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ON THE OSCULATORY BEHAVIOR OF SURFACE SCROLLS

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Dedicated to Silvio Greco in occasion of his 60-th birthday.

A lower bound for the dimensions of the second osculating spaces to any surface scroll is given, relying on the special feature of osculating hyperplane sections to such surfaces. Moreover a class of counterexamples to the even dimensional part of a conjecture of Piene-Tai is provided.

Introduction and statement of the results.

Let $S \subset \mathbb{P}^N$ be a non-degenerate smooth complex surface embedded in the projective space, let $L = (\mathcal{O}_{\mathbb{P}^N}(1))_S$ be the hyperplane line bundle and let V be the vector subspace of $H^0(L)$ giving rise to the embedding. For every integer $k \geq 0$ let $J_k L$ be the k-th jet bundle of L and let $j_k : V \otimes \mathcal{O}_S \to J_k L$ be the sheaf homomorphism sending any section $s \in V$ to its k-th jet $j_{k,x}(s)$ evaluated at x, for every $x \in S$. Then the k-th osculating space to S at x is defined as $\mathrm{Osc}^k_x(S) := \mathbb{P}(\mathrm{Im}(j_{k,x}))$. Identifying $\mathbb{P}^N$ with $\mathbb{P}(V)$ (the set of codimension 1 vector subspaces of V) we see that $\mathrm{Osc}^k_x(S)$ is a linear subspace of $\mathbb{P}^N$. To avoid that it fills up the whole ambient space we assume that N is large enough; for instance, for $k = 2$, a reasonable assumption is that $N \geq 6$ or even 5, depending

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on the regularity of the surface we are dealing with. Recalling that $J_k L$ has rank $\binom{k+2}{2}$, we have

$$\dim(\mathrm{Osc}^k_x(S)) \leq \binom{k+2}{2} - 1.$$

For $k \geq 2$ it may happen that this is a strict inequality for every point $x \in S$. Note that if this happens for $k = 2$, i. e., $\dim(\mathrm{Osc}^2_x(S)) \leq 4$ for all $x \in S$, then the homogeneous coordinates of the points of $\mathbb{P}^N$ lying on S (and hence any section $s \in V$) satisfy a second order linear partial differential equation in terms of local coordinates (a Laplace equation, in the classical terminology of projective differential geometry) [10]. Differentiating further up to the order k, this equation gives more relations and one can easily see that

$$(\#_k) \qquad \dim(\mathrm{Osc}^k_x(S)) \leq 2k \quad \text{for every } x \in S.$$

Of course, once N is fixed, this is meaningful only for $k \leq m := \left[\frac{N-1}{2}\right]$.

Note that this is exactly what happens for scrolls. Actually in this case there are local coordinates (u, v) around every point $x \in S$ such that the homogeneous coordinates x_i, $(i = 0, \ldots, N)$ of the points of S near x, locally, can be written as $x_i = a_i(u) + v b_i(u)$, where a_i and b_i are holomorphic functions of u. Since every section $s \in V$ is a linear combination $s = \sum_{i=0}^{N} \lambda_i x_i$ we thus see that the second derivative s_{vv} vanishes at every point. Thus $\dim(\mathrm{Osc}^2_x(S)) \leq 4$ for every $x \in S$, hence $(\#_k)$ holds for every k. Apart from scrolls, sporadic surfaces satisfying $(\#_k)$ for every k are known: they have been found by Togliatti [12], sec. 3, Dye [2], Theorem 4, and Perkinson [9], Theorem 3.2.

There is a conjecture of Piene and Tai [10], related to the inequalities $(\#_k)$, stating the following.

Let $S \subset \mathbb{P}^N$ $(N \geq 5)$ be a non-degenerate complex smooth surface such that $(\#_k)$ holds for every k and $(\#_m)$ is an equality, where m is defined above. Then (S, L, V) is either $(\mathbb{F}_0, [C_0 + mf], H^0)$ if $N = 2m + 1$ (balanced rational normal scroll), or $(\mathbb{F}_1, [C_0 + (m+1)f], H^0)$ if $N = 2m + 2$ (semibalanced rational normal scroll). Here $\mathbb{F}_e$ denotes the Segre-Hirzebruch surface of invariant $e \geq 0$, C_0 stands for a section of minimal self-intersection and f for a fibre.

For N odd the conjecture is true, as proved by Ballico, Piene and Tai [1], by using adjunction theory. In this paper I prove the following results.

Theorem A. *For any linearly normal elliptic scroll $S \subset \mathbb{P}^N$ $(N \geq 6)$ of invariant -1, we have* $\dim(\mathrm{Osc}^m_x(S)) = 2m$.

In particular, for N even the conjecture above is not true, even in the setting of scrolls (compare with the discussion in [9], end of p. 496 concerning the setting of toric surfaces).

Theorem B. *Let $S \subset \mathbb{P}^N$ $(N \geq 5)$ be any scroll over a smooth curve; then* $\dim(\mathrm{Osc}_x^2(S)) \geq 3$ *for every $x \in S$.*

The meaning of Theorem B is that the osculatory behavior of scrolls is not so bad, as we will see. The proof of both results simply relies on the consideration of the linear system of k-osculating hyperplane sections to a smooth projective surface and its special feature in case of a surface scroll. Finally I would like to note that both theorems can be easily rephrased in terms of Weierstrass schemes associated to the Wronski system coming from the jet bundles $J_k L$ (see [8], Section 4). I am indebted to Dan Laksov for drawing my attention to [8].

The paper is organized as follows. In Section 1 I discuss linear systems of k-osculating hyperplane sections and prove Theorem B in two different ways. Theorem A is proved in Section 2, where the subject is reconsidered with the help of the jumping sets of suitable ample and spanned line bundles. In Section 3 I describe a further pathology of the osculatory behavior of surfaces, which makes clear the meaning of Theorem B.

The word surface will always mean smooth complex projective surface. Let $S \subset \mathbb{P}^N$, L, V be as at the beginning. I denote by $|V|$ the linear system defined by the vector subspace $V \subseteq H^0(S, L)$ (which, in general, is not a complete linear system, in spite of the notation). Sometimes I refer to S as the abstract surface and to the pair (S, V) as the embedded surface. Accordingly, I say that (S, V) ((S, L) if $V = H^0(L)$) is a scroll to mean that S, L, V are as above with S a $\mathbb{P}^1$-bundle over a smooth curve, $|V|$ very ample, and $L_f = \mathcal{O}_{\mathbb{P}^1}(1)$ for every fibre f of S. I adopt the additive notation for the tensor product of line bundles and, with a little abuse, I do not distinguish between a line bundle and the corresponding invertible sheaf. In particular, if (S, V) is a scroll and f is a fibre, $L - f$ stands for the line bundle $L \otimes \mathcal{O}_S(-f)$; moreover I denote by $|V - f|$ the linear system $\{(s)_0 - f \mid s \in V \text{ and } (s)_0 \supset f\}$ and by $V(-f)$ the corresponding vector subspace of $H^0(S, L - f)$. Of course, up to adding f as a fixed component, $|V - f|$ can be identified with a linear subsystem of $|V|$.

1. Linear systems of osculating hyperplane sections.

Let S, L and V be as in the Introduction. Recall that a hyperplane $H \in \mathbb{P}^{N\vee}$ is said to be k-osculating to S at x if $H \supseteq \mathrm{Osc}_x^k(S)$. Identifying the dual projective space $\mathbb{P}^{N\vee}$ with the linear system $|V|$, H corresponds to the divisor $(s)_0$ of a section $s \in V$ and the fact that H is k-osculating to S at x is equivalent to the condition $j_{k,x}(s) = 0$, i. e., $(s)_0 \in |V - (k+1)x|$. In other words, the dual of $\mathbb{P}(\mathrm{Ker} j_{k,x})$ can be identified with the linear system $|V - (k+1)x|$ of hyperplane sections having a point of multiplicity $\geq (k+1)$ at x. From the equality $\dim V = \dim(\mathrm{Ker}(j_{k,x})) + \dim(\mathrm{Im}(j_{k,x}))$, we thus get for every $k \geq 1$,

$$(1.0_k) \qquad \dim(\mathrm{Osc}_x^k(S)) + \dim(|V - (k+1)x|) = N - 1.$$

Remark 1.1. Let $S \subset \mathbb{P}^N = \mathbb{P}(V)$ be a non-degenerate surface. Then

$$\dim(\mathrm{Osc}_x^k(S)) = 2 + \mathrm{codim}(|V - (k+1)x|, |V - 2x|).$$

Proof. Since $\mathrm{Osc}_x^1(S)$ is the projective tangent plane to S at x, the equality simply follows by subtracting (1.0_1) from (1.0_k). □

Now suppose that (S, V) is a scroll and let f_x be the fibre of S through a point $x \in S$. If $D \in |V - 2x|$ then $D = f_x + R$, where R is an effective divisor in the linear system $|V - f_x|$, passing through x, i. e., $R \in |V - f_x - x|$. This follows immediately from the fact that $Df_x = 1$ for every $D \in |V|$, since (S, L, V) is a scroll. Actually, if $D \in |V - 2x|$ would not contain f_x, then we would get

$$1 = Df_x \geq \mathrm{mult}_x(D)\,\mathrm{mult}_x(f_x) \geq 2,$$

a contradiction. Moreover, if $D \in |V - 3x|$, then R must have a double point at x, i. e., $R \in |V - f_x - 2x|$. But then, arguing as before we have $D = 2f_x + T$, where T is an effective divisor in the linear system $|V - 2f_x|$, passing through x, i. e., $T \in |V - 2f_x - x|$. More generally, iterating this argument we have

Remark 1.2. Let (S, V) be a scroll and let f_x be the fibre through any point $x \in S$. Then

$$|V - (k+1)x| = f_x + |V - f_x - kx| = \ldots = kf_x + |V - kf_x - x|.$$

In particular,

$$(1.2.1) \quad \dim(|V-(k+1)x|) = \dim(|V - f_x - kx|) = \ldots = \dim(|V - kf_x - x|).$$

Now let (S, V) be a scroll. We give two different proofs of Theorem B

1.3. *First proof of Theorem B*. In view of Remark (1.1) it is equivalent to show that $|V-2x| \neq |V-3x|$ for every $x \in S$. Since (S, V) is a scroll, by Remark (1.2) we know that $|V-3x| = 2f_x + |V-2f_x-x|$. Assume, by contradiction, that

$$|V-2x| = 2f_x + |V-2f_x-x|$$

for some point $x \in S$. Then every hyperplane tangent to S at x is tangent along the whole fibre f_x. As a consequence the tangent plane to S is constant along f_x. But this contradicts the finiteness of the Gauss map $\gamma : S \to \mathbb{G}(2, N)$ sending every point $y \in S$ to $\mathrm{Osc}^1_y(S)$, regarded as a point of the grassmannian $\mathbb{G}(2, N)$ of planes of $\mathbb{P}^N$ (e. g., see [13], Theorem 2.3, c), p. 21). □

The second proof of Theorem B relies on two lemmas of some interest in themselves. The former one will be helpful also in Section 2.

Lemma 1.4. *Let (S, V) be a scroll. Then $Bs(|V-f_x|) = \emptyset$ for every $x \in S$.*

Proof. (inspired by [11], Lemma 0.10.1) Let $y \in S$ and let D be the pull-back via the embedding given by V of a hyperplane of $\mathbb{P}^N$ containing f_x, but not containing y if $y \notin f_x$, and not containing the tangent plane to S at y if $y \in f_x$. In both cases we have that $D = f_x + R$, with $R \not\ni y$. □

Now, for any $x \in S$, let $\varphi_x : S - - \to \mathbb{P}$ be the map associated with the linear system $|V-f_x|$. Then Lemma 1.4 says that φ_x is a morphism. We have $\dim|V| \geq 3$, since $|V|$ is very ample, hence $\dim|V-f_x| \geq 1$ for every $x \in S$. Since $\varphi_x(S)$ is non-degenerate in the projective space $\mathbb{P}(V(-f_x))$, this says that $\dim \varphi_x(S) \geq 1$.

Lemma 1.5. *Let (S, V) be a scroll and let φ_x be the morphism defined above.*

i) $\dim \varphi_x(S) = 1$ *for some (equivalently every) point $x \in S$ if and only if* $(S, L, V) = (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1, 1), H^0(L))$.
Let $\dim \varphi_x(S) = 2$.
ii) *If $(S, L) = (\mathbb{F}_e, [C_0 + (e+1)f])$, $e > 0$ then every fibre of φ_x is either a finite set or a finite set plus the fundamental section.*
iii) *In any other case every fibre of φ_x is a finite set.*

Proof. If $\dim(\varphi_x(S)) = 1$ then φ_x contracts a positive dimensional family of curves. The proof will be done by analyzing which curves on S can be contracted by φ_x. Note that $|V-f_x|$, hence $|L-f_x|$, has no fixed components by Lemma 1.4. So, for any irreducible curve $C \subset S$ there exists a divisor $D \in |L-f_x|$ not containing C among its components, hence $DC \geq 0$. This shows that $L-f_x$ is nef. Let C_0 and f denote a fundamental section and a fibre of S, respectively. Since (S, V) is a scroll we have that $L \equiv [C_0 + mf]$

(numerical equivalence) for a suitable integer m. Let q and e denote the irregularity and the invariant of S. Since $L - f_x \equiv [C_0 + (m-1)f]$ is nef, we get

$$(1.5.1) \qquad m - 1 \geq \begin{cases} e, & \text{if } e \geq 0, \\ e/2, & \text{if } e < 0. \end{cases}$$

Now let $C \subset S$ be an irreducible curve contracted by φ_x. Then $(L - f_x)C = 0$; moreover $C^2 \leq 0$, since $\dim \varphi_x(S) \geq 1$. Since $(L - f_x)f = 1$, C cannot be a fibre: so there are two possibilities: either j) $C = C_0$, or jj) $C \equiv aC_0 + bf$ for some integers a, b satisfying the conditions:

$$(1.5.2) \qquad a > 0 \quad \text{and} \quad b \geq \begin{cases} ae, & \text{if } e \geq 0, \\ ae/2, & \text{if } e < 0, \end{cases}$$

by [3], p. 382. In case jj) we get

$$(1.5.3) \quad 0 = (L - f_x)C = (C_0 + (m-1)f)(aC_0 + bf) = -ae + b + (m-1)a.$$

If $e \geq 0$ both summands in the right hand being non negative by (1.5.1), (1.5.2), this implies $b = ae$ and $m = 1$, which, in view of (1.5.1) gives $e = 0$; hence $b = 0$ and then $C \equiv aC_0$. But this contradicts the fact that C is irreducible, unless we are in case j). On the other hand, if $e < 0$, we can continue (1.5.3) as follows:

$$0 = (-ae/2 + b) + a(m - 1 - e/2),$$

where both summands are non negative in view of (1.5.1), (1.5.2). We thus get $b = ae/2$, $m - 1 = e/2$, hence $[C] \equiv a(L - f)$. But this gives a contradiction, since $C^2 \leq 0$, while $(L - f)^2 = (L^2 - 2) \geq 0$, the equality implying that (S, V) is the quadric surface, i. e., $e = 0$, a contradiction. Now suppose we are in case j). Thus

$$0 = (L - f_x)C = (C_0 + (m-1)f)C_0 = -e + m - 1.$$

Due to (1.5.1) it cannot be $e < 0$; so $e \geq 0$ and $m = e + 1$. But then $\deg L_{C_0} = LC_0 = (C_0 + (e+1)f)C_0 = 1$. Since L is a very ample line bundle, this clearly implies $q = 0$. Thus $S = \mathbb{F}_e$ and $L = [C_0 + (e+1)f]$. If $e = 0$, then $L = [C_0 + f]$, hence $|V - f_x| = |L - f_x| = |C_0|$. In this case φ_x is just the projection of $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1 = C_0 \times f$ onto the second factor. On the other hand, if $e > 0$ then C_0 is the only curve contracted by φ_x. This proves all the assertions. $\square$

1.6. *Second proof of Theorem B*. As already noted, it is equivalent to show that $|V-2x| \neq |V-3x|$ for every $x \in S$. Since (S, V) is a scroll, by Remark 1.2 the linear system on the left corresponds to $|V - f_x - x|$, while that on the right corresponds to $|V - 2f_x - x|$. So we have the equality $|V-2x| = |V-3x|$ if and only if

$$f_x \subseteq \mathrm{Bs}(|V - f_x - x|). \tag{1.6.1}$$

But this cannot happen. To see this, consider the morphism $\varphi_x : S \to \mathbb{P}$, defined by the linear system $|V - f_x|$. Since $N \geq 5$, by Lemma 1.5 φ_x has a 2-dimensional image and all its fibres cut every fibre of the ruling projection at a finite set. On the other hand

$$\mathrm{Bs}(|V - f_x - x|) = \bigcap_{D \in |V - f_x|,\ D \ni x} \mathrm{supp}(D) \ = \varphi_x^{-1}(\varphi_x(x)).$$

Therefore the base locus of $|V - f_x - x|$ must intersect every fibre of the ruling of S (in particular f_x) at finite set only. This shows that (1.6.1) cannot occur. □

Remark 1.7. Let (S, V) be a scroll over a smooth curve B and let $\pi : S \to B$ be the projection. Then $S = \mathbb{P}(\mathcal{E})$, where $\mathcal{E}$ is the very ample vector bundle of rank 2 given by $\pi_* L$. Then the very ampleness of $\mathcal{E}$ is equivalent to the equality

$$h^0(\mathcal{E}(-\pi(x) - \pi(y))) = h^0(\mathcal{E}) - 4, \tag{1.7.1}$$

for every $x, y \in S$ (e. g., see [4], Lemma 1). On the other hand, since all elements of $|V|$ have intersection 1 with any fibre, we see that $|V - x - x'| = f_x + |V - f_x|$ for any $x' \in f_x$, $x' \neq x$. Hence, due to the very ampleness of $|V|$ we have $\dim(|V - f_x|) = \dim(|V|) - 2$. Now, let $y \in S$. For the same reason as before we see that $|V - f_x - y - y'| = f_y + |V - f_x - f_y|$, where y' is any point of f_y distinct from y. As in (1.6) we have

$$\mathrm{Bs}(|V - f_x - y|) = \bigcap_{D \in |V - f_x|, D \ni y} \mathrm{supp}(D) \ = \varphi_x^{-1}(\varphi_x(y)).$$

By Lemma 1.5 this set cuts out a finite (possibly empty) set on f_y. Thus there exists a point $y' \in f_y$ such that $y' \notin \mathrm{Bs}(|V - f_x - y|)$. Hence $|V - f_x - y - y'|$ has codimension 1 in $|V - f_x - y|$. On the other hand $|V - f_x - y|$ has codimension 1 in $|V - f_x|$, by Lemma 1.4. Putting everything together we get

$$\begin{aligned} &\dim(|V - f_x - f_y|) = \dim(|V - f_x - y - y'|) = \\ &= \dim(|V - f_x - y|) - 1 = \dim(|V - f_x|) - 2. \end{aligned}$$

Thus the very ampleness of $|V|$ implies that

(1.7.2) $$\dim|V - f_x - f_y| = \dim|V - f_x| - 2 = \dim|V| - 4.$$

Note that when $V = H^0(S, L)$ (1.7.2) is clearly equivalent to (1.7.1) in view of the isomorphism $H^0(S, L) \cong H^0(B, \mathcal{E})$. Thus (1.7.2) can be regarded as a generalization of (1.7.1) to non complete linear systems.

2. Linearly normal elliptic scrolls of invariant−1.

2.1. *Proof of Theorem A.* Let C be a smooth curve of genus 1. Recall that the $\mathbb{P}^1$ bundle of invariant -1 over C is the surface $S = \mathbb{P}(\mathcal{E})$, where $\mathcal{E}$ is the holomorphic vector bundle of rank 2 defined by the non-split extension

(2.1.1) $$0 \to \mathcal{O}_C \to \mathcal{E} \to \mathcal{L} \to 0,$$

where $\mathcal{L} \in \mathrm{Pic}(C)$ has degree 1. Let $\pi : S \to C$ be the ruling projection, let C_0 be the tautological section on S and let δ be a divisor on C of degree $m+1 \geq 3$. Then the line bundle $L := \mathcal{O}_S(C_0 + \pi^*\delta)$ is very ample (e. g., see [3], Ex. 2.12 (b), p. 385) and the map associated with $|L|$ embeds S as a scroll of degree $2m + 3$ in $\mathbb{P}^{2m+2}$. Set $V = H^0(L)$ and let x be any point of S. Then

(2.1.2) $$\dim(\mathrm{Osc}^m_x(S)) = 2m + 1 - \dim(|L - (m + 1)x|),$$

by (1.0_m). On the other hand, since (S, L) is a scroll we have

(2.1.3) $$|L - (m + 1)x| = mf_x + |L - mf_x - x|,$$

by Remark 1.2. Note that the line bundle $L - mf_x = \mathcal{O}_S(C_0 + \pi^*(\delta - m\pi(x)))$ is spanned, since $\deg(\delta - m\pi(x)) = 1$ (see [3], Ex. 2.12 (a), p. 385). Hence

(2.1.4) $$\dim(|L - mf_x - x|) = \dim(|L - mf_x|) - 1.$$

On the other hand, by twisting (2.1.1) by $\mathcal{O}_C(\delta - m\pi(x))$ we immediately see that $h^0(L - mf_x) = h^0(\mathcal{E}(\delta - m\pi(x))) = 3$. Combining this with (2.1.3) and (2.1.4) gives $\dim(|L - (m + 1)x|) = 1$ and then (2.1.2) shows that $\dim(\mathrm{Osc}^m_x(S)) = 2m$, for every point $x \in S$. □

Theorem A, especially case $m = 2$, can be seen from a slightly more general point of view, suggested by the discussion in Section 1. Actually, if (S, V) is a scroll, by combining Remark 1.1 with (1.2.1) we get

$$\begin{aligned}\dim(\mathrm{Osc}^2_x(S)) &= 2 + \mathrm{codim}(|V - 3x|, |V - 2x|)\\ &= 2 + \mathrm{codim}(|V - f_x - 2x|, |V - f_x - x|),\end{aligned}$$

for every $x \in S$. So $\dim(\mathrm{Osc}_x^2(S)) = 3$ if and only if $\dim(|V - f_x - 2x|) = \dim(|V - f_x - x|) - 1$. Now suppose that $L - f$ is ample for a fibre f of S. Since ampleness is a numerical condition, this means that $L - f$ is ample for every fibre f of S. By Lemma 1.4 we know that the vector subspace $V(-f) \subseteq H^0(L - f)$ spans $L - f$. Under the assumption above, fix a fibre f of S. Then, from [7], Proposition 3.1 we have the equality

$$\{x \in f \mid \dim(|V - f - 2x|) = \dim(|V - f - x|) - 1\} = \mathcal{J}_1(V(-f)) \cap f,$$

where $\mathcal{J}_1(V(-f))$ is the first jumping set of $(S, V(-f))$, i. e., the ramification locus of the morphism defined by $|V - f|$. This argument proves the following

Proposition 2.2. *Let (S, V) be a scroll and assume that $L - f$ is ample, where f is a fibre of S. Then*

$$\{x \in S \mid \dim(\mathrm{Osc}_x^2(S)) = 3\} = \bigcup_f \big(f \cap \mathcal{J}_1(V(-f))\big),$$

the union being taken over all fibres of S.

Recall that $\mathcal{J}_1(W) = \emptyset$ if the morphism defined by the linear system $|W|$ is an immersion [7], Remark 2.3.2. We thus get.

Corollary 2.3. *If (S, V) is a scroll and the morphism defined by $|V - f|$ is an immersion for every fibre f of S, then*

$$\dim(\mathrm{Osc}_x^2(S)) = 4 \quad \textit{for every} x \in S.$$

Note that the case of linearly normal elliptic scrolls of invariant -1 with $N \geq 6$ discussed in Theorem A fits into the Corollary above. Actually for the line bundle L defined in the proof of Theorem A it turns out that $L - f$ is very ample for every fibre f, by [3], Ex, 2.12 (b), p. 385. However, in principle there could be other scrolls, not linearly normal and of higher genus, satisfying the assumption in Corollary 2.3. They would provide further counterexamples in $\mathbb{P}^6$ to the even dimensional part of the conjecture of Piene-Tai.

An interpretation in terms of jumping sets can be extended also to Theorem B. Let (S, V) be a scroll and suppose that $L - f$ is ample for a (hence every) fibre f of S. By Lemma 1.4 $V(-f)$ spans $L - f$ for a given fibre f and then we can also consider the second jumping set $\mathcal{J}_2(V(-f))$ of $(S, V(-f))$ [7], Section 1. By definition the set $f \cap \mathcal{J}_2(V(-f))$ consists of the points $x \in f$ such that $|V - f - x| = |V - f - 2x|$. But Theorem B says that there there are no such points. We thus get the following

Corollary 2.4. *Let (S, V) be a scroll and assume that $L - f$ is ample, where f is a fibre of S. Then*

$$f \cap \mathcal{J}_2(V(-f)) = \emptyset,$$

for every fibre f of S.

3. Further pathology of osculation.

From Remark 1.1 we know that

$$\dim(\mathrm{Osc}_x^2(S)) = 2 + \mathrm{codim}(|V - 3x|, |V - 2x|). \tag{3.0.1}$$

Thus $\dim(\mathrm{Osc}_x^2(S)) = 2$ if and only if $|V - 3x| = |V - 2x|$ and Theorem B says that this cannot happen for scrolls. In fact there are surfaces for which $\dim(\mathrm{Osc}_x^2(S)) = 2$ for some point $x \in S$. This means that every tangent hyperplane at such a point x is osculating. An interesting example of this situation is the so-called Togliatti's Del Pezzo surface.

3.1. *Example*. Let $(S, L = -K_S)$ be the Del Pezzo surface with $K_S^2 = 6$. Call X the surface S embedded by $|L|$; then X is a smooth surface of degree 6 in $\mathbb{P}^6$. Recall that S is isomorphic to $\mathbb{P}^2$ blown-up at three non-collinear points p_0, p_1, p_2. Choose homogeneous coordinates (x_0, x_1, x_2) in $\mathbb{P}^2$ in such a way that $p_0 = (1 : 0 : 0)$, $p_1 = (0 : 1 : 0)$, $p_2 = (0 : 0 : 1)$ and fix the basis of $H^0(L)$ corresponding to the 7 cubic monomials

$$x_0^2x_1, x_0^2x_2, x_0x_1^2, x_0x_2^2, x_1^2x_2, x_1x_2^2, x_0x_1x_2.$$

Then X is the image of the rational map $\mathbb{P}^2 - - \to \mathbb{P}^6$ defined by these monomials. One can see that the secant variety of X is a cubic hypersurface of $\mathbb{P}^6$ not containing the point $c = (0 : \ldots : 0 : 1)$. E. g., one can write down the explicit equation of the secant variety by using MAPLE and then this property can be checked directly. Thus the projection $\pi_c : \mathbb{P}^6 - - \to \mathbb{P}^5$ from c defines an embedding of X in $\mathbb{P}^5$. Let $Y = \pi_c(X)$. Then Y is the image of the rational map $\mathbb{P}^2 - - \to \mathbb{P}^5$ defined by the 6 monomials

$$x_0^2x_1, x_0^2x_2, x_0x_1^2, x_0x_2^2, x_1^2x_2, x_1x_2^2. \tag{3.1.1}$$

A very interesting property of the surface Y discovered by Togliatti [12] is that its 2-osculating spaces have dimension ≤ 4 at every point. But, in fact there are points of Y where the 2-osculating space coincides with the tangent plane ([10], Example 2.4, [5], Proposition 4.3). To recognize them, let $\sigma : S \to \mathbb{P}^2$ be the blow-up at the three points p_i, let $e_i = \sigma^{-1}(p_i)$ and for $i < j$ let l_{ij} denote the proper transform on S of the line $\langle p_i p_j \rangle$ joining p_i and p_j. The six curves e_i, l_{ij} $(0 \le i < j \le 2)$ define a 1-cycle E on S, which is mapped to a skew hexagon on Y; let $\mathcal{V}$ be the set of the 6 vertices, i. e., the set of points at which two irreducible components of E meet. Then $\dim(\mathrm{Osc}^2_x(Y)) = 2$ for every $x \in \mathcal{V}$. Call V the subspace of $H^0(L)$ generated by the elements corresponding to the monomials in (3.1.1). Then the condition above can be rephrased as follows:

$$|V - 2x| = |V - 3x| \quad \text{for every } x \in \mathcal{V}.$$

Understanding this equality in terms of the linear system of plane cubics representing the hyperplane sections of Y is an instructive exercise. Here is a sketch of the argument. Recall that $L = \sigma^*\mathcal{O}_{\mathbb{P}^2}(3) - e_0 - e_1 - e_2$, fix a point $x \in \mathcal{V}$, e. g., $x = e_0 \cap l_{01}$, and consider an element $H \in |V - 2x|$. Since $He_0 = Hl_{01} = 1$ we see that H must contain both e_0 and l_{01} as components. Thus $H = \sigma^*\Gamma - e_1 - e_2 - e_3$, where the plane cubic Γ consists of the line $\langle p_0 p_1 \rangle$ and a conic γ containing p_0 and p_2. On the other hand, since $H \in |V|$, the polynomial defining Γ is a linear combination of the monomials in (3.1.1). Since $\langle p_0 p_1 \rangle$ corresponds to the factor x_2, this implies that the quadratic polynomial defining γ is a linear combination of x_0x_2, x_1^2, x_1x_2 (but not x_0x_1, since $x_0x_1x_2$ corresponds to an element not in V). Therefore $|V - 2x|$ corresponds to the linear system of plane cubics generated by $x_0x_2^2, x_1^2x_2, x_1x_2^2$. Let $\Gamma_1, \Gamma_2, \Gamma_3$ be the cubics defined by these 3 generators. It is easy to see that for $i = 1, 2, 3$ the element $H_i = \sigma^*\Gamma_i - e_0 - e_1 - e_2$ has a point of multiplicity ≥ 3 at x. E. g., $H_1 = \sigma^*(2\langle p_0 p_1 \rangle + \langle p_1 p_2 \rangle) - e_0 - e_1 - e_2 = 2l_{01} + e_0 + 2e_1 + l_{12}$. Since H is a linear combination of H_1, H_2, H_3, we thus conclude that $H \in |V - 3x|$.

There are more surfaces for which $\dim(\mathrm{Osc}^2_x(S)) = 2$ at a finite set of points x. In fact this happens also for the two new surfaces with inflectionary pathology recently discovered by Perkinson in the setting of toric varieties [9], Theorem 3.2, cases (4), (5). In these cases, as well as in Example 3.1, the linear system $|V|$ is not complete. I would like to mention however that this pathology can occur also when $|V|$ is a complete very ample linear system, as shown in [6], Lemma 4.1, i).

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