ON THE OSCULATORY BEHAVIOR OF SURFACE SCROLLS

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Dedicated to Silvio Greco in occasion of his 60-th birthday.

A lower bound for the dimensions of the second osculating spaces to any surface scroll is given, relying on the special feature of osculating hyperplane sections to such surfaces. Moreover a class of counterexamples to the even dimensional part of a conjecture of Piene-Tai is provided.

Introduction and statement of the results.

Let $S \subset \mathbb{P}^N$ be a non-degenerate smooth complex surface embedded in the projective space, let $L = (\mathcal{O}_{\mathbb{P}^N}(1))^S$ be the hyperplane line bundle and let $V$ be the vector subspace of $H^0(L)$ giving rise to the embedding. For every integer $k \geq 0$ let $J_k L$ be the $k$-th jet bundle of $L$ and let $j_k : V \otimes \mathcal{O}_S \to J_k L$ be the sheaf homomorphism sending any section $s \in V$ to its $k$-th jet $j_k \circ s$ evaluated at $x$, for every $x \in S$. Then the $k$-th osculating space to $S$ at $x$ is defined as $\text{Osc}_k^x (S) := \mathbb{P} (\text{Im} (j_k \circ s))$. Identifying $\mathbb{P}^N$ with $\mathbb{P}(V)$ (the set of codimension 1 vector subspaces of $V$) we see that $\text{Osc}_k^x (S)$ is a linear subspace of $\mathbb{P}^N$. To avoid that it fills up the whole ambient space we assume that $N$ is large enough; for instance, for $k = 2$, a reasonable assumption is that $N \geq 6$ or even 5, depending

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on the regularity of the surface we are dealing with. Recalling that $J_k L$ has rank \( \binom{k+2}{2} \), we have
\[
\dim(\text{Osc}_k^F(S)) \leq \binom{k+2}{2} - 1.
\]
For $k \geq 2$ it may happen that this is a strict inequality for every point $x \in S$. Note that if this happens for $k = 2$, i.e., $\dim(\text{Osc}_2^F(S)) \leq 4$ for all $x \in S$, then the homogeneous coordinates of the points of $\mathbb{P}^N$ lying on $S$ (and hence any section $s \in V$) satisfy a second order linear partial differential equation in terms of local coordinates (a Laplace equation, in the classical terminology of projective differential geometry) [10]. Differentiating further up to the order $k$, this equation gives more relations and one can easily see that
\[
(\#_k) \quad \dim(\text{Osc}_k^F(S)) \leq 2k \quad \text{for every } x \in S.
\]
Of course, once $N$ is fixed, this is meaningful only for $k \leq m := \left\lfloor \frac{N+1}{2} \right\rfloor$.

Note that this is exactly what happens for scrolls. Actually in this case there are local coordinates $(u, v)$ around every point $x \in S$ such that the homogeneous coordinates $x_i$, $(i = 0, \ldots, N)$ of the points of $S$ near $x$, locally, can be written as $x_i = a_i(u) + vb_i(u)$, where $a_i$ and $b_i$ are holomorphic functions of $u$. Since every section $s \in V$ is a linear combination $s = \sum_{i=0}^{N} \lambda_i x_i$ we thus see that the second derivative $s_{uv}$ vanishes at every point. Thus $\dim(\text{Osc}_2^F(S)) \leq 4$ for every $x \in S$, hence $(\#_k)$ holds for every $k$. Apart from scrolls, sporadic surfaces satisfying $(\#_k)$ for every $k$ are known: they have been found by Togniatti [12], sec. 3, Dye [2], Theorem 4, and Perkinson [9], Theorem 3.2.

There is a conjecture of Piene and Tai [10], related to the inequalities $(\#_k)$, stating the following.

Let $S \subset \mathbb{P}^N$ ($N \geq 5$) be a non-degenerate complex smooth surface such that $(\#_k)$ holds for every $k$ and $(\#_m)$ is an equality, where $m$ is defined above. Then $(S, L, V)$ is either $(\mathbb{P}_0, [C_0 + mf], H^0)$ if $N = 2m + 1$ (balanced rational normal scroll), or $(\mathbb{F}_1, [C_0 + (m + 1) f], H^0)$ if $N = 2m + 2$ (semibalanced rational normal scroll). Here $\mathbb{F}_e$ denotes the Segre-Hirzebruch surface of invariant $e \geq 0$, $C_0$ stands for a section of minimal self-intersection and $f$ for a fibre.

For $N$ odd the conjecture is true, as proved by Ballico, Piene and Tai [1], by using adjunction theory. In this paper I prove the following results.

**Theorem A.** For any linearly normal elliptic scroll $S \subset \mathbb{P}^N$ ($N \geq 6$) of invariant $-1$, we have $\dim(\text{Osc}_4^m(S)) = 2m$. 
In particular, for $N$ even the conjecture above is not true, even in the setting of scrolls (compare with the discussion in [9], end of p. 496 concerning the setting of toric surfaces).

**Theorem B.** Let $S \subset \mathbb{P}^N$ $(N \geq 5)$ be any scroll over a smooth curve; then $\dim(\text{Osc}_k(S)) \geq 3$ for every $x \in S$.

The meaning of Theorem B is that the osculatory behavior of scrolls is not so bad, as we will see. The proof of both results simply relies on the consideration of the linear system of $k$-osculating hyperplane sections to a smooth projective surface and its special feature in case of a surface scroll. Finally I would like to note that both theorems can be easily rephrased in terms of Weierstrass schemes associated to the Wronskian system coming from the jet bundles $J_kL$ (see [8], Section 4). I am indebted to Dan Laksov for drawing my attention to [8].

The paper is organized as follows. In Section 1 I discuss linear systems of $k$-osculating hyperplane sections and prove Theorem B in two different ways. Theorem A is proved in Section 2, where the subject is reconsidered with the help of the jumping sets of suitable ample and spanned line bundles. In Section 3 I describe a further pathology of the osculatory behavior of surfaces, which makes clear the meaning of Theorem B.

The word surface will always mean smooth complex projective surface. Let $S \subset \mathbb{P}^N$, $L$, $V$ be as at the beginning. I denote by $|V|$ the linear system defined by the vector subspace $V \subseteq H^0(S, L)$ (which, in general, is not a complete linear system, in spite of the notation). Sometimes I refer to $S$ as the abstract surface and to the pair $(S, V)$ as the embedded surface. Accordingly, I say that $(S, V)$ $(S, L)$ if $V = H^0(L)$ is a scroll to mean that $S, L, V$ are as above with $S$ a $\mathbb{P}^1$-bundle over a smooth curve, $|V|$ very ample, and $L_f = \mathcal{O}_S(1)$ for every fibre $f$ of $S$. I adopt the additive notation for the tensor product of line bundles and, with a little abuse, I do not distinguish between a line bundle and the corresponding invertible sheaf. In particular, if $(S, V)$ is a scroll and $f$ is a fibre, $L - f$ stands for the line bundle $L \otimes \mathcal{O}_S(-f)$; moreover I denote by $|V - f|$ the linear system $\{(s)_0 - f \mid s \in V \text{ and } (s)_0 \supset f\}$ and by $V(-f)$ the corresponding vector subspace of $H^0(S, L - f)$. Of course, up to adding $f$ as a fixed component, $|V - f|$ can be identified with a linear subsystem of $|V|$.

1. Linear systems of osculating hyperplane sections.

Let $S$, $L$ and $V$ be as in the Introduction. Recall that a hyperplane $H \in \mathbb{P}^{N-1}$ is said to be $k$- osculating to $S$ at $x$ if $H \supseteq \text{Osc}_k(S)$. Identifying the dual
projective space $\mathbb{P}^{N_V}$ with the linear system $|V|$, $H$ corresponds to the divisor 
$(s)_0$ of a section $s \in V$ and the fact that $H$ is $k$-osculating to $S$ at $x$ is equivalent to the condition $j_{k,x}(s) = 0$, i.e., $(s)_0 \in |V - (k + 1)x|$. In other words, the dual of $\mathbb{P}(\text{Ker} j_{k,x})$ can be identified with the linear system $|V - (k + 1)x|$ of hyperplane sections having a point of multiplicity $\geq (k + 1)$ at $x$. From the equality $\dim V = \dim(\text{Ker}(j_{k,x})) + \dim(\text{Im}(j_{k,x}))$, we thus get for every $k \geq 1$,

$$\dim(\text{Osc}_1^k(S)) + \dim(|V - (k + 1)x|) = N - 1. \quad (1.0_k)$$

**Remark 1.1.** Let $S \subset \mathbb{P}^N = \mathbb{P}(V)$ be a non-degenerate surface. Then

$$\dim(\text{Osc}_1^k(S)) = 2 + \text{codim}(|V - (k + 1)x|, |V - 2x|).$$

**Proof.** Since $\text{Osc}_1^k(S)$ is the projective tangent plane to $S$ at $x$, the equality simply follows by subtracting $(1.0_1)$ from $(1.0_k)$. \qed

Now suppose that $(S, V)$ is a scroll and let $f_x$ be the fibre of $S$ through a point $x \in S$. If $D \in |V - 2x|$ then $D = f_x + R$, where $R$ is an effective divisor in the linear system $|V - f_x|$, passing through $x$, i.e., $R \in |V - f_x - x|$. This follows immediately from the fact that $Df_x = 1$ for every $D \in |V|$, since $(S, L, V)$ is a scroll. Actually, if $D \in |V - 2x|$ would not contain $f_x$, then we would get

$$1 = Df_x \geq \text{mult}_x(D) \text{ mult}_x(f_x) \geq 2,$$

a contradiction. Moreover, if $D \in |V - 3x|$, then $R$ must have a double point at $x$, i.e., $R \in |V - f_x - 2x|$. But then, arguing as before we have $D = 2f_x + T$, where $T$ is an effective divisor in the linear system $|V - 2f_x|$, passing through $x$, i.e., $T \in |V - 2f_x - x|$. More generally, iterating this argument we have

**Remark 1.2.** Let $(S, V)$ be a scroll and let $f_x$ be the fibre through any point $x \in S$. Then

$$|V - (k + 1)x| = f_x + |V - f_x - kx| = \ldots = kf_x + |V - kf_x - x|.$$

In particular,

$$\dim(|V - (k + 1)x|) = \dim(|V - f_x - kx|) = \ldots = \dim(|V - kf_x - x|). \quad (1.2.1)$$

Now let $(S, V)$ be a scroll. We give two different proofs of Theorem B
1.3. First proof of Theorem B. In view of Remark (1.1) it is equivalent to show that \(|V - 2x| \neq |V - 3x|\) for every \(x \in S\). Since \((S, V)\) is a scroll, by Remark (1.2) we know that \(|V - 3x| = 2f_s + |V - 2f_s - x|\). Assume, by contradiction, that

\[|V - 2x| = 2f_s + |V - 2f_s - x|\]

for some point \(x \in S\). Then every hyperplane tangent to \(S\) at \(x\) is tangent along the whole fibre \(f_s\). As a consequence the tangent plane to \(S\) is constant along \(f_s\). But this contradicts the finiteness of the Gauss map \(\gamma : S \to \mathbb{G}(2, N)\) sending every point \(y \in S\) to \(\text{Osc}^1_y(S)\), regarded as a point of the Grassmannian \(\mathbb{G}(2, N)\) of planes of \(\mathbb{P}^N\) (e. g., see [13], Theorem 2.3, c), p. 21). \(\square\)

The second proof of Theorem B relies on two lemmas of some interest in themselves. The former one will be helpful also in Section 2.

Lemma 1.4. Let \((S, V)\) be a scroll. Then \(Bs(|V - f_s|) = \emptyset\) for every \(x \in S\).

Proof. (inspired by [11], Lemma 0.10.1) Let \(y \in S\) and let \(D\) be the pull-back via the embedding given by \(V\) of a hyperplane of \(\mathbb{P}^N\) containing \(f_s\), but not containing \(y\) if \(y \notin f_s\), and not containing the tangent plane to \(S\) at \(y\) if \(y \in f_s\). In both cases we have that \(D = f_s + R\), with \(R \not
\)

Now, for any \(x \in S\), let \(\varphi_x : S \to \mathbb{P}\) be the map associated with the linear system \(|V - f_s|\). Then Lemma 1.4 says that \(\varphi_x\) is a morphism. We have \(\dim |V| \geq 3\), since \(|V|\) is very ample, hence \(\dim |V - f_s| \geq 1\) for every \(x \in S\). Since \(\varphi_x(S)\) is non-degenerate in the projective space \(\mathbb{P}(V(-f_s))\), this says that \(\dim \varphi_x(S) \geq 1\).

Lemma 1.5. Let \((S, V)\) be a scroll and let \(\varphi_x\) be the morphism defined above.

i) \(\dim \varphi_x(S) = 1\) for some (equivalently every) point \(x \in S\) if and only if \((S, L, V) = (\mathbb{P}^1 \times \mathbb{P}^1, O_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1), H^0(L))\).

Let \(\dim \varphi_x(S) = 2\).

ii) If \((S, L) = (\mathbb{P}^e, [C_0 + (e + 1)f]), e > 0\) then every fibre of \(\varphi_x\) is either a finite set or a finite set plus the fundamental section.

iii) In any other case every fibre of \(\varphi_x\) is a finite set.

Proof. If \(\dim(\varphi_x(S)) = 1\) then \(\varphi_x\) contracts a positive dimensional family of curves. The proof will be done by analyzing which curves on \(S\) can be contracted by \(\varphi_x\). Note that \(|V - f_s|\), hence \(|L - f_s|\), has no fixed components by Lemma 1.4. So, for any irreducible curve \(C \subset S\) there exists a divisor \(D \in |L - f_s|\) not containing \(C\) among its components, hence \(DC \geq 0\). This shows that \(L - f_s\) is nef. Let \(C_0\) and \(f\) denote a fundamental section and a fibre of \(S\), respectively. Since \((S, V)\) is a scroll we have that \(L \equiv [C_0 + mf]\)
(numerical equivalence) for a suitable integer $m$. Let $q$ and $e$ denote the irregularity and the invariant of $S$. Since $L - f_x \equiv [C_0 + (m - 1)f]$ is nef, we get

$$m - 1 \geq \begin{cases} e, & \text{if } e \geq 0, \\ e/2, & \text{if } e < 0. \end{cases}$$

(1.5.1)

Now let $C \subset S$ be an irreducible curve contracted by $\varphi_x$. Then $(L - f_x)C = 0;$ moreover $C^2 \leq 0$, since $\dim \varphi_x(S) \geq 1$. Since $(L - f_x)f = 1$, $C$ cannot be a fibre: so there are two possibilities: either j) $C = C_0$, or jj) $C \equiv aC_0 + bf$ for some integers $a, b$ satisfying the conditions:

$$a > 0 \quad \text{and} \quad b \geq \begin{cases} ae, & \text{if } e \geq 0, \\ ae/2, & \text{if } e < 0, \end{cases}$$

by [3], p. 382. In case jj) we get

$$0 = (L - f_x)C = (C_0 + (m - 1)f)(aC_0 + bf) = -ae + b + (m - 1)a.$$  

(1.5.3)

If $e \geq 0$ both summands in the right hand being non negative by (1.5.1), (1.5.2), this implies $b = ae$ and $m = 1$, which, in view of (1.5.1) gives $e = 0$; hence $b = 0$ and then $C \equiv aC_0$. But this contradicts the fact that $C$ is irreducible, unless we are in case j). On the other hand, if $e < 0$, we can continue (1.5.3) as follows:

$$0 = (-ae/2 + b) + a(m - 1 - e/2),$$

where both summands are non negative in view of (1.5.1), (1.5.2). We thus get $b = ae/2, m - 1 = e/2$, hence $[C] \equiv a(L - f)$. But this gives a contradiction, since $C^2 \leq 0$, while $(L - f)^2 = (L^2 - 2) \geq 0$, the equality implying that $(S, V)$ is the quadric surface, i.e., $e = 0$, a contradiction. Now suppose we are in case j). Thus

$$0 = (L - f_x)C = (C_0 + (m - 1)f)C_0 = -e + m - 1.$$  

Due to (1.5.1) it cannot be $e < 0$; so $e \geq 0$ and $m = e + 1$. But then $\deg L_{C_0} = LC_0 = (C_0 + (e + 1)f)C_0 = 1$. Since $L$ is a very ample line bundle, this clearly implies $q = 0$. Thus $S = \mathbb{P}_e$ and $L = \langle C_0 + (e + 1)f \rangle$. If $e = 0$, then $L = \langle C_0 + f \rangle$, hence $|V - f_x| = |L - f_x| = |C_0|$. In this case $\varphi_x$ is just the projection of $\mathbb{P}_0 = \mathbb{P}^1 \times \mathbb{P}^1 = C_0 \times f$ onto the second factor. On the other hand, if $e > 0$ then $C_0$ is the only curve contracted by $\varphi_x$. This proves all the assertions. □
1.6. Second proof of Theorem B. As already noted, it is equivalent to show that
\(|V - 2x| \neq |V - 3x|\) for every \(x \in S\). Since \((S, V)\) is a scroll, by Remark 1.2
the linear system on the left corresponds to \(|V - f_x - x|\), while that on the right corresponds to \(|V - 2f_x - x|\). So we have the equality \(|V - 2x| = |V - 3x|\) if
and only if

\[
1.6.1 \quad f_x \subseteq 
\text{Bs}(|V - f_x - x|).
\]

But this cannot happen. To see this, consider the morphism \(\varphi_x : S \to \mathbb{P}\),
developed by the linear system \(|V - f_x|\). Since \(N \geq 5\), by Lemma 1.5 \(\varphi_x\) has a
2-dimensional image and all its fibres cut every fibre of the ruling projection at
a finite set. On the other hand

\[
\text{Bs}(|V - f_x - x|) = \bigcap_{D \in |V - f_x|, D \not
\in \mathbb{P}} \text{supp}(D) = \varphi_x^{-1}(\varphi_x(x)).
\]

Therefore the base locus of \(|V - f_x - x|\) must intersect every fibre of the ruling
of \(S\) (in particular \(f_x\)) at finite set only. This shows that \((1.6.1)\) cannot occur. \(\square\)

Remark 1.7. Let \((S, V)\) be a scroll over a smooth curve \(B\) and let \(\pi : S \to B\)
be the projection. Then \(S = \mathbb{P}(\mathcal{E})\), where \(\mathcal{E}\) is the very ample vector bundle of
rank 2 given by \(\pi_*L\). Then the very ampleness of \(\mathcal{E}\) is equivalent to the equality

\[
1.7.1 \quad h^0(\mathcal{E}(-\pi(x) - \pi(y))) = h^0(\mathcal{E}) - 4,
\]

for every \(x, y \in S\) (e. g., see [4], Lemma 1). On the other hand, since all elements of \(|V|\) have intersection 1 with any fibre, we see that \(|V - x - x'| = f_x + |V - f_x|\) for any \(x' \in f_x, x' \neq x\). Hence, due to the very ampleness of \(|V|
we have \(\text{dim}(|V - f_x|) = \text{dim}(|V|) - 2\). Now, let \(y \in S\). For the same reason
as before we see that \(|V - f_x - y - y'\) = \(f_y + |V - f_x - f_y|\), where \(y'\) is any
point of \(f_y\) distinct from \(y\). As in (1.6) we have

\[
\text{Bs}(|V - f_x - y|) = \bigcap_{D \in |V - f_x|, D \not \in \mathbb{P}y} \text{supp}(D) = \varphi_x^{-1}(\varphi_x(y)).
\]

By Lemma 1.5 this set cuts out a finite (possibly empty) set on \(f_y\). Thus there
exists a point \(y' \in f_y\) such that \(y' \not \in \text{Bs}(|V - f_x - y|)\). Hence \(|V - f_x - y - y'|\) has
codimension 1 in \(|V - f_x - y|\). On the other hand \(|V - f_x - y|\) has codimension
1 in \(|V - f_x|\), by Lemma 1.4. Putting everything together we get

\[
\text{dim}(|V - f_x - f_y|) = \text{dim}(|V - f_x - y - y'|) =
\text{dim}(|V - f_x - y|) - 1 = \text{dim}(|V - f_x|) - 2.
\]
Thus the very ampleness of \(|V|\) implies that

\[(1.7.2) \quad \dim |V - f_x - f_y| = \dim |V - f_x| - 2 = \dim |V| - 4.\]

Note that when \(V = H^0(S, L)\) \((1.7.2)\) is clearly equivalent to \((1.7.1)\) in view of the isomorphism \(H^0(S, L) \cong H^0(B, \mathcal{E})\). Thus \((1.7.2)\) can be regarded as a generalization of \((1.7.1)\) to non complete linear systems.

2. Linearly normal elliptic scrolls of invariant\(−1\).

2.1. Proof of Theorem A. Let \(C\) be a smooth curve of genus 1. Recall that the \(\mathbb{P}^1\) bundle of invariant \(-1\) over \(C\) is the surface \(S = \mathbb{P}(\mathcal{E})\), where \(\mathcal{E}\) is the holomorphic vector bundle of rank 2 defined by the non-split extension

\[(2.1.1) \quad 0 \to \mathcal{O}_C \to \mathcal{E} \to \mathcal{L} \to 0,\]

where \(\mathcal{L} \in \text{Pic}(C)\) has degree 1. Let \(\pi : S \to C\) be the ruling projection, let \(C_0\) be the tautological section on \(S\) and let \(\delta\) be a divisor on \(C\) of degree \(m + 1 \geq 3\). Then the line bundle \(L := \mathcal{O}_S(C_0 + \pi^*\delta)\) is very ample (e. g., see [3], Ex. 2.12 (b), p. 385) and the map associated with \(|L|\) embeds \(S\) as a scroll of degree \(2m + 3\) in \(\mathbb{P}^{2m+2}\). Set \(V = H^0(L)\) and let \(x\) be any point of \(S\). Then

\[(2.1.2) \quad \dim(\text{Osc}^m_x(S)) = 2m + 1 - \dim(|L - (m + 1)x|),\]

by \((1.0_m)\). On the other hand, since \((S, L)\) is a scroll we have

\[(2.1.3) \quad |L - (m + 1)x| = mf_x + |L - mf_x - x|,\]

by Remark 1.2. Note that the line bundle \(L - mf_x = \mathcal{O}_S(C_0 + \pi^*(\delta - m\pi(x)))\) is spanned, since \(\deg(\delta - m\pi(x)) = 1\) (see [3], Ex. 2.12 (a), p. 385). Hence

\[(2.1.4) \quad \dim(|L - mf_x - x|) = \dim(|L - mf_x|) - 1.\]

On the other hand, by twisting \((2.1.1)\) by \(\mathcal{O}_C(\delta - m\pi(x))\) we immediately see that \(h^0(L - mf_x) = h^0(\mathcal{E}(\delta - m\pi(x))) = 3\). Combining this with \((2.1.3)\) and \((2.1.4)\) gives \(\dim(|L - (m + 1)x|) = 1\) and then \((2.1.2)\) shows that \(\dim(\text{Osc}^m_x(S)) = 2m\), for every point \(x \in S\). \(\square\)

Theorem A, especially case \(m = 2\), can be seen from a slightly more general point of view, suggested by the discussion in Section 1. Actually, if \((S, V)\) is a scroll, by combining Remark 1.1 with \((1.2.1)\) we get

\[\dim(\text{Osc}^2_x(S)) = 2 + \text{codim}(|V - 3x|, |V - 2x|)\]
\[= 2 + \text{codim}(|V - f_x - 2x|, |V - f_x - x|),\]
for every \( x \in S \). So \( \dim(\text{Osc}_x^2(S)) = 3 \) if and only if \( \dim(|V - f| - 2x|) = \dim(|V - f - x|) - 1 \). Now suppose that \( L - f \) is ample for a fibre \( f \) of \( S \). Since ampleness is a numerical condition, this means that \( L - f \) is ample for every fibre \( f \) of \( S \). By Lemma 1.4 we know that the vector subspace \( V(-f) \subseteq H^0(L - f) \) spans \( L - f \). Under the assumption above, fix a fibre \( f \) of \( S \). Then, from [7], Proposition 3.1 we have the equality

\[
\{ x \in f \mid \dim(|V - f - 2x|) = \dim(|V - f - x|) - 1 \} = \mathcal{J}_1(V(-f)) \cap f,
\]

where \( \mathcal{J}_1(V(-f)) \) is the first jumping set of \((S, V(-f))\), i.e., the ramification locus of the morphism defined by \( |V - f| \). This argument proves the following

**Proposition 2.2.** Let \((S, V)\) be a scroll and assume that \( L - f \) is ample, where \( f \) is a fibre of \( S \). Then

\[
\{ x \in S \mid \dim(\text{Osc}_x^2(S)) = 3 \} = \bigcup_f (f \cap \mathcal{J}_1(V(-f))),
\]

the union being taken over all fibres of \( S \).

Recall that \( \mathcal{J}_1(W) = \emptyset \) if the morphism defined by the linear system \( |W| \) is an immersion [7], Remark 2.3.2. We thus get.

**Corollary 2.3.** If \((S, V)\) is a scroll and the morphism defined by \( |V - f| \) is an immersion for every fibre \( f \) of \( S \), then

\[
\dim(\text{Osc}_x^2(S)) = 4 \quad \text{for every} \ x \in S.
\]

Note that the case of linearly normal elliptic scrolls of invariant \(-1\) with \( N \geq 6 \) discussed in Theorem A fits into the Corollary above. Actually for the line bundle \( L \) defined in the proof of Theorem A it turns out that \( L - f \) is very ample for every fibre \( f \), by [3], Ex. 2.12 (b), p. 385. However, in principle there could be other scrolls, not linearly normal and of higher genus, satisfying the assumption in Corollary 2.3. They would provide further counterexamples in \( \mathbb{P}^n \) to the even dimensional part of the conjecture of Piene-Tai.

An interpretation in terms of jumping sets can be extended also to Theorem B. Let \((S, V)\) be a scroll and suppose that \( L - f \) is ample for a (hence every) fibre \( f \) of \( S \). By Lemma 1.4 \( V(-f) \) spans \( L - f \) for a given fibre \( f \) and then we can also consider the second jumping set \( \mathcal{J}_2(V(-f)) \) of \((S, V(-f))\) [7], Section 1. By definition the set \( f \cap \mathcal{J}_2(V(-f)) \) consists of the points \( x \in f \) such that \( |V - f - x| = |V - f - 2x| \). But Theorem B says that there there are no such points. We thus get the following
**Corollary 2.4.** Let \((S, V)\) be a scroll and assume that \(L - f\) is ample, where \(f\) is a fibre of \(S\). Then
\[
f \cap F_2(V(-f)) = \emptyset,
\]
for every fibre \(f\) of \(S\).

3. Further pathology of osculation.

From Remark 1.1 we know that
\[
\dim(\text{Osc}^2_x(S)) = 2 + \dim([V - 3x], [V - 2x]).
\]
Thus \(\dim(\text{Osc}^2_x(S)) = 2\) if and only if \([V - 3x] = [V - 2x]\) and Theorem B says that this cannot happen for scrolls. In fact there are surfaces for which \(\dim(\text{Osc}^2_x(S)) = 2\) for some point \(x \in S\). This means that every tangent hyperplane at such a point \(x\) is osculating. An interesting example of this situation is the so-called Togliatti’s Del Pezzo surface.

3.1. Example. Let \((S, L = -K_S)\) be the Del Pezzo surface with \(K_S^2 = 6\). Call \(X\) the surface \(S\) embedded by \(|L|\); then \(X\) is a smooth surface of degree 6 in \(\mathbb{P}^6\). Recall that \(S\) is isomorphic to \(\mathbb{P}^2\) blown-up at three non-collinear points \(p_0, p_1, p_2\). Choose homogeneous coordinates \((x_0, x_1, x_2)\) in \(\mathbb{P}^2\) in such a way that \(p_0 = (1 : 0 : 0), p_1 = (0 : 1 : 0), p_2 = (0 : 0 : 1)\) and fix the basis of \(H^0(L)\) corresponding to the 7 cubic monomials
\[
x_0^2x_1, x_0^2x_2, x_0x_1^2, x_0x_2^2, x_1^2x_2, x_1x_2^2, x_0x_1x_2.
\]
Then \(X\) is the image of the rational map \(\mathbb{P}^2 \dashrightarrow \mathbb{P}^6\) defined by these monomials. One can see that the secant variety of \(X\) is a cubic hypersurface of \(\mathbb{P}^6\) not containing the point \(c = (0 : \ldots : 0 : 1)\). E. g., one can write down the explicit equation of the secant variety by using MAPLE and then this property can be checked directly. Thus the projection \(\pi_c : \mathbb{P}^6 \dashrightarrow \mathbb{P}^5\) from \(c\) defines an embedding of \(X\) in \(\mathbb{P}^5\). Let \(Y = \pi_c(X)\). Then \(Y\) is the image of the rational map \(\mathbb{P}^2 \dashrightarrow \mathbb{P}^5\) defined by the 6 monomials
\[
x_0^2x_1, x_0^2x_2, x_0x_1^2, x_0x_2^2, x_1^2x_2, x_1x_2^2.
\]
A very interesting property of the surface \(Y\) discovered by Togliatti [12] is that its 2-osculating spaces have dimension \(\leq 4\) at every point. But, in fact there are points of \(Y\) where the 2-osculating space coincides with the tangent plane ([10], Example 2.4, [5], Proposition 4.3). To recognize them, let \(\sigma : S \dashrightarrow \mathbb{P}^2\) be the
blow-up at the three points \( p_i \), let \( e_i = \sigma^{-1}(p_i) \) and for \( i < j \) let \( l_{ij} \) denote the proper transform on \( S \) of the line \( \langle p_i, p_j \rangle \) joining \( p_i \) and \( p_j \). The six curves \( e_i, l_{ij} \) (\( 0 \leq i < j \leq 2 \)) define a 1-cycle \( E \) on \( S \), which is mapped to a skew hexagon on \( Y \); let \( V \) be the set of the 6 vertices, i.e., the set of points at which two irreducible components of \( E \) meet. Then \( \dim(\text{Osc}_2^2(Y)) = 2 \) for every \( x \in V \). Call \( V \) the subspace of \( H^0(L) \) generated by the elements corresponding to the monomials in (3.1.1). Then the condition above can be rephrased as follows:

\[
|V - 2x| = |V - 3x| \quad \text{for every } x \in V.
\]

Understanding this equality in terms of the linear system of plane cubics representing the hyperplane sections of \( Y \) is an instructive exercise. Here is a sketch of the argument. Recall that \( L = \sigma^*\Theta_{2L}(3) = e_0 - e_1 - e_2 \), fix a point \( x \in V \), e.g., \( x = e_0 \cap l_{01} \), and consider an element \( H \in |V - 2x| \). Since \( H e_0 = H l_{01} = 1 \) we see that \( H \) must contain both \( e_0 \) and \( l_{01} \) as components. Thus \( H = \sigma^* \Gamma = e_1 - e_2 - e_3 \), where the plane cubic \( \Gamma \) consists of the line \( \langle p_0, p_1 \rangle \) and a conic \( \gamma \) containing \( p_0 \) and \( p_2 \). On the other hand, since \( H \in |V| \), the polynomial defining \( \Gamma \) is a linear combination of the monomials in (3.1.1). Since \( \langle p_0, p_1 \rangle \) corresponds to the factor \( x_2 \), this implies that the quadratic polynomial defining \( \gamma \) is a linear combination of \( x_0 x_2, x_1^2, x_1 x_2 \) (but not \( x_0 x_1 \), since \( x_0 x_1 x_2 \) corresponds to an element not in \( V \)). Therefore \( |V - 2x| \) corresponds to the linear system of plane cubics generated by \( x_0 x_2^2, x_1^2 x_2, x_1 x_2^2 \).

Let \( \Gamma_1, \Gamma_2, \Gamma_3 \) be the cubics defined by these 3 generators. It is easy to see that for \( i = 1, 2, 3 \) the element \( H_i = \sigma^* \Gamma_i - e_0 - e_1 - e_2 \) has a point of multiplicity \( \geq 3 \) at \( x \). E.g., \( H_1 = \sigma^* \langle 2(p_0 p_1) + (p_1 p_2) \rangle - e_0 - e_1 - e_2 = 2l_{01} + e_0 + 2e_1 + l_{12} \). Since \( H \) is a linear combination of \( H_1, H_2, H_3 \), we thus conclude that \( H \in |V - 3x| \).

There are more surfaces for which \( \dim(\text{Osc}_2^2(S)) = 2 \) at a finite set of points \( x \). In fact this happens also for the two new surfaces with inflectionary pathology recently discovered by Perkinson in the setting of toric varieties [9], Theorem 3.2, cases (4), (5). In these cases, as well as in Example 3.1, the linear system \( |V| \) is not complete. I would like to mention however that this pathology can occur also when \( |V| \) is a complete very ample linear system, as shown in [6], Lemma 4.1, i).

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REFERENCES


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