds$$
$$\leq \frac{T_\varepsilon^2}{1+|\mu|^2(t-T_\varepsilon)^{2\alpha}} (1+\|Q\|^2_{\mathcal{L}(\mathbb{H})})\|f\|^2_\infty \underset{t\to\infty}{\to} 0\,.$$

and

$$(K)^2 \int_{T_\varepsilon}^{t} \frac{1}{1+|\mu|(t-s)^\alpha} ds \int_{T_\varepsilon}^{t} \frac{1}{1+|\mu|(t-s)^\alpha} E\|f(s+\omega)-Qf(s)\|^2 ds$$
$$\leq (K)^2 \int_0^{|\mu|(t-T_\varepsilon)^\alpha} \frac{z^{\frac{1}{\alpha}-1}}{\alpha|\mu|^{\frac{1}{\alpha}}(1+z)} dz \int_0^{|\mu|(t-T_\varepsilon)^\alpha} \frac{z^{\frac{1}{\alpha}-1}}{\alpha|\mu|^{\frac{1}{\alpha}}(1+z)} dz E\|f(s+\omega)-Qf(s)\|^2 ds$$
$$< (K)^2 \Big(\int_0^\infty \frac{z^{\frac{1}{\alpha}-1}}{\alpha|\mu|^{\frac{1}{\alpha}(1+z)}} dz\Big)^2 \times \frac{\varepsilon|\mu|^{\frac{1}{2\alpha}}\alpha^2 \sin^2(\frac{\pi}{\alpha})}{(K)^2\pi^2}$$
$$< \frac{(K)^2}{\alpha^2\mu|^{\frac{1}{2\alpha}}} \Big(\Gamma(\frac{1}{\alpha})\Gamma(1-\frac{1}{\alpha})\Big)^2 \times \frac{\varepsilon|\mu|^{\frac{1}{2\alpha}}\alpha^2 \sin^2(\frac{\pi}{\alpha})}{(K)^2\pi^2}$$
$$< \frac{(K)^2}{\alpha^2\mu|^{\frac{1}{2\alpha}}} \frac{\pi^2}{\sin^2(\frac{\pi}{\alpha})} \times \frac{\varepsilon|\mu|^{\frac{1}{2\alpha}}\alpha^2 \sin^2(\frac{\pi}{\alpha})}{(K)^2\pi^2}$$
$$< \varepsilon.$$

Thus, $\lim_{t\to+\infty} B(t) = 0$.

Hence, we get $\lim_{t\to+\infty} E\|z_f(t+\omega) - Qz_f(t)|^2 = 0$.

On the other hand, we show that $\lim_{t\to\infty} E\|z_g(t+\omega) - Qz_g(t)\|^2 = 0$. We have

$$E\|z_g(t+\omega) - Qz_g(t)\|^2$$
$$= E\Big\| \int_0^{t+\omega} \mathcal{J}_\alpha(t+\omega-s)g(s)dW(s) - Q\int_0^t \mathcal{J}_\alpha(t-s)g(s)dW(s)\Big\|^2$$
$$= E\Big\| \int_0^{t+\omega} \mathcal{J}_\alpha(t+\omega-s)g(s)dW(s) - Q\int_0^t \mathcal{J}_\alpha(t-s)g(s)dW(s)\Big\|^2$$
$$= E\Big\| \int_0^{\omega} \mathcal{J}_\alpha(t+\omega-s)g(s)dW(s) + \int_\omega^{t+\omega} \mathcal{J}_\alpha(t+\omega-s)g(s)dW(s)$$
$$\quad - Q\int_0^t \mathcal{J}_\alpha(t-s)g(s)dW(s)\Big\|^2$$
$$= E\Big\| \int_0^{\omega} \mathcal{J}_\alpha(t+\omega-s)g(s)dW(s) + \int_0^{t} \mathcal{J}_\alpha(t-s)g(s+\omega)dW(s+\omega)$$
$$\quad - Q\int_0^t \mathcal{J}_\alpha(t-s)g(s)dW(s)\Big\|^2.$$

For $\xi \in \mathbb{R}^+$, setting $\widetilde{W}(s) = W(s+\xi) - W(\xi)$ and thanks to Ito's isometry property, we get

$$\begin{aligned}
&E\|z_g(t+\omega) - Qz_g(t)\|^2\\
=&E\Big\| \int_0^{\omega} \mathcal{J}_\alpha(t+\omega-s)f(s)dW(s) + \int_0^t \mathcal{J}_\alpha(t-s)\big(g(s+\omega)-Qf(s)\big)d\tilde{W}(s)\Big\|^2\\
\leq&2E\Big\|\int_0^{\omega}\mathcal{J}_\alpha(t+\omega-s)f(s)dW(s)\Big\|^2+2E\Big\|\int_0^t\mathcal{J}_\alpha(t-s)\big(g(s+\omega)-Qf(s)\big)d\tilde{W}(s)\Big\|^2\\
\leq&2\int_0^{\omega}\|\mathcal{J}_\alpha(t+\omega-s)\|^2E\|f(s)\|^2ds+2\int_0^t\|\mathcal{J}_\alpha(t-s)\|^2E\|g(s+\omega)-Qg(s)\|^2ds\\
\leq&2A(t)+2B(t).
\end{aligned}$$

For $A(t)$, simple estimations yield that

$$A(t) \leq \frac{(K)^2\omega}{2(1+|\mu|^2t^{2\alpha})}\|f\|_\infty^2 \to 0 \text{ as } t \to \infty.$$

For $B(t)$, let $\varepsilon > 0$.
Since $f \in \mathcal{SAP}_{\omega,Q}(\mathbb{R}_+, L^2(\Omega,\mathbb{H}))$,then there exists $T_\varepsilon > 0$ such that, for $t \geq T_\varepsilon$,

$$E\|f(t+\omega) - Qf(t)\|^2 < \frac{2\varepsilon|\mu|^{\frac{1}{\alpha}}\sin(\frac{\pi}{2\alpha})}{(K)^2\frac{\pi}{2\alpha}}.$$

Then, we have

$$\begin{aligned}
B(t) =& \int_0^t \|\mathcal{J}_\alpha(t-s)\|^2E\|g(s+\omega)-Qg(s)\|^2ds.\\
=&\int_0^{T_\varepsilon}\|\mathcal{J}_\alpha(t-s)\|^2E\|g(s+\omega)-Qf(s)\|^2ds+\int_{T_\varepsilon}^t\|\mathcal{J}_\alpha(t-s)\|^2E\|g(s+\omega)-Qg(s)\|^2ds.
\end{aligned}$$

Note that

$$\int_0^{T_\varepsilon}\|\mathcal{J}_\alpha(t-s)\|^2E\|g(s+\omega)-Qg(s)\|^2ds \leq \frac{(K)^2\|g\|_\infty^2}{(1+|\mu|^2(t-T_\varepsilon)^{2\alpha})}(1+\|Q\|^2_{\mathcal{L}(\mathbb{H})})T_\varepsilon \underset{t\to\infty}{\to} 0$$

and

$$\begin{aligned}
\int_{T_\varepsilon}^t\|\mathcal{J}_\alpha(t-s)\|^2\|E\|g(s+\omega)-Qg(s)\|^2ds &< \frac{2\alpha\varepsilon|\mu|^{\frac{1}{\alpha}}\sin(\frac{\pi}{2\alpha})}{\pi}\int_0^{|\mu|^2(t-T_\varepsilon)^{2\alpha}}\frac{z^{\frac{1}{2\alpha}-1}}{2\alpha|\mu|^{\frac{1}{\alpha}}(1+z)}dz\\
&< \frac{\varepsilon\sin(\frac{\pi}{2\alpha})}{\pi}\frac{\pi}{\sin(\frac{\pi}{2\alpha})}\\
&< \varepsilon.
\end{aligned}$$

It follows that $\lim_{t\to+\infty}E\|z_g(t+\omega)-Qz_g(t)\|^2=0$. Therefore $z_g \in \mathcal{SAP}_{\omega,Q}(\mathbb{R}^+,L^2(\Omega,\mathbb{H}))$.

Consequently, since $z_f, z_g \in \mathcal{SAP}_{\omega,Q}(\mathbb{R}^+, L^2(\Omega,\mathbb{H}))$, we conclude that $z = z_f + z_g \in \mathcal{SAP}_{\omega,Q}(\mathbb{R}^+, L^2(\Omega,\mathbb{H}))$.

□

4. Square-mean *S*-asymptotically (ω-Q) periodic mild solutions

To give the definition of mild solutions to Eq.(1), we consider the following linear equation :

$$v'(t)=\int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}Av(r)dr+\ell(t),\ t\geq 0, v(0)=x_0\in\mathbb{H}. \tag{2}$$

where $\ell:[0,+\infty)\to\mathbb{H}$ is a continuous function and A generates an integrable solution operator $\mathcal{J}_\alpha(t)$. The variation of parameters formula allows us to write the mild solution of problem (2) as

$$v(t)=\mathcal{J}_\alpha(t)x_0+\int_0^t \mathcal{J}_\alpha(t-s)\ell(s)ds, \quad \text{for all } t\geq 0. \tag{3}$$

Motivated by this conclusion and by the previous related literature, we adopt the following concept for a mild solution to Equ.(1).

Definition 4.1. Let $\psi\in\mathcal{B}_0$. A stochastic process $v:[-q,+\infty)\to\mathbb{L}^2(\Omega,\mathbb{H})$ is referred as a mild solution of Equ. (1), if for any $t\in[0,\infty)$ the following conditions hold:

1. $v(t)$ is and $\mathcal{F}_t$-adapted stochastic process for $t\geq 0$;
2. $v(t)\in L^2(\Omega,\mathbb{H})$ has cádlág paths on $t\in[0,\infty)$ almost surely;
3. $v(t)=\psi(t)$ for $t\in[-q,0]$, and for each $t\geq 0$

$$\begin{aligned}v(t)=&\mathcal{J}_\alpha(t)\left[\psi(0)+h(0,\psi)\right]-h(t,v_t)+\int_0^t\mathcal{J}_\alpha(t-s)f(s,v_s)ds\\&+\int_0^t\mathcal{J}_\alpha(t-s)g(s,v_s)dW(s).\end{aligned} \tag{4}$$

Theorem 4.2. Assume that $A:D(A)\subset\mathbb{H}\to\mathbb{H}$ is a μ-sectorial operator of angle $\theta=\dfrac{(\alpha-1)\pi}{2}$ with $\mu<0$, $\alpha\in(1,2)$, $f,h\in\mathcal{SAP}_{\omega,Q}(\mathbb{R}^+\times\mathcal{B}_0,L^2(\Omega,\mathbb{H}))$ and $g\in\mathcal{SAP}_{\omega,Q}(\mathbb{R}^+\times\mathcal{B}_0,\mathcal{L}(\mathbb{K},\mathbb{H}))$. Assume further that there exist positive constants $L_f,L_g,L_h>0$ such that for $a,b\in\mathcal{B}_0$,

$$E\|f(t,a)-f(t,b)\|^2\leq L_f\|a-b\|^2_{\mathcal{B}_0},\quad E\|g(t,a)-g(t,b\|^2_{\mathcal{L}(\mathbb{K},\mathbb{H})}\leq L_g\|a-b\|^2_{\mathcal{B}_0},$$

and $E\|h(t,a)-h(t,b)\|^2\leq L_h\|a-b\|^2_{\mathcal{B}_0}$. Then the problem (1) has a unique S-asymptotically (ω,Q)-periodic mild solution provided that $3\Theta<1$ where

$$\Theta:=L_h+\frac{L_f(CM)^2}{(\alpha|\mu|^{\frac{1}{\alpha}})^2}\frac{\pi^2}{(\sin(\frac{\pi}{\alpha}))^2}+\frac{L_g(CM)^2}{4\alpha|\mu|^{\frac{1}{2\alpha}}}\frac{\pi}{\sin(\frac{\pi}{2\alpha})}.$$

Proof. Let $\mathcal{G} : \mathcal{SAP}_{\omega,Q}\Big([-q,+\infty), L^2(\Omega,\mathbb{H})\Big) \to \mathcal{SAP}_{\omega,Q}\Big([-q,+\infty), L^2(\Omega,\mathbb{H})\Big)$ defined by

$$(\mathcal{G}v)(t) = \begin{cases} \psi(t),\ t \in [-q,0] \\ \mathcal{J}_\alpha(t)\,[\psi(0)+h(0,\psi)]-h(t,u_t+m_t)+\int_0^t \mathcal{J}_\alpha(t-s)f(s,u_s+m_s)ds \\ \qquad + \int_0^t \mathcal{J}_\alpha(t-s)g(s,u_s+m_s)dW(s). \end{cases}$$

For $\psi \in \mathcal{B}_0$, we define the function $u \in \mathcal{BC}\Big([-q,+\infty), L^2(\Omega,\mathbb{H})\Big)$ by

$$u(t) = \begin{cases} \psi(t), t \in [-q,0], \\ \mathcal{J}_\alpha(t)\psi(0), t \geq 0, \end{cases}$$

and for $\widetilde{m} \in \mathcal{BC}\Big([-q,+\infty), L^2(\Omega,\mathbb{H})\Big)$ with $\widetilde{m}_0 = 0$, we denote by m the function

$$m(t) = \begin{cases} 0, t \in [-q,0], \\ \widetilde{m}(t), t \geq 0, \end{cases}$$

If $v(t) = (\mathcal{G}v)(t)$, we can decompose it as $v(t) = u(t)+m(t)$, $t \in [0,\infty)$, which provides $v_t = u_t + m_t$ for every $t \in [0,\infty)$ and $m(\cdot)$ verifies

$$\begin{aligned} m(t) =& \mathcal{J}_\alpha(t)h(0,\psi) - h(t,u_t+m_t) + \int_0^t \mathcal{J}_\alpha(t-s)f(s,u_s)+m_s)ds \\ &+ \int_0^t \mathcal{J}_\alpha(t-s)g(s,u_s+m_s)dW(s). \end{aligned}$$

Consider

$$\mathcal{SAP}^0_{\omega,Q} := \left\{ \begin{array}{r} m \in \mathcal{BC}\Big([-q,+\infty), L^2(\Omega,\mathbb{H})\Big) : m|_{[-q,0]} = 0 \text{ and} \\ m|_{[0,+\infty)} \in \mathcal{SAP}_{\omega,Q}\Big(\mathbb{R}^+, L^2(\Omega,\mathbb{H})\Big) \end{array} \right\}.$$

It is easily shown that $\Big(\mathcal{SAP}^0_{\omega,Q}, \|\cdot\|_\infty\Big)$ is a Banach space. We define the operator $\mathcal{P} : \mathcal{SAP}^0_{\omega,Q} \to \mathcal{SAP}^0_{\omega,Q}$ defined by

$$\begin{aligned} (\mathcal{P}m)(t) =& \mathcal{J}_\alpha(t)h(0,\psi) - h\big(t,u_t+m_t\big) + \int_0^t \mathcal{J}_\alpha(t-s)f\big(s,u_s+m_s\big)ds \\ &+ \int_0^t \mathcal{J}_\alpha(t-s)g\big(s,u_s+m_s\big)dW(s). \end{aligned} \tag{5}$$

For $m\in\mathcal{SAP}^0_{\omega,Q}$, we have $m_0=0$ and $m|_{[0,+\infty)}\in\mathcal{SAP}_{\omega,Q}\Big(\mathbb{R}^+,L^2(\Omega,\mathbb{H})\Big)$. By Lemma 3.4, we derive that $m_t\in\mathcal{SAP}_{\omega,Q}(\mathbb{R}^+,\mathcal{B}_0)$. Moreover, for $t\geq 0$, we have

$$
\begin{aligned}
\mathsf{E}\|u(t+\omega)-Qu(t)\|^2 &= \mathbb{E}\|\mathcal{J}_\alpha(t+\omega)\psi(0)-Q\mathcal{J}_\alpha(t)\psi(0)\|^2\\
&\leq 2\left[\left(\frac{CM}{1+|\mu|(t+\omega)^\alpha}\right)^2+\|Q\|^2_{\mathcal{L}(\mathbb{H})}\left(\frac{CM}{1+|\mu|t^\alpha}\right)^2\right]\mathbb{E}\|\psi(0)\|^2\\
&\to 0 \text{ as } t\to\infty.
\end{aligned}
$$

Since $u_0=\psi\in\mathcal{B}_0$ and $u|_{[0,\infty)}\in\mathcal{SAP}_{\omega,Q}\Big(\mathbb{R}^+,L^2(\Omega,\mathbb{H})\Big)$, by Lemma 3.4, we have $u_t\in\mathcal{SAP}_{\omega,Q}(\mathbb{R}^+,\mathcal{B}_0)$. Thus $u_t+m_t\in\mathcal{SAP}_{\omega,Q}(\mathbb{R}^+,\mathcal{B}_0)$. From Theorem (3.5), we derive that the functions $t\mapsto f(t,u_t+m_t)$, $t\mapsto g(t,u_t+m_t)$ and $t\mapsto h(t,u_t+m_t)$ belongs to $\mathcal{SAP}_{\omega,Q}(\mathbb{R}^+,L^2(\Omega,\mathbb{H}))$. Lemma 3.3, Theorem 3.5 and 3.8 yields that $\mathcal{P}m\in\mathcal{SAP}^0_{\omega,Q}$ and $\mathcal{P}$ is well defined. Furthermore, for each $v\in\mathcal{SAP}_{\omega,Q}\Big([-q,+\infty),L^2(\Omega,\mathbb{H})\Big)$, we can derive that $(\mathcal{G}v)(t)=u(t)+(\mathcal{P}m)(t)\in\mathcal{SAP}^0_{\omega,Q}\subset\mathcal{SAP}_{\omega,Q}\Big([-q,+\infty),L^2(\Omega,\mathbb{H})\Big)$. Hence, $\mathcal{G}v\in\mathcal{SAP}_{\omega,Q}\Big([-q,+\infty),L^2(\Omega,\mathbb{H})\Big)$ and $\mathcal{G}$ is well defined.

We prove that $\mathcal{P}$ is a strict contraction on $\mathcal{SAP}^0_{\omega,Q}$. Let $m,z\in\mathcal{SAP}^0_{\omega,Q}$.

$$
\begin{aligned}
&E\|(\mathcal{P}m)(t)-(\mathcal{P}z)(t)\|^2\\
&\leq 3L_h\|m_t-z_t\|^2_{\mathcal{B}_0}+3L_f\int_0^t\frac{CM}{1+|\mu|(t-s)^\alpha}ds\int_0^t\frac{CM}{1+|\mu|(t-s)^\alpha}\|m_s-z_s\|^2_{\mathcal{B}_0}ds\\
&\quad+3L_g\int_0^t\left(\frac{CM}{1+|\mu|(t-s)^\alpha}\right)^2\|m_s-z_s\|^2_{\mathcal{B}_0}ds\\
&\leq 3L_h\|m-z\|^2_\infty+3L_f\left(\int_0^t\frac{CM}{1+|\mu|(t-s)^\alpha}ds\right)^2\|m-z\|^2_\infty\\
&\quad+3L_g\int_0^t\frac{(CM)^2}{2(1+|\mu|^2(t-s)^{2\alpha})}ds\|m-z\|^2_\infty\\
&\leq 3L_h\|m-z\|^2_\infty+3L_f(CM)^2\left(\int_0^{|\mu|t^\alpha}\frac{z^{\frac{1}{\alpha}-1}}{\alpha|\mu|^{\frac{1}{\alpha}}(1+z)}dz\right)^2\|m-z\|^2_\infty\\
&\quad+\frac{3L_g(CM)^2}{2}\int_0^{|\mu|^2t^{2\alpha}}\frac{z^{\frac{1}{2\alpha}-1}}{\alpha|\mu|^{\frac{1}{\alpha}}(1+z)}dz\|m-z\|^2_\infty
\end{aligned}
$$

$$\leq 3L_h\|m-z\|_\infty^2+3L_f(CM)^2\Big(\int_0^\infty \frac{z^{\frac{1}{\alpha}-1}}{\alpha|\mu|^{\frac{1}{\alpha}}(1+z)}dz\Big)^2\|m-z\|_\infty^2$$
$$+\frac{3L_g(CM)^2}{2}\int_0^\infty \frac{z^{\frac{1}{2\alpha}-1}}{2\alpha|\mu|^{\frac{1}{\alpha}}(1+z)}dz\|m-z\|_\infty^2$$
$$=3L_h\|m-z\|_\infty^2+\frac{3L_f(CM)^2}{(\alpha|\mu|^{\frac{1}{\alpha}})^2}\Big(\int_0^\infty \frac{z^{\frac{1}{\alpha}-1}}{1+z}dz\Big)^2\|m-z\|_\infty^2$$
$$+\frac{3L_g(CM)^2}{4\alpha|\mu|^{\frac{1}{\alpha}}}\int_0^\infty \frac{z^{\frac{1}{2\alpha}-1}}{1+z}dz\|m-z\|_\infty^2$$
$$\leq 3\Big(L_h+\frac{L_f(CM)^2}{(\alpha|\mu|^{\frac{1}{\alpha}})^2}\frac{\pi^2}{(\sin(\frac{\pi}{\alpha}))^2}+\frac{L_g(CM)^2}{4\alpha|\mu|^{\frac{1}{2\alpha}}}\frac{\pi}{\sin(\frac{\pi}{2\alpha})}\Big)\|m-z\|_\infty^2.$$
$$\leq 3\Theta\|m-z\|_\infty^2.$$

Since $3\Theta = 3\left[L_h+\frac{L_f(CM)^2}{(\alpha|\mu|^{\frac{1}{\alpha}})^2}\frac{\pi^2}{(\sin(\frac{\pi}{\alpha}))^2}+\frac{L_g(CM)^2}{4\alpha|\mu|^{\frac{1}{2\alpha}}}\frac{\pi}{\sin(\frac{\pi}{2\alpha})}\right] < 1$, we deduce by the Banach contraction principle $\mathcal{P}$ has a unique fixed point $m^\star \in \mathcal{SAP}^0_{\omega,Q}$. Then $v^\star = u+m^\star$ is a fixed point of the operator $\mathcal{G}$, which is the unique mild solution of the problem (1). □

Theorem 4.3. Assume that $A : D(A) \subset \mathbb{H} \to \mathbb{H}$ is a sectorial of type $\mu < 0$ and $f,h \in \mathcal{SAP}_{\omega,Q}(\mathbb{R}^+ \times \mathcal{B}_0, L^2(\Omega,\mathbb{H}))$ and $g \in \mathcal{SAP}_{\omega,Q}(\mathbb{R}^+ \times \mathcal{B}_0, \mathcal{L}(\mathbb{K},\mathbb{H}))$. Assume further that there exist functions $L_f, L_g, L_h : \mathbb{R}^+ \to \mathbb{R}^+$ such that for each $\lambda > 0$ and for all $a,b \in L^2(\Omega,\mathbb{H})$ with $\|a\|^2_{\mathcal{B}_0} \leq \lambda$ and $\|b\|^2_{\mathcal{B}_0} \leq \lambda$,

$$E\|h(t,a)-h(t,b)\|^2 \leq L_h(\lambda)\|a-b\|^2_{\mathcal{B}_0},$$
$$E\|f(t,a)-f(t,b)\|^2 \leq L_f(\lambda)\|a-b\|^2_{\mathcal{B}_0},$$
$$E\|g(t,a)-g(t,b)\|^2_{\mathcal{L}(\mathbb{K},\mathbb{H})} \leq L_g(\lambda)\|a-b\|^2_{\mathcal{B}_0}$$

uniformly for all $t \in \mathbb{R}$. Then problem (1) admit a unique S-asymptotically (ω,Q)-periodic mild solution provided that

$$\sup_{\lambda>0}[\lambda-\lambda K(\lambda)] > \Delta_0 : \tag{6}$$

where

$$K(\lambda) := 8\left[L_h(\lambda)+L_f(\lambda)\frac{(CM)^2}{(\alpha|\mu|^{\frac{1}{\alpha}})^2}\frac{\pi^2}{\sin^2(\frac{\pi}{\alpha})}+L_g(\lambda)\frac{(CM)^2}{4\alpha|\mu|^{\frac{1}{\alpha}}}\frac{\pi}{\sin(\frac{\pi}{2\alpha})}\right]$$
$$\Delta_0 := (2CM)^2E\|h(0,\psi)\|^2+8\sup_{t\in\mathbb{R}^+}E\|h(t,u_t)\|^2$$
$$+\frac{8(CM)^2}{(\alpha|\mu|^{\frac{1}{\alpha}})^2}\frac{\pi^2}{\sin^2(\frac{\pi}{\alpha})}\sup_{t\in\mathbb{R}^+}E\|f(t,u_t)\|^2+8\frac{(CM)^2}{\alpha|\mu|^{\frac{1}{\alpha}}}\frac{\pi}{\sin(\frac{\pi}{2\alpha})}\sup_{t\in\mathbb{R}^+}E\|g(t,u_t)\|^2$$

and $u(\cdot)$ the process defined by $u_0=\psi$ and for $t\geq 0$, $u(t)=\mathcal{J}_\alpha(t)\psi(0)$.

Proof. By condition (6), we can infer that there exists $\lambda>0$ such that

$$\lambda-\lambda K(\lambda)>\Delta_0: \tag{7}$$

Now, similarly to the proof of Theorem 4.2, we consider the operator $\mathcal{P}$ defined in (5) and

$$\Sigma_\lambda=\{m\in\mathcal{SAP}^0_{\omega,Q}\,:\,\|m\|^2_\infty<\lambda\}.$$

To complete the proof, have just to show that $\mathcal{P}$ is a contraction in Σ_λ. We start by showing that the operator $\mathcal{P}(\Sigma_\lambda)\subset\Sigma_\lambda$. By similar arguments as in the proof Theorem 4.2, for each $m\in\Sigma_\lambda\subset\mathcal{SAP}^0_{\omega,Q}$, we have $\mathcal{P}m\in\mathcal{SAP}^0_{\omega,Q}$.

For all $t\in\mathbb{R}^+$, we have

$$\begin{aligned}
&E\|(\mathcal{P}m)(t)\|^2\\
&\leq 4\|\mathcal{J}_\alpha(t)\|^2E\|h(0,\psi)\|^2+4E\|h(t,u_t+m_t)-h(t,u_t)+h(t,u_t)\|^2\\
&\quad+4E\Big\|\int_0^t\mathcal{J}_\alpha(t-s)\Big(f(s,u_s+m_s)-f(s,u_s)+f(s,u_s)\Big)ds\Big\|^2\\
&\quad+4E\Big\|\int_0^t\mathcal{J}_\alpha(t-s)\Big(g(s,u_s+m_s)-g(s,u_s)+g(s,u_s)\Big)dW(s)\Big\|^2\\
&\leq\frac{(2CM)^2}{1+|\mu|^2t^{2\alpha}}E\|h(0,\psi)\|^2+8\left[L_h(\lambda)\|m_t\|^2_{\mathcal{B}_0}+E\|h(t,u_t)\|^2\right]\\
&\quad+8\int_0^t\frac{CM}{1+\|\mu\|(t-s)^\alpha}ds\int_0^t\frac{CM}{1+\|\mu\|(t-s)^\alpha}\left[L_f(\lambda)\|m_s\|^2_{\mathcal{B}_0}+E\|f(s,m_s)\|^2\right]ds\\
&\quad+8\int_0^t\Big(\frac{CM}{1+\|\mu\|(t-s)^\alpha}\Big)^2\left[L_g(\lambda)\|m_s\|^2_{\mathcal{B}_0}+E\|g(s,u_s)\|^2)\right]ds\\
&\leq(2CM)^2E\|h(0,\psi)\|^2+8\left[L_h(\lambda)\|m\|^2_\infty+\sup_{t\in\mathbb{R}^+}E\|h(t,u_t)\|^2\right]\\
&\quad+\frac{8(CM)^2}{(\alpha|\mu|^{\frac{1}{\alpha}})^2}\frac{\pi^2}{\sin^2(\frac{\pi}{\alpha})}\left[L_f(\lambda)\|m\|^2_\infty+\sup_{t\in\mathbb{R}}E\|f(t,u_t)\|^2\right]\\
&\quad+\frac{8(CM)^2}{\alpha|\mu|^{\frac{1}{\alpha}}}\frac{\pi}{\sin(\frac{\pi}{2\alpha})}\left[L_g(\lambda)\|m\|^2_\infty+\sup_{t\in\mathbb{R}}E\|g(t,u_t)\|^2\right]\\
&\leq\Delta_0+8\left[L_h(\lambda)+L_f(\lambda)\frac{(CM)^2}{(\alpha|\mu|^{\frac{1}{\alpha}})^2}\frac{\pi^2}{\sin^2(\frac{\pi}{\alpha})}+L_g(\lambda)\frac{(CM)^2}{4\alpha|\mu|^{\frac{1}{\alpha}}}\frac{\pi}{\sin(\frac{\pi}{2\alpha})}\right]\|m\|^2_\infty\\
&\leq\Delta_0+\lambda K(\lambda)
\end{aligned}$$

By inequality (7), we infer that $E\|(\mathcal{P}m)(t)\|^2\leq\lambda$. Thus $\mathcal{P}(\Sigma_\lambda)\subset\Sigma_\lambda$.

For each $m, z \in \Sigma_\lambda$ and all $t \in [0, \infty)$, similar calculations yields that

$$\begin{aligned} &E\|(\mathcal{P}m)(t-(\mathcal{P}z)(t)\|^2 \\ \leq &3\left[L_h(\lambda)+L_f(\lambda)\frac{(CM)^2}{(\alpha|\mu|^{\frac{1}{\alpha}})^2}\frac{\pi^2}{\sin^2(\frac{\pi}{\alpha})}+L_g(\lambda)\frac{(CM)^2}{4\alpha|\mu|^{\frac{1}{\alpha}}}\frac{\pi}{\sin(\frac{\pi}{2\alpha})}\right]\|m-z\|_\infty^2, \end{aligned}$$

Therefore $\|\mathcal{P}m-\mathcal{P}z\|_\infty^2 \leq K(\lambda)\|m-z\|_\infty^2$.

From condition (7), we have $\lambda - \lambda K(\lambda) > 0$ which implies that $K(\lambda) < 1$. Hence $\mathcal{P}$ is a contraction on Σ_λ and consequently there exists an unique $m^\star \in \Sigma_\lambda$ such that $\mathcal{P}m^\star = m^\star$ by Banach fixed point theorem. Thus $v^\star = u + m^\star$ is a fixed point of the operator $\mathcal{G}$, which is the unique square-mean S-asymptotically (ω, Q)-periodic mild solution of the problem (1). □

Remark 4.4. Definition 3.1, which introduces the space of square-mean S-asymptotically (ω, Q)-periodic processes for a general linear isomorphism Q, recovers the classical S-asymptotically ω-periodic processes when $Q = I$, and the (ω, c)-stochastic framework of [3] when $Q = cI$, but is strictly more general than both. Lemma 3.1 (completeness of the resulting space), Theorem 3.3 (convolution invariance), and Theorem 3.4 (superposition principle) are all new in the operator-Q setting: the analogues available in [3] cover only the scalar case $Q = cI$ and their proofs rely on the modulus estimate $|c| = \|Q\|_{L(H)}$, which breaks down for a general isomorphism. Theorems 4.1 and 4.2 on the existence and uniqueness of mild solutions extend the results of [29] (which treat $Q = I$) to the full operator-Q setting and to the neutral fractional stochastic framework with infinite delay simultaneously.

The proof strategies of Theorems 4.1 and 4.2 follow a fixed-point approach analogous to that of [29], adapted to the operator-Q setting; we acknowledge this in the revised text. The analytical results of Section 3, by contrast, require new arguments to handle the non-commutativity of Q with convolution operators and the loss of the modulus estimate $|c| = \|Q\|_{L(H)}$ that is available in the scalar case.

5. Examples

Example 1. Consider the following stochastic fractional differential equation:

$$\begin{cases} d\Big[y(t,x)+a\displaystyle\int_{t-q}^{t}\mathcal{R}(t-s)\Big(\sigma_1(y(s,x))+\frac{\sin(y(s,x))}{1+t^2}\Big)ds\Big] \\ =\Big[\displaystyle\int_0^t\frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}\Big(\frac{\partial^2}{\partial x^2}-\nu\Big)\Big(y(t,x)+a\int_{t-q}^{t}\mathcal{R}(t-s)\Big(\sigma_1(y(s,x))+\frac{\sin(y(s,x))}{1+t^2}\Big)ds \\ \qquad +a\displaystyle\int_{t-q}^{t}\mathcal{R}(t-s)\Big(\sigma_2(y(s,x))+\frac{\cos(y(s,x))}{e^t}\Big)ds\Big]dt \\ \quad +a\Big[\displaystyle\int_{t-q}^{t}\mathcal{R}(t-s)\Big(\sigma_3(y(s,x))+\frac{\sin(y(s,x))}{e^t}\Big)ds\Big]dW(t),\ (t,x)\in\mathbb{R}\times(0,\pi) \\ y(t,0)+a\displaystyle\int_{t-q}^{t}\mathcal{R}(t-s)\Big(\sigma_1(y(s,0))+\frac{\sin(y(s,0))}{1+t^2}\Big)ds=0 \\ y(t,\pi)+a\displaystyle\int_{t-q}^{t}\mathcal{R}(t-s)\Big(\sigma_1(y(s,\pi))+\frac{\sin(y(s,\pi))}{1+t^2}\Big)ds=0 \\ y(\theta,x)=y_0(\theta,x), \qquad -q\le\theta\le 0 \end{cases} \tag{8}$$

where $q,a,\nu>0$, $y_0:[-q,0]\times[0,\pi]\to\mathbb{R}$ is a continuous function, $W(t)$ is a one-sided and standard one-dimensional Brownian motion defined on the filtered probability space $(\Omega,\mathcal{F},\mathbb{P},\mathcal{F}_\tau)$. Let $\mathbb{H}=L^2([0,\pi])$, $Q:L^2([0,\pi])\to L^2([0,\pi])$ a linear isomorphism, $\omega>0$ and $\mathcal{R}:\mathbb{R}^+\longrightarrow\mathbb{R}$ is a continuous function with $\int_{-q}^{0}|\mathcal{R}(-\theta)|d\theta<1$ and for $i\in\{1,2,3\}$, $\sigma_i\mathcal{B}_0\to\mathbb{H}$ is continuous function and verifies the conditions $\sigma_i(Qz)=Q\sigma_i(z)$ and there exist function $M>0$ such that for all $m,z\in\mathcal{B}_0$, $E\|\sigma_i(m)-\sigma_i(z)\|^2\le M_i\|m-z\|^2_{\mathcal{B}_0}$.
Define $A:=\frac{\partial^2}{\partial x^2}-\nu$ with domain

$$\mathcal{D}(A)=\{x\in\mathbb{H},\quad x,x'\text{are absolutely continuous},\quad x''\in\mathbb{H},\ x(0)=x(\pi)\}.$$

It is well known that A is a sectorial operator of type $\mu=-\nu<0$. Therefore A generates an α-resolvent family $\{\mathcal{J}_\alpha(t)\}_{t\ge0}$ such that $\|\mathcal{J}_\alpha(t)\|\le\dfrac{CM}{1+|\mu|t^\alpha}$, $t\ge 0$ with $1<\alpha<2$.
Let $x\in[0,\pi]$, we set $y(t)(x)=y(t,x)$, $\psi(t)(x)=y_0(\theta,x)$, $\cos(y(x))=\cos(y)(x)$, $\sin(y(x))=\sin(y)(x)$,

$$h(t,y)(x)=a\int_{-q}^{0}\mathcal{R}(-\theta)\Big(\sigma_1(y(\theta))+\frac{\sin(y(\theta))}{1+t^2}\Big)(x)d\theta,$$

$f(t,y)(x) = a\int_{-q}^{0}\mathcal{R}(-\theta)\Big(\sigma_2(y(\theta)) + \dfrac{\cos(y(\theta))}{e^t}\Big)(x)d\theta$
and $g(t,y)(x) = a\int_{-q}^{0}\mathcal{R}(-\theta)\Big(\sigma_3(y(\theta)) + \dfrac{\sin(y(\theta))}{e^t}\Big)(x)d\theta.$
Then the problem (8) can be written into the form (1). It is easy to show that $f,g,h \in \mathcal{BC}(\mathbb{R}^+ \times \mathcal{B}_0, L^2(\Omega,\mathbb{H}))$ and we get following estimation:

$$
\begin{aligned}
&E\|f(t+\omega,y) - Qf(t,Q^{-1}y)\|^2\\
=&a^2E\Big\|\int_{-q}^{0}\mathcal{R}(-\theta)\Big(\sigma_2(y(\theta)) + \frac{\cos(y(\theta))}{e^{t+\omega}}\Big)d\theta\\
&\qquad -Q\int_{-q}^{0}\mathcal{R}(-\theta)\Big(\sigma_2(Q^{-1}y(\theta)) + \frac{\cos(Q^{-1}y(\theta))}{e^t}\Big)d\theta\Big\|^2\\
\leq&a^2\int_{-q}^{0}|\mathcal{R}(-\theta)|d\theta\int_{-q}^{0}|\mathcal{R}(-\theta)|E\Big\|\frac{\cos(y(\theta))}{e^{t+\omega}} - Q\frac{\cos(Q^{-1}y(\theta))}{e^t}\Big\|^2d\theta\\
\leq&2a^2\Big(\int_{-q}^{0}|\mathcal{R}(-\theta)|d\theta\Big)^2\Big(\Big|\frac{1}{e^{t+\omega}}\Big|^2 + |Q|^2\Big|\frac{1}{e^t}\Big|^2\Big)\\
\leq&2a^2\Big(\Big|\frac{1}{e^{t+\omega}}\Big|^2 + |Q|^2\Big|\frac{1}{e^t}\Big|^2\Big) \to 0 \text{ as } t \to +\infty.
\end{aligned}
$$

Hence $\lim_{t\to+\infty} E\|f(t+\omega,y) - Qf(t,Q^{-1}y)\|^2 = 0.$
Similarly we have $\lim_{t\to+\infty} E\|g(t+\omega,y) - Qg(t,Q^{-1}y)\|^2 = 0.$

$$
\begin{aligned}
&E\|h(t+\omega,y) - Qh(t,Q^{-1}y)\|^2\\
&\leq a^2\int_{-q}^{0}|\mathcal{R}(-\theta)|d\theta\int_{-q}^{0}|\mathcal{R}(-\theta)|E\Big\|\frac{\cos(y(\theta))}{1+(t+\omega)} - Q\frac{\cos(Q^{-1}y(\theta))}{1+t}\Big\|^2d\theta\\
&\leq 2a^2\Big(\int_{-q}^{0}|\mathcal{R}(-\theta)|d\theta\Big)^2\Big(\Big|\frac{1}{1+(t+\omega)}\Big|^2 + |Q|^2\Big|\frac{1}{1+t}\Big|^2\Big)\\
&\leq 2a^2\Big(\Big|\frac{1}{1+(t+\omega)}\Big|^2 + |Q|^2\Big|\frac{1}{1+t}\Big|^2\Big) \to 0 \text{ as } t \to +\infty.
\end{aligned}
$$

Therefore we have $\lim_{t\to+\infty} E\|h(t+\omega,y) - Qh(t,Q^{-1}y)\|^2 = 0$. Let $m,z \in \mathcal{B}_0$ and

$t \in \mathbb{R}^+$.

$$
\begin{aligned}
&E\|h(t,m)-h(t,z)\|^2 \\
&= a^2E\Big\|\int_{-q}^{0}\mathcal{R}(-\theta)\Big(\sigma_1(m(\theta))-\sigma_1(z(\theta))\Big)d\theta+\int_{-q}^{0}\frac{\mathcal{R}(-\theta)}{1+t}\Big(\sin(m(\theta))-\sin(z(\theta))\Big)d\theta\Big\|^2 \\
&\le 2a^2E\Big\|\int_{-q}^{0}\mathcal{R}(-\theta)\Big(\sigma_1(m(\theta))-\sigma_1(z(\theta))\Big)d\theta\Big\|^2 \\
&\qquad + \frac{2a^2}{1+t}E\Big\|\int_{-q}^{0}\mathcal{R}(-\theta)\Big(\sin(m(\theta))-\sin(z(\theta))\Big)d\theta\Big\|^2 \\
&\le 2a^2M_1\int_{-q}^{0}|\mathcal{R}(-\theta)|d\theta\int_{-q}^{0}|\mathcal{R}(-\theta)|E\|m(\theta)-z(\theta)\|^2d\theta \\
&\qquad + 2a^2\int_{-q}^{0}|\mathcal{R}(-\theta)|d\theta\int_{-q}^{0}|\mathcal{R}(-\theta)|E\|m(\theta)-z(\theta)\|^2d\theta \\
&\le 2a^2(M_1+1)\|m-z\|_{\mathcal{B}_0}^2\Big(\int_{-q}^{0}|\mathcal{R}(-\theta)|d\theta\Big)^2 \\
&\le 2a^2(M_1+1)\|m-z\|_{\mathcal{B}_0}^2.
\end{aligned}
$$

and

$$
\begin{aligned}
&E\|f(t,m)-f(t,z)\|^2 \\
&\le 2a^2E\Big\|\int_{-q}^{0}\mathcal{R}(-\theta)\Big(\sigma_2(m(\theta))-\sigma_2(z(\theta))\Big)d\theta\Big\|^2 \\
&\qquad + 2a^2E\Big\|\int_{-q}^{0}\frac{\mathcal{R}(-\theta)}{e^t}\Big(\cos(m(\theta))-\cos(z(\theta))\Big)d\theta\Big\|^2 \\
&\le 2a^2\int_{-q}^{0}|\mathcal{R}(-\theta)|d\theta\int_{-q}^{0}|\mathcal{R}(-\theta)|E\|\sigma_2(m(\theta))-\sigma_2(z(\theta))\|^2d\theta \\
&\quad +2a^2e^{-2t}\int_{-q}^{0}|\mathcal{R}(-\theta)|d\theta\int_{-q}^{0}|\mathcal{R}(-\theta)|E\|\cos(m(\theta))-\cos(z(\theta))\|^2d\theta \\
&\le 2a^2M_2\int_{-q}^{0}|\mathcal{R}(-\theta)|d\theta\int_{-q}^{0}|\mathcal{R}(-\theta)|E\|m(\theta)-z(\theta)\|^2d\theta \\
&\quad + 2e^{-2t}\int_{-q}^{0}|\mathcal{R}(-\theta)|\int_{-q}^{0}|\mathcal{R}(-\theta)|E\|m(\theta)-z(\theta)\|^2d\theta \\
&\le 2a^2(M_2+1)\|m-z\|_{\mathcal{B}_0}^2\Big(\int_{-q}^{0}|\mathcal{R}(-\theta)|d\theta\Big)^2 \\
&\le 2a^2(M_2+1)\|m-z\|_{\mathcal{B}_0}^2.
\end{aligned}
$$

Similarly, we show that $E\|g(t,m)-g(t,z)\|^2 \le 2a^2(M_3+1)\|m-z\|_{\mathcal{B}_0}^2$. Therefore, by Theorem 4.2, the problem (8) has a unique square-mean S-asymptotically (ω,Q)-periodic mild solution provided that a is small enough.

Example 2. We consider the following neutral stochastic integro-differential system:

$$\begin{cases} d\left[\begin{pmatrix} u(t,\xi)\\ v(t,\xi)\end{pmatrix} + b|\sin(3t)|\begin{pmatrix} u(t-q,\xi)\\ v(t-q,\xi)\end{pmatrix} + \begin{pmatrix} \left(\frac{1}{2}\right)^{\frac{3t}{2\pi}}\\ \cos(3t)\end{pmatrix}\right] \\ =\left[\int_0^t \frac{(t-r)^{\alpha-2}}{\gamma(\alpha-1)}\left(\frac{\partial^2}{\partial\xi^2}-\nu\right)\left[\begin{pmatrix} u(t,\xi)\\ v(t,\xi)\end{pmatrix} + b|\sin(3r)|\begin{pmatrix} u(r-q,\xi)\\ v(r-q,\xi)\end{pmatrix} + \begin{pmatrix} \left(\frac{1}{2}\right)^{\frac{3r}{2\pi}}\\ \cos(3r)\end{pmatrix}\right]dr \right.\\ \left. + \begin{pmatrix} e^{-3t}\sin 3t\\ b\,v(t-q,\xi)+e^{-3t}\cos 3t\end{pmatrix}\right]dt \\ + \begin{pmatrix} b\,u(t-q,\xi)|\cos\left(\frac{3t}{2\pi}\right)|+e^{-t}\cos 3t\\ 0\end{pmatrix}(dW_1(t),dW_2(t)),\ t>0 \\ \begin{pmatrix} u(t,0)\\ v(t,0)\end{pmatrix} = -b|\sin(3t)|\begin{pmatrix} u(t-q,0)\\ v(t-q,0)\end{pmatrix} - \begin{pmatrix} \left(\frac{1}{2}\right)^{\frac{3t}{2\pi}}\\ \cos(3t)\end{pmatrix},\ t>0 \\ \begin{pmatrix} u(t,\pi)\\ v(t,\pi)\end{pmatrix} = -b|\sin(3t)|\begin{pmatrix} u(t-q,\pi)\\ v(t-q,\pi)\end{pmatrix} - \begin{pmatrix} \left(\frac{1}{2}\right)^{\frac{3t}{2\pi}}\\ \cos(3t)\end{pmatrix},\ t>0 \\ \begin{pmatrix} u(\theta,\xi)\\ v(\theta,\xi)\end{pmatrix} = \begin{pmatrix} u_0(\theta,\xi)\\ v_0(\theta,\xi)\end{pmatrix}, -q\le\theta\le 0,\quad 0\le\xi\le\pi \end{cases} \tag{9}$$

where $b\in\mathbb{R}$, $q>0$, $u_0,v_0:[-q,0]\times[0,\pi]\to\mathbb{R}$ are continuous functions and $\alpha\in(1,2)$, $W_1(t),W_2(t)$ are two standard Wiener processes defined on a probability space $(\Omega,\mathcal{F},\mathbb{P})$. We set $X(t)(\xi)=\begin{pmatrix} u(t,\xi)\\ v(t,\xi)\end{pmatrix}=\begin{pmatrix} u(t)\\ v(t)\end{pmatrix}(\xi)$, $\psi(t)(\xi)=\begin{pmatrix} u_0(\theta,\xi)\\ v_0(\theta,\xi)\end{pmatrix}=\begin{pmatrix} u_0(\theta)\\ v_0(\theta)\end{pmatrix}(\xi)$, $Q=\begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix}$, $QX(t)=\begin{pmatrix} u(t)\\ -v(t)\end{pmatrix}$ and $\mathbb{H}=L^2([0,\pi],\mathbb{R})\times L^2([0,\pi],\mathbb{R})$ with the norm $\|(a,b)\|=\|a\|_{L^2([0,\pi],\mathbb{R})}+\|b\|_{L^2([0,\pi],\mathbb{R})}$. For each $t\ge 0$, $\xi\in[0,\pi]$, set

$$h(t,X)(\xi)=b|\sin(3t)|X(-q)(\xi)+\begin{pmatrix} \left(\frac{1}{2}\right)^{\frac{3t}{2\pi}}\\ \cos(3t)\end{pmatrix}=b|\sin(3t)|\begin{pmatrix} u(-q)(\xi)\\ v(-q)(\xi)\end{pmatrix}+\begin{pmatrix} \left(\frac{1}{2}\right)^{\frac{3t}{2\pi}}\\ \cos(3t)\end{pmatrix}$$

$$f(t,X)(\xi)=\begin{pmatrix} e^{-3t}\sin 3t\\ b\,v(-q)(\xi)+e^{-3t}\cos 3t\end{pmatrix} \text{ and}$$

$$g(t,X)(\xi)=\begin{pmatrix} b\,u(-q)(\xi)|\cos\left(\frac{3}{2}t\right)|+e^{-t}\cos 3t\\ 0\end{pmatrix}$$

We have $f,g,h\in\mathcal{BC}(\mathbb{R}^+\times\mathcal{B}_0,L^2(\Omega,\mathbb{H}))$ and for $\phi\in\mathcal{B}_0$

$$Qh(t,Q^{-1}\phi)=b|\sin(3t)|\phi(-q)+\begin{pmatrix}\left(\frac{1}{2}\right)^{\frac{3t}{2\pi}}\\ -\cos(3t)\end{pmatrix}$$
$$h(t+\frac{2\pi}{3},\phi)=b|\sin(3t)|\ \phi(-q)+\begin{pmatrix}\left(\frac{1}{2}\right)^{1+\frac{3t}{2\pi}}\\ -\cos(3t))\end{pmatrix}$$

and then

$$\begin{aligned}\mathbb{E}\left\|h\left(t+\frac{2\pi}{3},\phi\right)-Qh\left(t,Q^{-1}\phi\right)\right\|^2&=\mathbb{E}\left\|\begin{pmatrix}\frac{1}{2}\times\left(\frac{1}{2}\right)^{\frac{3t}{2\pi}}\\ \\ -\cos(3t)\end{pmatrix}-\begin{pmatrix}\left(\frac{1}{2}\right)^{\frac{3t}{2\pi}}\\ -\cos(3t)\end{pmatrix}\right\|^2\\ &=\mathbb{E}\left\|\begin{pmatrix}\frac{1}{2}\left(\frac{1}{2}\right)^{\frac{3t}{2\pi}}\\ \\ 0\end{pmatrix}\right\|^2=\mathbb{E}\int_0^\pi\left|\left(\frac{1}{2}\right)^{\frac{3t}{2\pi}+1}\right|^2d\xi\\ &=\frac{\pi}{4}\left(\frac{1}{2}\right)^{\frac{3t}{\pi}}\to 0\text{ as }t\to+\infty\end{aligned}$$

Now for $t\in\mathbb{R}_+$ and $\phi_1=\begin{pmatrix}u_1\\u_2\end{pmatrix},\phi_2=\begin{pmatrix}v_1\\v_2\end{pmatrix}\in\mathcal{B}_0$

$$\begin{aligned}\mathbb{E}\|h(t,\phi_1)-h(t,\phi_2)\|^2&=\mathbb{E}\|b|\sin(3t)|(\phi_1(-q)-\phi_2(-q))\|^2\\ &\le b^2\mathbb{E}\|\phi_1(-q)-\phi_2(-q)\|^2\le b^2\|\phi_1-\phi_2\|^2_{\mathcal{B}_0}\end{aligned}$$

Since for $\phi=\begin{pmatrix}u\\v\end{pmatrix}\in\mathcal{B}_0$, we have

$$g(t,\phi)=\begin{pmatrix}bu(-q)|\cos(3t)|+e^{-t}\cos 3t\\0\end{pmatrix}$$

then we have

$$g(t+\frac{2\pi}{3},\phi)=\begin{pmatrix}-bu(-q)\cos(3t)+e^{-t-\frac{2\pi}{3}}\cos 3t\\0\end{pmatrix}$$

and

$$Qg(t,Q^{-1}\phi)=\begin{pmatrix}bu(-q)|\cos(3t)|+e^{-t}\cos 3t\\0\end{pmatrix}.$$

Therefore

$$\begin{aligned}\mathbb{E}\|g(t+\frac{2\pi}{3},\phi)-Qg(t,Q^{-1}\phi)\|^2 &= \mathbb{E}\|\cos 3t\begin{pmatrix} e^{-t-\frac{2\pi}{3}}-e^{-t} \\ 0\end{pmatrix}\| \\ &= \mathbb{E}\Big(\int_0^{\pi}|\cos(3t)(e^{-t-\frac{2\pi}{3}}-e^{-t})|^2 d\xi\Big) \\ &\longrightarrow 0 \text{ as } t\to+\infty.\end{aligned}$$

Now let $t\in\mathbb{R}_+$ and $\phi_1=\begin{pmatrix}u_1\\u_2\end{pmatrix}, \phi_2=\begin{pmatrix}v_1\\v_2\end{pmatrix}\in\mathcal{B}_0$

$$\begin{aligned}&\mathbb{E}\|g(t,\phi_1)-g(t,\phi_2)\|^2 \\ &=\mathbb{E}\|\begin{pmatrix} bu_1(-q)|\cos\left(\frac{3t}{2\pi}\right))|-bu_2(-q)|\cos\left(\frac{3t}{2\pi}\right)| \\ 0\end{pmatrix}\|^2 \\ &=\mathbb{E}\|bu_1(-q)|\cos\left(\frac{3t}{2\pi}\right)|-bu_2(-q)|\cos\left(\frac{3t}{2\pi}\right)\|^2_{L^2([0,\pi],\mathbb{R})} \\ &\le b^2\mathbb{E}\|u_1(-q)-u_2(-q)\|^2_{L^2([0,\pi],\mathbb{R})}\le b^2\mathbb{E}\|\phi_1(-q)-\phi_2(-q)\|^2 \\ &\le b^2\|\phi_1-\phi_2\|^2_{\mathcal{B}_0}.\end{aligned}$$

Similarly, we have

$$f(t,\phi)=\begin{pmatrix} e^{-3t}\sin 3t \\ bv(-q)+e^{-3t}\cos 3t\end{pmatrix}, f(t+\frac{2\pi}{3},\phi)=\begin{pmatrix} e^{-3t-2\pi}\sin 3t \\ bv(-q)+e^{-3t-2\pi}\cos 3t\end{pmatrix}$$

and

$$Qf(t,Q^{-1}\phi)=\begin{pmatrix} e^{-3t}\sin 3t \\ bv(-q)-e^{-3t}\cos 3t\end{pmatrix}$$

Then, simple calculations provides that

$$\begin{aligned}&\mathbb{E}\|f(t+\frac{2\pi}{3},\phi)-Qf(t,Q^{-1}\phi)\|^2 \\ &=\mathbb{E}\|\begin{pmatrix} e^{-3t-2\pi}\sin 3t \\ bv(-q)+e^{-3t-2\pi}\cos 3t\end{pmatrix}-\begin{pmatrix} e^{-3t}\sin 3t \\ bv(-q)-e^{-3t}\cos 3t\end{pmatrix}\|^2 \\ &=\mathbb{E}\|\begin{pmatrix}\left[e^{-2\pi}-1\right]e^{-3t}\sin 3t \\ \left[e^{-2\pi}+1\right]e^{-3t}\cos 3t\end{pmatrix}\|^2 \\ &\le 2e^{-6t}\mathbb{E}\int_0^{\pi}\Big[(\left[e^{-2\pi}-1\right]\sin 3t)^2+(\left[e^{-2\pi}+1\right]\cos 3t)^2\Big]d\xi \\ &\le 4\pi\left[e^{-2\pi}+1\right]^2 e^{-6t}\to 0 \text{ as } t\to\infty.\end{aligned}$$

and

$$
\begin{aligned}
&\mathbb{E}\|f(t,\phi_1)-f(t,\phi_2)\|^2\\
=&\mathbb{E}\|\begin{pmatrix} e^{-3t}\sin 3t \\ b v_1(-q)+e^{-3t}\cos 3t\end{pmatrix}-\begin{pmatrix} e^{-3t}\sin 3t \\ b v_2(-q)+e^{-3t}\cos 3t\end{pmatrix}\|^2\\
=&\mathbb{E}\|\begin{pmatrix} 0 \\ b v_1(-q)-b v_2(-q)\end{pmatrix}\|^2=b^2\mathbb{E}\|v_1(-q)-v_2(-q)\|^2_{L^2([0,\pi],\mathbb{R})}\\
\leq& b^2\mathbb{E}\|\phi_1(-q)-\phi_2(-q)\|^2\leq b^2\mathbb{E}\|\phi_1-\phi_2\|^2_{\mathcal{B}_0}.
\end{aligned}
$$

As a consequence of Theorem 4.2, we have the following result

Proposition 5.1. *Let* $Q=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$. *Assume that*

$$
|b|<\frac{1}{\sqrt{3\left[1+\frac{(CM)^2}{(\alpha|\mu|^{\frac{1}{\alpha}})^2}\frac{\pi^2}{(\sin(\frac{\pi}{\alpha}))^2}+\frac{(CM)^2}{4\alpha|\mu|^{\frac{1}{2\alpha}}}\frac{\pi}{\sin(\frac{\pi}{2\alpha})}\right]}}.
$$

Then there exists a unique square-mean S-asymptotically $(\frac{2\pi}{3},Q)$*-periodic mild solution X of the problem* (9).

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