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# ON SEMI C-PERIODIC FUNCTIONS OF TYPE I AND APPLICATIONS

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In this paper, we study the (newly introduced) class of functions called semi-c-periodic functions of type I with values in a Banach space. We first investigate their basic properties, including a convolution result and invertibility of abstract convolution operators on suvh function space. We prove that an important subclass of such functions is a Banach space under the sup-norm. We then study the existence and uniqueness of semi-c-periodic mild solutions for both autonomous and non-autonomous linear evolution equations. We achieve the existence results using the Banach fixed point theorem and the method of reduction.

## 1. Introduction

Periodicity is a key concept in mathematics, with applications spanning multiple disciplines such as physics, engineering, biology, and economics. It is essential for modeling wave behavior, oscillatory motion, biological rhythms, and economic cycles. Due to its importance, various extensions of periodicity have been developed to broaden its use, including Bloch periodic, almost periodic, pseudo-periodic, c-periodic functions, etc.

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Bloch periodicity was studied by Hasler and N'Guérékata [11], generalizing both periodic and antiperiodic functions. See for instance [8] for recent and deeper developments.

The concept of almost periodicity is due to Bohr [20]. Bohr's contributions established the groundwork for numerous extensions of periodicity, which have been applied in harmonic analysis, differential equations, and mathematical physics.

An important extension is the one of c-periodicity, also referred to as ($\omega$,c) periodicity, introduced by Alvarez *et al.* [1], see also [1, 2, 15, 17]. This concept broadens the idea of periodicity by permitting a function to display periodic-like behavior with a multiplicative factor rather than an exact repetition. c-periodic functions have been extensively studied with applications to differential equations and functional analysis, as they naturally emerge in various practical and theoretical settings, see [1, 2, 13].

An additional refinement of this concept is the one of semi-c periodicity. It offers a more adaptable framework for examining functions that display periodic-like behavior but permit perturbations or variations. This generalization is especially valuable in applied mathematics, where strict periodicity can be too inflexible to represent real-world phenomena accurately, see [12].

In this work, we consider a complex Banach space $X$ with norm $||\cdot||$ and consider continuous functions $f:\mathbb{R}\to X$. Such a function $f(t)$ is called *c*-periodic if it satisfies:

$$f(t+P)=cf(t),$$

where $P>0$ is the c-period and $c\neq 0$ is a constant possibly complex scaling factor. If $c=1$, the function is classically periodic, with period $P$. If $c=-1$, the function is anti-periodic, meaning it alternates in sign every $P$. A c-periodic function (also called (P,c)-periodic function) can be expressed as:

$$f(t)=c^{t/P}u(t),$$

where $u(t)$ is a periodic function with period $P$. This means that instead of simply repeating with a constant factor, the function's amplitude may grow or decay over time, making it more flexible for mathematical modeling. Alvarez, Gomez, and Pinto [1] analyzed the fundamental properties of $(P,c)$-periodic functions, demonstrating their significance in fractional integro-differential equations. Furthermore, research by Larrouy and N'Guérékata[16] highlighted their role in ergodic processes and biological models. This concept enables the study of periodic functions under perturbations, making it highly relevant in functional analysis and applied mathematics. While $c$-periodicity requires an exact periodic structure with a multiplicative factor, semi-$c$-periodicity introduces a relaxed condition, allowing approximate periodicity. A function $f : \mathbb{R} \to X$ is semi-$c$-periodic if:

$$\forall \varepsilon > 0, \exists P > 0 \text{ such that } \|f(t+mP) - c^m f(t)\| \leq \varepsilon \quad \forall m \in \mathbb{Z}, \forall t \in \mathbb{R}. \tag{1}$$

This concept, introduced by Khalladi *et al.* [12], is particularly useful in analyzing functions that exhibit near-periodic behavior but allow for perturbations. Clearly every $c$-periodic function is also semi-$c$-periodic. Still, the converse is not always true unless additional constraints, such as uniform convergence, are imposed [18]. This distinction makes semi-$c$-periodicity particularly relevant in settings where periodicity is disturbed by external influences.

Several research works have advanced the development of semi-periodicity and its applications. For example, the book M. Kostić's [13] provides a rigorous examination of various types of periodic functions, including semi-c periodic functions, discussing their properties, functional spaces, and applications in differential equations. The paper by M.T. Khalladi, M. Kostić, M. Pinto, A. Rahmani, and D. Velinov, see [12], establishes the equivalence of semi-c-periodicity and c-periodicity when the absolute value of c is not equal to 1, analyzing their fundamental properties. Moreover, the notable contribution from H. Ounis and J.M. Sepulcre, see [20], where the study of semi-c- periodic functions extends to the complex plane, exploring their relationships with almost automorphic and c-uniformly recurrent functions and addressing an open problem related to semi-c-periodicity in the real domain. Despite advances in the study of semi-c-periodic functions, there are still gaps in the understanding of their stability in functional spaces and their applications in differential equations, which makes this study of particular importance. Driven by these contributions, this paper seeks to enhance the understanding of semi-c-periodic functions, concentrating on their theoretical properties and applications in differential equations. It specifically examines their stability under operations like convolution while also establishing new findings regarding the existence and uniqueness of semi-c-periodic solutions in differential equations.

The structure of the paper is as follows. In Section 2, we recall the definition of semi-c-periodic functions of type I and present several basic properties of this class of functions. We also define a suitable norm and prove that the resulting space is a Banach space. In Section 3, we investigate the convolution of semi-c-periodic functions and prove that the space is closed under convolution. In Section 4, we apply the theory to investigate the existence and uniqueness of semi-periodic mild solutions for both autonomous and non-autonomous linear differential equations.

## 2. Fundamental Properties of Semi-$c$-Periodicity

In this section, we recall the fundamental definition of semi-$c$-periodic functions with values in Banach spaces and study some of its properties.
Throughout the paper, $(X, ||\cdot||)$ will denote a complex Banach space and $c \in \mathbb{C} - \{0\}$.

**Definition 2.1.** [3, 5] A continuous function $f : \mathbb{R} \to X$ is said to be $c$**-periodic** if there exists $P > 0$ such that $f(t+P) = cf(t)$ for all $t \in \mathbb{R}$. Here $P$ is known as a $c$**-period** of $f$.

**Definition 2.2.** [13] (Semi-$c$-periodicity of type I)
Let $I = [0,\infty)$ and $S = \mathbb{N}$ or $I = \mathbb{R}$ and $S = \mathbb{Z}$. A continuous function $f : I \to X$ is said to be semi-$c$-periodic of type I if:

$$\forall \varepsilon > 0, \exists P > 0, \forall m \in S, \forall t \in I, \|f(t+mP) - c^m f(t)\| \leq \varepsilon.$$

P is aclled a semi-c-period of $f$.

**Example 2.3.** [12] Let $p$ and $q$ be odd natural numbers such that $p - 1 = 0(\text{mod } q)$, and let $c = e^{(i\pi/q)}$. The function

$$f(t) = \sum_{n=1}^{N} \frac{e^{(it/(2nq+1))}}{n^2}, \ t \in \mathbb{R}$$

is semi-c-periodic because it is the uniform limit of $[\pi \cdot (1+2q)...(1+Nq)]$ -periodic functions

$$f_N(t) = \sum_{n=1}^{N} \frac{e^{(it/(2nq+1))}}{n^2}, \ t \in \mathbb{R} \ (N \in \mathbb{N})$$

**Definition 2.4.** [13] (Semi-$c$-periodicity of type II)
Let $I = [0,\infty)$ and $S = \mathbb{N}$ or $I = \mathbb{R}$ and $S = \mathbb{Z}$. A continuous function $f : I \to X$ is said to be semi-$c$-periodic of type II if:

$$\forall \varepsilon > 0, \exists P > 0, \forall m \in S, \forall t \in I, \|c^{-m} f(t+mP) - f(t)\| \leq \varepsilon$$

We will assume that $c \neq 1$ and $c \neq -1$.
**Notation:** We will denote by $Sp_c(I,X)$ the space of all functions $f : I \to X$.that are semi-$c$-periodic of type I.

*Remark* 2.5. A semi-c-periodic function may be unbounded ([12]). Semi-$c$-periodic functions, as defined in this paper, are not necessarily bounded unless otherwise stated.

Indeed, without additional conditions, such as boundedness imposed explicitly, a semi-$c$-periodic function may exhibit unbounded behavior. For example, a function $f : \mathbb{R} \to X$ that satisfies the semi-$c$-periodic condition for $|c| \neq 1$ can grow exponentially, or logarithmically, or polynomially and be unbounded.

Therefore, in our analysis, we restrict attention to semi-$c$-periodic functions that are bounded unless otherwise stated. This restriction is natural and sufficient for the results concerning completeness and convolution in $Sp_c(\mathbb{R},X)$.

In the present work, we will consider only semi-c-periodic functions of type I.

We first start with characterizing $c$-periodic functions, Proposition 2.2 [1].

**Proposition 2.6.** A continuous function $f : \mathbb{R} \to X$ is $c$-periodic with c-period $P$ if and only if $f(t) = c^{t/P} g(t)$ where $g$ is a periodic function of period $P$.

**Theorem 2.7.** [13] Let $|c| = 1$. A continuous function $f : \mathbb{R} \to X$ is semi-c-periodic if and only if there exists a sequence of c-periodic functions $(f_n)$ such that $\lim_{n\to\infty} f_n(t) = f(t)$ uniformly in $\mathbb{R}$.

**Theorem 2.8.** Suppose that $f$, $f_1$, $f_2$ are semi-c-periodic and $\gamma$, $\delta$ are scalars, then the following are also semi-c-periodic:
(i) $\gamma f_1 + \delta f_2$
(ii) $Af$ where $A$ is a bounded linear operator $X \to X$
(iii) $f_\tau(t) := f(t+\tau)$, $\tau$ is a fixed real number.

*Proof.* Obvious. □

**Theorem 2.9.** Let $|c| \leq 1$ and $f_n$ be semi-c-periodic functions, $n = 1,2,...$ such that $\lim_{n\to\infty} f_n(t) = f(t)$ uniformly in $t \in \mathbb{R}$. Then $f$ is semi c-periodic.

*Proof.* Let $\varepsilon > 0$ be given. Then there exists $N \in \mathbb{N}$ such that

$$\|f_n(t) - f(t)\| < \varepsilon, \quad \forall t \in \mathbb{R}, \text{ if } n > N.$$

Since $f_n$ is semi-$c$-periodic, there exists $P > 0$ such that

$$\|f_n(t+mP) - c^m f_n(t)\| < \varepsilon, \quad \forall t \in \mathbb{R}, \forall m \in \mathbb{Z}.$$

Thus, we have:

$$\begin{aligned} \|f(t+mP) - f(t)\| &\leq \|f(t+mP) - f_n(t+mP)\| \\ &+ \|f_n(t+mP) - c^m f_n(t)\| \\ &+ \|c^m f_n(t) - c^m f(t)\|. \end{aligned}$$

Applying the given bounds:

$$\|f(t+mP) - f(t)\| < \varepsilon + \varepsilon + |c|^m \varepsilon \leq 3\ \varepsilon.$$

Since $\varepsilon$ is arbitrary, it follows that $f$ is semi-$c$-periodic. □

**Theorem 2.10.** Let $|c| \geq 1$ and suppose that $f : \mathbb{R} \to \mathbb{C}$ is semi c-periodic with $\inf|f(t)| > \gamma > 0$. Then the function $g : \mathbb{R} \to \mathbb{C}$ defined by $g(t) = \frac{1}{f(t)}$ is also semi c-periodic.

*Proof.* Let $\varepsilon > 0$ be given. Then there exists $P > 0$ such that:

$$\|f(t+mP) - c^m f(t)\| < \varepsilon, \quad \forall t \in \mathbb{R}, \forall m \in \mathbb{Z}.$$

Thus, we have:

$$\begin{aligned} \|g(t+mP) - c^m g(t)\| &= \left\| \frac{1}{f(t+mP)} - \frac{1}{c^m f(t)} \right\| \\ &= \left\| \frac{f(t+mP) - c^m f(t)}{c^m f(t) f(t+mP)} \right\| \\ &\leq \frac{\|f(t+mP) - c^m f(t)\|}{|c|^m \gamma^2}. \end{aligned}$$

Since $\|f(t+mP)-c^m f(t)\| < \varepsilon$, it follows that:

$$\|g(t+mP)-c^m g(t)\| < \frac{\varepsilon}{\gamma^2}.$$

Since $\varepsilon$ is arbitrary, this proves that $g$ is also semi-$c$-periodic.

□

**Lemma 2.1.** *Let $P>0$, $t\in[0,P]$, and $m\in\mathbb{Z}\setminus\{0\}$. Define*

$$c_m(t) = e^{t/(mP)}.$$

*Then for all such $m$ and $t\in[0,P]$, we have*

$$e^{-1} \le c_m(-t) \le e.$$

*Proof.* We consider two cases. If $m>0$, then $-t/(mP)\in[-1/m,0]$, so $e^{-1}\le e^{-t/(mP)}\le 1$. If $m<0$, then $-t/(mP)>0$ and still bounded above by $1/|m|$, so again $e^{-1}\le e^{-t/(mP)}\le e$. Therefore, for all $m\neq 0$ and $t\in[0,P]$, the inequality holds. □

Now for $f\in Sp_c(\mathbb{R},X)$, define

$$||f||^* = \sup_{m\in\mathbb{Z}\setminus\{0\}} \sup_{t\in[0,P]} ||c_m(-t)f(t)||..$$

**Proposition 2.11.** $\|\cdot\|^*$ is a norm on $Sp_c(\mathbb{R},X)$.

*Proof.* Clearly, for any scalar $\lambda\in\mathbb{C}$,

$$\begin{aligned}\|\lambda f\|^* &= \sup_{m\neq 0}\sup_{t\in[0,P]} \|c^m(-t)\lambda f(t)\| \\ &= |\lambda| \sup_{m\neq 0}\sup_{t\in[0,P]} \|c^m(-t)f(t)\| \\ &= |\lambda|\|f\|^*.\end{aligned}$$

The triangle inequality follows from that in $X$:

$$\begin{aligned}\|f+g\|^* &= \sup_{m\neq 0}\sup_{t\in[0,P]} \|c^m(-t)(f(t)+g(t))\| \\ &\le \sup_{m\neq 0}\sup_{t\in[0,P]} (\|c^m(-t)f(t)\| + \|c^m(-t)g(t)\|) \\ &\le \|f\|^* + \|g\|^*.\end{aligned}$$

Now suppose $\|f\|^* = 0$. Then,

$$\|c^m(-t)f(t)\| = 0, \quad \text{for all } m \in \mathbb{Z}\setminus\{0\}, t \in [0,P],$$

so $f(t) = 0$ for all $t \in [0,P]$. Let $t \in [P,2P]$. Then $t = \tau + P$ for some $\tau \in [0,P]$. By semi-$c$-periodicity of $f$, for every $\varepsilon > 0$,

$$\|f(t) - cf(\tau)\| < \varepsilon.$$

But $f(\tau) = 0$, so $\|f(t)\| < \varepsilon$ for every $\varepsilon > 0$. Hence $f(t) = 0$. Repeating this argument, we conclude that $f(t) = 0$ for all $t \geq 0$. The case $t < 0$ follows similarly using negative shifts. Thus, $f = 0$, and we have shown that $\|\cdot\|^*$ is a norm. □

We show using the above lemma, equivalence of norms.

**Proposition 2.12.** The norms

$$\|f\|_P := \sup_{t\in[0,P]} \|f(t)\| \quad \text{and} \quad \|f\|^* := \sup_{m\in\mathbb{Z}\setminus\{0\}} \sup_{t\in[0,P]} \|c^m(-t)f(t)\|$$

are equivalent on $Sp_c(\mathbb{R},X)$.

*Proof.* From Lemma 2.11, for all $m \in \mathbb{Z}\setminus\{0\}$ and $t \in [0,P]$, we have:

$$\begin{aligned}
\frac{1}{e}\|f(t)\| &\leq \|c^m(-t)f(t)\| \leq e\|f(t)\| \\
\frac{1}{e}\sup_{t\in[0,P]}\|f(t)\| &\leq \sup_{t\in[0,P]}\|c^m(-t)f(t)\| \leq e\sup_{t\in[0,P]}\|f(t)\| \\
\frac{1}{e}\|f\|_P &\leq \sup_{m\in\mathbb{Z}\setminus\{0\}}\sup_{t\in[0,P]}\|c^m(-t)f(t)\| \leq e\|f\|_P \\
\frac{1}{e}\|f\|_P &\leq \|f\|^* \leq e\|f\|_P
\end{aligned}$$

and hence the norms are equivalent. □

**Theorem 2.13.** Let $|c| \leq 1$. The space

$$BS_{pc}(\mathbb{R},X) := Sp_c(\mathbb{R},X) \cap BC(\mathbb{R},X)$$

is a Banach space when equipped with the supremum norm.

*Proof.* Let $(f_n) \subset BS_{pc}(\mathbb{R},X)$ be a Cauchy sequence. Then $(f_n)$ is a Cauchy sequence in $BC(\mathbb{R},X)$, so there exists $f \in BC(\mathbb{R},X)$ such that $f_n \to f$ uniformly on $\mathbb{R}$.

Let $\varepsilon > 0$. Then there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have

$$\|f_n(t) - f(t)\| < \frac{\varepsilon}{3}, \quad \forall t \in \mathbb{R}.$$

Since each $f_n \in Sp_c(\mathbb{R},X)$, there exists $P > 0$ such that for all $m \in \mathbb{Z}$, $t \in \mathbb{R}$,

$$\|f_n(t+mP) - c^m f_n(t)\| < \frac{\varepsilon}{3}.$$

Now consider:

$$\begin{aligned}
\|f(t+mP) - c^m f(t)\| &\leq \|f(t+mP) - f_n(t+mP)\| \\
&\quad + \|f_n(t+mP) - c^m f_n(t)\| \\
&\quad + \|c^m f_n(t) - c^m f(t)\| \\
&\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + |c^m| \cdot \|f_n(t) - f(t)\| \\
&\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

Hence, $f \in Sp_c(\mathbb{R},X)$, and since $f \in BC(\mathbb{R},X)$, we have $f \in BS_{pc}(\mathbb{R},X)$.

Therefore, $BS_{pc}(\mathbb{R},X)$ is complete with respect to the supremum norm. □

## 2.1. Uniformly close $c$-periodic functions on $\mathbb{R}$, with $|c| \neq 1$

In this section we will analyze under which conditions $c$-periodic functions $f_1$ and $f_2$ with $|c| \neq 1$ can be uniformly close, that is there exists $M < \infty$ so that for all $t$: $|f_1(t) - f_2(t)| \leq M$. We will argue that in all cases $f_1 = f_2$.

**Proposition 2.14.** Suppose $|c| \neq 1$ and that $f_1$ and $f_2$ are $c$-periodic, uniformly close, and have the same period $P$. Then $f_1 = f_2$.

*Proof.* Let $P_1, P_2$ be the c-periods of $f_1$ and $f_2$ respectively. Without loss of generality we may assume that $P_1 = P_2 = P = 1$. Using Proposition 2.6 we can write these functions as:

$$f_i(t) = c^t g_i(t),$$

with each $g_i$ periodic of period $P = 1$. Since $f_1$ and $f_2$ are uniformly close there exists $M < \infty$ such that

$$|f_1(t) - f_2(t)| \leq M \quad \text{for all } t$$

Therefore

$$|g_1(t) - g_2(t)| \leq c^{-t}M$$

If $|c| > 1$ we conclude that as $t \to \infty$, $|g_1(t) - g_2(t)|$ tends to 0. Since both $g_1$ and $g_2$ are periodic with period $P = 1$, we conclude that $|g_1(t) - g_2(t)| = 0$ for all $t$ and that therefore $f_1 = f_2$.
If $|c| < 1$ we conclude that as $t \to -\infty$ $|g_1(t) - g_2(t)|$ tends to 0. Since both $g_1$ and $g_2$ are periodic with period $P = 1$ we conclude that $|g_1(t) - g_2(t)| = 0$ for all $t$ and that therefore $f_1 = f_2$.

□

We next consider the case where $P_1$ and $P_2$ are different.

**Proposition 2.15.** Suppose $|c| \neq 1$ and that $f_1$ and $f_2$ are $c$-periodic, uniformly close, with different periods $P_1 \neq P_2$. Then $f_1 = f_2 = 0$.

*Proof.* Without loss of generality we may assume that $P_1 = 1$ and that $P_2 > P_1$. Using Proposition 2.6 we can write these functions as:

$$f_1(t) = c^t g_1(t),$$

with $g_1$ periodic of period $P_1 = 1$, and

$$f_2(t) = c^{t/P_2} g_1(t),$$

with $g_2$ periodic of period $P_2 > 1$. Therefore the difference between $g_1$ and $g_2$ is bounded, i.e. there exists $k > 0$ for that for all $t$:

$$|g_1(t) - g_2(t)| \leq k$$

With $f_1$ and $f_2$ uniformly close there exists $M < \infty$ so that

$$|f_1(t) - f_2(t)| \leq M \quad \text{for all } t$$

Suppose $|c| > 1$, then we obtain

$$|g_1(t) - c^{t/P_2 - t} g_2(t)| \leq |c|^{-t}M$$

Since $P_2 > 1$ and since $g_2$ is bounded we see that as $t \to \infty$, $g_1(t)$ tends to 0. Since $g_1$ has period 1, we conclude that $g_1(t) = 0$ for all $t$ and that therefore $f_1(t) = 0$ for all $t$. But then:

$$|c^{t/P_2} g_2(t)| \leq M$$

and therefore

$$|g_2(t)| \leq |c|^{-t/P_2} M$$

Letting $t \to \infty$ we again conclude that also $g_2(t) \to 0$ and therefore as $g_2$ has period $P_2$ also $g_2(t) = 0$ for all $t$. We conclude that also $f_2(t) = 0$ for all $t$.

If $|c| < 1$ we can let $t \to -\infty$ and draw the same conclusions: $f_1(t) = f_2(t) = 0$ for all $t$.

□

## 2.2. Convolution

**Theorem 2.16.** Suppose that $g \in L^1(\mathbb{R})$ has compact support and $f$ is c-periodic (resp. semi-c-periodic). Then the convolution $F = f * g$ defined by

$$F(t) = (f * g)(t) := \int_{-\infty}^{\infty} f(t-s)g(s)ds$$

is also c-periodic (resp. semi-c-periodic).

*Proof.* We first note that under the above assumptions, $F(t)$ is well-defined. Suppose that $f$ is $c$-periodic with $c$-period P . Then $f(t+P) = cf(t)$ for all $t$ and

$$F(t+P) = \int_{-\infty}^{\infty} f(t+P-s)g(s)ds = \int_{-\infty}^{\infty} cf(t-s)g(s)ds = cF(t)$$

Therefore $F = f * g$ is again $c$-periodic with $c$-period $P$.
Now suppose $f$ is semi c-periodic. Since $g$ has compact support, there exists $M < \infty$ for which $\int_{-\infty}^{\infty} |g(s)|ds < M$.

Let $\varepsilon > 0$, and choose $\varepsilon' > 0$ so that $\varepsilon' M < \varepsilon$. Then there exists a semi-period $P$ so that for all integers $m$:

$$||f(t+mP) - c^m f(t)|| < \varepsilon'$$

Then for all $t$ and integers $m$:

$$F(t+mP) - c^m F(t) = \int_{-\infty}^{\infty} (f(t+mP-s) - c^m f(t-s))g(s)ds$$

Therefore

$$||F(t+mP)-F(t)|| \leq \int_{-\infty}^{\infty} ||f(t+mP-s)-c^m f(t-s)|||g(s)|ds$$

$$||F(t+mP)-F(t)|| \leq \varepsilon' \int_{-\infty}^{\infty} |g(s)|ds \leq \varepsilon' M < \varepsilon$$

Therefore the convolution is again in $Sp_c(\mathbb{R},X)$.

□

*Remark* 2.17. Let $\varphi \in L^1(\mathbb{R})$ with compact support and $\lambda \in \mathbb{C}$. Consider the operator $A_{\lambda,\varphi}$ defined by

$$A_{\lambda,\varphi}u := \lambda u + \varphi \star u$$

Then it is clear that $A_{\lambda,\varphi}(Sp_c(\mathbb{R},X)) \subset Sp_c(\mathbb{R},X)$. Moreover $A_{\lambda,\varphi}$ acts continuously in $Sp_c(\mathbb{R},X)$, that is there exists a constant $C > 0$ such that

$$\|A_{\lambda,\varphi}u\| \leq C\|u\|, \forall u \in Sp_c(\mathbb{R},X).$$

Let's now present a result on the invertibility of the convolution operators in $Sp_c(\mathbb{R},X)$. Consider the Fourier Transform.

$$a(\xi) := \lambda + \hat{\varphi}(\xi)$$

where $\hat{\varphi}(\xi)$ is the Fourier transform of the function $\varphi$. $a(\xi)$ is the *symbol* of the operator $A_{\lambda,\varphi}$. And since $\lim_{\xi\to\infty} \hat{\varphi}(\xi) = 0$, the symbol $a(\xi)$ is a well defined continuous function on $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$, and $a(\infty) = \lambda$.

Now we state and prove

**Theorem 2.18.** Suppose $\varphi \in L^1(\mathbb{R})$. Then the operator $A_{\lambda,\varphi}$ is invertible in $Sp_c(\mathbb{R},X)$ if $a(\xi) \neq 0$ for all $\xi \in \overline{\mathbb{R}}$.

*Proof.* Suppose $a(\xi) \neq 0$ for all $\xi \in \overline{\mathbb{R}}$. Then the function $\frac{1}{a(\xi)}$ is well-defined on $\overline{\mathbb{R}}$ and in view of the classical Wiener's theorem, we get

$$\frac{1}{a(\xi)} = \frac{1}{\lambda} + \hat{\psi}(\xi),$$

where $\psi \in L^1(\mathbb{R})$. It is easy to check the $A_{\psi,\frac{1}{\lambda}}$ is the inverse to the operator $A_{\lambda,\varphi}$ which acts in $Sp_c(\mathbb{R},X)$ in view of the above remark.

□

*Remark* 2.19. The boundedness result for semi-$c$-periodic functions can be extended to all $|c| \leq 1$, as outlined in Kostić's framework on semi-$c$-periodicity. This extension holds under the given conditions of the space.

**Theorem 2.20.** Consider the Volterra integral equation. Suppose that $\alpha(s) \in L^1(\mathbb{R})$ decays exponentially fast: there exist constants $C > 0$ and $\lambda > 0$ such that:

$$|\alpha(s)| \leq Ce^{-\lambda|s|}.$$

Suppose that $\hat{\alpha}(\xi) - 1$ never vanishes and suppose that

$$\int_{-\infty}^{\infty} |\alpha(s)| ds < 1,$$

If $f(t) \in Sp_c(\mathbb{R}, X)$, then the solution $u(t)$ of the equation

$$u(t) = f(t) + \int_{-\infty}^{\infty} u(t-s)\,\alpha(s)\,ds$$

exists and is in $Sp_c(\mathbb{R}, X)$.

*Proof.* We define the operator $F$ as:

$$Fu(t) = f(t) + \int_{-\infty}^{\infty} u(t-s)\alpha(s)\,ds.$$

We analyze the difference between $Fu(t)$ and $Fv(t)$:

$$\|Fu(t) - Fv(t)\| = \left\| \int_{-\infty}^{\infty} (u(t-s) - v(t-s))\alpha(s)\,ds \right\|.$$

By applying the norm inside the integral:

$$\|Fu(t) - Fv(t)\| \leq \int_{-\infty}^{\infty} \|u(t-s) - v(t-s)\| |\alpha(s)|\,ds.$$

Let $C = \int_{-\infty}^{\infty} |\alpha(s)| ds$ then $C < 1$ and:

$$\|Fu - Fv\|_\infty \leq C\|u - v\|_\infty.$$

Therefore $F$ is a contraction. Since $F$ is a contraction mapping, Banach's Fixed Point Theorem guarantees the existence of a unique fixed point $u^*$, meaning:

$$u^*(t) = Fu^*(t) = f(t) + \int_{-\infty}^{\infty} u^*(t-s)\alpha(s)\,ds.$$

Thus, $u^*(t)$ is the unique solution to the given integral equation.

□

## 3. An application to some evolution equations

In this section, we consider semi-$c$-periodic mild solutions of abstract differential equations. Recall that a mild solution of the equation

$$u'(t) = Au(t) + f(t)$$

is a function $u(t)$ defined by the variation of constants formula

$$u(t) = \int_{-\infty}^{t} T(t-s)f(s)\,ds,$$

where $(T(t))_{t\geq 0}$ is the $C_0$-semigroup generated by the operator $A$. A semi-$c$-periodic mild solution is one where $f \in Sp_c(\mathbb{R},X)$ and the resulting function $u$ also belongs to $Sp_c(\mathbb{R},X)$.

**Definition 3.1.** [14] A strongly measurable family of operators $(T(t))_{t\geq 0} \subset B(X)$, the space of bounded linear operators on $X$, is said to be uniformly integrable if $|||T|||:=\int_0^\infty \|T(t)\|dt < \infty$.

Consider the linear differential equation

$$u'(t) = \lambda u(t) + f(t),\ \ t \in \mathbb{R},\ \lambda \in \mathbb{C} \tag{2}$$

**Theorem 3.2.** If $\mathcal{R}e\,\lambda < 0$ and $f \in Sp_c(\mathbb{R},X)$, then Eq.2 has a unique solution in $Sp_c(\mathbb{R},X)$.

*Proof.* It has been proved (cf. for instance [18]) that Eq.2 has a unique solution of the form

$$u(t) = \int_{-\infty}^{t} e^{\lambda(t-s)} f(s)ds,\ \ t \in \mathbb{R}$$

Let $\varepsilon > 0$ and let $P > 0$ be so that

$$||f(t+mP) - c^m f(t)|| < \varepsilon,\ \ \text{all m} \in \mathbb{Z}, \text{t} \in \mathbb{R}.$$

Then

$$\|u(t+mP)-c^m u(t)\| \leq \int_{-\infty}^{t} e^{\mathcal{R}e\,\lambda(t-s))}||f(s+mP)-c^m f(s)||ds \leq \varepsilon \int_{-\infty}^{t} e^{|\mathcal{R}e\,\lambda|(t-s))}ds$$

Therefore for all m $\in \mathbb{Z}$, t $\in \mathbb{R}$,

$$\|u(t+mP) - c^m u(t)\| \leq \frac{\varepsilon}{|\mathcal{R}e\,\lambda|}$$

and we conclude that $u(t)$ is also semi-c-periodic. The proof is complete. □

**Theorem 3.3.** Assume that the operator $A$ generates an uniformly integrable semigroup $(T(t)_{t\geq 0})$. Then for each $f$ semi-c-periodic, there exists a unique semi-c-periodic mild solution of the equation

$$u'(t) = Au(t) + f(t), \ \ t \in \mathbb{R} \tag{3}$$

*Remark* 3.4. Note that a function $u \in C^1(\mathbb{R}, X)$ is called a strong solution on $\mathbb{R}$ of Eq.(3) if $u \in C(\mathbb{R}, D(A))$ and Eq.(3) holds on $\mathbb{R}$. If merely $u(t) \in X)$ instead of $D(A)$, we say that $u$ is a mild solution of Eq.(3), and can be represented by $u(t) = \int_{-\infty}^{t} T(t-s)f(s)\,ds$ (cf. for instance [14]).

*Proof.* Let $u(t)$ be a mild solution of the above equation. Then we have

$$u(t) = \int_{-\infty}^{t} T(t-s)f(s)\,ds$$

Now let $\varepsilon > 0$ be given. There exists $P > 0$ such that for all $m \in \mathbb{Z}$ and all $t \in \mathbb{R}$, we have

$$\|f(t+mP) - c^m f(t)\| \leq \varepsilon.$$

So,

$$\begin{aligned}
\|u(t+mP) - c^m u(t)\| &= \left\| \int_{-\infty}^{t+mP} T(t+mP-s)f(s)\,ds - \int_{-\infty}^{t} T(t-s)f(s)\,ds \right\| \\
&= \left\| \int_{-\infty}^{t} T(t-s)(f(s+mP) - c^m f(s))\,ds \right\| \\
&\leq \int_{-\infty}^{t} \|T(t-s)\| \cdot \|f(s+mP) - c^m f(s)\|\,ds \\
&\leq \varepsilon \int_{-\infty}^{t} \|T(t-s)\|\,ds \\
&= \varepsilon \|T\|_I
\end{aligned}$$

for all integer $m$ and all real $t$, which proves the theorem. □

**Example 3.5.** Let $A = -\delta I$ with $\delta > 0$ and $I$ the identity operator. Then $T(t) = e^{-\delta t} I$. So

$$u'(t) = -\delta u(t) + f(t), \ \ t \in \mathbb{R} \tag{4}$$

has a unique strong solution defined by $u(t) = \int_{-\infty}^{t} e^{-\delta(t-s)} f(s) ds$ which is semi-c-periodic.

## 4. The nonautonomous linear equation

In this section, we assume that $\mathbb{X}$ is of finite dimension, say $X = \mathbb{C}^n$. This assumption simplifies the analysis, especially when transforming the system into its Jordan canonical form. We note that extending the results to infinite-dimensional Banach spaces would require additional conditions on the operator $A(t)$, such as generating a strongly continuous semigroup or being a closed, densely defined operator. Unless otherwise stated, the results in this section rely on the finite-dimensional setting. We consider inhomogeneous linear evolution equations of the form

$$u'(t) = A(t)u(t) + f(t), \quad t \in \mathbb{R}, u(t) \in X, \tag{5}$$

$A(\cdot)$ is a $\tau$-periodic linear matrix-valued function and $f$ is a $X$-valued semi-c-periodic function.

First, we note that by Floquet Theory of periodic ordinary differential equations, without loss of generality we may assume that $A$ is independent of $t$.

Next we will show that the problem can be reduced to the one-dimensional case (cf. for instance [18]). In fact, if $A$ is independent of $t$, by a change of variable if necessary, we may assume that $A$ is of Jordan normal form. In this direction, we can go further with the assumption that $A$ has only one Jordan block. That is we are dealing with the system of equations of the form

$$\begin{pmatrix} \dot{u}_1(t) \\ \dot{u}_2(t) \\ \vdots \\ \dot{u}_n(t) \end{pmatrix} = \begin{pmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \\ 0 & 0 & 0 & \dots & \lambda \end{pmatrix} + \begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{pmatrix}.$$

Let us consider the last equation for $u_n(t)$. We have

$$\dot{u}_n(t) = \lambda u_n(t) + f_n(t)$$

If $\Re\lambda < 0$, then we can easily check that

$$u_n(t) = \int_{-\infty}^{t} e^{\lambda(t-s)} f_n(s) ds$$

is the unique solution of Eq. (5). By Theorem 3.2, $u_n(t)$ is in $Sp_c(\mathbb{R}, X)$. Let us consider next the equation involving $u_{n-1}$ and $u_n$. That is

$$u'_{n-1}(t) = \lambda u_{n-1}(t) + u_n(t) + f_{n-1}(t)$$

Since $u_n$ is in $Sp_c(\mathbb{R},X)$, by repeating the above argument we can show that $u_{n-1}$ is also in $Sp_c(\mathbb{R},X)$. Continuing this process, we can show that all $u_k$ are in $Sp_c(\mathbb{R},X)$. That is $u(t)=(u_1(t),...,u_n(t)^T$ is in $Sp_c(\mathbb{R},X)$.

**Conclusion.** Note that in this paper we consider functions with the same semi-c-periods in order to obtain a vector space. We believe that the main result (Theorem 2.14) can be generalized. A future work must be conducted for unbounded functions in a more general setting to include completeness and further applications.

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## REFERENCES

[1] E. Alvarez, A. Gomez, M. Pinto Jiménez, *$(P,c)$-periodic functions and mild solutions to abstract fractional integro-differential equations*, Electronic Journal of Qualitative Theory of Differential Equations, vol.16, no.16, (2018), 1-8.

[2] E. Alvarez, S. Castillo, M. Pinto, *$(P,c)$-asymptotically periodic functions, first-order Cauchy problem, and Lasota-Wazewaska model with unbounded oscillating production of red cells*, Mathematical Methods in the Applied Sciences, vol.43, no.1, pp. 305-319, 2020.

[3] J. Andres, D. Pennequin, *Semi-periodic solutions of difference and differential equations*, Boundary Value Problems, 2012, 2012:141.

[4] A. Haraux, *Antiperiodic solutions of some nonlinear evolution equations*, Manuscripta Math. Vol.63 No.4 (1989), 479-505.

[5] I.D. Berg, Albert Wilansky, *Periodic, almost periodic and semiperiodic sequences*.

[6] H. Bounit, M. Kostić, R. Varga, *Semi-periodic functions and evolution equations with unbounded delays*, Mathematical Reports, vol.21, no.3, pp. 219-232, 2019.

[7] Y-K. Chang, H-N. Mao, *Measure Pseudo Asymptotically Bloch Periodic Solutions to Integro-Differential Equations in Thermodynamics of Memory Materials*, Journal of Integral Equations and Applications, 2024.

[8] Y-K. Chang,G.M. N'Guérékata, *R. Ponce, Bloch-type Periodic Functions; Theory and Applications, World Scientific 2021.*

[9] T. Diagana, *Almost Automorphic Type and Almost Periodic Type functions in Abstract Spaces*, Springer, New York, 2013.

[10] A. Haraux, P. Souplet, *An example of uniformly recurrent function which is not almost periodic*, Journal of Fourier Analysis and Applications, vol.10, no.2, pp. 217-220, 2004.

[11] M. Hasler, G.M. N'Guérékata, *Bloch periodic functions and some applications*, Nonlinear Studies, vol.21, no.1, pp. 21-30, 2014.

[12] M.T. Khalladi, M. Kostić, M. Pinto, A. Rahmani, and D. Velinov, *On semi-c-periodic functions*, Hindawi Journal of Mathematics, Vol. 2021, Art. ID 6620625, 5 pages.

[13] M. Kostić, *Selected Topics in Almost Periodicity*, De Gruyter Studies in Mathematics, 2021.

[14] C. Lizama, G.M. N'Guérékata, *Bounded mild solution for semilinear integrodifferential equations in Banach spaces*, Integral Equations and Operator Theory, 8 (2010), 207-227.

[15] J. Larrouy, G.M. N'Guérékata, *$(P,c)$-periodic and asymptotically $(P,c)$-periodic mild solutions to fractional Cauchy problems*, Applicable Analysis.

[16] J. Larrouy and G. M. N'Guérékata, "Measure $(P,c)$-Pseudo-Almost Periodic Functions and Lasota-Wazewska Model with Ergodic and Unbounded Oscillating Oxygen Demand," *Abstract and Applied Analysis*, vol. 2022, Article ID 9558928, 18 pages, 2022. DOI:

[17] G. Mophou, G.M. N'Guérékata, *An existence result of$(P,c)$-periodic mild solutions to some fractional differential equations*, Nonlinear Studies, Vol.27 No.1 (2020), 167-175.

[18] G.M. N'Guérékata, *Almost Periodic and Almost Automorphic Functions in Abstract Spaces*, Springer Cham, New York 2021, second edition.

[19] G.M. N'Guérékata, V. Valmorin, *Antiperiodic solutions of semilinear integrodifferential equations in Banach spaces*, Appl. Math. Comput. Vol.218 (2012), 11118-11124.

[20] H. Ounis and J.M. Sepulcre, *Semi-C-Periodicity, C-Uniform Recurrence and Almost Automorphy in the Complex Plane*, Quaestiones Mathematicae, 2024.

[21] S. Zaidman, *Almost Periodic Functions in Abstract Spaces*, Pitman Research Notes in Mathematics, vol. 126, Pitman Boston, 1985.

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