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DISJOINT WEAK* p-CONVERGENT OPERATORS ON BANACH LATTICES, $1 \leq p \leq \infty$

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We introduce and study the disjoint weak* p-convergent operators on Banach lattices, and we explore some characterizations of them in terms of disjoint sequences in the positive cones. As an application, we examine the domination and the duality properties of the class of positive disjoint weak* p-convergent operators. Next, we investigate the connections between our operators and disjoint p-convergent operators. Finally, we deduce some important results about the positive Schur property of order p.

1. Introduction

Note that the definition of a p-convergent operator was introduced by Castillo and Sanchez [3] as an operator from a Banach space X to a Banach space Y which carries weakly p-summable sequences in X to norm null sequences in Y. Many authors are interested in the study of these operators (see, for instance, [1], [4], [6], [13] and [16]). Recently, Fourie and Zeekoei [11] considered a weaker star version of p-convergent operators, the so-called weak* p-convergent operator. Following Fourie and Zeekoei, an operator from a Banach space X into a Banach space Y is weak* p-convergent if $f_n(T(x_n)) \to 0$ as $n \to \infty$ for every weakly p-summable sequence (x_n) in X, and for every weak* null sequence

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(f_n) in Y'. In the present paper, using the disjoint sequence techniques in Banach lattices, we consider the disjoint version of weak* p-convergent operator that we call disjoint weak* p-convergent operator (see Definition 3.1). This last operator generalizes the operators mentioned above and the almost weak* Dunford-Pettis operator that was introduced in our article mentioned in [15]. In particular, the almost weak* Dunford-Pettis operators are precisely the disjoint weak* ∞-convergent operators. Based on the study of almost weak* Dunford-Pettis, p-convergent and weak* p-convergent operators, we examine some interesting properties of our new operator in the framework of Banach lattices. More precisely, we characterize positive disjoint weak* p-convergent operators in terms of sequences in the positive cones, and we derive the domination and duality properties of the class of positive disjoint weak* p-convergent operators (see Corollary 3.3 and Theorem 3.5). Recall from [16] that an operator T from a Banach lattice E into a Banach lattice F is disjoint p-convergent if it maps disjoint weakly p-summable sequences in E^+ into norm null sequences in F, and note that every disjoint p-convergent operator is disjoint weak* p-convergent but the converse is not true in general (see Remark 3.7). Inspired by this fact, we characterize Banach lattices such that every disjoint weak* p-convergent operator is disjoint p-convergent (Theorem 3.8 and Corollary 3.10), and we deduce an interesting result about the characterization of the positive Schur property of order p (see Corollary 3.11).

2. Definitions and notations

To state our results, we need to fix some notations and recall some definitions. A Banach lattice is a Banach space $(E, \|\cdot\|)$ such that E is a vector lattice and its norm satisfies the following property: for each $x, y \in E$ such that $|x| \leq |y|$, we have $\|x\| \leq \|y\|$. If E is a Banach lattice, its topological dual E', endowed with the dual norm, is also a Banach lattice. A norm $\|\cdot\|$ of a Banach lattice E is order continuous if, for each generalized sequence (x_α) such that $x_\alpha \downarrow 0$ in E, the sequence (x_α) converges to 0 in the norm $\|\cdot\|$, where the notation $x_\alpha \downarrow 0$ means that the sequence (x_α) is decreasing, its infimum exists and $\inf(x_\alpha) = 0$. A Riesz space is said to be σ-Dedekind complete if every countable subset that is bounded above has a supremum; equivalently, whenever $0 \leq x_n \uparrow \leq x$, the supremum $\sup(x_n)$ exists.

Throughout the paper we use X, Y to denote Banach spaces. The identity operator on X is denoted by Id_X and the closed unit ball of X by B_X. As is customary, we use E, F to denote Banach lattices. The dual of a Banach space X is denoted by X'. We use the term operator $T : X \longrightarrow Y$ between two Banach spaces to mean a bounded linear mapping. It is positive if $T(x) \geq 0$ in F whenever $x \geq 0$ in E. Its adjoint operator T' is defined from Y' into X'

by $T'(f)(x) = f(T(x))$ for each $f \in Y'$ and each $x \in X$. Let $1 \leq p < \infty$. The conjugate number is denoted by p', i.e., $\frac{1}{p} + \frac{1}{p'} = 1$. The Banach space of p-summable scalar sequences is denoted by ℓ^p, and ℓ^∞ is the space of bounded scalar sequences. The closed subspace of ℓ^∞ consisting of the scalar sequences which are convergent with limit 0 is denoted by c_0. The unit vector basis of ℓ^p is denoted by (e_n). Recall from [8, p. 32] that a sequence (x_n) in X is weakly p-summable if $f(x_n) \in \ell^p$ for each $f \in X'$. The sequence (x_n) of a Banach lattice E is disjoint if $|x_n| \wedge |x_m| = 0$ for $n \neq m$.

Let $1 \leq p \leq \infty$. Recall that a Banach space X has the Dunford-Pettis* property of order p (DP^* property of order p) if $f_n(x_n) \to 0$ as $n \to \infty$ for every weakly p-summable sequence (x_n) in X, and for every weak* null sequence (f_n) in X' (see Theorem 2.4 of [12]). A Banach lattice E has the positive Schur property of order p if each disjoint weakly p-summable sequence in E^+ is norm null in E (see Proposition 3.3 of [16]). The reader is referred to Aliprantis-Burkinshaw [2], Diestel [7], Diestel, Jarchow and Tonge [8], and Dunford-Schwartz [10] for undefined notation and terminology.

3. Main results

Throughout this study we assume that $1 \leq p \leq \infty$, unless otherwise stated.

We begin this section with the definition of a disjoint weak* p-convergent operator between two Banach lattices.

Definition 3.1. An operator T from a Banach lattice E to a Banach lattice F is disjoint weak* p-convergent if $f_n(T(x_n)) \to 0$ as $n \to \infty$ for every disjoint weakly p-summable sequence (x_n) in E, and for every disjoint weak* null sequence (f_n) in F'.

Now, using the disjoint sequence technique in the positive cones, we characterize positive disjoint weak* p-convergent operators between two Banach lattices.

Theorem 3.2. *Let E and F be two Banach lattices such that F is σ-Dedekind complete. For every positive operator T from E into F, the following assertions are equivalent:*

1. *T is a disjoint weak* p-convergent operator.*
2. *For every disjoint weakly p-summable sequence $(x_n) \subset E^+$, and every disjoint weak* null sequence $(f_n) \subset (F')^+$, we have $f_n(T(x_n)) \to 0$ as $n \to \infty$.*
3. *For every disjoint weakly p-summable sequence $(x_n) \subset E^+$, and every weak* null sequence $(f_n) \subset F'$, we have $f_n(T(x_n)) \to 0$ as $n \to \infty$.*

4. *For every disjoint weakly p-summable sequence $(x_n) \subset E^+$, and every weak* null sequence $(f_n) \subset (F')^+$, we have $f_n(T(x_n)) \to 0$ as $n \to \infty$.*

5. *For every weakly p-summable sequence $(x_n) \subset E^+$, and every weak* null sequence $(f_n) \subset (F')^+$, we have $f_n(T(x_n)) \to 0$ as $n \to \infty$.*

Proof. (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (3) Assume, by way of contradiction, that there exist a disjoint weakly p-summable sequence $(x_n) \subset E^+$ and a weak* null sequence $(f_n) \subset F'$ such that $f_n(T(x_n)) \nrightarrow 0$ as $n \to \infty$. The inequality $|f_n(T(x_n))| \leq |f_n|(T(x_n))$ implies that $|f_n|(T(x_n))$ does not converge to 0 as $n \to \infty$. Then there exist $\varepsilon > 0$ and a subsequence of $|f_n|(T(x_n))$ (which we still denote by $|f_n|(T(x_n))$) satisfying $|f_n|(T(x_n)) > \varepsilon$ for all natural numbers n.

On the other hand, since (x_n) is a weakly p-summable sequence in E, then $T(x_n) \to 0$ weakly in F. An inductive argument proves that there exist a subsequence (z_n) of (x_n) and a subsequence (g_n) of (f_n) such that

$$|g_n|(T(z_n)) > \varepsilon$$

and

$$\left(4^n \sum_{i=1}^{n} |g_i|\right)(T(z_{n+1})) < \frac{1}{n}$$

for all $n \geq 1$. Put $h = \sum_{n=1}^{\infty} 2^{-n}|g_n|$ and $h_n = (|g_{n+1}| - 4^n \sum_{i=1}^{n} |g_i| - 2^{-n}h)^+$. By Lemma 4.35 of [2], the sequence (h_n) is disjoint. Since $0 \leq h_n \leq |g_{n+1}|$ for all $n \geq 1$ and (g_n) is weak* null in F', it follows from Lemma 2.2 of [5] that (h_n) is weak* null in F'. From the inequality

$$\begin{aligned} h_n(T(z_{n+1})) &\geq \left(|g_{n+1}| - 4^n \sum_{i=1}^{n} |g_i| - 2^{-n}h\right)(T(z_{n+1})) \\ &\geq \varepsilon - \frac{1}{n} - 2^{-n}h(T(z_{n+1})), \end{aligned}$$

we see that $h_n(T(z_{n+1})) \geq \frac{\varepsilon}{2}$ must hold for all sufficiently large n, because $2^{-n}h(T(z_{n+1})) \to 0$. This contradicts our hypothesis (2).

(3) $\Rightarrow$ (4) is obvious.

(4) $\Rightarrow$ (5) Assume, by way of contradiction, that there exist a weakly p-summable sequence $(x_n) \subset E^+$ and a weak* null sequence $(f_n) \subset (F')^+$ such that $f_n(T(x_n)) \nrightarrow 0$ as $n \to \infty$. Then there exist $\varepsilon > 0$ and a subsequence of $f_n(T(x_n))$ (which we still denote by $f_n(T(x_n))$) satisfying $f_n(T(x_n)) \geq \varepsilon$ for all natural numbers n.

On the other hand, as (f_n) is weak* null in F', we see that $T'(f_n) \to 0$ weak* in E'. An inductive argument shows that there exist a subsequence (z_n) of (x_n) and a subsequence (g_n) of (f_n) such that

$$T'(g_n)(z_n) > \varepsilon$$

and

$$T'(g_{n+1})\left(4^n \sum_{i=1}^{n} z_i\right) < \frac{1}{n}$$

for all $n \geq 1$. Put $z = \sum_{n=1}^{\infty} 2^{-n} z_n$ and $y_n = (z_{n+1} - 4^n \sum_{i=1}^{n} z_i - 2^{-n} z)^+$. By Lemma 4.35 of [2], the sequence (y_n) is disjoint. Since $0 \leq y_n \leq z_{n+1}$ for all $n \geq 1$ and (z_n) is weakly p-summable in E, it follows from Remark 1.3 of [16] that (y_n) is a weakly p-summable sequence in E. From the inequality

$$\begin{aligned} T'(g_{n+1})(y_n) &\geq T'(g_{n+1})\left(z_{n+1} - 4^n \sum_{i=1}^{n} z_i - 2^{-n} z\right) \\ &\geq \varepsilon - \frac{1}{n} - 2^{-n} T'(g_{n+1})(z), \end{aligned}$$

we see that $g_{n+1}(T(y_n)) = T'(g_{n+1})(y_n) \geq \frac{\varepsilon}{2}$ must hold for all sufficiently large n, because $2^{-n} T'(g_{n+1})(z) \to 0$. This contradicts our hypothesis (4).

(5) $\Rightarrow$ (1) Let (x_n) be a weakly p-summable sequence in E consisting of pairwise disjoint terms, and let (f_n) be a weak* null sequence in F' consisting of pairwise disjoint terms. From Lemma 2.2 of [5], we see that $(|f_n|)$ is weak* null in F', and from Proposition 2.2 of [16], we have that $(|x_n|)$ is weakly p-summable in E. Hence, by our hypothesis (5), $|f_n|(T|x_n|) \to 0$ as $n \to \infty$. Now, from the inequality $|f_n(T(x_n))| \leq |f_n|(T(|x_n|))$ for all natural numbers n, we conclude that $f_n(T(x_n)) \to 0$ as $n \to \infty$. This completes the proof. □

As a consequence of Theorem 3.2, we derive the domination property for the class of disjoint weak* p-convergent operators.

Corollary 3.3. *Let E and F be two Banach lattices such that F is σ-Dedekind complete. If S and T are two positive operators from E into F such that $0 \leq S \leq T$ and T is disjoint weak* p-convergent, then S is also disjoint weak* p-convergent.*

Proof. Let (x_n) be a weakly p-summable sequence in E^+ and let (f_n) be a weak* null sequence in $(F')^+$. According to assertion (5) of Theorem 3.2, it suffices to prove that $f_n(S(x_n)) \to 0$ as $n \to \infty$. Since T is disjoint weak* p-convergent, Theorem 3.2 implies that $f_n(T(x_n)) \to 0$ as $n \to \infty$. Now, by the inequality $0 \leq f_n(S(x_n)) \leq f_n(T(x_n))$ for each natural number n, we see that $f_n(S(x_n)) \to 0$ as $n \to \infty$. This completes the proof. □

For the proof of the next theorem, we need the following lemma, which is Lemma 2.8 of [16].

Lemma 3.4. *Let E be a Banach lattice with type q, $1 < q \leq 2$, and let $p \geq q'$. Each disjoint sequence (x_n) in the solid hull of a relatively weakly compact subset W of E is weakly p-summable in E. In particular, the sequence $(|x_n|)$ is weakly p-summable in E.*

Now, we establish the duality property of the class of positive disjoint weak* p-convergent operators.

Theorem 3.5. *Let T be a positive operator from a Banach lattice E into another Banach lattice F such that F is Grothendieck and F' has type q, $1 < q \leq 2$, and let $p \geq q'$. If the adjoint T' from F' into E' is disjoint weak* p-convergent, then T itself is disjoint weak* p-convergent.*

Proof. Let (x_n) be a disjoint weakly p-summable sequence in E^+, and let (f_n) be a disjoint weak* null sequence in $(F')^+$. Let $\tau : E \to E''$ be the canonical injection of E into its topological bidual E''. As τ is a lattice homomorphism, $(\tau(x_n))$ is a weakly null sequence in $(E'')^+$. On the other hand, since F is Grothendieck, we see that (f_n) is a disjoint weakly null sequence in $(F')^+$. Lemma 3.4 implies that (f_n) is a disjoint weakly p-summable sequence in F'. Since the adjoint T' is disjoint weak* p-convergent from F' into E', assertion (4) of Theorem 3.2 gives

$$(\tau(x_n))(T'(f_n)) \to 0$$

as $n \to \infty$. Since

$$f_n(T(x_n)) = (T'(f_n))(x_n) = (\tau(x_n))(T'(f_n))$$

holds for all natural numbers n, we deduce that $f_n(T(x_n)) \to 0$ as $n \to \infty$. This completes the proof. □

Proposition 3.6. *Let E, F be two Banach lattices and let G be a Banach space. If G has the DP^* property of order p, then each operator $T : E \to F$ that admits a factorization through the Banach space G is disjoint weak* p-convergent.*

Proof. Let $T_1 : E \to G$ and $T_2 : G \to F$ be two operators such that $T = T_2 \circ T_1$. Let (x_n) be a disjoint weakly p-summable sequence in E and let (f_n) be a disjoint weak* null sequence in F'. It is clear that $T_1(x_n)$ is weakly p-summable in G and $T_2'(f_n) \to 0$ weak* in G'. As G has the DP^* property of order p, we have

$$f_n(T(x_n)) = f_n(T_2 \circ T_1(x_n)) = (T_2'(f_n))(T_1(x_n)) \to 0$$

as $n \to \infty$. This proves that T is disjoint weak* p-convergent. □

Remark 3.7. Note that each disjoint p-convergent operator is disjoint weak* p-convergent, but the converse is not true in general. In fact, the identity operator $Id_{\ell^\infty} : \ell^\infty \to \ell^\infty$ is disjoint weak* p-convergent because ℓ^∞ has the DP^* property of order p (see Proposition 3.6), but Id_{ℓ^∞} is not disjoint p-convergent because (e_n) is a disjoint weakly p-summable sequence in ℓ^∞ and $\|e_n\|_{\ell^\infty} \nrightarrow 0$ as $n \to \infty$.

Theorem 3.8. *Let E and F be two Banach lattices such that F is σ-Dedekind complete. Then the following assertions are equivalent:*

1. *Each positive disjoint weak* p*-convergent operator T from E into F is disjoint p-convergent.*
2. *One of the following assertions is valid:*
   (a) *E has the positive Schur property of order p,*
   (b) *the norm of F is order continuous.*

Proof. (1) $\Rightarrow$ (2) Assume, by way of contradiction, that E does not have the positive Schur property of order p and the norm of F is not order continuous. We shall construct a positive disjoint weak* p-convergent operator which is not disjoint p-convergent. As E does not have the positive Schur property of order p, there exists a disjoint weakly p-summable sequence (x_n) in E^+ which is not norm null. By choosing a subsequence, we may suppose that there is $\varepsilon > 0$ with $\|x_n\| > \varepsilon > 0$ for all n. From the equality

$$\|x_n\| = \sup\left\{f(x_n) : f \in (E')^+, \|f\| = 1\right\},$$

there exists a sequence $(f_n) \subset (E')^+$ such that $\|f_n\| = 1$ and $f_n(x_n) \geq \varepsilon$ for all n. Now, consider the operator $R : E \to \ell^\infty$ defined by

$$R(x) = (f_n(x))_{n=1}^{\infty}.$$

On the other hand, since the norm of F is not order continuous, it follows from Theorem 4.51 of [2] that ℓ^∞ is lattice embeddable in F, i.e., there exists a lattice homomorphism $S : \ell^\infty \to F$ and there exist two positive constants M and m satisfying

$$m\|(\lambda_k)_k\|_\infty \leq \|S((\lambda_k)_k)\|_F \leq M\|(\lambda_k)_k\|_\infty$$

for all $\lambda \in \ell^\infty$. Put $T = S \circ R$, and note by Proposition 3.6 that T is a positive disjoint weak* p-convergent operator because ℓ^∞ has the DP^* property of order p. However, for the disjoint weakly p-summable sequence $(x_n) \subset E^+$, we have

$$\|T(x_n)\| = \|S((f_k(x_n))_k)\| \geq m\|(f_k(x_n))_k\|_\infty \geq m f_n(x_n) \geq m\varepsilon$$

for every natural number n. This shows that T is not a disjoint p-convergent operator.

$2(a) \Rightarrow (1)$ In this case, each operator $T : E \to F$ is disjoint p-convergent.

$2(b) \Rightarrow (1)$ Suppose that $T : E \to F$ is positive and disjoint weak* p-convergent. Let $(x_n) \subset E$ be a positive disjoint weakly p-summable sequence. We shall show that $\|T(x_n)\| \to 0$. By Corollary 2.6 of [9], it suffices to prove that $|T(x_n)| \xrightarrow{w} 0$ and $f_n(T(x_n)) \to 0$ for every disjoint and norm bounded sequence $(f_n) \subset (F')^+$. Let $f \in (F')^+$. By Theorem 1.23 of [2], there exists some $g \in [-f, f]$ with $f|Tx_n| = g(Tx_n)$. Since $x_n \xrightarrow{w} 0$, we have

$$f|Tx_n| = g(Tx_n) = (T'g)(x_n) \to 0.$$

Thus $|T(x_n)| \xrightarrow{w} 0$. On the other hand, let $(f_n) \subset (F')^+$ be a disjoint and norm bounded sequence. As the norm of F is order continuous, Corollary 2.4.3 of [14] gives $f_n \xrightarrow{w^*} 0$. Since T is positive disjoint weak* p-convergent, we see that $f_n(T(x_n)) \to 0$ as $n \to \infty$. This completes the proof. □

Remark 3.9. Let E be a Banach lattice with type q, $1 < q \leq 2$, and let $p \geq q'$. If E has the positive Schur property of order p, then the norm of E is order continuous (see Theorem 4.14 of [2], p. 190, another result in [2], p. 192, and Lemma 3.4).

As consequences of Theorem 3.8 and Remark 3.9, we have the following characterization.

Corollary 3.10. *Let E be a σ-Dedekind complete Banach lattice with type q, $1 < q \leq 2$, and let $p \geq q'$. Then the following assertions are equivalent:*

1. *Each positive disjoint weak* p-convergent operator $T : E \to E$ is disjoint p-convergent.*
2. *The norm of E is order continuous.*

Now, from Corollary 3.10 and Theorem 4.9 of [2], we obtain the following result.

Corollary 3.11. *Let E be a Banach lattice with type q, $1 < q \leq 2$, and let $p \geq q'$. Then the following assertions are equivalent:*

1. *E has the positive Schur property of order p.*
2. *Id_E is a disjoint weak* p-convergent operator, and the norm of E is order continuous.*

Proof. $(1) \Rightarrow (2)$ is obvious.

$(2) \Rightarrow (1)$ Since the norm of E is order continuous, it follows from Theorem 4.9 of [2] that E is σ-Dedekind complete. As $Id_E : E \to E$ is a disjoint weak* p-convergent operator, Corollary 3.10 implies that $Id_E : E \to E$ is disjoint p-convergent. This proves that E has the positive Schur property of order p. □

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