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ON THE LATTICE FORMULATION OF THE UNION-CLOSED SETS CONJECTURE

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The union-closed sets conjecture, also known as Frankl's conjecture, is a well-studied problem with various formulations. In terms of lattices, the conjecture states that every finite lattice L with more than one element contains a join-irreducible element that is less than or equal to at most half of the elements in L. In this work, we obtain several necessary conditions for any counterexample $\tilde{L}$ of minimum size.

1. Introduction

Let $\mathcal{A}$ be a finite family of distinct finite sets with at least one nonempty member set. The union-closed sets conjecture, also known as Frankl's conjecture, states that if $X, Y \in \mathcal{A}$ implies that $X \cup Y \in \mathcal{A}$, then there exists an element of $\bigcup_{A \in \mathcal{A}} A$ that is in at least $\frac{|\mathcal{A}|}{2}$ member sets of $\mathcal{A}$. With respect to this standard formulation of the conjecture, minimal counterexamples have been studied. For example, in [7] it was shown that for any counterexample $\tilde{\mathcal{A}}$ of minimum size $|\tilde{\mathcal{A}}|$, there must exist at least three elements of $\bigcup_{A \in \tilde{\mathcal{A}}} A$ that are each in exactly $\frac{|\tilde{\mathcal{A}}|-1}{2}$ member sets of $\tilde{\mathcal{A}}$. Another result, from [6] and [10], is that $|\tilde{\mathcal{A}}| \geq 4|\bigcup_{A \in \mathcal{A}^*} A| - 1$, where $\mathcal{A}^*$ is a counterexample, in this case, with minimum $|\bigcup_{A \in \mathcal{A}^*} A|$. In the present work, we consider the lattice formulation of the union-closed sets conjecture

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(found, for instance, in [8] and [11]), obtaining several necessary conditions for any counterexample of minimum size.

To begin, we recall some terminology. A *poset* (or *partially ordered set*) $\langle P; \leq_P \rangle$ is a set P equipped with a binary relation $\leq_P$ on P that is reflexive, antisymmetric, and transitive. For $x, y \in P$, if $x \leq_P y$, then we state that x is less than or equal to y (and y is greater than or equal to x) in P. Similarly, if $x \leq_P y$ and $x \neq y$ (denoted by $x <_P y$), then we state that x is less than y (and y is greater than x) in P. Two elements $x, y \in P$ are *comparable* in P if $x \leq_P y$ or $y \leq_P x$. Otherwise, x and y are *incomparable* in P, denoted by $x \parallel_P y$ (or $y \parallel_P x$). We define a *subposet* $\langle S; \leq_S \rangle$ of P to be a poset such that $S \subseteq P$ and, for all x and y in S, $x \leq_S y$ if and only if $x \leq_P y$. A poset is *linearly ordered* if every two of its elements are comparable, and a *chain* $\langle C; \leq_C \rangle$ in P is a linearly ordered subposet of P. We use the notation $(\uparrow x)_P = \{y \in P \mid x \leq_P y\}$, $(\downarrow x)_P = \{y \in P \mid y \leq_P x\}$, and $(\parallel x)_P = \{y \in P \mid x \parallel_P y\}$.

Two elements $x, y \in P$ have a *join* $\sup_P\{x, y\}$ in P if $\sup_P\{x, y\}$ is an element of P such that both $x \leq_P \sup_P\{x, y\}$ and $y \leq_P \sup_P\{x, y\}$, and $x \leq_P z$ and $y \leq_P z$ together imply that $\sup_P\{x, y\} \leq_P z$ for all $z \in P$. Similarly, x and y have a *meet* $\inf_P\{x, y\}$ in P if $\inf_P\{x, y\}$ is an element of P such that $\inf_P\{x, y\} \leq_P x$ and $\inf_P\{x, y\} \leq_P y$, and $z \leq_P x$ and $z \leq_P y$ imply that $z \leq_P \inf_P\{x, y\}$. An element x *upper covers* an element y (and y *lower covers* x) in P if $y <_P x$, and for all $z \in P$, $y <_P z \leq_P x$ implies that $z = x$. An element is *join-irreducible* in P if it upper covers exactly one element in P, *meet-irreducible* if it lower covers exactly one element, and *doubly irreducible* if it is both join and meet-irreducible. On the other hand, an element is *join-reducible* in P if it upper covers more than one element in P, *meet-reducible* if it lower covers more than one element, and *doubly reducible* if it is both join and meet-reducible.

A poset $\langle L; \leq_L \rangle$ is a *lattice* if every two elements $x, y \in L$ have both a join and meet in L. When L is finite, its greatest element is denoted by 1_L and least element by 0_L. An element that upper covers 0_L in L is an *atom*, and an element that lower covers 1_L is a *dual atom*. For more information on lattices, see [2]. The lattice formulation of the union-closed sets conjecture (phrased, as usual, according to the intersection-closed dual) is stated as follows:

Conjecture 1.1. *Any finite lattice L with more than one element contains a join-irreducible element j such that $|(\uparrow j)_L| \leq \frac{|L|}{2}$.*

In the next section, we characterize any counterexample $\tilde{L}$ to Conjecture 1.1 of minimum size $|\tilde{L}|$. We conclude the current section by considering some pertinent statements regarding lattices in general.

Lemma 1.2 (Agalave, Shewale, and Kharat [1]). *An element $x \in L$ is join or meet-irreducible in a finite lattice L with $|L| > 1$ if and only if the subposet $L \setminus \{x\}$ of L is a lattice.*

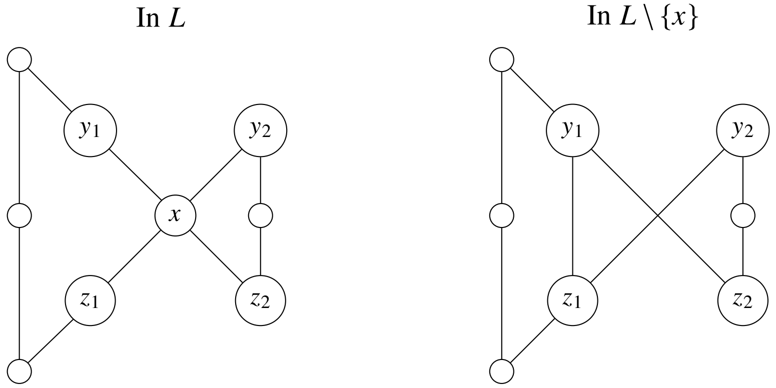
Proof. See Lemma 2.1 of [1]. We include here a proof using the present notation. First, we show that if x is join-irreducible in L, upper covering a unique element z in L, then the subposet $L \setminus \{x\}$ of L is a lattice. Consider any $y_1, y_2 \in L \setminus \{x\}$. If $\inf_L\{y_1, y_2\} = x$, then $\inf_{L\setminus\{x\}}\{y_1, y_2\} = z$. (Otherwise, there exists an element l less than or equal to y_1 and y_2 in $L \setminus \{x\}$ such that $z <_{L\setminus\{x\}} l$ or $z \parallel_{L\setminus\{x\}} l$ (with $l <_L x$ because $\inf_L\{y_1, y_2\} = x$ and $l \neq x$). However, $z <_{L\setminus\{x\}} l$ does not hold (because z lower covers x in L), and if $z \parallel_{L\setminus\{x\}} l$, then $l \leq_L w$ for some $w \in L$ such that $w \neq z$ and w lower covers x in L, contradicting the join-irreducibility of x in L.) Now, if $\inf_L\{y_1, y_2\} \neq x$, then $\inf_L\{y_1, y_2\} \in L \setminus \{x\}$ and $\inf_{L\setminus\{x\}}\{y_1, y_2\} = \inf_L\{y_1, y_2\}$. Also, $y_1, y_2 \in L \setminus \{x\}$ and x being join-irreducible in L together imply that $\sup_L\{y_1, y_2\} \in L \setminus \{x\}$, so we have that $\sup_{L\setminus\{x\}}\{y_1, y_2\} = \sup_L\{y_1, y_2\}$. Thus, a meet and join exist in $L \setminus \{x\}$ for y_1 and y_2, and $L \setminus \{x\}$ a lattice. If x is meet-irreducible in L, then $L \setminus \{x\}$ is a lattice by duality.

Next, we show by contraposition that for all $x \in L$, if the subposet $L \setminus \{x\}$ of L is a lattice, then x is join or meet-irreducible in L. If $x = 1_L$ and is join-reducible in L, or $x = 0_L$ and is meet-reducible in L, then there are, respectively, two dual atoms from L without a join in $L \setminus \{x\}$, or two atoms from L without a meet in $L \setminus \{x\}$, so $L \setminus \{x\}$ is not a lattice. If x is doubly reducible in L, then there exist two elements y_1, y_2 that upper cover x and two elements z_1, z_2 that lower cover x in L, all distinct. In this case, if $\sup_{L\setminus\{x\}}\{z_1, z_2\}$ exists, then $x = \sup_L\{z_1, z_2\} <_L \sup_{L\setminus\{x\}}\{z_1, z_2\} \leq_L y_1$ and $x = \sup_L\{z_1, z_2\} <_L \sup_{L\setminus\{x\}}\{z_1, z_2\} \leq_L y_2$. Because $y_1 \neq y_2$, we then have that $x <_L \sup_{L\setminus\{x\}}\{z_1, z_2\} <_L y_1$ or $x <_L \sup_{L\setminus\{x\}}\{z_1, z_2\} <_L y_2$, so y_1 or y_2 does not upper cover x in L, a contradiction. Thus, $\sup_{L\setminus\{x\}}\{z_1, z_2\}$ does not exist, and $L \setminus \{x\}$ is not a lattice, as illustrated in Figure 1.1. □

Lemma 1.3. *(i) If j_1 and j_2 are join-irreducible elements in a finite lattice L, then j_2 is join-irreducible in the subposet $L \setminus \{j_1\}$ of L; (ii) If m_1 and m_2 are meet-irreducible elements in a finite lattice L, then m_2 is meet-irreducible in the subposet $L \setminus \{m_1\}$ of L.*

Proof. For part (i), we denote by l_1 and l_2 the unique elements that respectively lower cover j_1 and j_2 in L. If $l_2 \neq j_1$, then l_2 is the only element that lower covers j_2 in $L \setminus \{j_1\}$. Else, $l_2 = j_1$, making l_1 the only element that lower covers l_2 in L. Together with l_2 being the only element that lower covers j_2 in L, this implies that l_1 is the only element that lower covers j_2 in $L \setminus \{j_1\}$. Thus, j_2 is also join-irreducible in $L \setminus \{j_1\}$. Part (ii) follows by duality. □

Figure 1.1: Example corresponding structures in L and (non-lattice) $L \setminus \{x\}$.



We note that the converse of Lemma 1.3 does not hold (for both parts (i) and (ii)). We consider part (i). Let j be a join-irreducible element in a finite lattice L, and x be an element that upper covers only j and one other element y in L such that y upper covers the unique element z that lower covers j in L. Here, the converse of the lemma is not satisfied in that, although x is join-irreducible in the subposet $L \setminus \{j\}$ of L, x is not join-irreducible in L.

Theorem 1.4. *(i) If J is a set of join-irreducible elements in a finite lattice L, then the subposet $L \setminus J$ of L is a lattice; (ii) If M is a set of meet-irreducible elements in a finite lattice L, then the subposet $L \setminus M$ of L is a lattice.*

Proof. Again, we consider only part (i). Without loss of generality, let $J = \{j_1, \cdots, j_{|J|}\}$. If $|J| = 1$, then the subposet $L \setminus J$ of L is a lattice by Lemma 1.2. If $|J| > 1$, then we first have that the subposet $L_1 = L \setminus \{j_1\}$ of L is a lattice by Lemma 1.2, and $j_2, \cdots, j_{|J|}$ are join-irreducible in L_1 by Lemma 1.3. Then, for all integers k such that $1 \leq k \leq |J| - 1$, if the subposet $L_k = L \setminus \{j_1, \cdots, j_k\}$ of L is a lattice and $j_{k+1}, \cdots, j_{|J|}$ are join-irreducible in L_k, then the subposet $L_{k+1} = L_k \setminus \{j_{k+1}\}$ of L_k is a lattice by Lemma 1.2 and, when $k < |J| - 1$, $j_{k+2}, \cdots, j_{|J|}$ are join-irreducible in L_{k+1} by Lemma 1.3. Thus, we have that the subposet $L \setminus J = L_{|J|} = L_{|J|-1} \setminus \{j_{|J|}\}$ of $L_{|J|-1}$ (and therefore of L) is a lattice. □

Theorem 1.4 does not extend, in general, to a finite lattice L and set I of join and meet-irreducible elements in L. A counterexample that demonstrates this is any finite lattice L with $I = \{j, m\}$ such that j is join-irreducible and meet-reducible in L, m is meet-irreducible and join-reducible in L, and j upper covers

m in L. In this case, the conclusion of Theorem 1.4 is not satisfied because the subposet $L \setminus I$ of L is not a lattice.

2. Characterizing a counterexample to Conjecture 1.1 of minimum size

We now obtain necessary conditions for any counterexample $\tilde{L}$ to Conjecture 1.1 such that no counterexample $\hat{L}$ exists with $|\hat{L}| < |\tilde{L}|$. We note that $\tilde{L}$ must contain more than two elements.

Theorem 2.1. *Every join-irreducible element j in $\tilde{L}$ (with the unique element that it upper covers in $\tilde{L}$ denoted by x) lower covers an element y in $\tilde{L}$, where y is not less than or equal to any join-irreducible element and y upper covers exactly one other element z in $\tilde{L}$ such that $x <_{\tilde{L}} z$.*

Proof. If not, then there exists a join-irreducible element j in $\tilde{L}$ such that any element y that upper covers j in $\tilde{L}$ satisfies at least one of the following criteria:

(i) y upper covers more than two elements in $\tilde{L}$;

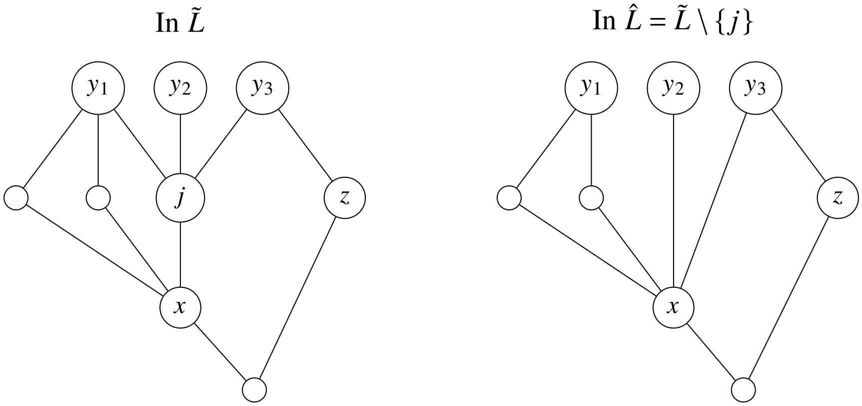
(ii) $y \leq_{\tilde{L}} j^*$ for some join-irreducible j^* in $\tilde{L}$;

(iii) y upper covers j and exactly one other element z in $\tilde{L}$ such that z is incomparable with x in $\tilde{L}$.

We consider the subposet $\hat{L} = \tilde{L} \setminus \{j\}$ of $\tilde{L}$, which is a lattice by Lemma 1.2, and consider any join-irreducible element $\hat{j}$ in $\hat{L}$. First, we assume that $\hat{j}$ is also join-irreducible in $\tilde{L}$. If $\hat{j} <_{\tilde{L}} j$, then $|(\uparrow \hat{j})_{\hat{L}}| \geq |(\uparrow j)_{\tilde{L}}| > \frac{|\tilde{L}|}{2} = \frac{|\hat{L}|+1}{2}$. Else, $|(\uparrow \hat{j})_{\hat{L}}| = |(\uparrow \hat{j})_{\tilde{L}}| > \frac{|\tilde{L}|}{2} = \frac{|\hat{L}|+1}{2}$. Next, we assume that $\hat{j}$ is join-reducible in $\tilde{L}$. In this case, $\hat{j}$ upper covers j in $\tilde{L}$, and so must satisfy at least one of the three criteria in question. (i) is not satisfied, as it implies that $\hat{j}$ is join-reducible in $\hat{L}$. (iii) is not satisfied, as it implies that $\hat{j}$ upper covers both x and z in $\hat{L}$, and thus again that $\hat{j}$ is join-reducible in $\hat{L}$. Hence, (ii) is satisfied, and $|(\uparrow \hat{j})_{\hat{L}}| = |(\uparrow \hat{j})_{\tilde{L}}| \geq |(\uparrow j^*)_{\tilde{L}}| > \frac{|\tilde{L}|}{2} = \frac{|\hat{L}|+1}{2}$. Therefore, any join-irreducible element $\hat{j}$ in $\hat{L}$ has $|(\uparrow \hat{j})_{\hat{L}}| > \frac{|\hat{L}|}{2}$, and $\hat{L}$ is a counterexample to Conjecture 1.1, contradicting the minimality of $|\tilde{L}|$. □

In Figure 2.1, y_1 satisfies the first criterion from the proof of Theorem 2.1, y_2 satisfies the second, and y_3 the third. By the theorem, element j of Figure 2.1 must also lower cover an element in $\tilde{L}$ that does not satisfy any of the three criteria.

Figure 2.1: Example corresponding structures in $\tilde{L}$ and $\hat{L} = \tilde{L} \setminus \{j\}$.



Corollary 2.2. *$0_{\tilde{L}}$ is meet-reducible in $\tilde{L}$.*

Proof. Otherwise, denote by a the unique atom of $\tilde{L}$. For all $x \in \tilde{L} \setminus \{a\}$, $\sup_{\tilde{L}}\{a,x\} \in \{a,x\}$. Because a is join-irreducible in $\tilde{L}$, we have by Theorem 2.1 that there exists an element b upper covering a and exactly one other element c in $\tilde{L}$, making $\sup_{\tilde{L}}\{a,c\} = b \notin \{a,c\}$, a contradiction. □

Theorem 2.3. *No meet-irreducible element is less than a join-irreducible element in $\tilde{L}$.*

Proof. Otherwise, there exist a meet-irreducible element m and join-irreducible element j in $\tilde{L}$ such that $m <_{\tilde{L}} j$. By Lemma 1.2, the subposet $\hat{L} = \tilde{L} \setminus \{m\}$ of $\tilde{L}$ is a lattice. Consider any join-irreducible element $\hat{j}$ in $\hat{L}$. We first assume that $\hat{j}$ is also join-irreducible in $\tilde{L}$. If $\hat{j} <_{\tilde{L}} m$, then $\hat{j} <_{\tilde{L}} j$ and $|(\uparrow\hat{j})_{\hat{L}}| = |(\uparrow\hat{j})_{\tilde{L}}| - 1 > |(\uparrow j)_{\tilde{L}}| > \frac{|\tilde{L}|}{2} = \frac{|\hat{L}|+1}{2}$. Else, $|(\uparrow\hat{j})_{\hat{L}}| = |(\uparrow\hat{j})_{\tilde{L}}| > \frac{|\tilde{L}|}{2} = \frac{|\hat{L}|+1}{2}$. Next, we instead assume that $\hat{j}$ is join-reducible in $\tilde{L}$, implying that $\hat{j}$ upper covers m in $\tilde{L}$. In this case, $\hat{j} <_{\tilde{L}} j$, and we have that $|(\uparrow\hat{j})_{\hat{L}}| = |(\uparrow\hat{j})_{\tilde{L}}| > |(\uparrow j)_{\tilde{L}}| > \frac{|\tilde{L}|}{2} = \frac{|\hat{L}|+1}{2}$. In both cases, $|(\uparrow\hat{j})_{\hat{L}}| > \frac{|\hat{L}|}{2}$, and so $\hat{L}$ is a counterexample to Conjecture 1.1 that is smaller than $\tilde{L}$. This contradicts the minimality of $|\tilde{L}|$. □

Corollary 2.4. *$1_{\tilde{L}}$ is join-reducible in $\tilde{L}$.*

Proof. If not, then $1_{\tilde{L}}$ is join-irreducible in $\tilde{L}$, upper covering exactly one dual atom. This contradicts Theorem 2.3, as all dual atoms of a lattice are meet-irreducible. □

Lemma 2.5. *If x is doubly irreducible in $\tilde{L}$, then $|(\uparrow x)_{\tilde{L}}| = \frac{|\tilde{L}|+1}{2}$.*

Proof. Assume otherwise, i.e. that there is a doubly irreducible element x in $\tilde{L}$ such that $|(\uparrow x)_{\tilde{L}}| > \frac{|\tilde{L}|+1}{2}$. By Lemma 1.2, the subposet $\hat{L} = \tilde{L} \setminus \{x\}$ of $\tilde{L}$ is a lattice. For any join-irreducible element $\hat{j}$ in $\hat{L}$, we first assume that $\hat{j}$ is also join-irreducible in $\tilde{L}$. If $\hat{j} <_{\tilde{L}} x$, then $|(\uparrow \hat{j})_{\hat{L}}| = |(\uparrow \hat{j})_{\tilde{L}}| - 1 \geq |(\uparrow x)_{\tilde{L}}| > \frac{|\tilde{L}|+1}{2} = \frac{|\hat{L}|+2}{2}$. Else, $|(\uparrow \hat{j})_{\hat{L}}| = |(\uparrow \hat{j})_{\tilde{L}}| > \frac{|\tilde{L}|}{2} = \frac{|\hat{L}|+1}{2}$. Now we assume that $\hat{j}$ is join-reducible in $\tilde{L}$. In this case, $\hat{j}$ upper covers x in $\tilde{L}$, and $|(\uparrow \hat{j})_{\hat{L}}| = |(\uparrow \hat{j})_{\tilde{L}}| = |(\uparrow x)_{\tilde{L}}| - 1 > \frac{|\tilde{L}|+1}{2} - 1 = \frac{|\hat{L}|}{2}$. In both cases, $|(\uparrow \hat{j})_{\hat{L}}| > \frac{|\hat{L}|}{2}$. Therefore, $\hat{L}$ is a counterexample to Conjecture 1.1, contradicting the minimality of $|\tilde{L}|$. □

Theorem 2.6. *There is at most one doubly irreducible element in $\tilde{L}$.*

Proof. If not, then there exist two distinct elements x and y that are doubly irreducible in $\tilde{L}$. We first assume that x and y are comparable in $\tilde{L}$, where $x <_{\tilde{L}} y$ without loss of generality. Then $|(\uparrow x)_{\tilde{L}}| > |(\uparrow y)_{\tilde{L}}|$, contradicting Lemma 2.5 which requires that $|(\uparrow x)_{\tilde{L}}| = |(\uparrow y)_{\tilde{L}}| = \frac{|\tilde{L}|+1}{2}$. Next, we assume that x and y are incomparable in $\tilde{L}$. We consider the subposet $\hat{L} = L \setminus \{x, y\}$ of $\tilde{L}$, which is itself a lattice by Theorem 1.4. Now, consider any join-irreducible element $\hat{j}$ in $\hat{L}$. First, we assume that $\hat{j}$ is also join-irreducible in $\tilde{L}$. If either $\hat{j} <_{\tilde{L}} x$ or $\hat{j} <_{\tilde{L}} y$, then $|(\uparrow \hat{j})_{\hat{L}}| = |(\uparrow \hat{j})_{\tilde{L}}| - 1 > \frac{|\tilde{L}|+1}{2} - 1 = \frac{|\hat{L}|+1}{2}$. If neither $\hat{j} <_{\tilde{L}} x$ nor $\hat{j} <_{\tilde{L}} y$, then $|(\uparrow \hat{j})_{\hat{L}}| = |(\uparrow \hat{j})_{\tilde{L}}| > \frac{|\tilde{L}|}{2} = \frac{|\hat{L}|+2}{2}$. If both $\hat{j} <_{\tilde{L}} x$ and $\hat{j} <_{\tilde{L}} y$, then $|(\uparrow \hat{j})_{\hat{L}}| = |(\uparrow \hat{j})_{\tilde{L}}| - 2 > \frac{|\tilde{L}|+3}{2} - 2 = \frac{|\hat{L}|+1}{2}$. Next, we assume that $\hat{j}$ is join-reducible in $\tilde{L}$. In this case, at least one of x or y lower covers $\hat{j}$ in $\tilde{L}$, and $|(\uparrow \hat{j})_{\hat{L}}| = |(\uparrow \hat{j})_{\tilde{L}}| = \frac{|\tilde{L}|+1}{2} - 1 = \frac{|\hat{L}|+1}{2}$. Therefore, in both cases $|(\uparrow \hat{j})_{\hat{L}}| > \frac{|\hat{L}|}{2}$, making $\hat{L}$ a counterexample to Conjecture 1.1. This contradicts the minimality of $|\tilde{L}|$. □

In [4], two incomparable doubly irreducible elements were also removed from a lattice, in that case with respect to the class of dismantlable lattices, in order to inductively prove that Conjecture 1.1 holds for all lattices therein. The *length* $\ell(L)$ of a lattice L is one less than the maximum size of a chain in L. Theorem 2.6 can be paired with a result of Rival (see Theorem 1 of [9]) to prove the following:

Theorem 2.7. *If j is a join-irreducible element in $\tilde{L}$, then $|(\uparrow j)_{\tilde{L}}| > \ell(\tilde{L})$.*

Proof. In [9], it was shown that $|L| \geq 2(\ell(L) + 1) - |\mathtt{Irr}(L)|$ for any finite lattice L, where $\mathtt{Irr}(L)$ is the set of doubly irreducible elements in L. We note that 0_L and 1_L are considered by definition in [9] to be, respectively, join and meet-irreducible in a finite lattice L, which is not the case in the present work. However, the set of elements that are doubly irreducible in $\tilde{L}$ is not affected by this difference because $0_{\tilde{L}}$ is meet-reducible in $\tilde{L}$ by Corollary 2.2, and $1_{\tilde{L}}$ is join-reducible in $\tilde{L}$ by Corollary 2.4, implying that, whether or not $0_{\tilde{L}}$ and $1_{\tilde{L}}$ are respectively considered join and meet-irreducible in $\tilde{L}$, neither are considered doubly irreducible. Now, by Theorem 2.6, $\mathtt{Irr}(\tilde{L}) \leq 1$. It follows that $|\tilde{L}| \geq 2(\ell(\tilde{L})+1)-1$, and so $\frac{|\tilde{L}|}{2} > \ell(\tilde{L})$. We recall that any join-irreducible element j in $\tilde{L}$ has $|(\uparrow j)_{\tilde{L}}| > \frac{|\tilde{L}|}{2}$, and thus also $|(\uparrow j)_{\tilde{L}}| > \ell(\tilde{L})$. □

Corollary 2.8. *Any doubly irreducible element is less than at least one doubly reducible element in $\tilde{L}$.*

Proof. By Theorem 2.3, every element greater than a doubly irreducible element x in $\tilde{L}$ is join-reducible in $\tilde{L}$, so we need only to show that one such element is also meet-reducible in $\tilde{L}$. We assume that every element $y \in (\uparrow x)_{\tilde{L}} \setminus \{1_{\tilde{L}}\}$ is meet-irreducible in $\tilde{L}$. Then there is a unique maximal chain C in $\tilde{L}$ such that $0_C = x$ and $1_C = 1_{\tilde{L}}$, and an element of $\tilde{L}$ is greater than x in $\tilde{L}$ if and only if it is greater than x in C. Further, $0_{\tilde{L}} <_{\tilde{L}} x$ implies that C is a subposet of a chain C_0 in $\tilde{L}$ such that $0_{\tilde{L}} \in C_0$. It follows that $|(\uparrow x)_{\tilde{L}}| = |C| < |C_0| \leq \ell(\tilde{L})+1$. Therefore, $|(\uparrow x)_{\tilde{L}}| \leq \ell(\tilde{L})$, contradicting Theorem 2.7. □

We observe that every element $x \in \tilde{L} \setminus \{0_{\tilde{L}}, 1_{\tilde{L}}\}$ is comparable with at least four elements, excluding itself, in $\tilde{L}$. These include $0_{\tilde{L}}$ and $1_{\tilde{L}}$, as well as two other distinct elements from $\tilde{L} \setminus \{x\}$. (Otherwise, x is doubly irreducible in $\tilde{L}$, and either lower covers or is equal to a dual atom in $\tilde{L}$, contradicting Corollary 2.8.)

Theorem 2.9. *If M is a nonempty set of elements that are meet-irreducible in $\tilde{L}$, then there exists a join-irreducible element j in $\tilde{L}$ such that $|(\uparrow j)_{\tilde{L}} \cap M| > \frac{|M|}{2}$.*

Proof. Otherwise, there exists a nonempty set M of meet-irreducible elements in $\tilde{L}$ such that $|(\uparrow j)_{\tilde{L}} \cap M| \leq \frac{|M|}{2}$ for every join-irreducible element j in $\tilde{L}$. We consider the subposet $\hat{L} = \tilde{L} \setminus M$ of $\tilde{L}$, which is a lattice by Theorem 1.4, and consider any join-irreducible element $\hat{j}$ in $\hat{L}$. If $\hat{j}$ is also join-irreducible in $\tilde{L}$, then $|(\uparrow \hat{j})_{\hat{L}}| > \frac{|\tilde{L}|}{2} - |(\uparrow \hat{j})_{\tilde{L}} \cap M| = \frac{|\hat{L}|+|M|}{2} - |(\uparrow \hat{j})_{\tilde{L}} \cap M| = \frac{|\hat{L}|}{2} + (\frac{|M|}{2} - |(\uparrow \hat{j})_{\tilde{L}} \cap M|) \geq \frac{|\hat{L}|}{2}$. If $\hat{j}$ is join-reducible in $\tilde{L}$, then there exists $m_1 \in M$ that lower covers $\hat{j}$ in $\tilde{L}$. It follows that $\{m_1\} \subseteq C = \{m_1, \cdots, m_{|C|}\} \subseteq M$, where C is any chain of maximum size in $\tilde{L}$ such that $1 \leq i < |C|$ implies that m_i upper covers m_{i+1} in $\tilde{L}$. We first assume that $m_{|C|}$ is join-irreducible in $\tilde{L}$, implying that $|(\uparrow m_{|C|})_{\tilde{L}}| > \frac{|\tilde{L}|}{2}$. Then, based on the definition of C, we have that $|(\uparrow \hat{j})_{\tilde{L}}| = |(\uparrow m_{|C|})_{\tilde{L}}| - |C| > \frac{|\tilde{L}|-2|C|}{2}$, and so $|(\uparrow \hat{j})_{\hat{L}}| = |(\uparrow \hat{j})_{\tilde{L}}| - |(\uparrow \hat{j})_{\tilde{L}} \cap M| > \frac{|\hat{L}|+|M|-2(|C|+|(\uparrow \hat{j})_{\tilde{L}} \cap M|)}{2}$. If $|C| + |(\uparrow \hat{j})_{\tilde{L}} \cap M| > \frac{|M|}{2}$, then $|(\uparrow m_{|C|})_{\tilde{L}} \cap M| >$

$\frac{|M|}{2}$, as $C \cup ((\uparrow \hat{j})_{\tilde{L}} \cap M) = (\uparrow m_{|C|})_{\tilde{L}} \cap M$ and $C \cap ((\uparrow \hat{j})_{\tilde{L}} \cap M) = \emptyset$. However, because $m_{|C|}$ is join-irreducible in $\tilde{L}$, we also have that $|(\uparrow m_{|C|})_{\tilde{L}} \cap M| \leq \frac{|M|}{2}$, a contradiction. Thus, $|C| + |(\uparrow \hat{j})_{\tilde{L}} \cap M| \leq \frac{|M|}{2}$, making $|(\uparrow \hat{j})_{\hat{L}}| > \frac{|\hat{L}|}{2}$. Now, we assume that $m_{|C|}$ is join-reducible in $\tilde{L}$, upper covering distinct elements $x, y \notin M$. In this case, because $\hat{j}$ is join-irreducible in $\hat{L}$, we have that $|C| > 1$, and there exists an element $z \notin C$ less than $\hat{j}$ in $\tilde{L}$ that upper covers m_k in $\tilde{L}$ for some $k \in \{2, \cdots, |C|\}$. This contradicts the meet-irreducibility of m_k in $\tilde{L}$. In conclusion, whether or not $\hat{j}$ is join-irreducible in $\tilde{L}$, we have that $|(\uparrow \hat{j})_{\hat{L}}| > \frac{|\hat{L}|}{2}$. Then $\hat{L}$ is a counterexample to Conjecture 1.1, contradicting the minimality of $|\tilde{L}|$. □

Corollary 2.10. *There is a join-irreducible element j in $\tilde{L}$ that is less than or equal to more than half of the meet-irreducible elements in $\tilde{L}$.*

Proof. We set M of Theorem 2.9 equal to the set of all meet-irreducible elements in $\tilde{L}$. □

Corollary 2.11. *For any two meet-irreducible elements m_1 and m_2 in $\tilde{L}$, there exists a join-irreducible element j in $\tilde{L}$ such that $j \leq_{\tilde{L}} m_1$ and $j \leq_{\tilde{L}} m_2$.*

Proof. We set M of Theorem 2.9 equal to $\{m_1, m_2\}$. □

By Corollary 2.11, any meet-irreducible atom is less than all other meet-irreducible elements in $\tilde{L}$.

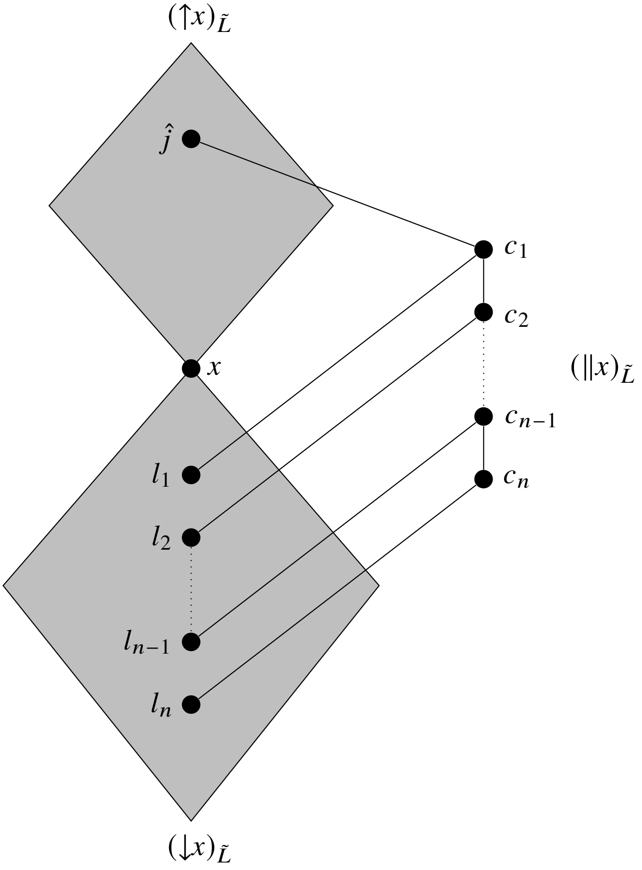
Theorem 2.12. *For every meet-irreducible element m in $\tilde{L}$, there exists a join-irreducible element j in $\tilde{L}$ such that $j \leq_{\tilde{L}} m$ and $|(\uparrow j)_{\tilde{L}}| = \frac{|\tilde{L}|+1}{2}$.*

Proof. Otherwise, there exists a meet-irreducible element m in $\tilde{L}$ such that any join-irreducible element j in $\tilde{L}$ with $j \leq_{\tilde{L}} m$ has $|(\uparrow j)_{\tilde{L}}| > \frac{|\tilde{L}|+1}{2}$. We consider the subposet $\hat{L} = \tilde{L} \setminus \{m\}$ of $\tilde{L}$, which is a lattice by Lemma 1.2. Any join-irreducible element $\hat{j}$ in $\hat{L}$ is also join-irreducible in $\tilde{L}$. (If not, then $\hat{j}$ upper covers m in $\tilde{L}$, and because $\hat{j}$ is join-irreducible in $\hat{L}$, we have that m is join-irreducible in $\tilde{L}$. It follows that $|(\uparrow m)_{\tilde{L}}| > \frac{|\tilde{L}|+1}{2}$, yet $|(\uparrow m)_{\tilde{L}}| = \frac{|\tilde{L}|+1}{2}$ by Lemma 2.5, a contradiction.) If $\hat{j} <_{\tilde{L}} m$, then $|(\uparrow \hat{j})_{\hat{L}}| = |(\uparrow \hat{j})_{\tilde{L}}| - 1 > \frac{|\tilde{L}|+1}{2} - 1 = \frac{|\hat{L}|}{2}$. Else, $|(\uparrow \hat{j})_{\hat{L}}| = |(\uparrow \hat{j})_{\tilde{L}}| > \frac{|\tilde{L}|}{2} = \frac{|\hat{L}|+1}{2}$. Therefore, $\hat{L}$ is a counterexample to Conjecture 1.1, contradicting the minimality of $|\tilde{L}|$. □

Theorem 2.13. *For all $x \in \tilde{L} \setminus \{0_{\tilde{L}}, 1_{\tilde{L}}\}$, the subposet $(\|x)_{\tilde{L}}$ of $\tilde{L}$ is not a chain in $\tilde{L}$.*

Proof. Assume otherwise. Then there exists an element $x \in \tilde{L} \setminus \{0_{\tilde{L}}, 1_{\tilde{L}}\}$ such that $(\|x)_{\tilde{L}} = \emptyset$ or $(\|x)_{\tilde{L}} = \{c_1, \cdots, c_n\}$, where $n = |(\|x)_{\tilde{L}}|$ and $1 \leq i \leq j \leq n$ implies that $c_j \leq_{\tilde{L}} c_i$ without loss of generality. No element from $(\|x)_{\tilde{L}}$ can upper cover an element from $(\uparrow x)_{\tilde{L}}$ or lower cover an element from $(\downarrow x)_{\tilde{L}}$ in $\tilde{L}$. Thus, if $|(\|x)_{\tilde{L}}| > 1$ and $1 \leq i < n$, then c_i upper covers c_{i+1} in $\tilde{L}$. Further, if $|(\|x)_{\tilde{L}}| > 1$ and $1 \leq i < j \leq n$, and two elements u_i and u_j (l_i and l_j) from $(\uparrow x)_{\tilde{L}}$ (from $(\downarrow x)_{\tilde{L}}$) respectively upper cover (lower cover) c_i and c_j in $\tilde{L}$, then $u_j <_{\tilde{L}} u_i$ ($l_j <_{\tilde{L}} l_i$). We consider the subposet $\hat{L} = (\uparrow x)_{\tilde{L}}$ of $\tilde{L}$. For all $y_1, y_2 \in \hat{L}$, $\sup_{\tilde{L}}\{y_1, y_2\} \in \hat{L}$ because $x \leq_{\tilde{L}} y_1 \leq_{\tilde{L}} \sup_{\tilde{L}}\{y_1, y_2\}$. Also, $\inf_{\tilde{L}}\{y_1, y_2\} \in \hat{L}$ because $x \leq_{\tilde{L}} y_1$ and $x \leq_{\tilde{L}} y_2$ together imply that $x \leq_{\tilde{L}} \inf_{\tilde{L}}\{y_1, y_2\}$. Thus, $\hat{L}$ is a lattice with $\sup_{\hat{L}}\{y_1, y_2\} = \sup_{\tilde{L}}\{y_1, y_2\}$ and $\inf_{\hat{L}}\{y_1, y_2\} = \inf_{\tilde{L}}\{y_1, y_2\}$. Now, consider any join-irreducible element $\hat{j}$ in $\hat{L}$. If $\hat{j}$ is also join-irreducible in $\tilde{L}$, then $|(\uparrow \hat{j})_{\hat{L}}| = |(\uparrow \hat{j})_{\tilde{L}}| > \frac{|\tilde{L}|}{2} = \frac{|\hat{L}|+|\tilde{L}\setminus\hat{L}|}{2} > \frac{|\hat{L}|}{2}$. If $\hat{j}$ is join-reducible in $\tilde{L}$, then $\hat{j}$ upper covers an element from $(\|x)_{\tilde{L}}$ in $\tilde{L}$, and we denote the unique element from $(\uparrow x)_{\tilde{L}}$ that upper covers c_1 in $\tilde{L}$ by u_1. We first assume that $n = 1$, or that $n > 1$ and c_1 only upper covers c_2 in $\tilde{L}$. In either case, c_1 is doubly irreducible in $\tilde{L}$, with $|(\uparrow c_1)_{\tilde{L}}| = \frac{|\tilde{L}|+1}{2}$ by Lemma 2.5, and $\hat{j} \leq_{\tilde{L}} u_1$ implies that $|(\uparrow \hat{j})_{\tilde{L}}| \geq |(\uparrow u_1)_{\tilde{L}}| = |(\uparrow c_1)_{\tilde{L}}| - 1 = \frac{|\tilde{L}|-1}{2} = \frac{|\hat{L}|+|\tilde{L}\setminus\hat{L}|-1}{2}$. It follows that $|(\uparrow \hat{j})_{\hat{L}}| > \frac{|\hat{L}|}{2}$, as $\{0_{\tilde{L}}, c_1\} \subseteq \tilde{L} \setminus \hat{L}$ implies that $|\tilde{L} \setminus \hat{L}| \geq 2$. Next, we assume that $n > 1$ and c_1 upper covers an element $l_1 \in (\downarrow x)_{\tilde{L}}$ in $\tilde{L}$. This implies that every element of $(\|x)_{\tilde{L}}$ is meet-irreducible in $\tilde{L}$. (Otherwise, there exists $m \in \{2, \ldots, n\}$ such that c_m lower covers some $u_m \in (\uparrow x)_{\tilde{L}}$ in $\tilde{L}$, and we have that $c_m <_{\tilde{L}} c_1$ and u_m upper covers c_m in $\tilde{L}$, yet $l_1 <_{\tilde{L}} u_m$ and c_1 upper covers l_1 in $\tilde{L}$, contradicting the existence of $\inf_{\tilde{L}}\{c_1, u_m\}$.) It follows that $\hat{j} = u_1$, and c_n is doubly irreducible in $\tilde{L}$. By Theorem 2.6, we then have that all elements $c_1, \cdots, c_{n-1}$ are join-reducible in $\tilde{L}$. Thus, for every $i \in \{1, \cdots, n\}$, c_i upper covers an element $l_i \in (\downarrow x)_{\tilde{L}}$ in $\tilde{L}$, and $1 \leq j < k \leq n$ implies that $l_k <_{\tilde{L}} l_j$, as illustrated in Figure 2.2. Then $|(\uparrow \hat{j})_{\hat{L}}| = |(\uparrow \hat{j})_{\tilde{L}}| = |(\uparrow c_n)_{\tilde{L}}| - n = \frac{|\tilde{L}|+1-2n}{2} = \frac{|\hat{L}|+|\tilde{L}\setminus\hat{L}|+1-2n}{2}$, making $|(\uparrow \hat{j})_{\hat{L}}| > \frac{|\hat{L}|}{2}$ because $\{c_1, l_1, \cdots, c_n, l_n\} \subseteq \tilde{L} \setminus \hat{L}$ implies that $|\tilde{L} \setminus \hat{L}| \geq 2n$. Therefore, $|(\uparrow \hat{j})_{\hat{L}}| > \frac{|\hat{L}|}{2}$ whether or not $\hat{j}$ is join-irreducible in $\tilde{L}$, and $\hat{L}$ is a counterexample to Conjecture 1.1, contradicting the minimality of $|\tilde{L}|$. □

Figure 2.2: $\hat{L} = (\uparrow x)_{\tilde{L}}$ must also be a counterexample to Conjecture 1.1.



Corollary 2.14. *Every element* $x \in \tilde{L} \setminus \{0_{\tilde{L}}, 1_{\tilde{L}}\}$ *is incomparable with at least three elements in* $\tilde{L}$.

Proof. Otherwise, there exists an element $x \in \tilde{L} \setminus \{0_{\tilde{L}}, 1_{\tilde{L}}\}$ such that $|(\|x)_{\tilde{L}}| \in \{0,1,2\}$. If $|(\|x)_{\tilde{L}}| \in \{0,1\}$, then $(\|x)_{\tilde{L}}$ is a chain in $\tilde{L}$, contradicting Theorem 2.13. If $|(\|x)_{\tilde{L}}| = 2$, then $(\|x)_{\tilde{L}}$ is either a chain in $\tilde{L}$, contradicting Theorem 2.13, or both elements of $(\|x)_{\tilde{L}}$ are doubly irreducible in $\tilde{L}$, contradicting Theorem 2.6. Therefore, $|(\|x)_{\tilde{L}}| \geq 3$. □

By similar reasoning, greater lower bounds can be proved for $|(\|x)_{\tilde{L}}|$. Further, the argument of Theorem 2.13 can be extended to show that the subposet $(\|x)_{\tilde{L}}$ of $\tilde{L}$ is not of the form $C_{(k)} = \bigcup_{1 \leq i \leq k} C_i$, where $\{C_i\}_{1 \leq j \leq k}$ is a sequence of k disjoint chains in $\tilde{L}$ such that $x \leq_{C_{(k)}} y$ implies that $x, y \in C_j$ for some $j \in \{1, \cdots, k\}$.

Theorem 2.15. *If L' is a subposet of $\tilde{L}$ and itself a lattice satisfying at least one of the following criteria:*

(i) $3 < |L'| < 8$;

(ii) $2 < |L'| < |\tilde{L}| - 2$ and $d \leq_{\tilde{L}} 1_{L'}$ for some dual atom d in $\tilde{L}$:

Then there exists an element $x \in L' \setminus \{0_{L'}, 1_{L'}\}$ that upper or lower covers some element $y \in \tilde{L} \setminus L'$ in $\tilde{L}$.

Proof. Otherwise, there exists a subposet L' of $\tilde{L}$ that is itself a lattice satisfying (i) or (ii) such that no element $x \in L' \setminus \{0_{L'}, 1_{L'}\}$ upper or lower covers any element $y \in \tilde{L} \setminus L'$ in $\tilde{L}$. First, we assume that L' satisfies (i), i.e. that $3 < |L'| < 8$. Provided by Kyuno in [5] are 2, 5, 15, and 53 Hasse diagrams for all unlabeled lattices L with, respectively, four, five, six, and seven elements, matching the numbers of unlabeled lattices obtained also in [3]. We observe from these diagrams that all lattices L with $3 < |L| < 8$ have at least two doubly irreducible elements. Therefore, there exist two doubly irreducible elements x_1 and x_2 in L'. It follows that x_1 and x_2 are also doubly irreducible in $\tilde{L}$, as neither x_1 nor x_2 upper or lower covers any element $x_3 \in \tilde{L} \setminus L'$ in $\tilde{L}$. This contradicts Theorem 2.6, and so L' does not satisfy (i).

Next, we assume that L' satisfies (ii), i.e. that $2 < |L'| < |\tilde{L}| - 2$ and $d \leq_{\tilde{L}} 1_{L'}$ for some dual atom d in $\tilde{L}$. If $1_{L'}$ is join-reducible in L', then we let $\hat{L}$ be the lattice L' and consider any join-irreducible element $\hat{j}$ in $\hat{L}$. For all $x \in \hat{L} \setminus \{0_{\hat{L}}, 1_{\hat{L}}\}$, if x upper covers an element y in $\tilde{L}$, then y belongs to $\hat{L}$. This implies that $\hat{j}$ is also join-irreducible in $\tilde{L}$. Also, if x lower covers an element y in $\tilde{L}$, then y belongs to $\hat{L}$. It follows that $|(\uparrow \hat{j})_{\hat{L}}| \geq |(\uparrow \hat{j})_{\tilde{L}}| - 1$, as $d \leq_{\tilde{L}} 1_{L'}$. Hence, $|(\uparrow \hat{j})_{\hat{L}}| > \frac{|\tilde{L}|}{2} - 1 = \frac{|\hat{L}| + |\tilde{L} \setminus \hat{L}| - 2}{2}$, and we have that $|(\uparrow \hat{j})_{\hat{L}}| > \frac{|\hat{L}|}{2}$, as $|L'| < |\tilde{L}| - 2$ and $\hat{L} = L'$ together imply that $|\tilde{L} \setminus \hat{L}| \geq 3$. Now, if $1_{L'}$ is join-irreducible in L', then we let $\hat{L}$ be the subposet $L' \setminus \{1_{L'}\}$ of L', which is a lattice by Lemma 1.2, and again consider any join-irreducible element $\hat{j}$ in $\hat{L}$. In this case, $\hat{j}$ is join-irreducible in $\tilde{L}$ and $|(\uparrow \hat{j})_{\hat{L}}| \geq |(\uparrow \hat{j})_{\tilde{L}}| - 2$. We have that $|(\uparrow \hat{j})_{\hat{L}}| > \frac{|\tilde{L}|}{2} - 2 = \frac{|\hat{L}| + |\tilde{L} \setminus \hat{L}| - 4}{2}$, and $|(\uparrow \hat{j})_{\hat{L}}| > \frac{|\hat{L}|}{2}$, as $|L'| < |\tilde{L}| - 2$ and $\hat{L} = L' \setminus \{1_{L'}\}$ imply that $|\tilde{L} \setminus \hat{L}| \geq 4$. Therefore, in both cases $\hat{L}$ is a counterexample to Conjecture 1.1. This contradicts the minimality of $|\tilde{L}|$, and so L' does not satsify (ii). Thus, L' satisfies neither (i) nor (ii). □

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