LE MATEMATICHE
Vol. LV (2000) – Fasc. II, pp. 437–446

PROJECTIVE MODULI SPACE OF SEMISTABLE PRINCIPAL SHEAVES FOR A REDUCTIVE GROUP

TOMÁS L. GÓMEZ - IGNACIO SOLS

Dedicated to Silvio Greco in occasion of his 60-th birthday.

1. Introduction.

This contribution to the homage to Silvio Greco is mainly an announcement of results to appear somewhere in full extent, explaining their development from our previous article [5] on conic bundles.

In [11] and [15] Narasimhan and Seshadri defined stable bundles on a curve and provided by the techniques of Geometric Invariant Theory (GIT) developed by Mumford [10] a projective moduli space of the stable equivalence classes of semistable bundles. Then Gieseker [4] and Maruyama [8] [9] generalized this construction to the case of a higher-dimensional projective variety, obtaining again a projective moduli space by also allowing torsion-free sheaves. Ramanathan [12] [13] has provided the moduli space of semistable principal bundles on a connected reductive group G, thus generalizing the Narasimhan and Seshadri notion and construction, which then becomes the particular case $G = Gl(n, \mathbb{C})$.

Faltings [3] has considered the moduli stack of principal bundles on semistable curves. For G orthogonal or symplectic he considers a torsion-free

Both authors are members of EAGER (EC FP5 Contract no. HPRN-CT-2000-00099). T. G. was supported by a postdoctoral fellowship of Ministerio de Educación y Cultura (Spain)

sheaf with a quadratic form, and he also defines a notion of stability. For general reductive group G he uses the approach of loop groups. Sorger [19] had considered a similar problem. He works on a curve C (not necessarily smooth) on a smooth surface S, and constructs the moduli space of torsion free sheaves on C together with a symmetric form taking values on the dualizing sheaf ω_C.

In the talk "open problems on principal bundles" closing the conference on "vector bundles on algebraic curves and Brill-Noether theory" at Bad Honeff 2000, prof. Narasimhan proposed the problem of generalizing the work of the late Ramanathan to the case of higher-dimensional varieties and to the case of positive characteristics. We solve the first problem by providing a suitable definition of principal sheaf on a higher-dimensional projective variety X over the complex field, and a definition of its (semi)stability, which in case $\dim X = 1$ is that of Ramanathan, and for which a projective moduli space can be obtained.

We start by recalling our notion of (semi)stable conic bundles, i.e. symmetric $(2,0)$-tensors $\varphi : E \otimes E \to \mathcal{O}_C$ of rank $E = 3$ on an algebraic curve C, and their projective moduli space, notion and moduli space which have been generalized to the case of $(s,0)$ tensors on a curve by Schmitt [14] with the purpose of dealing with (semi)stable objects $\varphi : E^\rho \to M$, where E^ρ is the vector bundle associated to a vector bundle E and an arbitrary representation ρ of $G = Gl(n,\mathbb{C})$, and M is a line bundle. In case the symmetric $(2,0)$ tensor is of maximal rank at all points and $\det(E) \cong \mathcal{O}_C$, i.e. the case when (E,φ) is just a principal $SO(3,\mathbb{C})$-bundle, our notion of (semi)stability is drastically simplified and becomes equivalent to Ramanathan's notion of (semi)stability. We then generalize to higher dimension, with techniques of Simpson [18] and Huybrechts-Lehn [7], the notion and coarse projective moduli space of (semi)stable $(s,0)$-tensors, by allowing E to be a torsion free sheaf and those symmetric or antisymmetric and nowhere degenerate provide thus the moduli space of principal sheaves on $G = O(n,\mathbb{C})$, $Sp(n,\mathbb{C})$, $SO(2n+1,\mathbb{C})$, the remaining classical group $SO(2n,\mathbb{C})$ requiring a special treatment which fortunately does not alter the notion of (semi)stability.

Then we cope with the problem of an arbitrary connected reductive group G, by defining principal sheaves as $(2,1)$ tensors, i.e. torsion free sheaves E and $\varphi : E \otimes E \to E^{**}$, which on the points of the open set U_E where E is locally free are isomorphic to the structure tensor $\varphi_{\mathfrak{g}'} : \mathfrak{g}' \otimes \mathfrak{g}' \to \mathfrak{g}'$ of the Lie algebra $\mathfrak{g}'$ tangent to the commutator $G' = [G,G]$, together with a $G \to Aut(\mathfrak{g}')$ reduction of the associated principal bundle on U_E. The (semi)stability is defined as the δ-(semi)stability of the $(2,1)$ tensor (E,φ) for a polynomial parameter δ with degree exactly $\dim X - 1$, and it leads to a coarse projective moduli space. Furthermore, it reduces to Ramanathan's (semi)stability and moduli space in case $\dim X = 1$.

This announcement note consists mainly of the precise definitions and statements of such objects and results.

2. Conic bundles.

Let X be a complete, smooth, connected curve, and fix a positive rational $\tau > 0$. A conic bundle on X of degree d is a rank 3 symmetric $(2,0)$-tensor on X, i.e. a vector bundle E of rank 3 and degree d, together with a nonzero homomorphism

$$\varphi : E^2 = Sym^2 E \to L$$

Where L is a line bundle. We say it is (semi)stable if

1) For any subbundle $F \subseteq E$, it is

$$\frac{\deg F - c_\varphi(F)\tau}{\operatorname{rank} F} \; (\leq) \; \frac{\deg E - 2\tau}{\operatorname{rank} E}$$

where

$$c_\varphi(F) = \begin{cases} 2, \text{ if } \varphi(F^2) \neq 0 \text{ (i.e. } F \text{ not isotropic)} \\ 1, \text{ if } \varphi(F^2) = 0 \text{ and } \varphi(FE) \neq 0 \\ 0, \text{ if } \varphi(FE) = 0 \text{ (i.e. } F \text{ singular)} \end{cases}$$

2) For all critical flags $F_1 \subseteq F_2 \subseteq E$,

$$\deg F_1 + \deg F_2 \; (\leq) \; \deg E,$$

where a critical flag is a flag with F_1 isotropic of rank 1, F_2 of rank 2, $\varphi(F_1 F) \neq 0$, $\varphi(F_2 F_2) \neq 0$ and $\varphi(F_1 F_2) = 0$ (at a general point of X these are a point of the conic and its tangent line). In other words, choosing a basis adapted to the filtration on the fiber of E over a general point, the matrix form of φ is of the form

$$\begin{pmatrix} 0 & 0 & \times \\ 0 & \times & \cdot \\ \times & \cdot & \cdot \end{pmatrix}$$

where "$\times$" is a nonzero entry, and "$\cdot$" is arbitrary.

By the expression (semi)stable we always mean both semistable and stable, and then by the symbol $(\leq)$ we mean $\leq$ and $<$, respectively. As usual, there is a notion of stable equivalence classes of semistable objects (see [5] for the definition), and then it is proved in [5], by the use of GIT, the following.

Theorem 1. *There is a projective coarse moduli space of stable equivalence classes of semistable conic bundles of degree d and parameter τ, on a smooth, complete, connected curve.*

If $\det(E) \cong \mathcal{O}_X$, $L \cong \mathcal{O}_X$ and φ is nowhere degenerate, i.e. such that rank $\varphi(x) = 3$ for all $x \in X$, which amounts to a *principal* $SO(3)-$ *bundle on* X, then the condition 2), independent of the parameter τ is enough for the definition of (semi)stability, thus leading to a projective coarse moduli space as in Theorem 1.

Recall that in [12], [13], a definition of (semi)stable principal bundle P on a curve, for a connected, reductive group G was already given: if for all reduction $P(H)$ of P to a maximal parabolic subgroup $H \subseteq G$, the vector bundle $P(H, \mathfrak{h})$ associated to P by the adjoint representation of H in its tangent Lie algebra $\mathfrak{h}$, has

$$\deg P(H, \mathfrak{h}) \ (\leq) \ 0$$

In fact Ramanathan obtains in [13] a projective coarse moduli space of stable equivalence classes of semistable principal G-bundles of fixed topological type and our result for $SO(3)$-bundles on X becomes a particular case of Ramanathan's result, because it is proved in [5] that condition 2 is equivalent to the notion of Ramanathan.

Rank 2 bundles correspond, after projectivization, to geometrically ruled surfaces, and properties of the (semi)stable objects have been largely studied since their definition in [11] and [15]. Our definition of (semi)stable conic bundles opens analogous problems. For instance we would like to express here the following *conjecture*. It has been proved in [2], for a semistable scroll of $\mathbb{P}^r$ of degree d and irregularity q which is special (i.e. r distinct from the Riemann-Roch number $d + 1 - 2q$), the existence of a hyperplane containing $r - 1$ lines of the ruling, which amounts to the upper bound $d - (r - 1)$ for the degree of a unisecant curve of the ruled surface, a problem posed by Severi in [17] (the analogous bound being trivial in the nonsemistable case). Most probably, for a special semistable conic bundle of $\mathbb{P}^r$ there is a hyperplane containing $\left[\frac{r-2}{2}\right]$ of its conics, thus leading to an analogous upper bound of the minimal degree of a bisecant curve of the surface (and so on).

3. Principal sheaves for a classical group.

Let X be a smooth, projective complex variety of dimension n.

Definition 2. *A tensor field, or just a tensor, on X, is a pair (E, φ) consisting* *of a torsion free sheaf E and an homomorphism*

$$\varphi : \otimes^s E \to \mathcal{O}_X,$$

the rank and Chern classes of the tensor being called those of E. Let δ be a positive rational polynomial of degree at most $n-1$ (i.e. rational coefficients, and positive leading coefficient). The tensor is said to be δ-(semi)stable if for all weighted filtration (E., m.) of E, i.e. subsheaves $E_1 \subset \ldots \subset E_t \subset E_{t+1} = E$ and positive integers $m_1, \ldots, m_t$, it is

$$\sum m_i (r\chi_{E_i} - r_i \chi E) + \delta\ \mu(E., m., \varphi)(\leq)0$$

where $r, r_i, \chi_E, \chi_{E_i}$ are the ranks and Hilbert polynomials of E, E_i, and μ is defined as

$$\mu = \min\{\lambda_{i_1} + \ldots + \lambda_{i_s} | \varphi(E_{i_1} \otimes \ldots \otimes E_{i_s}) \neq 0\}$$

where $\lambda_1 < \ldots < \lambda_s$ are integers with $\lambda_i - \lambda_{i-1} = m_i r$ and

$$\sum \lambda_i rank(E_i/E_{i-1}) = 0.$$

In [6] the definition is slightly more general, and we prove the following

Theorem 3. *There is a coarse projective moduli space of δ-stable equivalence classes of δ-semistable tensors on a projective variety X, of fixed Chern classes and rank.*

The proof has two parts: first, show that the family consisting of such objects is bounded (remark that for δ-semistable (E, φ), the torsion free sheaf E needs not be semistable). Second, proceed with the techniques of Simpson [15] and Huybrechts-Lehn [7], starting by considering an integer $m \gg 0$ such that all torsion free sheaves in the family are generated by global sections and have $H^0(E(m)) = \chi_E(m)$. For each member of the family choose an isomorphism β of $H^0(E(m))$ with a fixed complex vector space V of dimension $\chi(E(m))$, thus obtaining a quotient

$$V \otimes \mathcal{O}_X(-m) \simeq H^0(E(m)) \otimes \mathcal{O}_X(-m) \longrightarrow E$$

inducing, for l high enough, a quotient

$$q : V \otimes H^0(\mathcal{O}_X(l-m)) \longrightarrow H^0(E(l)).$$

Consider also the induced homomorphism

$$\psi : V^{\otimes s} \longrightarrow H^0(E(m)^{\otimes s}) \longrightarrow H^0(\mathcal{O}_X(sm))$$

We then obtain an element of

$$\mathbb{P}\Big(\bigwedge\nolimits^{\chi_E(l)}(V^* \otimes H^0(\mathcal{O}_X(l-m))^*)\Big) \times \mathbb{P}\Big(V^{*\otimes s} \otimes H^0(\mathcal{O}_X(sm)\Big)$$

which we consider included in projective space by the linear system $|\mathcal{O}(n_1, n_2)|$ with

$$\frac{n_2}{n_1} = \frac{\chi_E(l)\delta(m) - \delta(l)\chi_E(m)}{\chi_E(m) - s\delta(m)}$$

This assignation embeds in a projective space $\mathbb{P}$ the scheme R of triples (E, φ, β), with (E, φ) being a δ-semistable tensor of the given rank and Chern classes and β a choice of basis as above. Quotienting by GIT with the natural action of $Sl(V)$ on R, induced from its natural action on $\mathbb{P}$, we obtain the wanted projective coarse moduli space.

Definition 4. *Let $G = O(r, \mathbb{C})$ or $Sp(r, \mathbb{C})$. A principal G-sheaf on X is a tensor $\varphi : E \otimes E \longrightarrow \mathcal{O}_X$ symmetric or antisymmetric which induces an isomorphism $E_{|U} \longrightarrow E^*_{|U}$ on the open set U where E is locally free. We call it (semi)stable if for all isotropic subsheaves $F \subseteq E$ it is*

$$\chi_F + \chi_{F^\perp} (\leq) \chi_E$$

Theorem 5. *For any positive polynomial δ of degree exactly $n-1$, a principal G-sheaf on X ($G = O(r, \mathbb{C})$) or $Sp(r, \mathbb{C})$ is δ-(semi)stable if and only if it is (semi)stable, so there is a coarse projective moduli space of stable-equivalence classes of semistable principal G-sheaves.*

The remaining classical group. $G = SO(r, \mathbb{C})$. Define a principal $SO(r, \mathbb{C})$-sheaf to be a triple (E, φ, ψ), where (E, φ) is a principal $O(r, \mathbb{C})$-sheaf and φ is an isomorphism between det (E) and $\mathcal{O}_X$ such that det $(\varphi) = \psi^2$. Note that for each $O(r, \mathbb{C})$-sheaf (E, φ), there is at most two distinct $SO(r, \mathbb{C})$-sheaves, namely (E, φ, ψ) and $(E, \varphi, -\psi)$. If rank (E) is odd, these two objects are isomorphic. This is why for $SO(2m+1, \mathbb{C})$ we can forget the third datum ψ. But if rank (E) is even, these two objects might not be isomorphic. With the same definition of (semi)stability as in Definition 4 , Theorem 5 still holds in this case (i.e. the added datum does not alter the GIT notion of stability) so *we obtain a coarse projective moduli space in the case G is any classical group.*

4. Principal sheaves on a reductive group.

Tensors considered in Section 3 were all $(s, 0)$ tensors, but with the same machinery we could have worked with (semi)stability and coarse projective moduli space of $(s, 1)$-tensors. In particular we need in this section $(2, 1)$-tensors $\varphi : E \otimes E \longrightarrow E^{**}$, for which δ-(semi)stability is defined by the fact that for all weighted filtration $(E_1 \subset \ldots \subset E_t, m_1, \ldots, m_t > 0)$ of E, it is

$$\sum m_i (r\chi_{E_i} - r_i \chi_E) + \delta\ \mu(E., m., \varphi)\ (\leq)\ 0$$

where

$$\mu = \min\{\lambda_i + \lambda_j - \lambda_k | 0 \neq \overline{\varphi} : E_i \otimes E_j \longrightarrow E^{**}/E^{**}_{k-1}\}$$

For fixed value of rank and Chern classes, there is a projective coarse moduli space of stable equivalence classes of δ-semistable $(2, 1)$ tensors on X.

Definition 6. *Let X be a projective variety, and G an algebraic group. A principal G-sheaf $\mathcal{P}$ is a triple (E, φ, ξ) where (E, φ) is a $(2, 1)$-tensor on X*

$$\varphi : E \otimes E \longrightarrow E^{**}$$

such that for the points x of the open set U_E where E is locally free, $\varphi(x)$ is isomorphic to the structure tensor $\varphi_{\mathfrak{g}'} : \mathfrak{g}' \otimes \mathfrak{g}' \longrightarrow \mathfrak{g}'$ of the Lie algebra $\mathfrak{g}'$ tangent to the commutator subgroup $G' = [G, G]$ (in particular, there is an associated $Aut(\mathfrak{g}')$-bundle P_{U_E} on U_E), and ξ is a reduction of P_{U_E} to G, via $Ad : G \longrightarrow Aut(\mathfrak{g}')$.

Obviously, if E is locally free, we recover the usual notion of principal G-bundle.

Definition 7. *Let G be a connected reductive group. We say that a principal G-sheaf $\mathcal{P} = (E, \varphi, \xi)$ is (semi)stable if the tensor (E, φ) is δ-(semi)stable, where δ is a polinomial of degree exactly $n - 1$.*

In order to characterize this notion, we define the Hilbert polynomial $\chi_{E_\bullet}$ of a filtration $E_\bullet$ of E (understood as $\mathbb{Z}$-indexed, with $E_{-\infty} = 0$ and $E_{+\infty} = E$) as

$$\chi_{E_\bullet} = \sum_{i \in \mathbb{Z}} \big(\mathrm{rank}(E)\chi_{E_i} - \mathrm{rank}(E_i)\chi_E\big)$$

and say the filtration is balanced if $\sum_{i \in \mathbb{Z}} i\,\mathrm{rank}(E_i/E_{i-1}) = 0$

Proposition 8. *A principal G-sheaf $\mathcal{P} = (E, \varphi, \xi)$ is (semi)stable if and only if for all balanced algebra filtrations $E_\bullet \subseteq E$ it is*

$$\chi_{E_\bullet} (\leq) 0$$

Theorem 9. *There is a projective coarse moduli space of stable-equivalence classes of semistable principal G-sheaves on X of fixed topological type.*

Comment on the proof. It is a long proof, parallel to the proof of Ramanathan [13], which will appear published elsewhere. Because of the nondegeneracy of the Killing form of the semisimple Lie algebra $\mathfrak{g}'$, the factor $\mu(E., m., \varphi)$ is always nonpositive. Although our notion of (semi)stability is equivalent to the δ-(semi)stability of the (2, 1) tensor (E, φ), it does not assure the existence of a moduli space, because it must also be checked that the extra datum of reduction ξ does not alter the (semi)stability in the sense of GIT of the corresponding point of the $Sl(V)$-acted projective space, which is the main bulk of the proof.

The case dim X=1. Finally, we need some considerations on root spaces in order to show that our notion of (semi)stability coincides with Ramanathan's when $\dim X = 1$. Recall from [1] that a $\mathfrak{t}'$-root decomposition

$$\mathfrak{g}' = \bigoplus_{\alpha \in R_{\mathfrak{t}'} \cup \{0\}} \mathfrak{g}'^{\alpha}$$

of the Lie algebra $\mathfrak{g}'$ arises whenever a toral algebra $\mathfrak{t}' \subseteq \mathfrak{g}'$ is given, not necessarily a Cartan algebra, in particular for the center $\mathfrak{t}' = \mathfrak{z}(\mathfrak{l}(\mathfrak{h}'))$ of the Levi component $\mathfrak{l}(\mathfrak{h}')$ of any parabolic subalgebra $\mathfrak{h}' \subset \mathfrak{g}'$. In this case a system of simple $\mathfrak{t}'$-roots (or decomposition $R_{\mathfrak{t}'} = R^+_{\mathfrak{t}'} \cup R^-_{\mathfrak{t}'}$) is naturally given, so the set $R_{\mathfrak{t}'} \cup \{0\}$ has a natural partial ordering ($\alpha \leq \beta$ if β is the sum of α with a sum of simple $\mathfrak{t}'$-roots). Denote $\mathfrak{g}'_{(\leq\alpha)} = \oplus_{\beta\leq\alpha}\mathfrak{g}'^{\beta}$ and analogously $\mathfrak{g}'_{(<\alpha)}$. We also write $R_{\mathfrak{h}'}$ for $R_{\mathfrak{t}'}$. Both are invariant by the adjoint action of $\mathfrak{h}'$, thus by the inner automorphism action of the corresponding parabolic subgroup H' of the group G', so the analogous subalgebras $\mathfrak{g}'_{(\leq\alpha)}$ and $\mathfrak{g}'_{(<\alpha)}$ of the Lie algebra $\mathfrak{g}'$ are also H'-invariant.

Let $\mathcal{P} = (E, \varphi, \xi)$ be a principal G-bundle on X, having a further $H \hookrightarrow G$ reduction to a parabolic subgroup H, let $H' = H \cap G'$, and let $\alpha \in R_{\mathfrak{h}'} \cup \{0\}$ where $\mathfrak{h}' = \mathrm{Lie}(H')$ as before. We define $E_{(\leq\alpha)}$ and $E_{(<\alpha)}$ as the subbundle of E associated to this reduction by the above representation of H' on $\mathfrak{g}'_{(\leq\alpha)}$ and $\mathfrak{g}'_{(<\alpha)}$, and define E^{α} as $E_{(\leq\alpha)}/E_{(<\alpha)}$.

Proposition 10. *A principal G-bundle $\mathcal{P} = (E, \varphi, \xi)$ on a curve is semistable if and only if E is semistable. It is furthermore stable if there is no reduction $P(H)$ of $\mathcal{P}$ to a parabolic subgroup H of G such that deg $E^{\alpha} = 0$ for all roots $\alpha \in R_{\mathfrak{h}'} \cup \{0\}$, i.e. such that the degree of the line bundle associated to the principal H-bundle $P(H)$ by any of the characters of H is zero.*

Corollary 11. *In case* $\dim X = 1$*, a principal G-bundle is (semi)stable, in our sense, if and only if it is (semi)stable in the sense of Ramanathan [12, 13] .*

REFERENCES

[1] A. Borel, *Linear algebraic groups,* New York: W. A. Benjamin Inc. 1969.

[2] J. Cilleruelo - I. Sols, *The Severi bound on sections of rank two semistable bundles on a Riemann surface,* to appear in Ann. of Math., Nov. 2001.

[3] G. Faltings, *Moduli stacks for bundles on semistable curves,* Math. Ann., 304 (1996), pp. 489–515.

[4] D. Gieseker, *On the moduli of vector bundles on an algebraic surfaces,* Ann. Math., 106 (1977), pp. 45–60.

[5] T. Gómez - I. Sols, *Stability of conic bundles,* Internat. J. Math., 11 (2000), pp. 1027–1055.

[6] T. Gómez - I. Sols, *Stable tensor fields and moduli space of principal G-sheaves for classical groups,* (math. AG/0103150).

[7] D. Huybrechts - M. Lehn, *Framed modules and their moduli,* Internat. J. Math., 6 (1995), pp. 297–324.

[8] M. Maruyama, *Moduli of stable sheaves. I,* J. Math. Kyoto Univ., 17 (1977), pp. 91–126.

[9] M. Maruyama, *Moduli of stable sheaves. II,* J. Math. Kyoto Univ., 18 (1978), pp. 557–614.

[10] D. Mumford - J. Fogarty - F. Kirwan, *Geometric invariant theory,* (Third edition) Ergebnisse der Mathematik und ihrer Grenzgebiete (2), 34, Springer-Verlag, Berlin, 1994.

[11] M.S. Narasimhan - C.S. Seshadri, *Stable and unitary vector bundles on a compact Riemann surface,* Ann. Math., 82 (1965), pp. 540–567.

[12] A. Ramanathan, *Stable principal bundles on a compact Riemann surface,* Math. Ann., 213 (1975), pp. 129–152.

[13] A. Ramanathan, *Moduli for principal bundles over algebraic curves: I and II,* Proc. Indian Acad. Sci. (Math. Sci.), 106 (1996), pp. 301–326 and pp. 421–449.

[14] A. Schmitt, *A universal construction for moduli spaces of decorated vector bundles over curves,* preprint 2000, math. AG/0006029.

[15] C.S. Seshadri, *Space of unitary vector bundles on a compact Riemann surface,* Ann. Math., 85 (1967), pp. 303–336.

[16] C.S. Seshadri, *Quotient spaces modulo reductive algebraic groups,* Ann. Math., 95 (1972), pp. 511–556.

[17] F. Severi, *Sulla classificazione delle rigate algebriche,* Rend. Mat., 2 (1941), pp. 1–32.

[18] C. Simpson, *Moduli of representations of the fundamental group of a smooth projective variety I,* Publ. Math. I.H.E.S., 79 (1994), pp. 47–129.

[19] C. Sorger, *Thêta-charactéristiques des courbes tracées sur une surface lisse ,* J. reine angew. Math., 435 (1993), pp. 83–118.

Tomás L. Gómez
School of Mathematics,
Tata Institute of Fundamental Research,
Bombay 400 005 (INDIA)
e-mail: tomas@math.tifr.res.in

Ignacio Sols
Departamento de Algebra,
Facultad de Matemáticas,
Universidad Complutense de Madrid,
28040 Madrid (SPAIN)
e-mail: sols@eucmax.sim.ucm.es