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ON MINIMUM EMBEDDINGS OF $\mathbf{P_k(u,\lambda)}$ INTO $\mathbf{KS(u+w,\mu)}$

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Let G be a simple finite graph and H be a subgraph of G. We say that an H-design $\Sigma = (U,\mathcal{C})$ of order u and index λ is *embedded* into a G-design $\Sigma' = (V,\mathcal{B})$ of order $u+w$ and index μ, $\lambda \leq \mu$, if there is an injective function $f : \mathcal{C} \to \mathcal{B}$ such that C is a subgraph of $f(C)$ for every $C \in \mathcal{C}$. The mapping f is called the *embedding* of $\Sigma = (U,\mathcal{C})$ into $\Sigma' = (V,\mathcal{B})$. If w attains the minimum possible value, then f is a *minimum* embedding. In this paper we study the minimum embedding of a $P_k(u,\lambda)$ into a $KS(u+w,\mu)$, for $k = 3,4$.

1. Introduction

Let Γ be a simple finite graph and G be a subgraph of Γ. A *G-decomposition of* $\lambda\Gamma$ (λ copies of Γ) is a pair $(X,\mathcal{B})$ where X is the vertex set of Γ and $\mathcal{B}$ is a nonempty collection of graphs (*blocks*), each isomorphic to G, which partitions the edges of $\lambda\Gamma$. If Γ is the complete graph K_v on v vertices, then we also refer to this as a *G-design* or *G-system of order v and index λ*. If a G-design of order v and index λ exists, then $\lambda\frac{v(v-1)}{2m} \in \mathbb{N}$ (the number of blocks), where m is the number of edges of G.

A path design $P_k(v,\lambda)$ is a P_k-design of order v and index λ, where P_k is the simple path with $k-1$ edges, $[a_1,a_2,\ldots,a_k] = \{\{a_1,a_2\},\{a_2,a_3\},\ldots,$

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$\{a_{k-1}, a_k\}\}$. It is well-known that: *i*) a $P_3(v,\lambda)$ exists if and only if $v \geq 3$ for $\lambda \equiv 0 \pmod 2$, and $v \equiv 0,1 \pmod 4$, $v \geq 4$, for $\lambda \equiv 1 \pmod 2$; *ii*) a $P_4(v,\lambda)$ exists if and only if $v \geq 4$ for $\lambda \equiv 0 \pmod 3$, and $v \equiv 0,1 \pmod 3$, $v \geq 4$, otherwise.

A *kite* is a simple graph, usually indicated by $K_3 + e$, on four vertices con-

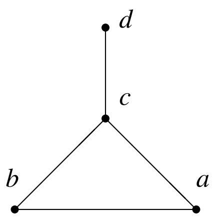

Figure 1: the kite (a,b,c)-d

sisting of a triangle and a single edge (tail) sharing one common vertex (see Figure 1). We denote a kite by $(a,b,c)-d$ where (a,b,c) is the triangle having *base* $\{a,b\}$, *tail* $\{c,d\}$ and *laterals* $\{a,c\},\{b,c\}$. A kite system KS(v,λ) is a G-design of order v and index λ, where the graph G is a kite. It is well-known that a KS(v,λ) exists if and only: *i*) $v \equiv 0,1 \pmod 8$, $v \geq 8$, for $\lambda \equiv 1 \pmod 2$; *ii*) $v \equiv 0,1 \pmod 4$, $v \geq 4$, for $\lambda \equiv 2 \pmod 4$; *iii*) $v \geq 4$ for $\lambda \equiv 0 \pmod 4$.

Let H be a subgraph of G. We say that an H-design $\Sigma = (U,\mathcal{C})$ of order u and index λ is *embedded* into a G-design $\Sigma' = (V,\mathcal{B})$ of order $u+w$ and index μ, $\lambda \leq \mu$, if there is an injective function $f : \mathcal{C} \to \mathcal{B}$ such that C is a subgraph of $f(C)$ for every $C \in \mathcal{C}$. The mapping f is called the *embedding* of Σ into Σ'. If w attains the minimum possible value, then f is a *minimum* embedding; if $f : \mathcal{C} \to \mathcal{B}$ is bijective, the embedding is called *exact* ([9]).

The embedding problem has been studied for $\mu = \lambda = 1$ and for many graphs G, H. The embedding of path designs into kite systems for $\mu = \lambda = 1$ are studied in [1, 8]. Embedding problems with $\mu \geq \lambda \geq 1$ are studied in [2, 5–7, 10, 11]. In particular, in [11] the following result is proved.

Theorem 1.1 ([11]). *There exists a minimum embedding of a KS(u,λ) into a KS$(u+w,\mu)$ if the conditions in the table below are satisfied.*

λ	$u \geq 4$	$\mu \geq \lambda$	w
any	$0,1 \pmod 8$	any	0
even	$4,5 \pmod 8$	even	0
$0 \pmod 4$	$2,3 \pmod 4$	$0 \pmod 4$	0
$0 \pmod 4$	$4k+h, h=2,3$	$2 \pmod 4, \mu \geq 3\lambda/2$	$4-h$
$0 \pmod 4$	$8k+h,\ 2 \leq h \leq 7$	odd, $\mu \geq 5\lambda/4$	$8-h$
$2 \pmod 4$	$8k+h, h=4,5$	odd, $\mu \geq 5\lambda/4$	$8-h$

In this paper we study the minimum embedding of a $P_k(u,\lambda)$ into a $KS(u+w,\mu)$, for $k=3$ (Section 2), $k=4$ (Section 3). We are specifically interested in the case where w is the smallest number $0 \le w < 8$ such that a $KS(u+w,\mu)$ exists, i.e., $\mu\frac{(u+w)(u+w-1)}{8} \in \mathbb{N}$. More precisely, we want that $w=0$ (when $\mu\frac{u(u-1)}{8} \in \mathbb{N}$) or w is the minimum non negative integer in the congruence class $[-u]_n$, with $n=\frac{8}{gcd(\mu,4)}$ (which implies that w depends only on the congruence class of u modulo n and does not increase as u increases). The minimum non negative integer in the congruence class $[-u]_n$ will be denoted by $(-u)_n$. For $k=3$, the above restriction on w will imply that $\mu > \lambda$. The results on the case $w=0$ are presented in [12], that is an extended abstract appeared in Proceedings of Combinatorics 2012.

To obtain our results we will make a massive use of the *difference method*. Let D_u denote the following set with elements from $\mathbb{Z}_u$:

$$D_u = \begin{cases} d : 1 \le d \le \frac{u}{2} & \text{if } u \text{ is even;} \\ d : 1 \le d \le \frac{u-1}{2} & \text{if } u \text{ is odd.} \end{cases}$$

The elements of D_u are called *differences* of $\mathbb{Z}_u$. For any $d \in D_u$, if $d \ne \frac{u}{2}$, then we can form a single 2-factor $\{\{i, d+i\} : i \in \mathbb{Z}_u\}$, if u is even and $d=\frac{u}{2}$, then we can form a 1-factor $\{\{i,\frac{u}{2}+i\} : 0 \le i \le \frac{u}{2}-1\}$. It is also worth remarking that 2-factors obtained from distinct differences are disjoint from each other and from the 1-factor. For later use we quote the following results.

Lemma 1.2 ([3])**.** *Let u and k be integers such that $u > 8k$. Then there exists a cyclic partial kite system of order u, whose base blocks contains every difference $d \in \{1,2,\dots,4k\}$ exactly once.*

Lemma 1.3 ([4])**.** *Let u and k be integers such that $u > 4k$. Then there exists a cyclic partial kite system of order u, whose base blocks contains every difference $d \in \{1,2,\dots,2k\}$ exactly twice.*

Finally, we observe what follows.

Remark 1.4. For $k=3,4$, if $\Sigma_1=(U,\mathcal{B}_1)$ is a $KS(u,\mu)$ which embeds a $P_k(u,\lambda)$ and $\Sigma_2=(U,\mathcal{B}_2)$ is a $KS(u,\nu)$, then $\Sigma'=(U,\mathcal{B}_1\cup\mathcal{B}_2)$ is trivially a $KS(u,\mu+\nu)$ which embeds a $P_k(u,\lambda)$. In addition, if Σ_2 embeds a $P_k(u,\lambda')$, then Σ' embeds a $P_4(u,\lambda+\lambda')$. As a consequence of the above, the union of ρ copies of a $KS(u,\mu)$ embedding a $P_k(u,\lambda)$ gives a $KS(u,\rho\mu)$ which embeds a $P_k(u,\rho\lambda)$

2. P$_3$-designs

To begin with, we study the embedding of a $P_3(u,\lambda)$ into a $KS(u,\mu)$. In this case ($w=0$), as a consequence of the following lemma a necessary condition is $\mu \ge 2\lambda$.

Lemma 2.1. *If a $P_3(u,\lambda)$ is embedded into a $KS(u+w,\mu)$, then*

$$\mu \geq 2\lambda \, \frac{u-1}{u-1+2w}. \tag{1}$$

Proof. It follows from the trivial inequality $(\mu-\lambda)\binom{u}{2}+\mu uw \geq \lambda\binom{u}{2}$. □

Note that if $\lambda=\mu$, then it follows $w \geq (u-1)/2$ and so the minimum w increases as u increases. Since we want $0 \leq w < 8$, we will assume $\mu > \lambda$.

Lemma 2.2. *For every $u = 8k+h$, with $h = 0,1,4,5$, $u \geq 4$, there exists a $KS(u,2)$ which embeds exactly a $P_3(u,1)$.*

Proof. For every $u = 8k+h$, with $h = 0,1,4,5$, $u \geq 4$, construct as follows a $KS(u,2)$ $\Sigma' = (U,\mathcal{B})$ which embeds a $P_3(u,1)$ $\Sigma = (U,\mathcal{C})$, where $\mathcal{C}$ is the collection of copies of P_3 obtained by considering the laterals of each kite in $\mathcal{B}$.
Case $h=0$. Set $U=\mathbb{Z}_{8k-1}\cup\{\infty\}$ and place in $\mathcal{B}$ the translates of the base blocks $(2+i,\ 4k-1-i,\ 0)-(4k+1+2i)$, for $i=0,1,\dots,2k-2$, and $(1,\ \infty,\ 0)-(4k-1)$.
Case $h=1$. Set $U=\mathbb{Z}_{8k+1}$ and place in $\mathcal{B}$ the translates of the base blocks $(1+i,\ 4k-i,\ 0)-(4k+1+2i)$, for $i=0,1,\dots,2k-1$.
Case $h=4$. Set $U=\mathbb{Z}_{8k+3}\cup\{\infty\}$ and place in $\mathcal{B}$ the translates of the base blocks $(2+i,\ 4k+1-i,\ 0)-(4k+3+2i)$, for $i=0,1,\dots,2k-1$ (only for $k>1$), and $(1,\ \infty,\ 0)-(4k+2)$.
Case $h=5$. Set $U=\mathbb{Z}_{8k+5}$ and place in $\mathcal{B}$ the translates of the base blocks $(1+i,\ 4k+2-i,\ 0)-(4k+3+2i)$, for $i=0,1,\dots,2k$. □

Lemma 2.3. *For every $u \geq 4$, there exists a $KS(u,4)$ which embeds exactly a $P_3(u,2)$.*

Proof. We give a solution by defining a set of base blocks $G_i=(i,-i,0)-2i$, $G_\infty=(0,\infty,-1)-1$ and $H_\infty=(\infty,1,0)-2$ depending on the parity of the integer u.
If u is odd, on $U=\mathbb{Z}_u$ consider the translates of the base blocks $G_i=(i,-i,0)-2i$, for $i=1,2,\dots,\frac{u-1}{2}$, except for the case $u\equiv 3 \pmod 6$, where we take the same G_i by swapping the tails of $G_{\frac{u}{3}}$ and G_1.
If u is even, on $U=\mathbb{Z}_{u-1}\cup\{\infty\}$ consider the translates of the base blocks $G_i=(i,-i,0)-2i$, for $i=2,\dots,\frac{u-2}{2}$ (only for $u\geq 6$), G_∞, H_∞, except for the case $u\equiv 4 \pmod 6$, where we take the same G_i by swapping the tails of $G_{\frac{u-1}{3}}$ and G_2. □

Proposition 2.4. *There exists a $KS(u,\mu)$ which embeds a $P_3(u,\lambda)$ if and only if u,λ,μ are admissible and $\mu \geq 2\lambda$. For $\mu=2\lambda$ the embedding is exact.*

Proof. The necessity follows from Lemma 2.1. Now, we prove the sufficiency. If $\mu = 2\lambda$, then it is sufficient to take λ copies of a $KS(u,2)$ from Lemma 2.2 for $u \equiv 0,1 \pmod 4$ and $\lambda/2$ copies of a $KS(u,4)$ form Lemma 2.3 for $u \equiv 2,3 \pmod 4$ (note that in both cases the embedding is exact). If $\mu > 2\lambda$, by Remark 1.4 it is sufficient to take the union of a copy of a $KS(u,2\lambda)$ which embeds a $P_3(u,\lambda)$ and a copy of a $KS(u,\mu-2\lambda)$. □

Theorem 2.5. *There exists a minimum embedding of a $P_3(u,\lambda)$ into a $KS(u+w,\mu)$, with $\mu > \lambda$, if the conditions in the table below are satisfied.*

λ	$u \pmod 8$	$\mu > \lambda$	w
any	$0,1$	$\mu \geq 2\lambda$	0
any	$4,5$	even, $\mu \geq 2\lambda$	0
even	$2,3,6,7$	$0 \pmod 4$, $\mu \geq 2\lambda$	0
even	$2,3,6,7$	$2 \pmod 4$, $\mu \geq 3\lambda$	$(-u)_4$
even	$2,3,6,7$	odd, $\mu \geq 5\lambda/2$	$(-u)_4$
any	$4,5$	odd, $\mu \geq 5\lambda/2$	$(-u)_8$

Proof. In the first three cases the conclusion follows by Proposition 2.4. For the remaining cases, by means of Proposition 2.4 embed a $P_3(u,\lambda)$ into a $KS(u,2\lambda)$ and by using Theorem 1.1 embed the $KS(u,2\lambda)$ into a $KS(u+w,\mu)$, where $\mu \geq 5\lambda/2$, if μ is odd, or $\mu \geq 3\lambda$, if $\mu \equiv 2 \pmod 4$. □

3. P$_4$-designs

Throughout the section, in order to describe a $KS(u+w,\mu)$ $\Sigma' = (U \cup W, \mathcal{B})$ embedding a $P_4(u,\lambda)$ $\Sigma = (U,\mathcal{C})$ we always denote by $\mathcal{B}_e$ the subcollection of $\mathcal{B}$ such that $f(\mathcal{C}) = \mathcal{B}_e$, where $f : \mathcal{C} \to \mathcal{B}$ is the injective function defined by $f([a,b,c,d]) = (a,b,c) - d$. Note that when $\mathcal{B}_e = \mathcal{B}$, the embedding is exact.

To begin with, we observe that embedding $P_4(u,\lambda)$ into a $KS(u+w,\mu)$ requires $\mu > \lambda$ because to complete the graph P_4 to a kite we need to add an edge joining two vertices of P_4 and so the index must increase. The following lemma gives a lower bound for the index μ.

Lemma 3.1. *If a $P_4(u,\lambda)$ is embedded into a $KS(u+w,\mu)$, then*

$$\mu \geq \frac{4\lambda u(u-1)}{3u(u-1)+3w(w-1)-2uw}. \qquad (2)$$

Proof. Taking into account that if $\Sigma' = (U \cup W, \mathcal{B})$ is a $KS(u+w,\mu)$ which embeds a $P_4(u,\lambda)$, then each block in $\mathcal{B} \setminus \mathcal{B}_e$ contains at most three edges between U and W, it follows

$$\frac{\mu uw}{3} \leq \frac{\mu}{4}\binom{u+w}{2} - \frac{\lambda}{3}\binom{u}{2},$$

which gives the inequality. □

Firstly, we study the embedding of a $P_4(u,\lambda)$ into a $KS(u,\mu)$. In this case ($w=0$), Lemma 3.1 implies $\mu \geq \frac{4}{3}\lambda$.

Lemma 3.2. *For every $u \geq 4$, there exists a $KS(u,4)$ which embeds exactly a $P_4(u,3)$.*

Proof. Consider the kite system given in the proof of Lemma 2.3. □

Lemma 3.3. *For every $u = 12k+h$, with $h = 0,1,4,9$, $u \geq 4$, there exists a $KS(u,2)$ which embeds a $P_4(u,1)$.*

Proof. By Proposition 3.2, it is sufficient to prove the assertion for $l = 0$. For each $u = 12k+h$, $h \in \{0,1,4,9\}$, construct a $KS(u,2)$ $\Sigma' = (U,\mathcal{B})$ where $\mathcal{B}$ is partitioned into the subcollections $\mathcal{B}_e$ and $\mathcal{B}'$ as follows.
Case $h=0$. Let $U = \mathbb{Z}_{12k-1} \cup \{\infty\}$. Place in $\mathcal{B}_e$ the translates of the base blocks $(6k-1-i,\ 2+i,\ 0)-(6k-2-2i)$, for $i=0,1,\dots,2k-2$, and $(\infty,\ 1,\ 0)-(6k-1)$, and apply Lemma 1.2 in order to settle the remaining differences $1,2,\dots,4k$ and obtain $\mathcal{B}'$.
Case $h=1$. Let $U = \mathbb{Z}_{12k+1}$. Place in $\mathcal{B}_e$ the translates of the base blocks $(6k-1-i,\ 2+i,\ 0)-(6k-2-2i)$, for $i=0,1,\dots,2k-2$, and $(6k,\ 1,\ 0)-(6k+1)$, and apply Lemma 1.2 in order to settle the remaining differences $1,2,\dots,4k$ and obtain $\mathcal{B}'$.
Case $h=4$. Let $U = \mathbb{Z}_{12k+4}$. Consider the kites defined as follows

$$G = (3k,\ 1,\ 0)-(9k+4),$$

$$G_i = (9k+2-i,\ 3k+2+i,\ 0)-(6k+1-2i),$$

$$H_i = (3k-1-i,\ 2+i,\ 0)-(3k-2-2i),$$

$$G'_i = (6k+1-i,\ 4k+2+i,\ 0)-(2k-2i),$$

and the two set of kites $\mathcal{K} = \{(6k+2+i,\ 9k+3+i,\ 3k+1+i)-i, (3k+1+i,\ 6k+2+i,\ i)-(9k+3+i) : i=0,1,\dots,3k\}$ and $\mathcal{K}' = \{(i,\ 6k+2+i,\ 9k+3+i)-(3k+1+i) : i=0,1,\dots,3k\}$. Now, construct $\mathcal{B}_e$ and $\mathcal{B}'$ as follows:

- for $k=0$, put $\mathcal{B}_e = \mathcal{K}$ and $\mathcal{B}' = \mathcal{K}'$;

- for $k=1$, place in $\mathcal{B}_e$ the translates of the base blocks G and G_0 along with the kites of $\mathcal{K}$ and in $\mathcal{B}'$ the translates of G'_0 along with the kites of $\mathcal{K}'$;

- for $k \geq 2$, place in $\mathcal{B}_e$ the translates of the base blocks G, G_i, for $i = 0,1,\ldots,k-1$, and H_i, for $i = 0,1,\ldots,k-2$, along with the kites of $\mathcal{K}$ and in $\mathcal{B}'$ the translates of the base blocks G'_i, for $i = 0,1,\ldots,k-1$, along with the kites of $\mathcal{K}'$.

Case $h = 9$. Let $U = \mathbb{Z}_{12k+8} \cup \{\infty\}$. Consider the kites defined as follows

$$G_\infty = (\infty,\ 0,\ 3k+1) - (9k+4),$$

$$G_i = (3k+1-i,\ 1+i,\ 0) - (3k-1-2i),$$

$$H_i = (9k+5-i,\ 3k+3+i,\ 0) - (6k+1-2i),$$

$$G'_i = (6k+2-i,\ 4k+3+i,\ 0) - (2k-2i),$$

$$G' = (5k+3,\ 1,\ 0) - (6k+3),$$

$$H' = (5k+4,\ 3,\ 0) - (2k+1),$$

and the set of kites $\mathcal{K} = \{(6k+4+i,\ 9k+6+i,\ 3k+2+i) - i, (3k+2+i,\ 6k+4+i,\ i) - (9k+6+i) : i = 0,1,\ldots,3k+1\}$, $\mathcal{K}'_1 = \{(i,\ 6k+4+i,\ 9k+6+i) - (3k+2+i) : i = 0,1,\ldots,3k+1\}$ and $\mathcal{K}'_2 = \{(4i,\ 4+4i,\ 2+4i) - (6+4i), (1+4i,\ 5+4i,\ 3+4i) - (7+4i) : i = 0,1,\ldots,3k+1\}$. Now, construct $\mathcal{B}_e$ and $\mathcal{B}'$ as follows:

- for $k = 0$, place in $\mathcal{B}_e$ the translates of G_∞, along with the kites of $\mathcal{K}$, and take $\mathcal{B}' = \{(1+i,4+i,i) - (5+i) : i = 0,1,2,3\} \cup \{(4,6,5) - 7, (6,0,7) - 1\}$;

- for $k = 1$, place in $\mathcal{B}_e$ the translates of the base blocks G_∞, G_0 and H_0, along with the kites of $\mathcal{K}$, and take $\mathcal{B}' = \mathcal{K}'_1 \cup \{(2+2i, 9+2i, 1+2i) - (10+2i), (2i, 3+2i, 1+2i) - (8+2i), (8+2i, 17+2i, 2i) - (2+2i) : i = 0,1,2,\ldots 9\}$;

- for $k \geq 2$, place in $\mathcal{B}_e$ the translates of the base blocks G_∞, G_i and H_i, for $i = 0,1,\ldots,k-1$, along with the kites of $\mathcal{K}$, and place in $\mathcal{B}'$ the translates of the base blocks G'_i, for $i = 0,1,\ldots,k-3$ (only for $k \geq 3$), G' and H', along with the kites of $\mathcal{K}'_1 \cup \mathcal{K}'_2$.

□

Lemma 3.4. *For every $u = 24k+h$, with $h = 0,1,9,16$, $u \geq 9$, there exists a $KS(u,3)$ which embeds a $P_4(u,2)$.*

Proof. By Proposition 3.2, it is sufficient to prove the assertion for $l = 0$. For each $u = 24k+h$, $h \in \{0,1,9,16\}$, construct a $KS(u,3)$ $\Sigma' = (U, \mathcal{B})$ where $\mathcal{B}$ is partitioned into the subcollections $\mathcal{B}_e$ and $\mathcal{B}'$ as follows.

Case $h=0$. Let $U=\mathbb{Z}_{24k-1}\cup\{\infty\}$. Place in $\mathcal{B}_e$ the translates of the base blocks $(12k-2-i,\ 1+i,\ 0)-(12k-4-2i)$, for $i=0,1,\ldots,4k-2$, $(4k+1+i,\ 12k-2-i,\ 0)-(8k-2-2i)$, for $i=0,1,\ldots,4k-3$, $(8k-1,\ 8k,\ 0)-(12k-1)$, $(0,\ 12k+1,\ 12k-1)-4k$, and $(12k-1,\ 0,\ \infty)-1$, and apply Lemma 1.2 in order to settle the remaining differences $1,2,\ldots,4k$ and obtain $\mathcal{B}'$.
Case $h=1$. Let $U=\mathbb{Z}_{24k+1}$. Place in $\mathcal{B}_e$ the translates of the base blocks $(4k+1+i,\ 12k-i,\ 0)-(8k-2i)$, for $i=0,1,\ldots,4k-1$, $(12k-1-i,\ 2+i,\ 0)-(12k-2-2i)$, for $i=0,1,\ldots,4k-2$, and $(12k,\ 1,\ 0)-(12k+1)$, and apply Lemma 1.2 in order to settle the remaining differences $1,2,\ldots,4k$ and obtain $\mathcal{B}'$.
Case $h=9$. Let $U=\mathbb{Z}_{24k+9}$. Consider the kites defined as follows

$$G=(12k+5,\ 0,\ 12k+4)-(4k+1),$$
$$H=(12k+4,\ 1,\ 0)-(12k+2),$$
$$G_i=(12k+2-i,\ 2+i,\ 0)-(12k+1-2i),$$
$$H_i=(4k+2+i,\ 12k+3-i,\ 0)-(8k+2-2i),$$
$$G'_i=(4k-i,\ 2k+1+i,\ 0)-(2k-2i),$$
$$G'=(3k-1,\ 3k+2,\ 0)-3k,$$
$$H'=(12k+3,\ 4k+1,\ 0)-(3k+1),$$

and the two set of kites $\mathcal{K}=\{(3i,1+3i,2+3i)-(3+3i),(3i,2+3i,4+3i)-(6+3i):i=0,1,\ldots,8k+2\}$ and $\mathcal{K}'=\{(2+3i,6+3i,4+3i)-(8+3i):i=0,1,\ldots,8k+2\}$. Now, construct $\mathcal{B}_e$ and $\mathcal{B}'$ as follows:

- for $k=0$, place in $\mathcal{B}_e$ the translates of the base blocks G and $(3,1,0)-6$, along with the kites of $\mathcal{K}$, and take $\mathcal{B}'=\mathcal{K}'$;

- for $k=1$, place in $\mathcal{B}_e$ the translates of the base blocks G_0, H_0, along with the kites of $\mathcal{K}$, and place in $\mathcal{B}'$ the translates of $(15,5,0)-3$, along with the kites of $\mathcal{K}'$;

- for $k\geq 2$, place in $\mathcal{B}_e$ the translates of the base blocks G, H, G_i and H_i, for $i=0,1,\ldots,4k-1$, along with the kites of $\mathcal{K}$, and place in $\mathcal{B}'$ the translates of the base blocks G'_i, for $i=0,1,\ldots,k-3$ (only for $k\geq 3$), G' and H', along with the kites of $\mathcal{K}'$.

Case $h=16$. Let $U=\mathbb{Z}_{24k+15}\cup\{\infty\}$. Consider the kites defined as follows

$$G_\infty=(12k+6,\ \infty,\ 0)-(12k+7),$$
$$G=(8k+4,\ 8k+5,\ 0)-(12k+7),$$

$H = (12k+7,\ 2,\ 0) - (12k+9),$

$G_i = (12k+5-i,\ 2+i,\ 0) - (12k+4-2i),$

$H_i = (4k+3+i,\ 12k+6-i,\ 0) - (8k+4-2i),$

$G'_i = (4k-i,\ 2k+1+i,\ 0) - (2k-2i),$

$G' = (3k-1,\ 3k+2,\ 0) - (4k+1),$

$H' = (3k,\ 3k+1,\ 0) - (4k+2),$

and the two set of kites $\mathcal{K} = \{(3i, 1+3i, 2+3i) - (3+3i) : i = 0, 1, \ldots, 8k+4\}$ and $\mathcal{K}' = \{(2+3i, 6+3i, 4+3i) - (8+3i), (3i,\ 4+3i,\ \infty) - (2+3i) : i = 0, 1, \ldots, 8k+4\}$. Now, construct $\mathcal{B}_e$ and $\mathcal{B}'$ as follows:

- for $k = 0$, place in $\mathcal{B}_e$ the translates of the base blocks H, G_0, H_0, $(6, 5, 0) - 7$ and $(1, \infty,\ 0) - 7$ along with the kites of $\mathcal{K}$, and take $\mathcal{B}' = \mathcal{K}'$;
- for $k = 1$, place in $\mathcal{B}_e$ the translates of the base blocks G_∞, G, H, G_i and H_i, for $i = 0, 1, \ldots, 4$, along with the kites of $\mathcal{K}$, and place in $\mathcal{B}'$ the translates of $(6, 1, 0) - 3$, along with the kites of $\mathcal{K}'$;
- for $k \geq 2$, place in $\mathcal{B}_e$ the translates of the base blocks G_∞, G, H, G_i and H_i, for $i = 0, 1, \ldots, 4k$, along with the kites of $\mathcal{K}$, and place in $\mathcal{B}'$ the translates of the base blocks G'_i, for $i = 0, 1, \ldots, k-3$ (only for $k \geq 3$), G' and H', along with the kites of $\mathcal{K}'$.

□

Lemma 3.5. *For every $u = 6k+h$, with $h = 0, 1, 3, 4$, $u \geq 4$, there exists a $KS(u, 4)$ which embeds a $P_4(u, 1)$.*

Proof. By Proposition 3.2, it is sufficient to prove the assertion for $l = 0$. For each $u = 6k+h$, $h \in \{0, 1, 3, 4\}$, construct a $KS(u, 4)$ $\Sigma' = (U, \mathcal{B})$ where $\mathcal{B}$ is partitioned into the subcollections $\mathcal{B}_e$ and $\mathcal{B}'$ as follows.
Case $h = 0$. Let $U = \mathbb{Z}_{6k-1} \cup \{\infty\}$. Place in $\mathcal{B}_e$ the translates of the base blocks $(3k-1-i,\ 2+i,\ 0) - (3k-2-2i)$, for $i = 0, 1, \ldots, k-2$, and $(\infty,\ 1,\ 0) - (3k-1)$. In order to obtain $\mathcal{B}'$ duplicate $\mathcal{B}_e$ and apply Lemma 1.3 to settle the remaining differences $1, 2, \ldots, 2k$.
Case $h = 1$. Let $U = \mathbb{Z}_{6k+1}$. Place in $\mathcal{B}_e$ the translates of the base blocks $(3k-1-i,\ 2+i,\ 0) - (3k-2-2i)$, for $i = 0, 1, \ldots, k-2$ if $k \geq 2$, and $(3k,\ 1,\ 0) - (3k+1)$. In order to obtain $\mathcal{B}'$ duplicate $\mathcal{B}_e$ and apply Lemma 1.3 to settle the remaining differences $1, 2, \ldots, 2k$.
Case $h = 3$. Let $U = \mathbb{Z}_{6k+3}$. Place in $\mathcal{B}_e$ the translates of the base blocks $(3k+2-i,\ 2+i,\ 0) - (3k+1-2i)$, for $i = 0, 1, \ldots, k-1$, along with the kites $(3i, 1+$

$3i, 2+3i) - (3+3i)$, $i = 0, 1, \dots, 2k$. Now, place in $\mathcal{B}'$ the translates of the base blocks $(3k+1-i,\ 3+i,\ 0) - (3k-1-2i)$, $(2k-i,\ 2+i,\ 0) - (4k+4+2i)$, for $i = 0, 1, \dots, k-2$ (only for $k \geq 2$), $(2k+1,\ 2k+2,\ 0) - (3k+3)$ and $(2k,\ 4k+1,\ 0) - (5k+2)$, along with the kites $(1+3i, 2+3i, 3+3i) - (4+3i)$ and $(2+3i, 3+3i, 4+3i) - (5+3i)$, $i = 0, 1, \dots, 2k$.
Case $h = 4$. Let $U = \mathbb{Z}_{6k+3} \cup \{\infty\}$. Place in $\mathcal{B}_e$ the translates of the base blocks $(3k+2-i,\ 2+i,\ 0) - (3k+1-2i)$, for $i = 0, 1, \dots, k-1$ (only for $k \geq 1$), along with the kites $(2+3i, \infty, 3i) - (1+3i)$ and $(3+3i, 2+3i, 1+3i) - \infty$, $i = 0, 1, \dots, 2k$. Now, place in $\mathcal{B}'$ the translates of the base blocks $(3k+1-i,\ 3+i,\ 0) - (3k-1-2i)$, for $i = 0, 1, \dots, k-2$ (only for $k \geq 2$), $(2k+1,\ 2k+2,\ 0) - (3k+3)$ (only for $k \geq 1$) and $(\infty,\ 2k+1,\ 0) - (2k+2)$, along with the kites $(2+3i, 4+3i, \infty) - 3i$, $i = 0, 1, \dots, 2k$, and finally apply Lemma 1.3 to settle the remaining differences $1, 2, \dots, 2k$. □

Lemma 3.6. *For every* $u = 6k+h$, *with* $h = 0, 1, 3, 4$, $u \geq 4$, *there exists a* $KS(u, 4)$ *which embeds a* $P_4(u, 2)$.

Proof. By Proposition 3.2, it is sufficient to prove the assertion for $l = 0$. For each $u = 6k+h$, $h \in \{0, 1, 3, 4\}$, construct a $KS(u, 4)$ $\Sigma' = (U, \mathcal{B})$ where $\mathcal{B}$ is partitioned into the subcollections $\mathcal{B}_e$ and $\mathcal{B}'$ as follows.
Case $h = 0$. Let $U = \mathbb{Z}_{6k-1} \cup \{\infty\}$. Place in $\mathcal{B}_e$ the translates, repeated twice, of the base blocks $(3k-1-i,\ 2+i,\ 0) - (3k-2-2i)$, for $i = 0, 1, \dots, k-2$ (only for $k \geq 2$), and $(\infty,\ 1,\ 0) - (3k-1)$. In order to settle the remaining differences $1, 2, \dots, 2k$ and obtain $\mathcal{B}'$ apply Lemma 1.3.
Case $h = 1$. Let $U = \mathbb{Z}_{6k+1}$. Place in $\mathcal{B}_e$ the translates, repeated twice, of the base blocks $(3k-1-i,\ 2+i,\ 0) - (3k-2-2i)$, for $i = 0, 1, \dots, k-2$ (only for $k \geq 2$), and $(3k,\ 1,\ 0) - (3k+1)$. In order to settle the remaining differences $1, 2, \dots, 2k$ and obtain $\mathcal{B}'$ apply Lemma 1.3.
Case $h = 3$. Let $U = \mathbb{Z}_{6k+3}$. Consider the kites defined as follows

$$G = (3, 3k+1, 0) - (3k-1),$$

$$H = (2, 0, 3) - (3k+3),$$

$$G_i = (3k+2-i,\ 2+i,\ 0) - (3k+1-2i),$$

$$H_i = (3k-i,\ 4+i,\ 0) - (3k-3-2i),$$

$$G'_i = (2k-i,\ 2+i,\ 0) - (4k+4+2i),$$

$$G' = (k, 2k+2, 0) - (2k+1),$$

$$H' = (2k,\ 4k+1,\ 0) - (k+1),$$

$$K' = (k-1, k+3, 0) - (6k-2),$$

and the two set of kites $\mathcal{K} = \{(3i, 1+3i, 2+3i) - (3+3i), (1+3i, 2+3i, 3+3i) - (4+3i) : i = 0, 1, \ldots, 2k\}$ and $\mathcal{K}' = \{(2+3i, 3+3i, 4+3i) - (5+3i) : i = 0, 1, \ldots, 2k\}$. Now, construct $\mathcal{B}_e$ and $\mathcal{B}'$ as follows:

- for $k = 1$, place in $\mathcal{B}_e$ the translates of the base blocks G_0 and $(2,0,3) - 7$, along with the kites of $\mathcal{K}$, and place in $\mathcal{B}'$ the translates of $(2,5,0) - 3$, along with the kites of $\mathcal{K}'$;

- for $k = 2$, place in $\mathcal{B}_e$ the translates of the base blocks G, H, G_0 and G_1, along with the kites of $\mathcal{K}$, and place in $\mathcal{B}'$ the translates of G' and H', along with the kites of $\mathcal{K}'$;

- for $k \geq 3$, place in $\mathcal{B}_e$ the translates of the base blocks G, H, G_i, for $i = 0, 1, \ldots, k-1$, H_i, for $i = 0, 1, \ldots, k-3$, along with the kites of $\mathcal{K}$, and place in $\mathcal{B}'$ the translates of the base blocks G'_i, for $i = 0, 1, \ldots, k-4$ (only for $k \geq 4$), G', H' and K', along with the kites of $\mathcal{K}'$.

Case $h = 4$. Let $U = \mathbb{Z}_{6k+3} \cup \{\infty\}$. Consider the kites defined as follows

$$G = (6k+1, 3k, 0) - (3k+1),$$

$$H = (3k+1, 3k, 0) - (3k-1),$$

$$G_\infty = (3k, \infty, 0) - (3k-1),$$

$$G_i = (3k-1-i,\ 2+i,\ 0) - (3k-2-2i),$$

$$G'_i = (2k-i,\ 3+i,\ 0) - (2k-2-2i),$$

$$G' = (3k+1, k+2, 0) - (k+1),$$

$$G'_\infty = (3k, \infty, 0) - 2k,$$

and the two set of kites $\mathcal{K} = \{(3i, 1+3i, 2+3i) - (3+3i) : i = 0, 1, \ldots, 2k\}$ and $\mathcal{K}' = \{(1+3i, 2+3i, 3+3i) - (4+3i), (2+3i, 3+3i, 4+3i) - (5+3i) : i = 0, 1, \ldots, 2k\}$. Now, construct $\mathcal{B}_e$ and $\mathcal{B}'$ as follows:

- for $k = 0$, let $\mathcal{B}_e = \{(2,1,0) - \infty, (0,2,1) - \infty, (1,0,\infty) - 2, (0,2,\infty) - 1\}$ and $\mathcal{B}' = \{(1,0,\infty) - 2, (2,1,\infty) - 0\}$;

- for $k = 1$, place in $\mathcal{B}_e$ the translates of the base blocks G, H and G_∞, along with the kites of $\mathcal{K}$, and place in $\mathcal{B}'$ the translates of $(3,\infty,0) - 4$, along with the kites of $\mathcal{K}'$;

- for $k = 2$, place in $\mathcal{B}_e$ the translates of the base blocks G_0 (repeated twice), G, H and G_∞, along with the kites of $\mathcal{K}$, and place in $\mathcal{B}'$ the translates of G' and H', along with the kites of $\mathcal{K}'$;

- for $k \geq 3$, place in $\mathcal{B}_e$ the translates of the base blocks G_i, for $i = 0, 1, \ldots, k-2$ (repeated twice), G, H and G_∞, along with the kites of $\mathcal{K}$, and place in $\mathcal{B}'$ the translates of the base blocks G'_i, for $i = 0, 1, \ldots, k-3$ (only for $k \geq 3$), G' and H', along with the kites of $\mathcal{K}'$.

□

Proposition 3.7. *For every $l \geq 1$ and $u \geq 4$, there exists a $KS(u, 4l)$ which embeds exactly a $P_4(u, 3l)$.*

Proof. It follows by Lemma 3.2 and Remark 1.4. □

Theorem 3.8. *There exists a minimum embedding of a $P_4(u, \lambda)$ into a $KS(u + w, \mu)$ if the conditions in the table below are satisfied.*

λ	$u \pmod{24}$	$\mu \geq \frac{4}{3}\lambda$	w
$3l$	$8, 17$	*any*	0
$3l$	$5, 20$	even	0
$3l$	$2, 11, 14, 23$	$0 \pmod 4$	0
any	$0, 1, 9, 16$	*any*	0
any	$4, 12, 13, 21$	even	0
$3l+1, 3l+2$	$3, 6, 7, 10, 15, 18, 19, 22$	$0 \pmod 4$	0
$3l$	$2, 11, 14, 23$	$2 \pmod 4$, $\mu \geq 6l$	$(-u)_4$
$3l+1, 3l+2$	$3, 6, 7, 10, 15, 18, 19, 22$	$2 \pmod 4$, $\mu \geq 6l+6$	$(-u)_4$
$3l$	$2, 4, 5, 11, 12, 13, 14, 20, 21, 23$	odd, $\mu \geq 5l$	$(-u)_8$
$3l+1$	$4, 12, 13, 21$	odd, $\mu \geq 5l+3$	$(-u)_8$
$3l+2$	$4, 12, 13, 21$	odd, $\mu \geq 5l+5$	$(-u)_8$
$3l+1, 3l+2$	$3, 6, 7, 10, 15, 18, 19, 22$	odd, $\mu \geq 5l+5$	$(-u)_8$

Proof. Firstly, we settle the cases where $w = 0$. For $\lambda \equiv 0 \pmod 3$, they follow by Proposition 3.7. For $\lambda \not\equiv 0 \pmod 3$, by Remark 1.4 it will be sufficient to prove that there exists a $P_4(u, \lambda)$ embedded into a $KS(u, \bar{\mu})$, where $\bar{\mu}$ is the minimum admissible index $\mu \geq \frac{4}{3}\lambda$. We distinguish three cases depending on the order $u \equiv 0, 1 \pmod 3$.
Case $u \equiv 0, 1, 9, 16 \pmod{24}$. If $\lambda = 3l + 1$, then $\bar{\mu} = 4l + 2$ and we can paste a $KS(u, 2)$ from Lemma 3.3 with a $KS(u, 4l)$ from Proposition 3.7. If $\lambda = 3l + 2$, then $\bar{\mu} = 4l + 3$ and we can apply Lemma 3.4 and Proposition 3.7.
Case $u \equiv 4, 12, 13, 21 \pmod{24}$. Here the index $\bar{\mu}$ must be even for the existence of a $KS(u, \bar{\mu})$. Therefore, if $\lambda = 3l + 1$ (or $3l + 2$), then $\bar{\mu} = 4l + 2$ (or $4l + 4$, respectively) and we can paste a $KS(u, 2)$ from Lemma 3.3 (or a $KS(u, 4)$ from Lemma 3.6, respectively) with a $KS(u, 4l)$ from Proposition 3.7.
Case $u \equiv 3, 6, 7, 10 \pmod{12}$. In this case $\bar{\mu}$ must be doubly even for the existence of a $KS(u, \bar{\mu})$. Therefore, if $\lambda = 3l + 1, 3l + 2$, then $\bar{\mu} = 4l + 4$ and we

can paste a $KS(u,4)$ from Lemma 3.5 or 3.6 (depending on whether $\lambda = 3l+1$ or $\lambda = 3l+2$, respectively) with a $KS(u,4l)$ from Proposition 3.7.
For the remaining cases ($w \neq 0$), embed a $P_4(u,\lambda)$ into a $KS(u,\bar{\mu})$, where $\bar{\mu}$ is the minimum admissible index $\mu \geq \frac{4}{3}\lambda$, and by means of Theorem 1.1 embed the $KS(u,\bar{\mu})$ into a $KS(u+w,\mu)$, where $\mu \geq 5\bar{\mu}/4$, if μ is odd, or $\mu \geq 3\bar{\mu}/2$, if $\mu \equiv 2 \pmod 4$. □

4. Conclusion

By Propositions 2.4 and 3.7, the minimum embedding problem for path designs $P_k(u,\lambda)$, $k = 3,4$, into kite systems $KS(u,\mu)$ ($w = 0$) is completely solved, i.e., the necessary and sufficient conditions are determined. For $w \neq 0$, a complete solution to our problem is not reached because in the proof of Theorems 2.5 and 3.8 we make use of Theorem 1.1, which does not give a complete solution to the minimum embedding problem for a $KS(u,\lambda)$ into a $KS(u+w,\mu)$ and leaves some cases open (see Conclusion section in [11]). For the cases when $w \neq 0$, we conjecture that there exists a minimum embedding of a $P_k(u,\lambda)$, $k = 3,4$, into a $KS(u+w,\mu)$ with $\mu > \lambda$ and $w = (-u)_n$, $n = \frac{8}{gcd(\mu,4)}$, for every admissible u, λ and μ satisfying the inequality 1 or 2, respectively.

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