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ELASTIC BEAM EQUATIONS WITH VARIABLE COEFFICIENTS: MULTIPLE SOLUTIONS UNDER MIXED NONLINEARITIES

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This paper investigates the existence of multiple solutions for a fourth-order differential equation modelling an elastic beam, where the coefficients are variable, and the nonlinearities exhibit both concave and convex characteristics. Our approach is based on variational methods and critical point theorems, particularly those formulated by Ricceri, which provide a powerful framework for proving the existence of solutions in reflexive Banach spaces. By leveraging these mathematical tools, we establish that the considered problem admits at least three distinct weak solutions under specific conditions. To validate our theoretical findings, we present an illustrative example demonstrating how our results can be applied in practice.

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1. Introduction

We study the existence of at least three distinct solutions for the following problem

$$\begin{cases} (p(\varsigma)z'')'' - (q(\varsigma)z')' + r(\varsigma)z = \gamma h(\varsigma, z(\varsigma)) + \mu g(\varsigma, z(\varsigma)), & \varsigma \in [0,1], \\ z(a) = z(b) = 0, \\ z''(a) = z''(b) = 0 \end{cases} \tag{P^h}$$

where $\lambda > 0$, $\mu \geq 0$ h, g are continuous functions, $p \in C^2[0,1]$, $q \in C^1[0,1]$, $r \in C[0,1]$ are regular functions with $p^- = \operatorname{ess\,inf}_{\varsigma \in [0,1]} p(\varsigma) > 0$.

The analysis of such equations becomes particularly relevant when considering real-world materials with varying mechanical properties, where standard assumptions about constant or sign-definite coefficients no longer hold. In recent years, the study of multiple solutions for boundary value problems involving nonlinear differential equations has gained significant attention. Several mathematical techniques have been developed to investigate the existence and uniqueness of solutions. Among these, variational methods have proven to be particularly powerful, allowing differential equations to be analyzed using functional analysis and critical point theory. Ricceri's three critical points theorem provides a useful tool for proving the existence of at least three distinct weak solutions under appropriate conditions. This theorem, formulated within the framework of reflexive Banach spaces, has been successfully applied in various contexts, including nonlinear elasticity and phase transition models (Ricceri [17, 18]). Previous studies have explored the existence of solutions for fourth-order differential equations with different types of nonlinearities. For instance, Bonanno et al. [5] employed critical point theory to demonstrate the existence of of a sequence of solutions for a class of fourth-order elastic beam equations. Similarly, Bonanno, Di Bella, and O'Regan [6] extended these results by considering variable coefficients and more general forms of nonlinearity, obtaining additional multiplicity results. Other works, such as those by Heidarkhani et al. [11], have used variational techniques to study impulsive perturbations of elastic beam equations, demonstrating the existence of multiple weak solutions. These contributions highlight the importance of developing refined mathematical tools to tackle the challenges posed by high-order differential equations, especially when coefficients exhibit complex behavior. The motivation behind studying such equations lies in their ability to capture intricate interactions that simpler models fail to represent. Fourth-order elastic beam equations are crucial in structural mechanics, where they provide a more accurate representation of deformations in beams, plates, and other flexible structures. These equations are also widely used in physics, particularly in problems related to quantum mechanics, thin plate theory, and hydrodynamics, where higher-order terms become necessary to describe interactions that cannot be effectively captured using second-order models. This paper investigates the existence of multiple solutions for a fourth-order elastic beam equation with variable coefficients incorporating the combined effects of concave and convex nonlinearities. The main innovation of this paper is our assumption that the coefficients p, q and r can change in sign, distinguishing it from the existing literature. For example, we can analyze the well-known Laguerre differential equation:

$$(\frac{3-\varsigma}{\varsigma}z'')'' - (\frac{2-\varsigma}{\varsigma}z')' + \frac{1}{6\varsigma}z = \gamma h(\varsigma, z(\varsigma)) + \mu g(\varsigma, z(\varsigma)),\ \varsigma \in (-4,-3). \tag{1.1}$$

Indeed, equation (1.1) represents a complete Sturm-Liouville differential equation with its coefficients defined as follows:
$p(\varsigma) = \dfrac{3-\varsigma}{\varsigma}$, $q(\varsigma) = \dfrac{2-\varsigma}{\varsigma}$ and $r(\varsigma) = \dfrac{1}{6\varsigma}$ which are negative and meet our established hypotheses.

The contributions of this study can be summarized as follows: (i) we prove the existence of at least three solutions under appropriate conditions, extending previous results that typically guarantee one or two solutions; (ii) we relax traditional sign constraints on the coefficients, making our results applicable to a broader class of problems; (iii) we apply advanced variational methods to obtain new multiplicity results for fourth-order elastic beam equations; and (iv) we provide an explicit example illustrating the applicability of our theoretical findings. Our approach also offers a new perspective on the role of critical point theory in studying nonlinear differential equations, highlighting the interplay between functional analysis and differential equation theory. An example highlights the effectiveness of our approach in identifying multiple solutions and showcases the role of parameter selection in determining the behaviour of the system. The numerical computations and qualitative analysis further reinforce the relevance of our theoretical contributions.

2. Preliminaries

Our main tool is a theorem due to Ricceri, who is recalled below in Lemma 2.1 and has been obtained in [17, Theorem 2]. Let X be a real Banach space, and as in [17], we denote by $\mathcal{W}_X$ the class of all functionals $\Theta : X \to \mathbb{R}$ possessing the following property: If $\{z_n\}$ is a sequence in X converging weakly to $z \in X$ with $\liminf_{n\to\infty} \Theta(z_n) \leq \Theta(z)$, then $\{z_n\}$ has a subsequence converging strongly to z. For example, if X is uniformly convex and $g : [0,\infty) \to \mathbb{R}$ is a continuous and strictly increasing function, then the functional $z \to g(\|z\|)$ belongs to the class $\mathcal{W}_X$.

Lemma 2.1. *Let X be a separable and reflexive real Banach space, let $\Theta : X \to \mathbb{R}$ be a coercive, sequentially weakly lower semicontinuous C^1-functional, belonging to $\mathcal{W}_X$, bounded on each bounded subset of X and whose derivative admits a continuous inverse on X^*, and let $J : X \to \mathbb{R}$ be a C^1-functional with compact derivative. Assume that Θ has a strict local minimum z_0 with $\Theta(z_0) = J(z_0) = 0$. Finally, setting*

$$\rho = \max\left\{0, \limsup_{\|z\|\to\infty} \frac{J(z)}{\Theta(z)}, \limsup_{u\to z_0} \frac{J(z)}{\Theta(z)}\right\},$$

$$\sigma = \sup_{z\in\Theta^{-1}(0,\infty)} \frac{J(z)}{\Theta(z)},$$

and assume that $\rho < \sigma$. Then for each compact interval $[c,d] \subset (1/\sigma, 1/\rho)$ (with the conventions that $1/0 = \infty$ and $1/\infty = 0$), there exists $R > 0$ with the following property: for every $\gamma \in [c,d]$ and every C^1-functional $\Upsilon : X \to \mathbb{R}$ with compact derivative, there exists $\gamma > 0$ such that for each $\mu \in [0,\gamma]$, the equation

$$\Theta'(z) = \gamma J'(z) + \mu \Upsilon'(z)$$

has at least three solutions in X whose norms are less than R.

We refer the reader to the papers [8, 11–13, 20, 21] in which Lemma 2.1 was successfully employed to ensure the existence of at least three solutions for boundary value problems.

The following two results of Ricceri are taken from [18, Theorem 1] and [19, Proposition 3.1], respectively.

Lemma 2.2. *Let X be a reflexive real Banach space, $I \subseteq \mathbb{R}$ an interval; let $\Theta : X \to \mathbb{R}$ be a sequentially weakly lower semicontinuous C^1 functional, bounded on each bounded subset of X, whose derivative admits a continuous inverse on X^*, $J : X \to \mathbb{R}$ functional with compact derivative. Assume that*

$$\lim_{\|z\|\to\infty} (\Theta(z) - \gamma J(z)) = \infty \quad \textit{for all } \gamma \in I,$$

and that there exists $\rho \in \mathbb{R}$ such that

$$\sup_{\gamma\in I} \inf_{z\in X} (\Theta(z) - \alpha(\rho - J(z))) < \inf_{z\in X} \sup_{\gamma\in I} (\Theta(z) - \alpha(\rho - J(z))).$$

Then there exists a nonempty open set $A \subseteq I$ and a positive number R with the following property: for every $\gamma \in A$ and every C^1 functional $\Upsilon : X \to \mathbb{R}$ with compact derivative, there exists $\delta > 0$ such that, for each $\mu \in [0,\delta]$, the equation

$$\Theta'(z) - \gamma J'(z) - \mu \Upsilon'(z) = 0$$

has at least three solutions in X whose norms are less than R.

Lemma 2.3. *Let X be a nonempty set and Θ and J two real functions on X. Assume that there are $s > 0$ and $z_0, z_1 \in X$ such that*

$$\Theta(z_0) = J(z_0) = 0,\ \Theta(z_1) > s, \quad \sup_{z\in\Theta^{-1}(-\infty,s]} J(z) < s\frac{J(z_1)}{\Theta(z_1)}.$$

Then for each ρ satisfying

$$\sup_{z\in\Theta^{-1}(-\infty,s]} J(z) < \rho < s\frac{J(z_1)}{\Theta(z_1)},$$

one has

$$\sup_{\gamma\geq 0}\inf_{z\in X}(\Theta(z) - \alpha(\rho - J(z))) < \inf_{z\in X}\sup_{\gamma\geq 0}(\Theta(z) - \alpha(\rho - J(z))).$$

We refer the reader to the paper [20] in which Lemma 2.2 was successfully employed to ensure the existence of at least three solutions for boundary value problems.

In this section, we present fundamental notations and supporting results to incorporate the equation (P^h) into a variational framework. Let E denote the Sobolev space $W^{2,2}([0,1]) \cap W_0^{1,2}([0,1])$, equipped with the norm

$$\|z\| = \left(\|z''\|_2^2 + \|z'\|_2^2 + \|z\|_2^2\right)^{\frac{1}{2}} \tag{2.1}$$

for every $z \in$ E where $\|\cdot\|_2$ is the usual norm in $L^2[0,1]$. It is well known that $\|\cdot\|$ is induced by the inner product

$$\int_0^1 \left(z''(\varsigma)v''(\varsigma) + z'(\varsigma)v'(\varsigma) + z(\varsigma)v(\varsigma)\right) \mathrm{d}\varsigma$$

for every $z, v \in$ E.

We highlight the following Poincaré-type inequalities, which can be found in the works of [14].

Proposition 2.1. For every $z \in$ E, if $k = \dfrac{1}{\pi^2}$, one has

$$\|z^{(i)}\|_2^2 \leq k^{j-i}\|z^{(j)}\|_2^2, \ \ i = 0,1, \ \ j = 1,2 \ \text{with} \ i < j. \tag{2.2}$$

Now, defining p^- as previously mentioned, and letting $q^- = \operatorname{ess\,inf}_{[0,1]} q$ and $r^- = \operatorname{ess\,inf}_{[0,1]} r$, we will examine the following set of conditions based on the signs of these quantities:

(H_1) $p^- > 0, q^- \geq 0, r^- \geq 0,$

(H_2) $p^- > 0, q^- < 0, r^- \geq 0$ and $p^- + q^- k > 0$,

(H_3) $p^- > 0, q^- \geq 0, r^- < 0$ and $p^- + r^- k > 0$,

(H_4) $p^- > 0, q^- < 0, r^- < 0$ and $p^- + q^- k + r^- k > 0$.

Additionally, take into account the following condition:

(H) $\min\{p^- + q^- k,\ p^- + r^- k,\ p^- + q^- k + r^- k\} > 0$.

Put

$$\sigma = \min\{p^-,\ p^- + q^- k,\ p^- + r^- k,\ p^- + q^- k + r^- k\}.$$

Clearly, assuming the condition (H) implies that $\sigma > 0$. Furthermore, a simple calculation reveals the following result.

Proposition 2.2. Condition (H) holds if and only if at least one of the conditions (H_1) through (H_4) is met.

We will now introduce a useful norm, which is equivalent to $\|\cdot\|$ and still ensures that E remains a Hilbert space. Therefore, for the fixed values of p, q and r mentioned earlier, we define the function $N : \mathrm{E} \to \mathbb{R}$ as follows

$$N(z) = \int_0^1 (p(\varsigma)|z''(\varsigma)|^2 + q(\varsigma)|z'(\varsigma)|^2 + r(\varsigma)|z(\varsigma)|^2)\mathrm{d}\varsigma$$

holds for any $z \in \mathrm{E}$. We have the following proposition, which will be helpful in confirming that $\sqrt{N(\cdot)}$ is a norm equivalent to the standard one.

Proposition 2.3. [6, Proposition 2.3] Assume (H) holds. Then, there exits $m > 0$ such that

$$N(z) \geq m\|z\|^2 \tag{2.3}$$

for any $z \in \mathrm{E}$, with $m = \dfrac{\sigma}{1+k+k^2}$. Moreover, one has

$$N(z) \geq \sigma\|z''\|_2^2 \tag{2.4}$$

for any $z \in \mathrm{E}$.

Proposition 2.4. [6, Proposition 2.4] Assume that condition (H) is satisfied and define

$$\|\cdot\|_{\mathrm{E}} = \sqrt{N(\cdot)}$$

for any $z \in \mathrm{E}$. Then, $\|\cdot\|_{\mathrm{E}}$ is a norm equivalent to the usual one defined in (2.2) and $(\mathrm{E}, \|\cdot\|_{\mathrm{E}})$ is a Hilbert space.

Now, assuming again (H), put

$$\delta=\sqrt{\sigma}=\left(\min\{p^-,\ p^-+q^-k,\ p^-+r^-k,\ p^-+q^-k+r^-k\}\right)^{\frac{1}{2}}.$$

The constant δ is well-defined, given that $\sigma>0$ holds under condition (H).

The following proposition will be useful in the next section.

Proposition 2.5. [6, Proposition 2.5] Assume that (H) holds. One has

$$\|z\|_\infty \leq \frac{1}{2\pi\delta}\|z\|_{\mathrm{E}} \tag{2.5}$$

for any $z\in \mathrm{E}$.

Definition 2.1. A function $z\in \mathrm{E}$ is called a weak solution of problem (P^h), if

$$\begin{aligned}&\int_0^1 \left(p(\varsigma)z''(\varsigma)v''(\varsigma)+q(\varsigma)z'(\varsigma)v'(\varsigma)+r(\varsigma)z(\varsigma)v(\varsigma)\right)\mathrm{d}\varsigma\\ =&\gamma\int_0^1 h(\varsigma,z(\varsigma))\mathrm{d}\varsigma+\mu\int_0^1 g(\varsigma,z(\varsigma))\mathrm{d}\varsigma\end{aligned}$$

holds for any $v\in \mathrm{E}$.

Now for every $z\in \mathrm{E}$, we define

$$\Theta(z)=\frac{1}{2}\|z\|_{\mathrm{E}}^2, \tag{2.6}$$

$$J(z)=\int_0^1 H(\varsigma,z(\varsigma))\mathrm{d}\varsigma \tag{2.7}$$

and

$$\Upsilon(z)=\int_0^1 G(\varsigma,z(\varsigma))\mathrm{d}\varsigma \tag{2.8}$$

where

$$H(\varsigma,\zeta)=\int_0^\zeta h(\varsigma,x)\mathrm{d}x \text{ for any } (\varsigma,\zeta)\in[0,1]\times\mathbb{R}$$

and

$$G(\varsigma,\zeta)=\int_0^\zeta g(\varsigma,x)\mathrm{d}x \text{ for any } (\varsigma,\zeta)\in[0,1]\times\mathbb{R}.$$

Proposition 2.6. [6, Proposition 2.5] Function z is a weak solution of (P^h) if only if z is a critical point of $\Theta-\mu\Upsilon-\gamma J$.

We need the following Proposition for existence our main results.

Proposition 2.7. Let $S : \mathrm{E} \longrightarrow \mathrm{E}^*$ be the operator defined by

$$S(z)(v) = \int_0^1 \big(p(t)z''(t)v''(t) + q(t)z'(t)v'(t) + r(t)z(t)v(t)\big)\,\mathrm{d}t$$

for every $z, v \in \mathrm{E}$. Then, S admits a continuous inverse on E^*.

Proof. It is obvious that

$$S(z)(z) = \int_0^1 \big(p(t)|z''(t)|^2 + q(t)|z'(t)|^2 + r(t)|z(t)|^2)\big)\,\mathrm{d}t = \|z\|_{\mathrm{E}}.$$

This follows that S is coercive. Owing to our assumptions on the data, one has

$$\langle S(z) - S(v), z - v\rangle \geq C\|z - v\|_{\mathrm{E}}^2 > 0$$

for some $C > 0$, for every $z, v \in \mathrm{E}$, which means that S is strictly monotone. Moreover, since E is reflexive, for $z_n \longrightarrow z$ strongly in E as $n \to +\infty$, one has $S(z_n) \to S(z)$ weakly in E^* as $n \to \infty$. Hence, S is demicontinuous, so by [22, Theorem 26.A(d)], the inverse operator S^{-1} of S exists and it is continuous. Indeed, let e_n be a sequence of E^* such that $e_n \to e$ strongly in E^* as $n \to \infty$. Let z_n and u in E such that $S^{-1}(e_n) = z_n$ and $S^{-1}(e) = z$. Taking into account that S is coercive, one has that the sequence z_n is bounded in the reflexive space E. For a suitable subsequence, we have $z_n \to \hat{z}$ weakly in E as $n \to \infty$, which concludes

$$\langle S(z_n) - S(z), z_n - \hat{z}\rangle = \langle e_n - e, z_n - \hat{z}\rangle = 0.$$

Note that if $z_n \to \hat{z}$ weakly in E as $n \to +\infty$ and $S(z_n) \to S(\hat{z})$ strongly in E^* as $n \to +\infty$, one has $z_n \to \hat{z}$ strongly in E as $n \to +\infty$, and since S is continuous, we have $z_n \to \hat{z}$ weakly in E as $n \to +\infty$ and $S(z_n) \to S(\hat{z}) = S(z)$ strongly in E^* as $n \to +\infty$. Hence, taking into account that S is an injection, we have $z = \hat{z}$. □

3. Main results

In this section, we state and prove our main results. First, put

$$\mathscr{B} = \frac{1}{2}\left(\frac{4096}{27}p^- + \frac{64}{9}q^- + \frac{13}{20}r^-\right)$$

and

$$\mathscr{D} = \frac{1}{2}\left(\frac{4096}{27}p^+ + \frac{64}{9}q^+ + \frac{13}{20}r^+\right)$$

where p^+, q^+ and r^+ are the ess sup in $[0,1]$ of the functions p, q and r respectively. Let

$$\theta_1 = \inf_{z\in \mathrm{E}\setminus\{0\}} \left\{ \frac{\frac{1}{2}\|z\|_{\mathrm{E}}^2}{\int_0^1 H(\varsigma, z(\varsigma))\mathrm{d}\varsigma} : z \in \mathrm{E},\ \int_0^1 H(\varsigma, z(\varsigma))\mathrm{d}\varsigma > 0 \right\}$$

and

$$\theta_2 = \frac{1}{\max\{0,\, \theta_0,\, \theta_\infty\}},$$

where

$$\theta_0 = \limsup_{z\to 0} \left(\frac{\int_0^1 H(\varsigma, z(\varsigma))\mathrm{d}\varsigma}{\frac{1}{2}\|z\|_{\mathrm{E}}^2} \right)^{-1}$$

and

$$\theta_\infty = \limsup_{\|z\|_{\mathrm{E}}\to\infty} \left(\frac{\int_0^1 H(\varsigma, z(\varsigma))\mathrm{d}\varsigma}{\frac{1}{2}\|z\|_{\mathrm{E}}^2} \right)^{-1}.$$

Theorem 3.1. Assume that

$(\mathcal{A}_1)$ there exists a constant $\varepsilon > 0$ such that

$$\max\left\{ \limsup_{z\to 0} \frac{H(\varsigma, z)}{|z|^2},\ \limsup_{|z|\to\infty} \frac{\max_{\varsigma\in[0,1],} H(\varsigma, z)}{|z|^2} \right\} < \varepsilon,$$

$(\mathcal{A}_2)$ there exists a function $w \in \mathrm{E}$ such that

$$\frac{1}{2}\|w\|_{\mathrm{E}}^2 \neq 0$$

and

$$\frac{\int_0^1 H(\varsigma, w(\varsigma))\mathrm{d}\varsigma}{\frac{1}{2}\|w\|_{\mathrm{E}}^2} > \frac{\varepsilon}{2\pi^2\delta^2}.$$

Then for each compact interval $[c,d] \subset (\theta_1, \theta_2)$, there exists $R > 0$ such that for every $\gamma \in [c,d]$ and every continuous functions $g : [0,1]\times\mathbb{R} \to \mathbb{R}$, there exists $\gamma > 0$ such that for every $\mu \in [0,\gamma]$, the problem (P^h) has at least three weak solutions whose norms in E are less than R.

Remark 3.1. Under conditions $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$, it is true that $\theta_1 < \theta_2$ as shown in the proof of Theorem 3.1 given below.

Proof. Our aim is to apply Lemma 2.1 to the problem (P^h). Clearly, E is a separable and uniformly convex Banach space. Let the functionals Θ, J and Υ be as given in (2.6), (2.7), and (2.8), respectively. Standard arguments show that $\Theta - \mu\Upsilon - \lambda J$ is a Gâteaux differentiable functional whose Gâteaux derivative at the point $z \in \mathrm{E}$ is given by

$$(\Theta' - \mu\Upsilon' - \lambda J')(z)(v) = \int_0^1 \big(p(\varsigma)z''(\varsigma)v''(\varsigma) + q(\varsigma)z'(\varsigma)v'(\varsigma) + r(\varsigma)z(\varsigma)v(\varsigma)\big)\,\mathrm{d}\varsigma$$
$$-\gamma\int_0^1 h(\varsigma, z(\varsigma))v(\varsigma)\mathrm{d}\varsigma - \mu\int_0^1 g(\varsigma, z(\varsigma))v(\varsigma)\mathrm{d}\varsigma$$

for each $v \in \mathrm{E}$. Furthermore, Θ, J and Υ are C^1-functions. By utilizing the definition of Θ, it follows that

$$\lim_{\|z\|_{\mathrm{E}} \to +\infty} \Theta(z) = +\infty,$$

which indicates that Θ is coercive, while Proposition 2.7 gives that Θ admits a continuous inverse on E^*. Now, let A be a bounded subset of E. Then there exist constants $c > 0$, such that $\|z\|_{\mathrm{E}} \le c$ for each $z \in A$. So from (2.5), $\max_{\varsigma\in[0,1]} |z(\varsigma)| \le \frac{1}{2\pi\delta} c$ for all $z \in A$. Then, we have

$$\Theta(z) \le \left(\frac{1}{2}\right) c^2.$$

Hence, Θ is bounded on each bounded subset of E. We now prove $\Theta \in \mathcal{W}_X$. To this end, let the sequence $\{z_k\} \subset \mathrm{E}$, $z \subset \mathrm{E}$, $z_k \rightharpoonup z$ and $\liminf_{k\to\infty} \Theta(z_k) \le \Theta(z)$. Below, we show that $\{z_k\}$ has a subsequence converging strongly to z. Assume, to the contrary, that $\{z_k\}$ does not have a subsequence converging strongly to z. Then, there exist $\varepsilon > 0$ and a subsequence of z_k, still denoted by itself, such that

$$\left\|\frac{z_k - z}{2}\right\| \ge \varepsilon \quad \text{for all } k \in \mathbb{N}.$$

Note that $\{z_k\}$ converges uniformly to z by [15, Proposition 1.2]. Then, in view of the definition of $\|\cdot\|$, there exists $\varepsilon_1 > 0$ such that

$$\Theta\left(\frac{z_k - z}{2}\right) \ge \varepsilon_1 \quad \text{for all } k \in \mathbb{N}.$$

Now, the sequentially weakly lower semicontinuity of Θ implies that $\liminf_{k\to\infty}\Theta(z_k) = \Theta(z)$. So there exists a subsequence of z_n, still denoted by itself, such that

$$\lim_{n\to\infty}\Theta(z_n) = \Theta(z).$$

Since $\{z_k\}$ converges uniformly to z, we reach that

$$\lim_{k\to\infty}\Theta(z_k) = \Theta(z).$$

It is obvious that Θ is sequentially weakly lower semicontinuous and that $(z_k + z)/2 \rightharpoonup z$. Then, we obtain that

$$\Theta(z) \leq \liminf_{k\to\infty}\Theta\left(\frac{z_k+z}{2}\right). \tag{3.1}$$

We assume by contradiction that z_k does not converge to z in E. Hence, there exist $\varepsilon_0 > 0$ and a subsequence $\{z_m\}$ of $\{z_k\}$ such that

$$\left\|\frac{z_m - z}{2}\right\| > \varepsilon_0 \quad \text{for all } m\in\mathbb{N}.$$

Then there exists $\varepsilon_1 > 0$ such that

$$\Theta\left(\frac{z_m-z}{2}\right) > \varepsilon_1 \quad \text{for all } m\in\mathbb{N}.$$

On the other hand,

$$\frac{1}{2}\Theta(z_m)+\frac{1}{2}\Theta(z)-\Theta\left(\frac{z_m+z}{2}\right) \geq \Theta\left(\frac{z_m-z}{2}\right) > \varepsilon_1 \quad \text{for all } m\in\mathbb{N}$$

(see [16]). Taking limit superior as $k\to\infty$ in the above inequality, we get that

$$\Theta(z)-\varepsilon_1 \geq \limsup_{k\to\infty}\Theta\left(\frac{z_m+z}{2}\right),$$

which contradicts (3.1). This shows that $\{z_k\}$ has a subsequence converging strongly to z. Thus, $\{z_k\}$ has a subsequence converging strongly to z. Therefore, $\Theta \in \mathcal{W}_X$.

The functionals J and Υ are two C^1-functionals with compact derivatives. Moreover, Θ has a strict local minimum 0 with $\Theta(0) = J(0) = \Upsilon(0) = 0$. Therefore, the regularity assumptions on Θ, J and Υ, as requested in Lemma 2.1, are verified. In view of $(\mathcal{A}_1)$, there exist τ_1, τ_2 with $0 < \tau_1 < \tau_2$ such that

$$H(\varsigma, z) \leq \varepsilon |z|^2 \tag{3.2}$$

for every $\varsigma \in [0,1]$, and every z with $|z| \in [0,\tau_1) \cup (\tau_2,\infty)$. Since $H(\varsigma,z)$ is continuous on $[0,1]\times\mathbb{R}$, $H(\varsigma,z)$ is bounded on $\varsigma \in [0,1]\times[\tau_1,\tau_2]$. Thus we can choose $\eta > 0$ and $\upsilon > 2$ such that

$$H(\varsigma,z) \leq \varepsilon|z|^2 + \eta|z|^{\upsilon}$$

for all $(\varsigma,z) \in [0,1]\times\mathbb{R}$. Then, from (2.5), we have

$$J(z) \leq \varepsilon\frac{1}{4\pi^2\delta^2}\|z\|_{\mathrm{E}}^2 + \eta\frac{1}{2^{\upsilon}\pi^{\upsilon}\delta^{\upsilon}}\|z\|_{\mathrm{E}}^{\upsilon} \tag{3.3}$$

for all $z \in \mathrm{E}$. Hence, from (3.3), we have

$$\limsup_{|z|\to 0}\frac{J(z)}{\Theta(z)} \leq \frac{\varepsilon\frac{1}{4\pi^2\delta^2}}{\frac{1}{2}}. \tag{3.4}$$

Moreover, by (3.2), for each $z \in \mathrm{E}\setminus\{0\}$, we obtain that

$$\begin{aligned}\frac{J(z)}{\Theta(z)} &= \frac{\int_{|z(\varsigma)|\leq\tau_2} H(\varsigma,z(\varsigma))\mathrm{d}\varsigma}{\Theta(z)} + \frac{\int_{|z(\varsigma)|>\tau_2} H(\varsigma,z(\varsigma))\mathrm{d}\varsigma}{\Theta(z)}\\ &\leq \frac{\sup_{\varsigma\in[0,1],|z(\varsigma)|\in[0,\tau_2]} H(\varsigma,z(\varsigma))}{\Theta(z)} + \frac{\varepsilon\frac{1}{4\pi^2\delta^2}\|z\|_{\mathrm{E}}^2}{\Theta(z)}\\ &\leq \frac{\sup_{\varsigma\in[0,1],|z(\varsigma)|\in[0,\tau_2]} H(\varsigma,z(\varsigma))}{\frac{1}{2}\|z\|_{\mathrm{E}}^2} + \frac{\varepsilon}{2\pi^2\delta^2}.\end{aligned}$$

So, we get that

$$\limsup_{\|z\|_{\mathrm{E}}\to\infty}\frac{J(z)}{\Theta(z)} \leq \frac{\varepsilon}{2\pi^2\delta^2}. \tag{3.5}$$

In view of (3.4) and (3.5), we have

$$\rho = \max\left\{0, \limsup_{\|z\|\to\infty}\frac{J(z)}{\Theta(z)}, \limsup_{z\to 0}\frac{J(z)}{\Theta(z)}\right\} \leq \frac{\varepsilon}{2\pi^2\delta^2}. \tag{3.6}$$

Assumption $(\mathcal{A}_2)$ in conjunction with (3.6) yields

$$\begin{aligned}\sigma &= \sup_{z\in\Theta^{-1}(0,\infty)}\frac{J(z)}{\Theta(z)} = \sup_{\mathrm{E}\setminus\{0\}}\frac{J(z)}{\Theta(z)}\\ &\geq \frac{\int_0^1 H(\varsigma,w(\varsigma))\mathrm{d}\varsigma}{\Theta(w(\varsigma))} \geq \frac{\int_0^1 H(\varsigma,w(\varsigma))\mathrm{d}\varsigma}{\frac{1}{2}\|w\|_{\mathrm{E}}^2}\end{aligned}$$

$$> \frac{\varepsilon}{2\pi^2\delta^2} \geq \rho.$$

Thus, all the hypotheses of Lemma 2.1 are satisfied. Clearly, $\theta_1 = 1/\sigma$ and $\theta_2 = 1/\rho$. Then, by Lemma 2.1, for each compact interval $[c,d] \subset (\theta_1, \theta_2)$, there exists $R > 0$ such that for every $\gamma \in [c,d]$ and every continuous function $g : [0,1] \times \mathbb{R} \to \mathbb{R}$, there exists $\gamma > 0$ such that for each $\mu \in [0,\gamma]$, the problem (P^h) has at least three weak solutions whose norms in E are less than R. □

Another application of Lemma 2.1 reads as follows.

Theorem 3.2. Assume that

$$\max_{z\in \mathrm{E}} \left\{ \limsup_{z\to 0} \frac{H(\varsigma,z)}{|z|^2}, \limsup_{|z|\to\infty} \frac{H(\varsigma,z)}{|z|^2} \right\} \leq 0 \tag{3.7}$$

and

$$\sup_{z\in\mathrm{E}} \frac{\int_0^1 H(\varsigma, z(\varsigma))\mathrm{d}\varsigma}{\frac{1}{2}\|z\|_{\mathrm{E}}^2} > 0. \tag{3.8}$$

Then for each compact interval $[c,d] \subset (\theta_1, \infty)$, there exists $R > 0$ such that for every $\gamma \in [c,d]$ and every continuous function $g : [0,1] \times \mathbb{R} \to \mathbb{R}$, there exists $\gamma > 0$ such that for each $\mu \in [0,\gamma]$, the problem (P^h) has at least three weak solutions whose norms in E are less than R.

Proof. For any $\varepsilon > 0$, (3.7) implies that there exist τ_1 and τ_2 with $0 < \tau_1 < \tau_2$ such that

$$H(\varsigma,z) \leq \varepsilon|z|^2$$

for every $\varsigma \in [0,1]$ and every z with $|z| \in [0,\tau_1) \cup (\tau_2, \infty)$. Since $H(\varsigma,z)$ is continuous on $[0,1]\times\mathbb{R}$, $H(\varsigma,z)$ is bounded on $[0,1]\times[\tau_1,\tau_2]$. Thus, we can choose $\eta > 0$ and $\upsilon > 2$ so that

$$H(\varsigma,z) \leq \varepsilon|z|^2 + \eta|z|^\upsilon$$

for all $(\varsigma,z) \in [0,1]\times\mathbb{R}$. Then by the same process as in the proof of Theorem 3.1, we obtain (3.4) and (3.5). Since ε is arbitrary, (3.4) and (3.5) give

$$\max\left\{0, \limsup_{\|z\|_{\mathrm{E}}\to+\infty} \frac{J(z)}{\Theta(z)}, \limsup_{z\to 0} \frac{J(z)}{\Theta(z)} \right\} \leq 0.$$

Then, with ρ and σ defined in Lemma 2.1, we have $\rho = 0$. By (3.8), we have $\sigma > 0$. In this case, clearly $\theta_1 = 1/\sigma$ and $\theta_2 = \infty$. Thus, by Lemma 2.1 the result is achieved. □

Remark 3.2. In Assumption $(\mathcal{A}_2)$ of Theorem 3.1, if we choose w as follows

$$w(\varsigma)=\begin{cases} -\dfrac{64\sigma}{9}\left(\varsigma^2-\dfrac{3}{4}\varsigma\right), & \text{if } \varsigma\in[0,\frac{3}{8}),\\ \sigma, & \text{if } \varsigma\in[\frac{3}{8},\frac{5}{8}]\\ -\dfrac{64\sigma}{9}\left(\varsigma^2-\dfrac{5}{4}\varsigma+\dfrac{1}{4}\right), & \text{if } \varsigma\in(\frac{5}{8},1]. \end{cases} \tag{3.9}$$

where $\sigma>0$. Clearly, $w\in \mathrm{E}$. Obviously, one has

$$\begin{aligned}
&\left(\frac{4096}{27}p^-+\frac{64}{9}q^-+\frac{13}{20}r^-\right)\sigma^2\\
&\leq\int_0^{\frac{3}{8}}p(t)\frac{16384}{81}\sigma^2\mathrm{d}t+\int_{\frac{5}{8}}^1 p(t)\frac{16384}{81}\sigma^2\mathrm{d}t+\int_0^{\frac{3}{8}}q(t)\left(-\frac{64\sigma}{9}\left(2t-\frac{3}{4}\right)\right)^2\mathrm{d}t\\
&+\int_{\frac{5}{8}}^1 q(t)\left(-\frac{64\sigma}{9}\left(2t-\frac{5}{4}\right)\right)^2\mathrm{d}t+\int_0^{\frac{3}{8}}r(t)\left(-\frac{64\sigma}{9}\left(t^2-\frac{3}{4}t\right)\right)^2\mathrm{d}t\\
&+\int_{\frac{5}{8}}^1 r(t)\left(-\frac{64\sigma}{9}\left(t^2-\frac{5}{4}t+\frac{1}{4}\right)\right)^2\mathrm{d}t+\int_{\frac{3}{8}}^{\frac{5}{8}}r(t)\sigma^2\mathrm{d}t\\
&=\int_0^1(p(t)|z''(t)|^2+q(t)|z'(t)|^2+r(t)|z(t)|^2)\mathrm{d}t\\
&=\|w_\sigma\|_{\mathrm{E}}^2\leq\left(\frac{4096}{27}p^++\frac{64}{9}q^++\frac{13}{20}r^+\right)\sigma^2.
\end{aligned}$$

Then, we have $\Theta(0)=\Upsilon(0)=0$ and

$$\frac{1}{2}\left(\frac{4096}{27}p^-+\frac{64}{9}q^-+\frac{13}{20}r^-\right)\sigma^2\leq\Theta(w)=\frac{1}{2}\|w\|_{\mathrm{E}}^2\leq\frac{1}{2}\left(\frac{4096}{27}p^++\frac{64}{9}q^++\frac{13}{20}r^+\right)\sigma^2.$$

Clearly, $w\in \mathrm{E}$ and $(\mathcal{A}_2)$ now takes the following form:

$(\mathcal{A}_3)$ there exists a positive constant σ such that

$$\mathscr{D}\sigma^2\neq 0$$

and

$$\frac{\varepsilon}{2\pi^2\delta^2}<\frac{\int_0^1 H(\varsigma,w(\varsigma))\mathrm{d}\varsigma}{\mathscr{D}\sigma^2}.$$

Now, we point out some results in which the function h is separable. To be precise, we consider the problem

$$\begin{cases} -(p(\varsigma)z'')''-(q(\varsigma)z')'+r(\varsigma)z=\gamma\beta(\varsigma)e(\zeta)+\mu g(\varsigma,z(\varsigma)), & \varsigma\in[0,1],\\ z(a)=z(b)=0 \end{cases} \tag{$\phi^{\beta}_{\gamma,\mu}$}$$

where $\beta : [0,1] \to \mathbb{R}$ is a nonzero function such that $\beta \in L^1([0,1])$ and $e : \mathbb{R} \to \mathbb{R}$ is a continuous function and $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ is as introduced for the problem (P^h) in the Introduction.

Let $H(\varsigma, x) = \beta(\varsigma)E(x)$ for every $(\varsigma, x) \in [0,1] \times \mathbb{R}$, where

$$E(x) = \int_0^x e(\xi)d\xi \quad \text{for all } x \in \mathbb{R}.$$

The following existence results are consequences of Theorem 3.1.

Theorem 3.3. Assume that

$(\mathcal{A}_4)$ there exists a constant $\varepsilon > 0$ such that

$$\sup_{\varsigma \in [0,1]} \beta(\varsigma) \max\left\{ \limsup_{z \to 0} \frac{E(z)}{|z|^2}, \limsup_{|z| \to \infty} \frac{E(z)}{|z|^2} \right\} < \varepsilon,$$

$(\mathcal{A}_5)$ there exists a positive constant σ such that

$$\mathscr{D}\sigma^2 \neq 0$$

and

$$\frac{\varepsilon}{2\pi^2\delta^2} < \frac{\int_0^1 \beta(\varsigma)E(w)\mathrm{d}\varsigma}{\mathscr{D}\sigma^2}$$

where w is defined by (3.9).

Then for each compact interval $[c,d] \subset (\theta_3, \theta_4)$, where θ_3 and θ_4 are the same as θ_1 and θ_2, but $\int_0^1 H(\varsigma, w)\mathrm{d}\varsigma$ is replaced by $\int_0^1 \beta(\varsigma)E(w)\mathrm{d}\varsigma$, respectively, there exists $R > 0$ such that for every $\gamma \in [c,d]$ and every continuous function $g : [0,1] \times \mathbb{R} \to \mathbb{R}$, there exists $\gamma > 0$ such that for each $\mu \in [0, \gamma]$, the problem $(\phi_{\gamma,\mu}^{\beta})$ has at least three weak solutions whose norms in E are less than R.

Theorem 3.4. Assume that there exists a positive constant σ such that

$$\mathscr{D}\sigma^2 > 0$$

and

$$\int_0^1 \beta(\varsigma)E(w)\mathrm{d}\varsigma > 0 \tag{3.10}$$

where w is given by (3.9). Moreover, suppose that

$$\limsup_{z \to 0} \frac{E(z)}{|z|^2} = \limsup_{|z| \to \infty} \frac{E(z)}{|z|^2} = 0. \tag{3.11}$$

Then for each compact interval $[c,d] \subset (\theta_3,\infty)$, where θ_3 is the same as θ_1 but $\int_0^1 H(\varsigma,z(\varsigma))\mathrm{d}\varsigma$ is replaced by $\int_0^1 \beta(\varsigma)E(z(\varsigma))\mathrm{d}\varsigma$, there exists $R>0$ such that for every $\gamma\in[c,d]$ and every continuous function $g:[0,1]\times\mathbb{R}\to\mathbb{R}$, there exists $\gamma>0$ such that for each $\mu\in[0,\gamma]$, the problem $(\phi^{\beta}_{\gamma,\mu})$ has at least three weak solutions whose norms in E are less than R.

Proof. From (3.11), we easily observe that the assumption $(\mathcal{A}_4)$ is satisfied for every $\varepsilon>0$. Moreover, using (3.10), by choosing $\varepsilon>0$ small enough, one can derive the assumption $(\mathcal{A}_5)$. Hence, the conclusion follows from Theorem 3.3. □

Now, we exhibit an example in which the hypotheses of Theorem 3.4 are satisfied.

Example 3.1. We consider the problem

$$\begin{cases} z^{(4)}-z''+z=\lambda e(z(\varsigma))+\mu g(\varsigma,z(\varsigma)), & \varsigma\in[0,1],\\ z(0)=z(1)=0, \\ z''(0)=z''(1)=0 \end{cases} \tag{3.12}$$

where

$$e(\zeta)=\begin{cases} 4\zeta^3, & |\zeta|\le 1,\\ 4\zeta, & 1<|\zeta|\le 2,\\ 8, & |\zeta|\ge 2. \end{cases}$$

Then, it is easy to check that

$$E(\zeta)=\begin{cases} \zeta^4, & |\zeta|\le 1,\\ 2\zeta^2-1, & 1<|\zeta|\le 2,\\ 8\zeta-9, & \zeta>2,\\ 8\zeta+23, & \zeta<-2. \end{cases}$$

By choosing $\sigma=1$, $w(\varsigma)$ has the form 3.1, if we choose w as follows

$$w(\varsigma)=\begin{cases} -\dfrac{64}{9}\left(\varsigma^2-\dfrac{3}{4}\varsigma\right), & \text{if } \varsigma\in[0,\frac{3}{8}),\\ 1, & \text{if } \varsigma\in[\frac{3}{8},\frac{5}{8}]\\ -\dfrac{64}{9}\left(\varsigma^2-\dfrac{5}{4}\varsigma+\dfrac{1}{4}\right), & \text{if } \varsigma\in(\frac{5}{8},1]. \end{cases}$$

Considering that $\delta=1$ and $\mathscr{D}=\dfrac{86111}{1080}$, since

$$\mathscr{D}\sigma^2=\frac{86111}{1080}>0,$$

$$\int_0^1 E(w)\mathrm{d}\varsigma > 0$$

and

$$\lim_{z\to 0}\frac{E(z)}{|z|^2} = \lim_{|z|\to\infty}\frac{E(z)}{|z|^2} = 0.$$

Hence, by Theorem 3.4, for each compact interval $[c,d]\subset(0,\infty)$, there exists $R>0$ such that for every $\gamma\in[c,d]$ and every continuous function $g:[0,1]\times\mathbb{R}\to\mathbb{R}$, there exists $\gamma>0$ such that for each $\mu\in[0,\gamma]$, the problem (3.12) has at least three weak solutions whose norms in E are less than R.

The following theorem is a consequence of Lemma 2.3.

Theorem 3.5. Assume that there exist three positive constants $1\le\zeta<2$, β, and σ with

$$\beta < \sqrt{\frac{\mathscr{B}}{2}}\frac{\sigma}{\pi\delta} \tag{3.13}$$

such that

$(\mathcal{B}_1)$ $h(\varsigma,\vartheta)\ge 0$ for each $(\varsigma,\vartheta)\in\left([0,\frac{3}{8})\cup(\frac{5}{8},1]\right)\times\mathbb{R}$,

$(\mathcal{B}_2)$

$$\frac{\int_0^1 \max_{|\zeta|\le\beta} H(\varsigma,\zeta)\mathrm{d}\varsigma}{\beta^2} < \frac{2\delta^2\pi^2}{\mathscr{D}}\frac{\int_{\frac{3}{8}}^{\frac{5}{8}} H(\varsigma,\sigma)\mathrm{d}\varsigma}{\sigma^2},$$

$(\mathcal{B}_3)$ there exists $p>0$ and a positive constant q such that

$$|H(\varsigma,z)|\le p|z|^{\zeta}+q \quad \text{for all } (\varsigma,z)\in[0,1]\times\mathbb{R},$$

$(\mathcal{B}_4)$ there exists $l>0$ and a function $\rho\in\mathbb{R}$ such that

$$G(\varsigma,z)\le lz^{\zeta}+\rho \quad \text{for all } (\varsigma,z)\in[0,1]\times\mathbb{R}.$$

Then there exist a nonempty open set $A\subset[0,\infty)$ and a positive number $R>0$ such that for every $\gamma\in A$ and every continuous function $g:[0,1]\times\mathbb{R}\to\mathbb{R}$, there exists $\delta>0$ such that for each $\mu\in[0,\delta]$, the problem (P^h) has at least three weak solutions whose norms in E are less than R.

Proof. For any $\gamma\ge 0$, $z\in$ E, by $(\mathcal{B}_3)$ and $(\mathcal{B}_4)$, we have

$$\Theta(z)-\gamma J(z)-\mu\Upsilon(z) = \frac{1}{2}\|z\|_{\mathrm{E}}^2-\gamma\left(\int_0^1 H(\varsigma,z(\varsigma))\mathrm{d}\varsigma+\frac{\mu}{\gamma}\int_0^1 G(\varsigma,z(\varsigma))\mathrm{d}\varsigma\right)$$

$$\geq \frac{1}{2}\|z\|_{\mathrm{E}}^2 - \gamma\left(p\frac{1}{2^\zeta\pi^\zeta\delta^\zeta}\|z\|_{\mathrm{E}}^\zeta + q\right)$$
$$-\mu\left(l\frac{1}{2^\zeta\pi^\zeta\delta^\zeta}\|z\|_{\mathrm{E}}^\zeta + \rho\right)$$
$$\geq \frac{1}{2}\|z\|_{\mathrm{E}}^2 - \left(\gamma p\frac{1}{2^\zeta\pi^\zeta\delta^\zeta} + \mu l\frac{1}{2^\zeta\pi^\zeta\delta^\zeta}\right)\|z\|_{\mathrm{E}}^\zeta$$
$$-\gamma q - \mu\rho$$

where C_0 and C_1 are positive constants. Since $\zeta < 2$, one has

$$\lim_{\|z\|_{\mathrm{E}}\to+\infty} \Theta(z) - \gamma J(z) - \mu\Upsilon(z) = \infty \quad \text{for all } \gamma > 0.$$

Let w be defined by (3.9) with σ given in the condition and $s = 2\delta^2\pi^2\beta^2$. Owing to Proposition 2.5, one has

$$\|z\|_\infty \leq \frac{1}{2\pi\delta}\|z\|_{\mathrm{E}} \leq \frac{1}{2\pi\delta}\sqrt{2s} = \beta.$$

Taking into account that $|z(\varsigma)| \leq \beta$ for all $z \in \mathrm{E}$ such that $\|z\|_{\mathrm{E}}^2 < 2s$, hence, we have

$$\sup_{\Theta(z)<s} J(z) \leq \int_0^1 \max_{|\zeta|\leq\beta} H(\varsigma,\zeta)\mathrm{d}\varsigma.$$

By using condition $(\mathcal{B}_1)$, we have

$$J(w) \geq \int_{\frac{3}{8}}^{\frac{5}{8}} H(\varsigma,\sigma)\mathrm{d}\varsigma$$

where w is defined by (3.9). Thus, from the assumption $(\mathcal{B}_2)$, we have

$$\begin{aligned} s\frac{J(w)}{\Theta(w)} &= \frac{s}{\Theta(w)}\left(\int_0^1 H(\varsigma,w(\varsigma))\mathrm{d}\varsigma\right) \\ &\geq \frac{2\delta^2\pi^2\beta^2\left(\int_{\frac{3}{8}}^{\frac{5}{8}} H(\varsigma,\sigma)\mathrm{d}\varsigma\right)}{\mathscr{D}\sigma^2} \\ &> \int_0^1 \max_{|\zeta|\leq\beta} H(\varsigma,\zeta)\mathrm{d}\varsigma \geq \sup_{z\in\Theta^{-1}((-\infty,s])} J(z). \end{aligned}$$

Then, we can fix ρ such that

$$\sup_{z\in\Theta^{-1}((-\infty,s])} J(z) < \rho < s\frac{J(w)}{\Theta(w)}.$$

From Lemma 2.3, we obtain

$$\sup_{\gamma\geq 0}\inf_{z\in \mathrm{E}}(\Theta(z)-\alpha(\rho-J(z))<\inf_{z\in \mathrm{E}}\sup_{\gamma\geq 0}(\Theta(z)-\alpha(\rho-J(z)).$$

Hence, by Lemma 2.2, for each compact interval $[c,d]\subset(\theta_1,\theta_2)$, there exists $R>0$ such that for every $\gamma\in[c,d]$, and every continuous function $g:[0,1]\times\mathbb{R}\to\mathbb{R}$ there exists $\delta>0$ such that, for each $\mu\in[0,\delta]$, $\Theta'(z)-\gamma J'(z)-\mu\Upsilon'(z)=0$ has at least three solutions in E. Hence, the problem (P^h) has at least three weak solutions whose norms are less than R. □

Remark 3.3. If h is non-negative then the weak solution guaranteed by Theorem 3.5 is also non-negative.

Remark 3.4. We note that the preceding theorems also hold for a Carathéodory function h alongside $p\in W^{2,1}([0,1])$, $q\in W^{1,1}([0,1])$ and $r\in L^\infty([0,1])$. It is evident that, in this context, the solutions are weak (see [3]).

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