LE MATEMATICHE
Vol. LV (2000) – Fasc. II, pp. 419–435

ON THE HARTSHORNE-RAO MODULE OF CURVES ON RATIONAL NORMAL SCROLLS

ROBERTA DI GENNARO

Dedicated to Silvio Greco in occasion of his 60-th birthday.

We study the Hartshorne–Rao module of curves lying on a rational normal scroll S_e of invariant $e \geq 0$ in $\mathbb{P}^{e+3}$.

We calculate the Rao function, we characterize the aCM curves on S_e. By using a result of Gorenstein liaison theory, we reduce all curves to two kinds: those consisting of distinct fibers and those with a "few" of fibers. In such a way, we find a set of minimal generators and the Buchsbaum index of each curve on S_e.

Finally, we give an algorithm to check if a curve is aCM or not and, in the second case, to calculate the Rao function.

Introduction.

In the last years there has been a great interest on the Hartshorne - Rao module $H^1_*(\mathcal{I}_C) = \bigoplus_{j\in} H^1(\mathcal{I}_C(j))$ of curves, because it gives many geometric information. For curves in $\mathbb{P}^3$ there are many results about both the Rao function $h^1(\mathcal{I}_C(j))$ and the structure of the module starting with the book [6].

Supported by GNSAGA - INDAM

1991 *Mathematics Subject Classification* [2000] 14J26, 14C20

Key words and phrases Curve, Cohomology group

Instead, the knowledge on this subject about general curves in projective space of dimension ≥ 4 is very small and only on the Rao function (cf. [2] and [11]). An idea to proceed is e.g. to consider curves lying on very well known surfaces which is the idea of Giuffrida and Maggioni (cf. [3]) in the study of the Hartshorne-Rao module of curves on a quadric or a cubic surface in $\mathbb{P}^3$. We begin our study of curves in projective space of dimension ≥ 4 lying on a surface by considering a smooth reduced normal scroll S_e of invariant $e \geq 0$ in $\mathbb{P}^{e+3}$ (cf. [4]). The first one of these surfaces is the quadric in $\mathbb{P}^3$ (for $e = 0$) and our results coincide with those in the Appendix of [3].

On those particular surfaces we can get many information on the Rao module of each curve. Our work proceeds as follows: after giving some general results on curves on aCM surfaces in the first section, in the second one we calculate the Rao function of a curve on S_e, we get the optimal bounds for it and we characterize the aCM curves on S_e. In this section we do not use the liaison theory as in [3] about the curves on the quadric. In the last section we investigate the multiplicative structure of the Rao module using a theorem of Gorenstein liaison theory (cf. [5]) which allows to "shift" the Rao module of a curve and to reduce our study to two kinds of curves: those consisting of fibers only and those having "few" fibers. In such a way we find a set of minimal generators for non-aCM curves and their Buchsbaum index. Buchsbaum and arithmetically Cohen Macauly property of divisors on rational normal scrolls of any dimension are studied also by U. Nagel in [10], by M. Casanellas in her PhD Thesis [1] and by Miyazaki in [9]. Our results, founded in a different way arguing on minimal generators, are the same in the case of surfaces.

Finally we give an algorithm which, by giving as input the invariant e of the surface and the two parameters of the curve, says if the curve is aCM and, if the curve is non-aCM, gives the positive values of the Rao function and it says the kind of simplified curve we get by liaison.

We are very grateful to Prof. S. Greco for important help, to R. Notari for interesting conversations about liaison theory and to the referee for his notes.

1. Something on curves on aCM surfaces.

Some notation 1.1. We work over an algebraically closed field k (of arbitrary characteristic) and we use the standard notation and results contained in Hartshorne's book [4]. S is an arithmetically Cohen - Macauly (briefly "aCM") surface in the projective space $\mathbb{P}^n$ and C is a *curve* (that is a non trivial effective divisor) on S of degree d.

We denote by $\mathcal{I}_C$ and $\overline{\mathcal{I}_C} \cong \mathcal{O}_S(-C)$ the ideal sheaves of C in $\mathbb{P}^n$ and S respectively, by H a general hyperplane section and by K a canonical divisor of S; p_C and p_S denote the arithmetic genus of C and S respectively. Moreover, we omit the environment in the cohomolgy groups, e.g. we use $H^0(\mathcal{O}_S(j))$ instead of $H^0(S, \mathcal{O}_S(j))$.

The Hartshorne–Rao module of C is the graded $k[x_0, \dots, x_n]$- module

$$H^1_*(\mathcal{I}_C) := \bigoplus_{j\in} H^1(\mathcal{I}_C(j))$$

We denote by $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ respectively the largest integer less or equal and the smallest integer greater or equal to the number in the bracket.

Some general results 1.2. In this section we give some calculations to find, for all $j \in \mathbb{Z}$, the Rao function $h^1(\mathcal{I}_C(j))$ of C.

Remark 1.1. We begin observing that we consider aCM surfaces because this property allows us to move our attention from the Rao module to the module $\bigoplus_{j\in\mathbb{Z}} H^1(\mathcal{O}_S(-C)(j))$. In fact there is the isomorphism of graded modules

$$H^1_*(\mathcal{I}_C) \cong H^1_*(\mathcal{O}_S(-C)),$$

as we prove by considering the long cohomology sequence of

$$0 \to \mathcal{I}_S \to \mathcal{I}_C \to \mathcal{O}_S(-C) \to 0$$

and by recalling that, since S is aCM, $H^1_*(\mathcal{I}_S) = 0 = H^2_*(\mathcal{I}_S)$

Now, our study consists to find the set of the values of twist where we can calculate the Rao function.

Proposition 1.2. *If the linear systems $|jH - C|$ and $|K + C - jH|$ are non-effective, we have*

$$h^1(\mathcal{I}_C(j)) = -p_S + \frac{1}{2}(jH - C)\cdot(K + C - jH) - 1$$

$$= -p_S - p_C - \frac{1}{2}j^2\, deg\,(S) + \frac{1}{2}\, jH\cdot K + j\, deg\,(C)$$

Proof. The first equality comes by the Riemann - Roch theorem applied to the divisor $jH - C$ of S:

$$h^0(\mathcal{O}_S(-C)(j)) - h^1(\mathcal{O}_S(-C)(j)) + h^2(\mathcal{O}_S(-C)(j)) =$$

$$= \frac{1}{2}(jH - C) \cdot (jH - C - K) + 1 + p_S.$$

Now, $h^0(\mathcal{O}_S(-C)(j)) = h^0(\mathcal{O}_S(jH - C)) = 0$, since $jH - C$ is non-effective and similarly, by Serre duality, $h^2(\mathcal{O}_S(-C)(j)) = h^0(\mathcal{O}_S(K + C - jH)) = 0$; so the thesis.
To prove the second equality, it is enough to note that $p_C = 1 + \frac{1}{2}C \cdot (C + K)$, by the adjunction formula, and deg $(jH \cap C) = C \cdot jH = j$ deg (C). □

Now, to study the set of the values of " j" where at least one of the two linear systems $|jH - C|$, $|K + C - jH|$ is effective, we use the classical method of characteristic series.

Proposition 1.3. *If at least one of the linear systems $|jH-C|$ and $|K+C-jH|$ is effective and irreducible and its intersection with the canonical divisor K is negative, then $h^1(\mathcal{I}_C(j)) = 0$.*

Proof. We begin observing that if D is an effective irreducible divisor and $D \cdot D > 2p_D - 2$, then $H^1(\mathcal{O}_S(D)) = 0$, as we get by considering the cohomology sequence of $0 \to \mathcal{O}_S(-D) \to \mathcal{O}_S \to \mathcal{O}_D \to 0$, tensorized by $\mathcal{O}_S(D)$, and recalling that $H^1(\mathcal{O}_S) = H^2(\mathcal{I}_S) = 0$. On the other hand, by the adjunction formula, for a divisor D on S, $D \cdot D > 2p_D - 2$ iff $D \cdot K < 0$.

So, if the hypotheses hold for $jH - C$, immediately

$$h^1(\mathcal{I}_C(j)) = h^1(\mathcal{O}_S(jH - C)) = 0;$$

if these hold for the divisor $K + C - jH$, it is enough to apply Serre duality. □

The above Propositions 1.2 and 1.3 do not consider all possible value of j; indeed the linear systems $|jH - C|$ or $|K + jH - C|$ might be effective but reducible. We will fill up this gap for rational normal scrolls in the next section.

2. The Rao function of curves on a rational normal scroll.

From now on, $S := S_e \subset \mathbb{P}^{e+3}$ is a rational normal scroll of invariant $e \geq 0$, namely the embedding of a rational geometrically ruled surface F_e (called Hirzebruch surface (cf. [7])) of invariant e via the very ample linear system $|C_0+(e+1)\mathfrak{f}|$, which is then the linear system of the hyperplane sections, where C_0 is a line of sel-intersection $C_0^2 = -e$ and $\mathfrak{f}$ is a fiber, so $f^2 = 0$ and $C_0 \cdot \mathfrak{f} = 1$ (cf. [4], ch. V). By this embedding, S_e is an aCM surface.

Each divisor C on S_e is linearly equivalent to $aC_0 + b\mathfrak{f}$, with $a, b \in \mathbb{Z}$ and it is effective if and only if $a, b \geq 0$ and $a+b \neq 0$. The degree of $C \sim aC_0 + b\mathfrak{f}$ is $a + b$, the arithmetic genus is $p_a(C) = 1 + ab - a - b - \frac{1}{2}ea(a-1)$. The canonical divisor is $K \sim -2C_0 - (e+2)\mathfrak{f}$.

Finally, we recall that (cf. [4] V - Cor. 2.18) a general curve $C \sim aC_0 + b\mathfrak{f}$ on S_e is irreducible if and only if $C \sim C_0$ or $C \sim \mathfrak{f}$ or $a, b > 0$ and $b \geq ae$.

The first example of rational normal scroll is the quadric in $\mathbb{P}^3$ (e=0) and the results of this paper generalized to any invariant $e \geq 0$ those appearing in [3], Appendix C.

We note that some of the following results appear in [7], where the author gives the values of $h^1(\mathcal{O}_S(D))$, with D effective divisor on S, but we found a misleading missprint.

Proposition 2.1. *Let $C \sim aC_0 + b\mathfrak{f}$. We have:*

1. *If $j \leq \min\left\{b - ae + e - 2, a - 2, \left\lfloor \frac{b-(e+2)}{e+1} \right\rfloor\right\}$, then $h^1(\mathcal{I}_C(j)) = 0$;*

2. *If $b - ae + e - 2 < j \leq \min\left\{a - 2, \left\lfloor \frac{b-(e+2)}{e+1} \right\rfloor\right\}$ and $\alpha := \left\lfloor \frac{b-j-2}{e} \right\rfloor$, then*
$$h^1(\mathcal{I}_C(j)) = (a - \alpha - 1)\left[\frac{e}{2}(a + \alpha) - b + j + 1\right];$$

3. *If $\min\left\{a - 2, \left\lfloor \frac{b-(e+2)}{e+1} \right\rfloor\right\} < j < \max\left\{a, \left\lceil \frac{b}{e+1} \right\rceil\right\}$, then*
$$h^1(\mathcal{I}_C(j)) = j(a + b) - p_C + 1 - \frac{1}{2}(j+1)[j(e+2) + 2];$$

4. *If $\max\left\{a, \left\lceil \frac{b}{e+1} \right\rceil\right\} \leq j < b - ae$ and $\alpha := \left\lfloor \frac{j-b}{e} \right\rfloor$, then*
$$h^1(\mathcal{I}_C(j)) = (a + \alpha)\left[j - b + 1 + \frac{e}{2}(a - \alpha - 1)\right]$$

5. *If $j \geq \max\{a, \left\lceil \frac{b}{e+1} \right\rceil, b - ae\}$, then $h^1(\mathcal{I}_C(j)) = 0$.*

Remark 2.2.

i) We note that if $e = 0$ there is no j such that $b - ae + e - 2 < j \leq \min\left\{a - 2, \left\lfloor \frac{b-(e+2)}{e+1} \right\rfloor\right\}$; so we do not have to calculate α and we do not consider the division by e.

ii) If C is a reducible curve, $b - ae < 0$ and so there is no j such that $\max\left\{a, \left\lceil \frac{b}{e+1} \right\rceil\right\} \leq j \leq b - ae - 1$ and $h^1(\mathcal{I}_C(j)) = 0$ for all $j \geq \max\left\{a, \left\lceil \frac{b}{e+1} \right\rceil\right\}$

Proof. Item 3. The interval of the third item corresponds to the case of Proposition 1.2, so we have only to calculate $j \deg(C) = j(a+b)$ and $jH \cdot K$.

To complete this proof, we need Lemma 2.4 and 2.3 below.

Lemma 2.3. *Let $C \sim b'\mathfrak{f}$ then $H^1(\mathcal{O}_S(b'\mathfrak{f})) = 0$*

Proof. We argue by induction on b'. It is clear, by Riemann-Roch Theorem, that $H^1(\mathcal{O}_S(\mathfrak{f})) = 0$. Let $b' > 1$. By considering the cohomology sequence of

$$0 \to \mathcal{O}_S(-\mathfrak{f}) \to \mathcal{O}_S \to \mathcal{O}_\mathfrak{f} \to 0$$

tensorized by $\mathcal{O}_S(b'\mathfrak{f})$, we get the thesis by induction, because $\mathcal{O}(\mathfrak{f} \cdot b'\mathfrak{f}) = \mathcal{O}_\mathfrak{f}$ and $H^1(\mathcal{O}_\mathfrak{f}) = 0$, since $\mathfrak{f}$ is a rational curve. □

Lemma 2.4. *If $|a'C_0 + b'\mathfrak{f}|$ is a reducible linear system on S_e, $a' > 0$ and $\alpha' = \left\lfloor \frac{b'}{e} \right\rfloor$, then*

$$h^1(\mathcal{O}_{S_e}(a'C_0 + b'\mathfrak{f})) = (a' - \alpha')\left[\frac{e}{2}(a' + \alpha' + 1) - b' - 1\right]$$

Proof. We begin by observing that, if $b' \neq 0$, $\alpha' = \lfloor \frac{b'}{e} \rfloor = \max\{i \leq a' | iC_0 + b'\mathfrak{f}$ is irreducible $\}$. So, $h^1(\mathcal{O}_{S_e}(\alpha'C_0 + b'\mathfrak{f})) = 0$. The same, if $b' = 0$ $h^1(\mathcal{O}_{S_e}(\alpha'C_0 + b'\mathfrak{f})) = h^1(\mathcal{O}_{S_e}) = 0$ This suggest to proceed by induction on a'.

Let $a' = \alpha' + 1$. We argue on the sequence

$$0 \to \mathcal{O}_{S_e}(-C_0) \to \mathcal{O}_{S_e} \to \mathcal{O}_{C_0} \to 0 \tag{2.1}$$

By considering the long cohomology sequence of (2.1) tensored by $\mathcal{O}_{S_e}((\alpha' + 1)C_0 + b'\mathfrak{f})$ we get

$$\cdots \to 0 \to H^1(\mathcal{O}_{S_e}((\alpha'+1)C_0 + b'\mathfrak{f})) \to H^1(\mathcal{O}_{C_0}(((\alpha'+1)C_0 + b'\mathfrak{f}) \cdot C_0)) \to 0$$

because $H^1(\mathcal{O}_{S_e}(\alpha' C_0 + b'\mathfrak{f})) = 0$ and $H^2(\mathcal{O}_{S_e}(\alpha' C_0 + b'\mathfrak{f})) = 0$ by Serre duality. So, we have only to calculate $h^1(\mathcal{O}_{C_0}(((\alpha'+1)C_0 + b'\mathfrak{f}) \cdot C_0))$. By Riemann - Roch theorem applied to the divisor $(\alpha'+1)C_0 + b'\mathfrak{f} \cap C_0$ on the line C_0, we get

$$h^1(\mathcal{O}_{S_e}((\alpha'+1)C_0 + b'\mathfrak{f})) = (\alpha'+1)e - b' - 1.$$

Now, let $a' > \alpha' + 1$. By proceeding with the same argument, tensoring (2.1) by $\mathcal{O}_{S_e}(a'C_0 + b'\mathfrak{f})$, we get $h^1(\mathcal{O}_{S_e}(a'C_0 + b'\mathfrak{f})) = h^1(\mathcal{O}_{S_e}((a'-1)C_0 + b'\mathfrak{f})) + h^1(\mathcal{O}_{C_0}((a'C_0 + b'\mathfrak{f}) \cdot C_0))$ and by induction and some calculations, the thesis follows. □

Now we can complete the proof of Proposition 2.1.

Item 1. and Item 5. The intervals of the first and the fifth items correspond respectively to $|K + C - jH|$ and $|jH - C|$ being effective and irreducible or, at most, to $|K + C - jH|$ and $|jH - C|$ being an union of fibers, respectively when $j = \min\left\{b - ae + e - 2, a - 2, \left\lfloor \frac{b-(e+2)}{e+1} \right\rfloor\right\} = a - 2$ and $j = \max\left\{a, \left\lceil \frac{b}{e+1} \right\rceil, b - ae\right\} = a$.

So, in the first case, we can apply Proposition 1.3. To apply it, it is enough to note, with a simple calculation, that for each irreducible curve $D \neq C_0$, $D \cdot K < 0$. Since $K - jH + C = C_0$ if $j = a - 3 = \left\lfloor \frac{b-e-2}{e+1} \right\rfloor$ and this value is not in the interval of Item 1 and since $jH - C = C_0$ if $j = a + 1 = \left\lceil \frac{b}{e+1} \right\rceil$ and this value is not in the interval of Item 5, then we have the hypotheses of Proposition 1.3 and the thesis follows.

Finally, in the cases of unions of fibers, we can apply Lemma 2.3.

Item 2. and Item 4. Recalling that $K + C - jH \sim (a - j - 2)C_0 + (b - je - j - e - 2)\mathfrak{f}$ and $jH - C \sim (j - a)C_0 + (j(e+1) - b)\mathfrak{f}$, the result follows immediately by Lemma 2.4. □

Remark 2.5. We note that if the curve is "large" enough, that means if $\min\left\{b - ae + e - 2, a - 2, \left\lfloor \frac{b-(e+2)}{e+1} \right\rfloor\right\} \geq 0$ the Rao function is zero in non-positive degree.

With simple calculations, we give a characterization of aCM curves on S_e.

Proposition 2.6. *A curve $C \sim aC_0 + b\mathfrak{f}$ on S_e is aCM if and only if* $(a-1)(e+1) \leq b \leq a(e+1) + 1$

Proof. First, if $(a-1)(e+1)-1 \leq b \leq a(e+1)$, we can write $b = ae+a-e+l$ with $0 \leq l \leq e$. In this case, $m = \min\left\{a-2, \left\lfloor \frac{b-(e-2)}{e+1} \right\rfloor\right\}$ which compare in Proposition 2.1 is $m = a-2$, $M = \max\left\{a, \left\lceil \frac{b}{e+1} \right\rceil\right\}$ is $M = a$ and there is no j such that $b - ae + e - 2 < j \leq m$ or such that $a \leq j \leq b - ae$. So we get that C is aCM, noting that $h^1(\mathcal{I}_C(a-1)) = 0$.

If $b = (a-1)(e+1)$, we have $m = a-3$ and $M = a$, so the previous arguments hold, noting that $h^1(\mathcal{I}_C(a-1)) = 0 = h^1(\mathcal{I}_C(a-2))$.

If $b = a(e+1)+1$, we have $m = a-2$ and $M = a+1$, so the previous arguments hold, noting that $h^1(\mathcal{I}_C(a-1)) = 0 = h^1(\mathcal{I}_C(a))$.

Now, if $b > ae+a+1$, let $b = ae+a+l$ with $l \geq 2$. With a simple calculation we get $h^1(\mathcal{I}_C(a)) = l-1 \neq 0$, so C is not aCM.

Finally, if $b < ae+a-e-1$, let $b = ae+a-e-l$ with $l \geq 2$. We get $h^1(\mathcal{I}_C(a-2)) = l-1 \neq 0)$ and again C is not aCM. □

Finally, we get the following optimal bounds.

Corollary 2.7. *Let $C \sim aC_0 + b\mathfrak{f}$ be a non-aCM curve on S_e, then there are the following optimal bounds:*

1. *If* $b < ae+a-e-1$

$$h^1(\mathcal{I}_C(j)) = 0 \qquad \textit{for all } j \leq b-ae+e-1 \textit{ and } j \geq a-1$$

2. *If* $b > ae+a+1$,

$$h^1(\mathcal{I}_C(j)) = 0 \qquad \textit{for all } j \leq a-1 \textit{ and } j \geq b-ae-1.$$

Proof. It is enough to use Proposition 2.1 to get some bounds.

In the case $b < ae+a-e-1$, we get $h^1(\mathcal{I}_C(j)) = 0$ for $j \leq b-ae+e-2$ and $j \geq a$, since there is no j such that $a \leq j < b-ae$. They are optimized by verifying that the Rao function vanishes for $j = b-ae+e-1$ and for $j = a-1$. Finally, by the arguments of proof of Proposition 2.6, these bounds are optimal.

The same if $b > ae+a+1$: by Proposition 2.1 we get $h^1(\mathcal{I}_C(j)) = 0$ for $j \leq a-2$ and $j \geq b-ae$. Again we verify that the Rao function vanishes for $j = a-1$ and for $j = b-ae-1$, and $h^1(\mathcal{I}_C(b-ae-2)) = 1$. □

3. Minimal generators.

In this section, we find a set of minimal generators for the Rao module of a non-aCM curve on a rational normal scroll of invariant $e \geq 0$.

The idea is to reduce the study of many curves to a certain number of fibers on S_e in general position, thanks to Theorem below originated from the Gorenstein liaison theory (cf. [5]). Then, we reduce the study of the remaining curves to some curves with a "little" number of fibers.

Theorem. (cf. [8], Corollary 5.3.4) *Let S be a smooth, aCM subscheme of $\mathbb{P}^n$. Let V be a divisor on S, i.e. a pure codimension one subscheme with no embedded components. Let V' be any element of the linear system $|V + kH|$, where H is the hyperplane section class and $k \in \mathbb{Z}$. Then, for $1 \leq i \leq dimV$,*

$$H^i_*(V') \cong H^i_*(V)(-k)$$

Proposition 3.1. *Let $C \sim aC_0 + b\mathfrak{f}$ be a non-aCM curve on S_e.*

1. *If $b > a(e+1)+1$ then*

$$H^1_*(\mathcal{I}_C) \cong H^1_*(\mathcal{I}_{C'}(-a))$$

where C' is the union of $b' := b - a(e+1) > 1$ fibers on S_e.

2. *If $b < (a-1)(e+1)$, then*

$$H^1_*(\mathcal{I}_C) \cong H^1_*\Big(\mathcal{I}_{C^*}\Big(-\Big\lfloor \frac{b}{e+1} \Big\rfloor\Big)\Big)$$

where $C^* \sim a'C_0 + r\mathfrak{f}$ with $a' = a - \left\lfloor \frac{b}{e+1} \right\rfloor \geq 2$ and $0 \leq r \leq e$ and r is the reminder of the division between b and $e+1$.

Proof. Item 1. If the number of fibers in C is very "large", C contains a hypersurface section of S_e. In particular, in our hypothesis on b, we have that $C \geq aH$. So the divisor $C' = C - aH \sim b'\mathfrak{f}$ is effective and by Theorem, we get the thesis.

Item 2. In this case again C contains a hypersurface section, but we have to calculate the degree of this hypersurface.

By considering the Euclidean division between b and $e+1$, we can write $C \sim aC_0 + \big[\lfloor \frac{b}{e+1} \rfloor (e+1) + r\big]\mathfrak{f}$ where r is the reminder of the division. By hypothesis, $a > \lfloor \frac{b}{e+1} \rfloor$, so C contains the hyperplane section of degree $\lfloor \frac{b}{e+1} \rfloor$. Again, by liaison theory we get the result. □

The following picture (Fig. 1) is an example on how the Rao module of a "large" curve "shifted" to the left corresponds to the Rao module of distinct fibers.

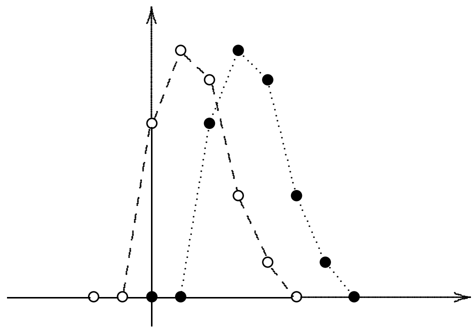

Fig. 1: $\bullet = C \sim 2C_0 + 10\mathfrak{f}$
$\circ = C \sim 6\mathfrak{f}$

At this point, we can find the degrees of the minimal generators of the Rao module of any curve C on S_e.

We recall a consequence of a general result proved by E. Schlesinger (cf. [12], Th. 3.2). Recall that the index of speciality of C is $\epsilon := \max\{j | h^1(\mathcal{O}_C(j)) \neq 0\}$.

Schlesinger's Theorem. *Let $C \subset \mathbb{P}^n$ be a curve with index of speciality ϵ. Then the Rao module of C does not have minimal generators in degree n, for $n \geq \epsilon + 3$*

Theorem 3.2. *Let $C \sim aC_0 + b\mathfrak{f}$ be a non-aCM curve on S_e. By Proposition 2.6, we have two possibilities:*

1. *If $b > a(e+1) + 1$ then the Rao module has a set of minimal generators consisting of $b - 1$ elements of degree a.*
2. *If $b < (a-1)(e+1)$ and $e > 0$, then, denoting by r the reminder of the Euclidean division between b and $e+1$, the Rao module of C has a set of minimal generators consisting of $a - \left\lfloor \frac{b}{e+1} \right\rfloor - 1$ elements, each one of degree $r - je$, for each $1 \leq j \leq a - \left\lfloor \frac{b}{e+1} \right\rfloor - 1$.*

Remark 3.3. We note that it is not restrictive to assume $e > 0$ in the second item because on the quadric in $\mathbb{P}^3$ (i.e. $e = 0$) the coefficients a and b are symmetric and so all possible cases are reconducted to the first item.

Proof. By Proposition 3.1, it is sufficient to prove that

1′. If C is the union of $b \geq 2$ distinct fibers on S_e, then the Rao module has a set of minimal generators consisting of $b-1$ elements of degree 0.

2′. If $C \sim aC_0 + r\mathfrak{f}$ with $a \geq 2$ and $0 \leq r \leq e$, the Rao module of C has a set of minimal generators consisting of $a-1$ elements, each one of degree $r - je$, for each $1 \leq j \leq a-1$.

Item 1′. We begin noting that the index of speciality of b fibers on S_e is $\epsilon = -2$. In fact the index of speciality of a line is -2 and the cohomology distributes the direct sums, so, since we can consider C as a disjoint union of lines, the index of speciality of C is also -2.

So by Schlesinger's Theorem, there are no generators in degree greater or equal than 1.

By Corollary 2.7, we get that the minimal generators can have just degree 0 and since $h^1(\mathcal{I}_C) = h^0(\mathcal{O}_C) - 1 = b - 1$, we get $b-1$ generators.

Item 2′. First of all, we recall (see Corollary 2.7) that the non trivial components of the Rao module of the curve $C \sim aC_0 + r\mathfrak{f}$ occur in degree greater or equal than $r - ae + e$ and less or equal than $a - 2$.

Now, the proof proceeds by induction on a.

To prove that the Rao module of the curve $2C_0 + r\mathfrak{f}$ has just one minimal generator in degree $r - e$, we twist the sequence

$$0 \to \mathcal{O}_S(-C_0) \to \mathcal{O}_S \to \mathcal{O}_{C_0} \to 0 \tag{3.1}$$

by $\mathcal{O}_S(-C_0 - r\mathfrak{f})$.
Since $C_0 + r\mathfrak{f}$ is aCM (Proposition 2.6), we get

$$0 \to H^0_*(\mathcal{O}_S(-2C_0 - r\mathfrak{f})) \to H^0_*(\mathcal{O}_S(-C_0 - r\mathfrak{f})) \to H^0_*(\mathcal{O}_{C_0}(e-r)) \to$$

$$\xrightarrow{\varphi} H^1_*(\mathcal{O}_S(-2C_0 - r\mathfrak{f})) \to 0$$

Now, $H^1_*(\mathcal{O}_S(-2C_0 - r\mathfrak{f})) = \bigoplus_{j \geq r-e} H^1(\mathcal{I}_C(j))$.

Since C_0 is a line in $\mathbb{P}^n$, with $n = e+3$ we can choose a homogeneous coordinate system such that C_0 has the equations $x_2 = \ldots = x_n = 0$. So, if we denote t_0 and t_1 the restrictions to C_0 of x_0 and x_1 respectively, we get $H^0_*(\mathcal{O}_{C_0}(e-r)) \cong k[t_0, t_1](r-e)$, the $k-$module of the homogeneous polynomials in two variables shifted of the degree $r - e$. This module has a unique minimal generator G_{r-e} in degree $r-e$, so also $H^1_*(\mathcal{O}_S(-2C_0 - r\mathfrak{f}))$ has a unique minimal generator $F_{-1} = \varphi(G_{r-e})$ in degree $r - e$, because φ is surjective.

The same arguments allow us to conclude the proof.

In fact, using again the sequence 3.1 tensored by $\mathcal{O}_S(-(a-1)C_0 - r\mathfrak{f})$, we get

$$\cdots \to k[t_0, t_1](r - e(a-1)) \xrightarrow{\varphi} H^1_*(\mathcal{O}_S(-aC_0 - r\mathfrak{f})) \to$$

$$\xrightarrow{\psi} H^1_*(\mathcal{O}_S(-(a-1)C_0 - r\mathfrak{f})) \to H^1_*(\mathcal{O}_{C_0}((a-1)e - r)) \to \cdots,$$

We note that $H^1_*(\mathcal{O}_S(-(a-1)C_0 - r\mathfrak{f})$ is non trivial in degree greater than $r - (a-2)e$, but in this interval $H^1_*(\mathcal{O}_{C_0}(r - (a-1)e))$ is trivial, because its non-trivial components occur only in degree less than $r - (a-1)e - 1$; so ψ is surjective.

Now, we assume by induction that $H^1_*(\mathcal{O}_S(-(a-1)C_0 - r\mathfrak{f}))$ has minimal generators G_{r-je} of degree $r - je$ for $1 \le j \le a-2$.

We are going to show that the minimal generators of $H^1_*(\mathcal{O}_S(-aC_0 - r\mathfrak{f}))$ are the image $F_{r-(a-1)e}$ by φ of the unique minimal generator $G_{r-(a-1)e}$ of $k[t_0, t_1](r - e(a-1))$ in degree $r - e(a-1)$ and, for each $1 \le j \le a-2$, F_{r-je} such that $\psi(F_{r-je}) = G_{r-je}$ (such elements exist because ψ is surjective). It is clear that $F_{r-(a-1)e}, \ldots, F_{r-e}$ generate $H^1_*(\mathcal{O}_S(-aC_0 - r\mathfrak{f}))$.

Now, $F_{r-(a-1)e}$ generates ker ψ and so it is linearly independent by the other generators; moreover, the generators F_{r-je} for $0 \le j \le a-2$ are independent, because such are the G_{r-je} for $0 \le j \le a-2$; then none of the above generators can be omitted. □

Remark 3.4. We can get easily that the index of speciality of a curve $C \sim aC_0 + r\mathfrak{f}$ with $a \ge 2$ and $1 \le r \le e$ is $\epsilon = \left\lfloor \frac{r-e-2}{e+1} \right\rfloor = -1 - \delta_{0,r}$, where again $\delta_{i,j}$ is the Kronecker symbol. So, by Schlesinger's Theorem the Rao module of C is generated in degree less or equal than $1 - \delta_{0,r}$. In Theorem 3.2 we got that the greatest degree of a minimal generator is $r - e \le 0$, so the bound of Schlesinger's theorem is non optimal in this case.

We give a picture (Fig. 2) to show the difference between the Rao module of a multiple of a fiber and a multiple of C_0 on a scroll with invariant e. We can see that the Rao functions coincide iff $e = 0$, that means iff we consider two fibers on a quadric in $\mathbb{P}^3$.

In Fig. 3, we show the Rao function of a curve of the "second" type, which has a "little" number of fiber and of an union of fibers.

We can note how the slope of the Rao function of this kind of curves increases by 1 every e steps while j increases by $b - ae + e - 1$ to -1. In these degrees we find a new minimal generator.

In Figure 4 we put the Rao function of an union of fibers.

We can note that the slope of this Rao function decreases by 1 every e steps while j decreases by $b - 1$ to $\left\lceil \frac{b}{e+1} \right\rceil$

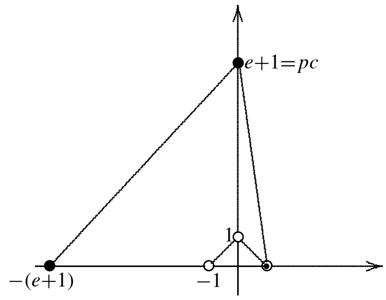

Fig. 2: $\bullet = C \sim 2C_0$
$\circ = C \sim 2\mathfrak{f}$

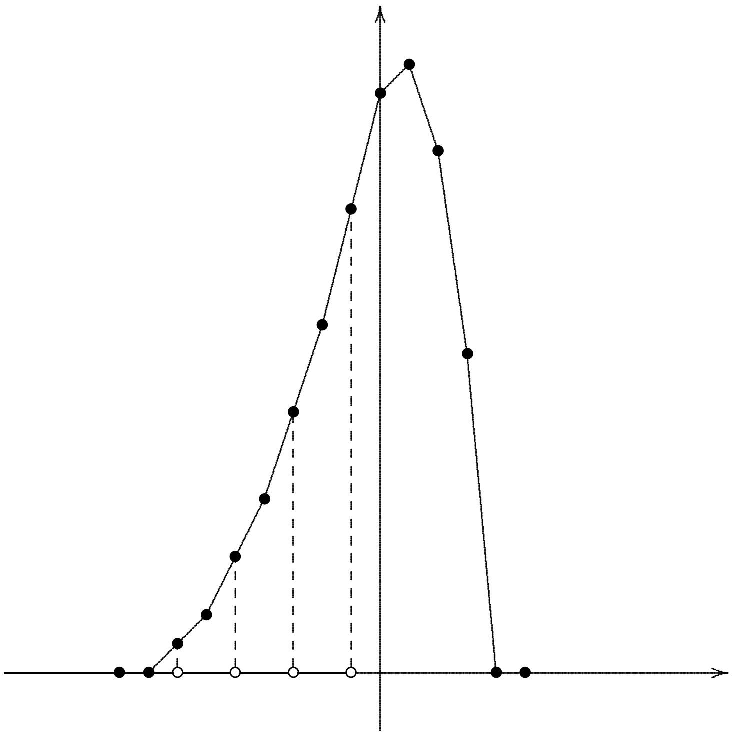

Fig. 3: $\bullet = C \sim 5C_0 + 2\mathfrak{f}$; $e = 2$
$\circ =$ degree of minimal generators

We denote by ρ and σ respectively the smallest and the largest integer such that $h^1(\mathcal{I}_C(j)) \neq 0$ and $\operatorname{diam}(C) := \sigma - \rho + 1$; moreover the Buchsbaum index

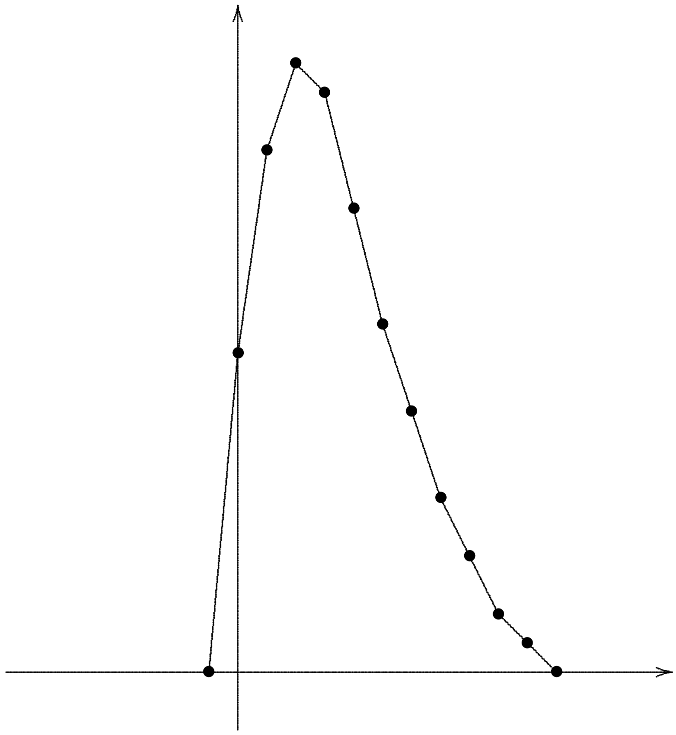

Fig. 4: $\bullet = C \sim 12\mathfrak{f}$; $e = 2$

of C is the smallest integer $k(C)$ such that $(x_0, \ldots, x_n)^i \cdot M(C) = 0$. If the Buchsbaum index is 1 the curve is called arithmetically Buchsbaum (aB). In this notation we can prove the following.

Corollary 3.5. *For a non-aCM curve* $C \sim aC_0 + b\mathfrak{f}$, *the Buchsbaum index is the maximum, that is diam*(C). *In particular C is* aB *if and only if*

$$b = (a-1)(e+1) - 1 \qquad or \qquad b = a(e+1) + 1.$$

Proof. As in the previous theorem and with the same notation, we reduce our study to the curves $b\mathfrak{f}$ and $aC_0 + r\mathfrak{f}$ with $0 \le r \le e$. For the first ones the result is known. For the last ones, we are going to show that the minimal generator $F_{r-(a-1)e}$, generates $h^1(\mathcal{I}_C(\sigma))$. In fact, we note that in this case $\rho = r - (a-1)e$ and $\sigma = a - 2$ and, since $h^1(\mathcal{I}_{C-C_0}(a-2)) = 0$, $H^1(\mathcal{I}_C(a-2)) \subset \ker \psi$, so it is generated by $F_{r-(a-1)e}$.

Finally, to find the relations between a and b such that $\mathrm{diam}(C) = 1$ it is enough to calculate ρ and σ by Corollary 2.7. $\square$

Finally, we give an algorithm such that, by giving as input the invariant e and the coefficients a and b, says if the curve $aC_0 + b\mathfrak{f}$ on S_e is aCM or not; in the second case, it says the kind of curve we get by shifting the Rao module and the positive values of the Rao function.

In the algorithm, we use the note about the slope of the Rao function.

```
Program Rao function of C ~ aC_0 + bf
  integer a, b, e, d, j, i, m, h1, int
  read a, b
  read e
  if b ≤ a * (e + 1) + 1 then
    if b ≥ (a − 1) * (e + 1) then
      print ''C is aCM''
    else
      int := b/(e + 1)
      b := b − int * (e + 1)
      a := a − int
      j := b − a * e + e − 1
      m := 0
      print "The Rao function of C is the same of the curve", a, "C_0+",
      b, "f shifted of", int, "units".
    repeat
      m := m + 1
      j := j + 1
      for i = 1, e
      h1 := h1 + m
      print "The Rao function in degree" j "is" h1
      j := j + 1
    endfor
    until m < a − 1 and j < 0
    d := a + b − 1
    e := e + 2
    for i = 1, a − 2
      h1 := h1 − i * e + d
      print "The Rao function in degree" j "is" h1
    endfor
    endif
    else
      int := b/(e + 1)
      if b − int * (e + 1) > 0 then int := int + 1
```

```
j := b − a ∗ e − 2
b := b − a ∗ (e + 1)
print "The Rao function of C is the same of the union of",
b,"distinct fibers shifted of", a, "units."
m := 0
repeat
m := m − 1
for i = 1, e
h1 := h1 − m
print "The Rao function in degree" j "is" h1
j := j − 1
endfor
until j > int
int := b/(e + 1)
if b − int ∗ (e + 1) = 0 then int := int − 1
h1 := 0
j := a
e := e + 2
b := b − 1
for i = 0, int
h1 := h1 − i ∗ e + b
print "The Rao function in degree" j "is" h1
j := j + 1
endfor
endif
end
```

REFERENCES

[1] M. Casanellas, *Teoria de liaison en codimensió arbitrària,* PhD Thesis 2001.

[2] N. Chiarli - S. Greco - U. Nagel, *On the genus and Hartshorne-Rao module of projective curves,* Math. Z., 229 (1998), pp. 695–724.

[3] S. Giuffrida - R. Maggioni, *On the Rao module of a Curve lying on a Smooth Cubic Surface in* $\mathbb{P}^3$*,* Comm. in Algebra, 18 - 7 (1990), pp. 2039–2061.

[4] R. Hartshorne, *Algebraic geometry,* Springer-Verlag, Berlin - Heidelberg - New York, 1977.

[5] J.O. Kepple - J.C. Migliore - R. Miró-Roig - U. Nagel - C. Peterson, *Gorenstein Liaison, Complete Intersection Liaison Invariants and Unobstructedness,* Mem. Amer. Math. Soc. (to appear).

[6] M. Martin-Deschamps - D. Perrin, *Sur la classification des courbes gauches,* Astérisque, (1990), pp. 184–185.

[7] Miyanishi, *Algebraic Geometry,* Trans. Math. Monographs, 136 AMS, 1994.

[8] J.C. Migliore, *Introduction to Liaison Theory and Deficiency Modules,* Progress in Mathematics 165, Birkhäuser, Boston-Basel-Berlin, 1998.

[9] C. Miyazaki, *Sharp bounds on Castelnuovo-Mumford regularity,* Trans. Amer. Math. Soc., 352 - 4 (2000), pp. 1675–1686.

[10] U. Nagel, *Arithmetically Buchsbaum divisors on varieties of minimal degree,* Trans. Amer. Math. Soc., 351 (1999), pp 4381–4409.

[11] R. Notari - M.L. Spreafico, *On curves of* $\mathbb{P}^n$ *with extremal Hartshorne-Rao module in positive degrees,* J. Pure Appl. Algebra, 156 (2001), pp. 95–114.

[12] E. Schlesinger, *On the spectrum of certain subschemes of* $\mathbb{P}^N$, J. Pure Appl. Algebra, 136 - 3 (1999), pp. 267–283.

Dipartimento Matematica e Applicazioni "R. Caccioppoli",
Complesso Universitario Monte S. Angelo,
Via Cinthia,
Napoli (ITALY)
email: digennar@unina.it