WHAT IS POSITIVE GEOMETRY?

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This article serves as an introduction to the special volume on Positive Geometry in the journal Le Matematiche. We attempt to answer the question in the title by describing the origins and objects of positive geometry at this early stage of its development. We discuss the problems addressed in the volume and report on the progress. We also list some open challenges.

1. Introduction

The volume *Positive Geometry* collects a series of articles which report on collaborations that were initiated during the school *Combinatorial Algebraic Geometry from Physics*, held at the Max Planck Institute for Mathematics in the Sciences, Leipzig, in May 2024. Several projects were inspired by the lectures of Michael Borinsky and Thomas Lam at that workshop. The topics of the articles align with the current goals of positive geometry, hence the volume title. Though it might be too early to spell out a definition of the field, the present article explains the editors' view on what is currently meant by positive geometry.

Positive geometry is an interdisciplinary subject inspired by fundamentally new ideas in particle physics and cosmology. Its origins lie in the observation that certain physical observables, called *scattering amplitudes* and *cosmological correlators*, can be recovered from the geometry of elegant mathematical objects, such as *amplituhedra* [7] and *cosmological polytopes* [5]. This has motivated physicists to search for such geometric structures in a range of quantum field theories. In positive geometry, we put this on a solid mathematical footing.

A first step in this direction was the formal definition of *positive geometries* in [4]. Building *Positive Geometry* as a subject is the next step. A positive geometry is a tuple consisting of a complex algebraic variety X, a semi-algebraic subset $X_{\geq 0}$ of its real points, and a meromorphic top-form $\Omega(X_{\geq 0})$. This tuple satisfies recursive axioms prescribed by physics. These are listed in [4, Section 2.1], [21, Definition 1] and [22, Definition 1.20]. Here, rather than spelling out the axioms, we present examples highlighting the topics featured in this volume.

Example 1.1. Let $X_{\geq 0} = P \subset \mathbb{R}^n$ be a convex polytope of dimension n. For an interior point $x \in \text{int}(P)$, let $(P - x)^\circ$ be the polar dual of the translated polytope $P - x$. The volume of $(P - x)^\circ$ depends on x. It defines a meromorphic form

$$\Omega(P) = \text{vol}(P - x)^\circ \, \mathrm{d}x$$

on $X = \mathbb{P}^n$, called the canonical form of P in [4]. The poles of $\Omega(P)$ are simple, and they coincide with the arrangement of hyperplanes defined by the facets of P. The zeros of $\Omega(P)$ lie along the adjoint hypersurface of P. If P is a cosmological polytope, then the cosmological correlator corresponding to P is $\int_\Gamma \Omega(P)$ for a suitable integration cycle Γ. In that context, the facet hyperplanes of P depend on parameters, here denoted by z. The function $z \mapsto \int_\Gamma \Omega(P(z))$ satisfies linear differential equations, best encoded via holonomic D-modules.

In particle physics, one replaces $\mathbb{P}^n$ with a complex Grassmannian variety $X = \text{Gr}_\mathbb{C}(k,n)$, and the polytope P by a semi-algebraic subset $\mathcal{A} = X_{\geq 0}$ of its real points, called the (tree) amplituhedron. Scattering amplitudes in $N = 4$ Super Yang–Mills theories are integrals of the canonical differential form $\Omega(\mathcal{A})$. In this setting, the facet hyperplane arrangement is replaced with a divisor in $\text{Gr}_\mathbb{C}(k,n)$, obtained as the algebraic boundary of the amplituhedron. It consists of a union of Schubert divisors coming from the faces of a cyclic polytope.   ◇

We should mention that, in recent work [12], Brown and Dupont formalize the construction of canonical differential forms using mixed Hodge theory. Let X be an n-dimensional projective variety and fix a divisor $Y \subset X$ containing the singular points of X. We assume (X, Y) to have *genus zero* [12, Section 3]. The divisor Y plays the role of the hyperplane and Schubert arrangements in Example 1.1. Brown and Dupont identify a canonical linear map sending a relative cycle $\sigma \in H_n(X, Y)$ (e.g., the polytope P) to a logarithmic n-form $\omega_\sigma \in \Omega^n_{\log}(X \setminus Y)$ (e.g., $\Omega(P)$). This suggests a new definition, and a new mechanism for constructing canonical forms. The ingredients remain roughly the same.

Example 1.1 showcases a direct connection between positive geometry and (real) algebraic geometry via complex varieties, semi-algebraic sets and their algebraic boundary divisors. The varieties X seen in physics are typically of

special interest to algebraic geometers as well. For instance, X is often a moduli space, such as $\mathrm{Gr}_{\mathbb{C}}(k,n)$ or $\overline{\mathcal{M}}_{0,n}$ [22]. On the other hand, hyperplane arrangements, polytopes, and Schubert arrangements in the Grassmannian should appeal to combinatorialists. Indeed, studying interesting objects in positive geometry requires tools from several established areas of mathematics, including, but not limited to, algebraic geometry and combinatorics. Other relevant areas include tropical geometry, (algebraic) analysis, and numerical mathematics.

The name *positive geometry* is slightly deceiving in that it suggest that one is only interested in real, positive points. In fact, a feature of the subject is that one often combines the complex, real and tropical viewpoint on each of its objects. A key example is the Grassmannian $\mathrm{Gr}_{\mathbb{C}}(k,n)$. The positive part of $\mathrm{Gr}_{\mathbb{R}}(k,n)$ is an important instance of a positive geometry. It projects linearly to the amplituhedron $\mathcal{A}$, and the images of its positroid cells form the analogue of a triangulation of $\mathcal{A}$. The *tropical Grassmannian* $\mathrm{Trop}(\mathrm{Gr}_{\mathbb{C}}(2,n))$ is the space of trees with n labeled leaves. These trees correspond to tree-level Feynman diagrams in particle scattering. Tropical Grassmannians with $k > 2$ are only understood for small n. Their relevance in physics led to *CEGM theory* [13].

A more visual example is the elliptic curve. Its associated Riemann surface is a torus, shown in the middle of Figure 1. The cycle that indicates a genus one object can be seen both in the left curve (real) and in the right curve (tropical).
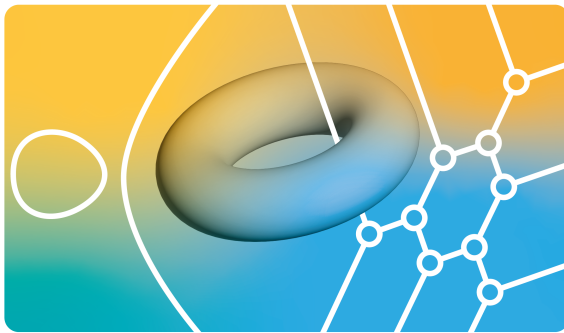

Figure 1: Positive geometry rests on three flavors of algebraic geometry: *real*, *complex* and *tropical*. The diagram shows an elliptic curve in these three flavors.

Though the inspiration for positive geometry comes from physics, its structures appear in other contexts too. For instance, a convex positive region P represents the feasible set of an optimization problem. Here, the canonical dual volume function $\mathrm{vol}(P-x)^{\circ}$ from Example 1.1 is the universal barrier function of P. In algebraic statistics, the positive points $\mathbb{P}^n_{\geq 0}$ of projective space $\mathbb{P}^n$ are identified with the n-dimensional probability simplex; the space of probability distributions on a random variable with $n+1$ states. A statistical model is a com-

plex projective variety $X \subset \mathbb{P}^n$, whose intersection with the probability simplex $X_{\geq 0} = X \cap \mathbb{P}^n_{\geq 0}$ is nonempty. Marginal likelihood is computed from Bayesian integrals on $X_{\geq 0}$ whose structure is similar to that of Feynman integrals. These settings justify the expectation that positive geometry will be widely applicable.

In summary, positive geometry revolves around positive regions $P = X_{\geq 0}$ in complex algebraic varieties X, their boundary arrangements in X, existence and uniqueness of the canonical differential form $\Omega(X_{\geq 0})$, and the associated period integrals $\int_\Gamma \Omega(X)$. It uses a variety of techniques from combinatorics and complex, real and tropical geometry. We reiterate that this description and the articles in this volume form a snapshot of the field in its current nascent stage.

In what follows, we explain some of the concepts above in more detail. Our exposition is guided by examples and by the contributions we are referring to. We highlight connections between the cited papers, and we explain how they fit the scope of positive geometry. The material is subdivided into three sections:

- Scattering equations and binary geometries [2, 8, 10, 14, 19, 22, 24, 28],

- (Semi-)algebraic subsets of the Grassmannian [1, 15–17, 20, 24, 27, 30],

- Integrals and differential equations [9, 18, 22, 23, 26].

Our article concludes with a collection of open problems and future directions.

2. Scattering equations and binary geometries

The framework of Cachazo–He–Yuan (CHY) for computing scattering amplitudes in bi-adjoint scalar ϕ^3 theories illustrates many prominent concepts in positive geometry. The lecture notes [22] provide an excellent mathematical exposition of this framework. It also contains open questions, some of which are answered in this volume [1, 10]. This section presents a sketch of the CHY formalism. We work with $n = 5$ particles for concreteness, offering insight into what happens for general n. The $n = 5$ *CHY amplitude* is the rational function

$$A_5 \;=\; \frac{1}{s_{12}s_{45}} + \frac{1}{s_{23}s_{15}} + \frac{1}{s_{12}s_{34}} + \frac{1}{s_{23}s_{45}} + \frac{1}{s_{15}s_{34}} \tag{1}$$

in the *Mandelstam variables* $s_{ij} = s_{ji}$, $1 \leq i, j \leq 5$, with $s_{ii} = 0$. The s_{ij} depend on the momenta of five massless particles. They satisfy $\sum_j s_{ij} = 0$ for all i because of momentum conservation. Additionally, there can be a rank constraint on the symmetric matrix $(s_{ij})_{i,j}$ with zeros on the diagonal. These conditions define the *Mandelstam variety* of physically meaningful data s_{ij}. Different representations of these kinematic data lead to other determinantal varieties, studied in [28].

Each term in (1) corresponds to a triangulation of the pentagon or, dually, to a planar trivalent tree with five labeled leaves. The moduli space of planar trees
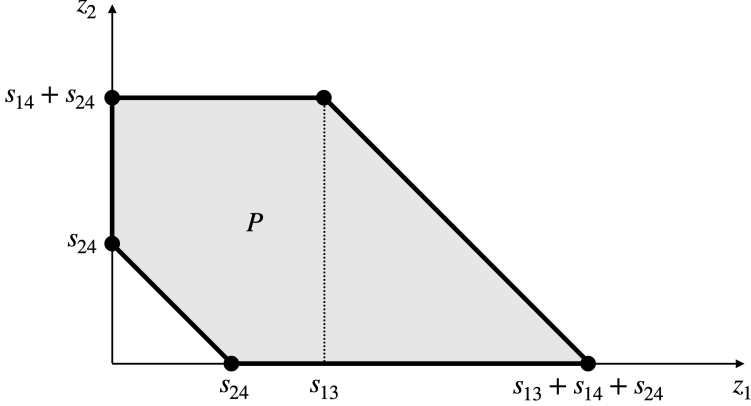
Figure 2: The amplitude A_5 is the dual volume of an associahedron.

is the positive tropical Grassmannian $\mathrm{Trop}^+\mathrm{Gr}(2,n)$, and (1) can be viewed as a sum over its maximal cones. Motivated by the analog of this observation for more general CEGM amplitudes, [2] studies the *chirotropical Grassmannian*.

We now relate the function A_5 to the positive geometries from [4] mentioned in the Introduction. We assume that s_{13}, s_{14} and s_{24} are positive real numbers. Let $X = \mathbb{P}^2$ and take $X_{\geq 0} = P$ to be the pentagon in Figure 2. Following Example 1.1, its canonical form is $\Omega(P) = \mathrm{vol}(P - z)^\circ \mathrm{d}z$. We claim that substituting $z = (z_1, z_2) = (-s_{23}, -s_{34})$ into the canonical function $\mathrm{vol}(P - z)^\circ$ gives (1). The pentagon P is a realization of the two-dimensional associahedron [3], whose vertices are the triangulations mentioned in the previous paragraph.

The CHY formula (4) writes (1) as a sum over the complex critical points of a logarithmic potential function. Let $(p_{ij})_{i<j}$ be the 2×2 minors of

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1+x & 1+x+y & 1 \end{pmatrix}. \tag{2}$$

For $x, y \in \mathbb{C}^2 \setminus \{\prod_{i<j} p_{ij} = 0\}$, this matrix represents a point on the moduli space $\mathcal{M}_{0,5}$ of configurations of five distinct points $\{0, 1, 1+x, 1+x+y, \infty\}$ on $\mathbb{P}^1$. That is, we model $\mathcal{M}_{0,5}$ as the complement of a *hyperplane arrangement* $\{(x,y) : \prod_{i<j} p_{ij} = 0\} \subset \mathbb{C}^2$. The *scattering equations* on $\mathcal{M}_{0,5}$ are given by

$$\frac{\partial L}{\partial x} = \frac{\partial L}{\partial y} = 0 \quad \text{where} \quad L = \log \prod_{i<j} p_{ij}^{s_{ij}}. \tag{3}$$

The paper [8] solves such equations by compactifying $\mathcal{M}_{0,5}$ and using numerical Gröbner degenerations. The *logarithmic discriminant* is a polynomial in the

s_{ij} characterizing when two solutions collide. This is studied in [19]. In our example, it has a nice symmetric determinantal expression [19, Example 8.1].

Denote the Euler operators in x and y by $\theta_x = x \frac{\partial}{\partial x}$ and $\theta_y = y \frac{\partial}{\partial y}$. We have

$$A_5 = \sum_{(x,y) \in \text{Crit}(L)} \det \begin{pmatrix} \theta_x^2 L & \theta_x \theta_y L \\ \theta_x \theta_y L & \theta_y^2 L \end{pmatrix}^{-1}. \tag{4}$$

The sum is over the set $\text{Crit}(L)$ of solutions to (3). Hence, the coordinates (x, y) are algebraic functions in s_{ij}. In our case, that set consists of only two points. The reader is encouraged, by [22, Exercise 3.40], to verify the identity (4).

We will reformulate (3) in different coordinates on $\mathcal{M}_{0,5}$. The cross ratios

$$[ij|kl] = \frac{p_{ik} p_{jl}}{p_{il} p_{jk}}, \quad \{i,j\}, \{k,l\} \in \binom{[5]}{2}, \ \{i,j\} \cap \{k,l\} = \emptyset$$

are regular functions on $\mathcal{M}_{0,5}$. The *dihedral coordinates*, as introduced by Francis Brown [11, Section 2], are the cross ratios $[i, i+1 | j, j+1]$. These are labeled by diagonals (ij) of a convex pentagon which are not edges. We denote these diagonals by $\Delta(5) = \{(13), (14), (24), (25), (35)\}$. The map

$$\mathcal{M}_{0,5} \longrightarrow (\mathbb{C}^*)^5, \quad (x,y) \longmapsto ([i, i+1 | j+1, j])_{(ij) \in \Delta(5)}.$$

is a closed embedding. The image $U \simeq \mathcal{M}_{0,5}$ is defined by five equations:

$$u_{13} + u_{24} u_{25} = 1, \quad u_{24} + u_{13} u_{35} = 1, \quad u_{35} + u_{14} u_{24} = 1,$$
$$u_{14} + u_{25} u_{35} = 1, \quad u_{25} + u_{13} u_{14} = 1.$$

Its closure in $\mathbb{C}^5$ is an instance of a *binary geometry* [22, Section 2.1]. The paper [10] discovers new binary geometries arising from *pellytopes*. In [14], the authors study tropicalizations of binary geometries associated with cyclohedra.

The potential function L is given in dihedral coordinates by

$$L = \log \prod_{(ij) \in \Delta(5)} u_{ij}^{X_{ij}}, \quad \text{where} \quad X_{ij} = s_{i,j+1} + s_{i+1,j} - s_{ij} - s_{i+1,j+1}.$$

One checks that its critical points on U satisfy the *scattering equations*

$$X^\top \begin{pmatrix} 0 & 1 - u_{14} - u_{25} & 1 - u_{13} & u_{13} - 1 & u_{24} + u_{35} - 1 \\ u_{14} + u_{35} - 1 & 0 & 1 - u_{13} - u_{25} & 1 - u_{24} & u_{24} - 1 \\ u_{35} - 1 & u_{14} + u_{25} - 1 & 0 & 1 - u_{13} - u_{24} & 1 - u_{35} \\ 1 - u_{14} & u_{14} - 1 & u_{13} + u_{25} - 1 & 0 & 1 - u_{24} - u_{35} \\ 1 - u_{14} - u_{35} & 1 - u_{25} & u_{25} - 1 & u_{13} + u_{24} - 1 & 0 \end{pmatrix} = 0,$$

where X^T is the row vector $(X_{13}, X_{24}, X_{35}, X_{14}, X_{25})$. It would be interesting to study scattering equations in dihedral coordinates for general $n \geq 5$.
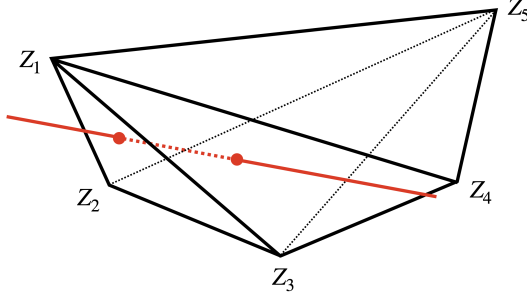
Figure 3: A cyclic polytope in $\mathbb{RP}^3$ with a stabbing line.

3. (Semi-)algebraic subsets of the Grassmannian

As mentioned in the Introduction, the discovery of the *amplituhedron* in [7] lies at the origin of positive geometry. Its canonical differential form is the integrand of the scattering amplitude in $N = 4$ Super Yang-Mills theory. The contribution of [15] is a detailed explanation of this assertion, aimed at mathematicians.

Fix integer parameters k, m, n satisfying $0 < k \leq k + m \leq n$, and let $Z \in \mathbb{R}^{n \times (k+m)}$ be a real matrix, all of whose ordered $(k+m) \times (k+m)$ minors are positive. The map $\tilde{Z} : \mathrm{Gr}(k,n)_{\geq 0} \to \mathrm{Gr}(k, k+m)$ is that induced by matrix multiplication: $[X] \mapsto [X \cdot Z]$ for $X \in \mathbb{R}^{k \times n}$ of rank k with non-negative $k \times k$ minors.

The *tree-level amplituhedron* $\mathcal{A}_{n,k,m}(Z)$ is the image of the map $\tilde{Z}$:

$$\mathcal{A}_{n,k,m}(Z) = \tilde{Z}(\mathrm{Gr}(k,n)_{\geq 0}) \subset \mathrm{Gr}(k, k+m).$$

It is a semi-algebraic subset of the real locus $\mathrm{Gr}_{\mathbb{R}}(k, k+m)$ in $\mathrm{Gr}(k, k+m)$.

For a concrete example, set $k = m = 2$, $n = 5$. To make sure all 4×4 minors of $Z \in \mathbb{R}^{5 \times 4}$ are positive, we can pick points on the twisted cubic for its rows:

$$Z_i = \begin{pmatrix} 1 & i & i^2 & i^3 \end{pmatrix}, \quad i = 1, \dots, 5. \tag{5}$$

These are the vertices of a cyclic polytope $\mathcal{A}_{5,1,3}(Z) = \mathrm{Conv}(Z)$ in $\mathbb{RP}^3$. The combinatorial type of this polytope is shown in Figure 3. Since $\mathcal{A}_{5,2,2}(Z)$ is a full-dimensional semi-algebraic subset of $\mathrm{Gr}_{\mathbb{R}}(2,4)$, it is described by polynomial inequalities in the Plücker coordinates $p_{12}, p_{13}, \dots, p_{34}$ on $\mathrm{Gr}(2,4)$. This description uses the standard notation $\langle ABij \rangle$ for the linear form in p_{ij} vanishing on lines $AB \in \mathrm{Gr}(2,4)$ which intersect the line $Z_i Z_j$. It was conjectured in [6], and proved in [25], that the lines $AB \in \mathcal{A}_{5,2,2}(Z)$ are characterized as follows:

$\langle AB12 \rangle, \langle AB23 \rangle, \langle AB34 \rangle, \langle AB45 \rangle, \langle AB15 \rangle$ have the same sign,
the sequence $\langle AB12 \rangle, \langle AB13 \rangle, \langle AB14 \rangle, \langle AB15 \rangle$ has exactly two sign flips.

Here zeros are ignored when counting sign flips. The red line AB in Figure 3 passes through the interior of the facets $Z_1Z_2Z_3$ and $Z_1Z_3Z_4$. It belongs to the *2-stabbing set* of $\text{Conv}(Z)$, but it does not belong to the amplituhedron: $\langle AB12 \rangle$ and $\langle AB34 \rangle$ have opposite signs. However, one can show that each line in the amplituhedron stabs the cyclic polytope. The paper [30] introduces and studies more general stabbing sets and their relation to tree amplituhedra.

The algebraic boundary of $\mathcal{A}_{5,2,2}(Z)$ is a union of five hyperplane sections of $\text{Gr}(2,4)$, given by the equation $\langle AB12 \rangle \langle AB23 \rangle \cdots \langle AB15 \rangle = 0$. Such arrangements in the Grassmannian are natural generalizations of hyperplane arrangements in $\mathbb{P}^n$, which are extensively studied in (algebraic) combinatorics and which we encountered in Section 2. The paper [24] investigates the real and complex topology of hyperplane arrangement complements in $\text{Gr}(k,n)$.

The *adjoint hypersurface* of our tree-level amplituhedron $\mathcal{A}_{5,2,2}(Z)$, for the specific Z in (5), is given by a linear equation in Plücker coordinates [29]:

$$593\, p_{12} - 330\, p_{13} + 143\, p_{14} + 49\, p_{23} - 30\, p_{24} + 5\, p_{34} = 0. \qquad (6)$$

It is uniquely determined by the following interpolation conditions. The adjoint contains the unique line $(1|23|45) \in \text{Gr}(2,4)$ passing through Z_1 and intersecting the lines Z_2Z_3 and Z_4Z_5. Similarly, it contains $(2|34|15)$, $(3|45|12)$, $(4|15|23)$ and $(5|12|34)$. The linear form (6) is the numerator of the integrand of the corresponding $N = 4$ Super Yang-Mills amplitude. The authors of [20] seek to uniquely determine the numerator of *MHV gravity amplitudes* via similar interpolation conditions on a flag variety. Except for some small cases, this description remains conjectural. It is natural to ask whether this numerator is adjoint to a positive geometry – an "amplituhedron" for MHV amplitudes.

Tree amplituhedra are a family of candidate positive geometries inside $X = \text{Gr}(k,k+m)$ by considering $X_{\geq 0} = \mathcal{A}_{n,k,m}(Z)$, $n \geq k+m$. When $k = 1$, $\mathcal{A}_{n,k,m}(Z)$ is a cyclic polytope in $\mathbb{P}^{k+m-1}$, and hence a positive geometry (Example 1.1). When $k + m = n$, $\mathcal{A}_{n,k,m}(Z)$ is isomorphic to $\text{Gr}(k,n)_{\geq 0}$, which is known to be a positive geometry. Recent progress for $k = m = 2$ is found in [29]. For other n,k,m, the amplituhedron $\mathcal{A}_{n,k,m}(Z)$ is only conjectured to be a positive geometry [4, Conjecture 9]. *Loop-level amplituhedra*, as opposed to tree-level amplituhedra, are known to *not* be positive geometries in the sense of [4]. They still have a meaningful adjoint hypersurface, as illustrated by the case study [16].

Other interesting positive geometries are semi-algebraic subsets of subvarieties of the Grassmannian. This arises in *ABJM scattering amplitudes*, with the relevant subvariety being the *orthogonal Grassmannian*. The authors of [17] examine the *positive orthogonal Grassmannian*, and prove that it is a positive geometry in some cases. A different subvariety, called *ABCT variety*, is studied in [1], which solves part of [22, Problem 4.18]. Lam's conjecture [22, Conjecture 4.10] regarding a positive geometry inside the ABCT variety remains open.

Inspired by the amplituhedron map $\tilde{Z}$, the paper [27] investigates whether a $(km-1)$-dimensional subvariety of $\mathrm{Gr}(k,n)_{\geq 0}$ can be recovered from its linear projections $[X] \mapsto [X \cdot Z] \in \mathrm{Gr}(k, k+m)$ for all possible Z. This generalizes the classical construction of recovering a projective variety from its *Chow form*.

4. Integrals and differential equations

Observables in physics are usually expressed as integrals. Such integrals include scattering amplitudes in particle physics and correlators in cosmology. The following Euler integral serves as a blueprint for cosmological correlators:

$$\phi(c) = \int_\Gamma \frac{1}{(c_1\alpha_1 + c_2\alpha_2 + c_3)(c_4\alpha_1 + c_5)(c_6\alpha_2 + c_7)} \alpha_1^{\varepsilon+1} \alpha_2^{\varepsilon+1} \frac{d\alpha_1}{\alpha_1} \frac{d\alpha_2}{\alpha_2}.$$

Here Γ is a twisted cycle in $\mathbb{C}^2$. The integral is a function ϕ of $c = (c_1, c_2, \dots, c_7)$. It satisfies a system of linear differential equations known as the *GKZ system*. These equations come in two flavors. First, ϕ is annihilated by Euler operators

$$\begin{aligned} c_1\partial_1 + c_2\partial_2 + c_3\partial_3 + 1, \; c_4\partial_4 + c_5\partial_5 + 1, \; c_6\partial_6 + c_7\partial_7 + 1, \\ c_1\partial_1 + c_4\partial_4 + (\varepsilon+1), \quad c_2\partial_2 + c_6\partial_6 + (\varepsilon+1). \end{aligned} \tag{7}$$

Here, ∂_i denotes the operator $\frac{\partial}{\partial c_i}$. And, second, we have the toric operators

$$\partial_1\partial_5 - \partial_3\partial_4 \quad \text{and} \quad \partial_2\partial_7 - \partial_3\partial_6. \tag{8}$$

The operators in (7)-(8) generate a holonomic D-ideal of rank 4. This means in particular that the solution space is 4-dimensional. This GKZ system and its role for the two-site chain model in cosmology are explored in [18, Section 3.1].

In most applications, the coefficients c_i of the parenthesized linear functions are not independent unknowns, but they satisfy special constraints. This makes the derivation of annihilating differential operators more difficult. One scenario, of relevance to cosmology, is studied in the article [26]. Here, the linear forms are fixed, and only the constant terms are allowed to vary. For instance, consider

$$\psi(u) = \int_\Gamma (\alpha_1 + \alpha_2 + u_1)^{s_1} (\alpha_1 + u_1)^{s_2} (\alpha_2 + u_3)^{s_3} \alpha_1^{v_1} \alpha_2^{v_2} \frac{d\alpha_1}{\alpha_1} \frac{d\alpha_2}{\alpha_2}.$$

The five exponents s_1, s_2, s_3, v_1, v_2 are parameters. The D-module for the function $\psi(u)$ in three variables $u = (u_1, u_2, u_3)$ is worked out in [26, Section 4.5].

The CHY amplitude we saw in (1) also has an analytic representation by means of an integral. Namely, A_5 arises as a limit from the *string amplitude*

$$\phi_\varepsilon(s) = \varepsilon^2 \int_{\mathbb{R}_+^2} \exp(\varepsilon \cdot L) \frac{dx}{x} \frac{dy}{y} = \varepsilon^2 \int_{\mathcal{M}_{0,5}^+} \frac{1}{p_{12}p_{23}p_{34}p_{45}p_{51}} \prod_{1 \leq i < j < 5} p_{ij}^{\varepsilon \cdot s_{ij}} \, dp.$$

Here $L(x,y)$ is the scattering potential from (3), which depends on the Mandelstam variables $s = (s_{12}, \ldots, s_{45})$. The positive moduli space $\mathcal{M}_{0,5}^+$ is the region $\mathbb{R}_+^2$ in the hyperplane arrangement mentioned prior to (3). The integrand is the *Koba–Nielsen potential* from [22, Section 5.1]. The limit of $\phi_\varepsilon(s)$, as ε tends to zero, is the rational function A_5 in (1) or (4). This is the content of [22, Theorem 5.12]. In string theory, the parameter $\varepsilon = \alpha'$ is the *inverse string tension*.

Iterated integrals arise in particle physics, number theory, and many other settings. In stochastic analysis, these are defined for paths $X : [0,1] \to \mathbb{R}^d$. Every sequence $i_1, i_2, \ldots, i_k$ of indices in $\{1, 2, \ldots, d\}$ specifies an iterated integral

$$\sigma_{i_1 i_2 \cdots i_k}(X) \;=\; \int_{\Delta_k} \mathrm{d}X_{i_1}(t_1)\, \mathrm{d}X_{i_2}(t_2) \cdots \mathrm{d}X_{i_k}(t_k).$$

Here Δ_k denotes the *k*-dimensional simplex given by $0 \le t_1 \le t_2 \le \cdots \le t_k \le 1$. These real numbers are the entries in a *k*-dimensional tensor σ of format $d \times d \times \cdots \times d$. This tensor is *k-th signature* of the path X. The article [23] concerns piecewise linear paths X, whose breakpoints are the vertices of a cyclic polytope. It characterizes linear combinations of the tensor entries $\sigma_{i_1 i_2 \cdots i_k}(X)$ that express invariant, geometric properties of the polytope. Cyclic polytopes are important in particle physics because they are special cases of amplituhedra.

The article [9] grew out of the lecture series *Counting graphs using quantum field theory*, given by Michael Borinsky at the May 2024 school in Leipzig. The integrals studied in that article are similar to those seen in quantum field theory:

$$I(z) \;=\; \frac{z}{2\pi} \int_D \exp(z\,g(x,y))\, \mathrm{d}x\,\mathrm{d}y.$$

Here, D is a subset of $\mathbb{R}^2$ and $g : D \to \mathbb{R}$ is an appropriate function. Notice the resemblance to the string amplitude $\phi_\varepsilon(s)$, setting $g = L$ and $z = \varepsilon$. The asymptotic expansion of $I(z)$ for large z serves as a generating function for the number of regular edge-bicolored graphs. Methods from analysis and combinatorics are combined to determine the limit behavior of these numbers. Analogous results in one variable had huge implications for the cohomology of moduli spaces, both classical and tropical, and we are similarly optimistic for the impact of [9].

In the Introduction, we emphasized the physical importance of the volume function $x \mapsto \mathrm{vol}(P - x)^\circ$. As in Example 1.1, P is a convex polytope or a more general positive geometry. Of course, such a volume is an integral. If P depends on parameters, then so does the function $x \mapsto \mathrm{vol}(P - x)^\circ$. It therefore makes sense to apply the *differential equations method*, which represents such a holonomic function by a D-module. The study of integrals and differential equations is thus an essential ingredient for the emerging field of positive geometry.

5. Open questions

In this section we present one open problem for each of the articles in this collection. In each case, the problem we chose is either stated in the article, in which case a direct citation is given, or it is inspired by results from the article.

Problem [1]: The image of the positive Grassmannian $\mathrm{Gr}(2,n)_{\geq 0}$ in the ABCT variety $V(d+1,n)$ is a semialgebraic set. Is that image a positive geometry? Is $V(d+1,n)$ with the relevant boundary a genus zero pair, in the sense of [12]?

Problem [2]: Consider a uniform oriented matroid M whose realization space is diffeomorphic to an open ball. Is the chirotropical Dressian of M equal to the chirotropical Grassmannian of M? This is conjectured in [2, Conjecture 6.2].

Problem [8]: A weak formulation of the CHY scattering equations leads to the intersection of the reciprocal variety of the moduli space $\mathcal{M}_{0,n}$ with a linear space. Determine the scheme structure of this intersection [8, Conjecture 5.6].

Problem [9]: How to best generalize the results from this paper to graphs with three or more colors? Now we must integrate over a space D of dimension ≥ 3.

Problem [10]: Does the Pellspace lead to an interesting compactification of the moduli space $\mathcal{M}_{0,n}$? Develop this from the perspective of algebraic geometry.

Problem [14]: Does the space of axially symmetric phylogenetic trees coincide with the tropicalized cluster configuration space in type C? [14, Conjecture 7.5].

Problem [15]: Is the amplituhedron a positive geometry, in the sense of [4]? This is a guiding conjecture, going back to the original paper by Arkani-Hamed and Trnka [7]. See [15, Section 6]. Can the paper [12] shed new light on this?

Problem [16]: Show that the two-loop amplituhedron $\mathcal{A}_4^{(2)}$ is a weighted positive geometry. Following [16, Section 5.2], some residues need to be computed.

Problem [17]: Determine a cell decomposition for the positive orthogonal Grassmannian. What should be added to the orthopositroids? [17, Problem 5.4].

Problem [18]: Prove the partial fraction decomposition suggested in [18, Conjecture 3.5] for the flat space wave function ψ_{flat} of an arbitrary Feynman graph.

Problem [19]: Show that the logarithmic discriminant of the moduli space $\mathcal{M}_{0,n}$ is an irreducible hypersurface, and determine its degree. [19, Conjecture 1].

Problem [20]: Establish the unique representation of the n-point MHV amplitude. This is stated as a commutative algebra problem in [20, Conjecture 2.1].

Problem [22]: Study spinor-helicity polytopes, with focus on their parametrizations and associated cohomology classes. Solve [22, Problems 4.22 and 4.23].

Problem [23]: Determine the ring of volume invariants. This refers to signatures of piecewise-linear paths on cyclic polytopes. Prove [23, Conjecture 4.1].

Problem [24]: What is the maximal number of regions in the complement of an arrangement of d Schubert hyperplanes in the real Grassmannian $\mathrm{Gr}_{\mathbb{R}}(k,n)$?

Problem [26]: Determine a set of generators for the annihilator $\mathrm{Ann}_{D(s,V)}(\phi)$ of the correlator function ϕ of an arbitrary hyperplane arrangement V.

Problem [27]: Let $\mathcal{V}$ be a subvariety of the Grassmannian $\mathrm{Gr}_{\mathbb{C}}(k,n)$ which is defined by $k(n-r)+1$ generic linear forms in the Plücker coordinates. Prove that $\mathcal{V}$ coincides with its Chow-Lam recovery $\mathcal{W}_{\mathcal{V}}$.   [27, Conjecture 6.6].

Problem [28]: Show that the kinematic variety for spinor brackets of order ≤ 3 is the variety of tensors with multilinear rank $\leq (2,4,2)$.   [28, Conjecture 6.2].

Problem [30]: Show that the cells in the k-face Schubert arrangement of any convex polytope are determined by their sign vectors.   [30, Conjecture 3.5].

We hope that these open problems will inspire mathematical progress in the development of positive geometry, and that this will have an impact in physics.

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