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# AN UNRAMIFIED REAL PLANE CURVE IS A CONIC

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*Dedicated to Silvio Greco in occasion of his 60-th birthday.*

## 1. Introduction.

Let $n$ be a natural integer satisfying $n \geq 2$. Let $C$ be a smooth geometrically integral real algebraic curve in real projective space $\mathbb{P}^n$ [2]. For readers less familiar with the theory of schemes: $C$ is the zero set in $\mathbb{P}^n$ of a finite number of homogeneous polynomials $F_1, \ldots, F_k$ belonging to $\mathbb{R}[X_0, \ldots, X_n]$. The set of complex points $C(\mathbb{C})$ of $C$ is the zero set of $F_1, \ldots, F_k$ in $\mathbb{P}^n(\mathbb{C})$, and is a Riemann surface. It has the property that it is stable for complex conjugation on $\mathbb{P}^n(\mathbb{C})$. The set of closed points of the scheme $C$ is nothing but the quotient of $C(\mathbb{C})$ by the action of complex conjugation. The set of real points $C(\mathbb{R})$ of $C$ is the zero set of $F_1, \ldots, F_k$ in $\mathbb{P}^n(\mathbb{R})$. The set $C(\mathbb{R})$ is exactly the set of fixed points of $C(\mathbb{C})$ with respect to complex conjugation. Since $C$ is smooth, each of the connected components of $C(\mathbb{R})$ is homeomorphic to the unit circle. Since $\mathbb{P}^n(\mathbb{R})$ is compact and since $C(\mathbb{R})$ is a closed smooth submanifold of $\mathbb{P}^n(\mathbb{R})$, the number of connected components of $C(\mathbb{R})$ is finite.

The curve $C$ is *nondegenerate* if $C$ is not contained in a real hyperplane of $\mathbb{P}^n$. Equivalently, $C$ is nondegenerate if and only if the Riemann surface $C(\mathbb{C})$ is not contained in a complex hyperplane of $\mathbb{P}^n(\mathbb{C})$. Suppose that $C$ is nondegenerate. Let $H$ be a real hyperplane of $\mathbb{P}^n$. Since $C$ is nondegenerate,

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the intersection product $H \cdot C$ is a well defined divisor on $C$. Recall that a divisor on $C$ is simply a finite formal sum of closed points of $C$. If $P$ is a real point of $C$, then the multiplicity of $P$ in $H \cdot C$ is equal to the order of tangency of $H(\mathbb{R})$ to $C(\mathbb{R})$ at $P$, increased by 1. If $P$ is a nonreal closed point of $C$, then $P$ defines a pair of complex conjugate points $Q$ and $\overline{Q}$ of $C(\mathbb{C})$. The multiplicity of $P$ in $H \cdot C$ is then equal to the order of tangency of $H(\mathbb{C})$ to $C(\mathbb{C})$ at $Q$, say, increased by 1.

Let $D$ be any effective divisor on $C$. Write

$$D = \sum_{i=1}^{\ell} m_i P_i,$$

where $P_i \neq P_j$ if $i \neq j$. The degree of $D$ is equal to

$$\sum_{i=1}^{\ell} m_i \deg (P_i),$$

where $\deg (P_i) = 1$ if $P_i$ is a real point, and $\deg (P_i) = 2$ if $P_i$ is a nonreal closed point of $C$. With this definition, the intersection product $H \cdot C$ is a divisor of degree $d$ on $C$, where $d$ is the degree of $C$. The reduced divisor $D_{\mathrm{red}}$ associated to $D$ is the divisor

$$D_{\mathrm{red}} = \sum_{i=1}^{\ell} P_i.$$

We say that $C$ is *unramified* [3] if, for all real hyperplanes $H$ of $\mathbb{P}^n$, one has

$$\deg(H \cdot C) - \deg(H \cdot C)_{\mathrm{red}} \leq n - 1,$$

In particular, taking $H$ the osculating hyperplane at a real point of $C$, an unramified real curve does not have real inflection points. The converse, however, does not hold.

The corresponding notion of an unramified complex algebraic curve in complex projective space is well understood. Indeed, any unramified complex algebraic curve is a rational normal curve and conversely [1]. For real algebraic curves, the situation seems to be much more interesting. In [3], it is shown that there are unramified real curves of any genus in any odd dimensional projective space. It is, however, conjectured that, in even dimensional projective spaces, all unramified real curves are rational normal curves. The object of this paper is to prove that conjecture for the projective plane:

**Theorem 1.** *Let $C$ be an unramified real plane curve. Then, $C$ is a conic.*

The method of the proof that we propose, is essentially topological. Some of the ideas of the proof may be useful for other problems in real algebraic geometry as well.

Jean-Philippe Monnier has informed me that Theorem 1 is also a consequence of Klein's Equation [5]. In fact, the idea of proof of Theorem 1 that we present here, can be used to give yet another proof of Klein's Equation. Details are postponed to a forthcoming paper.

## 2. A proof of Theorem 1.

Throughout this section, let $C$ be an unramified real plane curve. In particular, $C$ is a proper smooth geometrically integral real algebraic curve. The set of real points $C(\mathbb{R})$ of $C$ is a–possibly empty and not necessarily connected–manifold of dimension 1 without boundary. Hence, each connected component of $C(\mathbb{R})$ is, topologically, a circle.

Let $B$ be a connected component of $C(\mathbb{R})$. Since the fundamental group of $\mathbb{P}^2(\mathbb{R})$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, the submanifold $B$ of $\mathbb{P}^2(\mathbb{R})$ may be contractable or not. In the former case, $B$ is an *oval* of $C$. In the latter case, $B$ is a *pseudo-line* of $C$. Another way to characterize ovals and pseudo-lines is the following. Let $L$ be a real projective line in $\mathbb{P}^2$. If $B$ is an oval, then the divisor $L \cdot C$ has even degree on $B$. If $B$ is a pseudo-line, then $L \cdot C$ has odd degree on $B$.

For our proof of Theorem 1, we need to derive some preliminary lemmas. First, we show that all connected components of $C(\mathbb{R})$ are ovals:

**Lemma 1.** *$C$ does not have any pseudo-lines.*

*Proof.* Suppose that $C$ has a pseudo-line. We show that $C$ has a real inflection point. Since any two pseudo-lines in $\mathbb{P}^2(\mathbb{R})$ intersect and since $C$ is smooth, $C$ has only one pseudo-line. It follows that $C$ is of odd degree. Let $d$ be the degree of $C$. The Hessian curve $H$ of $C$ is of degree $3(d-2)$. In particular, the curve $H$ is of odd degree as well. By Bezout's Theorem, $H$ and $C$ intersect in a real point. Therefore, $C$ has a real inflection point. It follows that $C$ is ramified. Contradiction. $\square$

Let $P \in C(\mathbb{R})$. Denote by $T_P C$ the real projective line in $\mathbb{P}^2$ that is tangent to $C$ at $P$.

Let $B$ be a connected component of $C$ and let $P \in B$. For our proof of Theorem 1, we need to show that any tangent line $T_P C$ of $C$ intersects $B$ only in $P$. This is the statement of Lemma 5. Before we can prove that lemma, we need some preparation.

**Lemma 2.** *Let $B$ be a connected component of $C(\mathbb{R})$ and let $P$ and $Q$ be two distinct points of $B$. Suppose that $Q$ belongs to $T_PC$. Then, the multiplicity of $P$ in the intersection product $T_PC \cdot C$ is equal to* 2 *and the multiplicity of $Q$ in $T_PC \cdot C$ is equal to* 1.

*Proof.* Let $m$ and $n$ be the multiplicities of $P$ and $Q$, respectively, in $T_PC \cdot C$. Since $T_PC$ is tangent to $C$ at $P$, one has $m \geq 2$. Since $Q$ belongs to $T_PC$, one has $n \geq 1$. On the other hand, since $C$ is unramified,

$$(m-1)+(n-1) \leq \deg(T_PC \cdot C) - \deg(T_PC \cdot C)_{\mathrm{red}} \leq 1.$$

Hence, $m+n \leq 3$. Therefore, $m=2$ and $n=1$. □

**Lemma 3.** *Let $B$ be a connected component of $C(\mathbb{R})$. Let $A$ be the subset of $B^2$ defined by*

$$A = \{(P, Q) \in B^2 | Q \in T_PC\}.$$

*Then, $A$ is a–not necessarily connected–closed* 1*-dimensional submanifold of $B^2$. Moreover, $p_1|_A$ and $p_2|_A$ are topological coverings of $B$, where $p_1$ and $p_2$ are the projections from $B^2$ onto $B$.*

*Proof.* Let $T$ be the closed submanifold of $B \times \mathbb{P}^2(\mathbb{R})$ defined by

$$T = \{(P, Q) \in B \times \mathbb{P}^2(\mathbb{R}) | Q \in T_PC\}.$$

The manifold $T$ is a locally trivial $\mathbb{P}^1(\mathbb{R})$-bundle over $B$. This fiber bundle admits a section $\delta : B \to T$ defined by $\delta(P) = (P, P)$. Let $q$ be the restriction to $T$ of the projection from $B \times \mathbb{P}^2(\mathbb{R})$ onto the second factor $\mathbb{P}^2(\mathbb{R})$. Since $C$ is unramified, $C$ does not have any real inflection points. Then, a local study reveals that $q$ is a local homeomorphism away from $\delta(B)$, and that $q$ is a simple topological fold along $\delta(B)$. Since $q(\delta(B)) = B$, the inverse image $q^{-1}(B)$ is a closed 1-dimensional topological submanifold of $T$. Since $q^{-1}(B) = A$, the subset $A$ is a closed 1-dimensional topological submanifold of $T$. It follows that $A$ is a closed 1-dimensional topological submanifold of $B \times \mathbb{P}^2(\mathbb{R})$, and then also of $B^2$.

Next, we show that the maps $p_1|_A$ and $p_2|_A$ are topological coverings of $B$. Since $q$ is a local homeomorphism away from $\delta(B)$, its restriction to $q^{-1}(B) \setminus \delta(B)$ is a topological covering map of $B$. Obviously, the restriction of $q$ to $\delta(B)$ is also a topological covering map of $B$. Hence, the restriction of $q$ to $q^{-1}(B)$ is a topological covering map of $B$. Since $p_2|_A$ is equal to the restriction of $q$ to $q^{-1}(B)$, the map $p_2|_A$ is a topological covering map of $B$.

Let $p$ be the restriction to $T$ of the projection from $B \times \mathbb{P}^2(\mathbb{R})$ onto the first factor $B$. By Lemma 2, the restriction of $p$ to $q^{-1}(B) \setminus \delta(B)$ is a topological covering map of $B$. Obviously, the restriction of $p$ to $\delta(B)$ is also a topological covering map of $B$. Hence, the restriction of $p$ to $q^{-1}(B)$ is a topological covering map of $B$. Since $p_1|_A$ is equal to the restriction of $p$ to $q^{-1}(B)$, the map $p_1|_A$ is a topological covering map of $B$. □

**Lemma 4.** *Let $B$ and $A$ be as in Lemma 3. Suppose that $(P, Q) \in A$. Then, there is a unique continuous map $\varphi : B \longrightarrow B$ such that $\varphi(P) = Q$ and such that the graph* $\mathrm{graph}(\varphi)$ *of $\varphi$ is contained in $A$. Moreover, $\varphi$ is an orientation preserving homeomorphism.*

*Proof.* Let us first show the uniqueness of $\varphi$. By Lemma 3, $p_1|_A$ is a topological covering of $B$. Therefore, the uniqueness of $\varphi$ follows.

Next, we show the existence of $\varphi$. If $P = Q$ then $\varphi = \mathrm{id}_B$ satisfies clearly the conditions. Therefore, we may assume that $P \neq Q$.

Let $K$ be the connected component of $A$ containing $(P, Q)$. By Lemma 3, $K$ is a closed 1-dimensional submanifold of $B^2$. Also by Lemma 3, $p_1|_K$ and $p_2|_K$ are topological covering maps. In particular, the 1-dimensional submanifold $K$ of $B^2$ realizes a nonzero homology class $\kappa$ in $H_1(B^2, \mathbb{Z})/\{\pm 1\}$. Denote by $\Delta$ the diagonal in $B^2$. Clearly, $\Delta$ is also a connected component of $A$. Since $(P, Q) \in K \setminus \Delta$, one has $K \cap \Delta = \emptyset$. It follows that the homology class $\kappa$ is a nonzero multiple of the homology class of $\Delta$ in $H_1(B^2, \mathbb{Z})/\{\pm 1\}$. Since $K$ is a closed connected submanifold of $B^2$, the homology class $\kappa$ is not divisible [4]. Hence, $\kappa$ is equal to the homology class of $\Delta$ in $H_1(B^2, \mathbb{Z})/\{\pm 1\}$. It follows that $p_1|_K$ and $p_2|_K$ are homeomorphisms. Therefore, the subset $K$ of $B^2$ is the graph of a homeomorphism $\varphi : B \to B$. Clearly, $\varphi(P) = Q$. Moreover, $\varphi$ is orientation preserving since $\kappa$ is equal to the homology class of $\Delta$ in $H_1(B^2, \mathbb{Z})/\{\pm 1\}$. □

**Lemma 5.** *The subset $A$ of $B^2$ is equal to the diagonal $\Delta$ of $B^2$, i.e., $T_PC$ intersects $B$ only in $P$, for all $P \in B$.*

*Proof.* Let $P \in B$ be such that $T_PC$ intersects $B$ in at least one other point. By Lemma 1, $B$ is an oval. Hence, the real projective line $T_PC$ intersects $B$ in an even number of points when counted with multiplicities. By Lemma 2, there are two distinct points $Q_0$ and $Q_1$ of $B$, both different from $P$, that belong to $T_PC$.

Choose an orientation of $B$. It then makes sense to speak about the closed interval $[Q_0, Q_1]$. Indeed, $[Q_0, Q_1]$ is the closure of the unique connected component of $B \setminus \{Q_0, Q_1\}$ having the following property. There is an orientation-preserving homeomorphism from $[0, 1]$ onto $[Q_0, Q_1]$ that maps 0 to $Q_0$ and 1 to $Q_1$.

Since $T_PC$ intersects $B$ in a finite number of points, we may assume that the points $Q_0$ and $Q_1$ are chosen in such a way that $P$ belongs to the interval

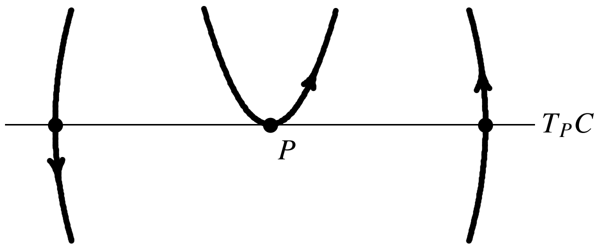


Figure 1: The germs of $B$ at the points $Q_0$, $P$ and $Q_1$

$[Q_0, Q_1]$ and that the points $Q_0$, $P$ and $Q_1$ are the only intersection points of $T_PC$ with $[Q_0, Q_1]$.

According to Lemma 4, there are unique continuous maps

$$\varphi_0, \varphi_1 : B \longrightarrow B$$

such that $\varphi_0(P) = Q_0$ and $\phi_1(P) = Q_1$ and such that $\varphi_0(R)$ and $\varphi_1(R)$ belong to $T_RC$ for all $R \in B$.

Choose a line at infinity such that, in the corresponding affine plane $\mathbb{R}^2$, $T_PC$ is the $x$-axis, $P$ is the origin, the germ of $B$ at $P$ lies in the upper half plane, its orientation induces the standard orientation on the $x$-axis, and the points $Q_0$ and $Q_1$ are on either side of the origin on the $x$-axis. By Lemma 4, $\varphi_0$ and $\varphi_1$ are orientation-preserving homeomorphisms. Hence, the orientations of the germs of $B$ at $Q_0$ and $Q_1$ are as indicated in Figure 1.

Now, we derive the contradiction we are looking for. There are two cases to consider: $Q_0$ is either situated to the left of $P$, or to the right of $P$. If $Q_0$ is situated to the left of $P$, then the interval $[Q_0, P]$ of $B$ has to pass through the line at infinity. This is because $[Q_0, P]$ intersects $T_PC$ only in $Q_0$ and $P$. The same holds for the interval $[P, Q_1]$ of $B$. It follows that they intersect each other in an interior point. This contradicts the fact that $B$ is a submanifold of $\mathbb{P}^2(\mathbb{R})$. If $Q_0$ is situated to the right of $P$, then, again, the two intervals $[Q_0, P]$ and $[P, Q_1]$ of $B$ have to intersect in an interior point. We arrive again at a contradiction. □

Before we give a proof of Theorem 1, we need yet some more preparation.

Let $C^{(2)}$ denote the symmetric square of $C$. It is well known that $C^{(2)}$ is a proper smooth geometrically integral real algebraic surface. The set of real points $C^{(2)}(\mathbb{R})$ of $C^{(2)}$ can be identified with the set of effective divisors of degree 2 on $C$. Let $X$ be the connected component of $C^{(2)}(\mathbb{R})$ containing all effective divisors of degree 2 on $C$ that have even degree on each real branch of $C$. Topologically, $X$ can be described as follows.

Let $C_{\mathrm{top}}$ be the Euclidean topology on the set of closed points of $C$. In fact, $C_{\mathrm{top}}$ is nothing else but the topological quotient $C(\mathbb{C})/\operatorname{Gal}(\mathbb{C}/\mathbb{R})$. In particular, $C_{\mathrm{top}}$ is a compact connected topological surface with–possibly empty–boundary. Its boundary is just the set of real points $C(\mathbb{R})$ of $C$.

Let $B_1, \ldots, B_s$ be the connected components of $C(\mathbb{R})$. Then, the symmetric square $B_i^{(2)}$ of $B_i$ is topologically a closed Möbius strip. Its boundary is nothing else but $B_i$. Therefore, the disjoint union

$$B = \coprod_{i=1}^{s} B_i^{(2)}$$

also has boundary $C(\mathbb{R})$.

Now, it is easy to see that $X$ is homeomorphic to the topological space obtained by gluing $C_{\mathrm{top}}$ and $B$ along their common boundary $C(\mathbb{R})$. Indeed, let

$$h : C_{\mathrm{top}} \coprod_{C(\mathbb{R})} B \longrightarrow X$$

be the map defined as follows. For $P \in C_{\mathrm{top}}$, let $h(P) = 2/\deg(P) \cdot P$. For $(P, Q) \in B_i^{(2)}$, let $h(P, Q) = P + Q$. Then, $h$ is continuous and bijective. Since $C_{\mathrm{top}} \coprod B$ is compact and since $X$ is Hausdorff, $h$ is a homeomorphism. Therefore, $X$ is homeomorphic to the topological space obtained by gluing $C_{\mathrm{top}}$ and $B$ along $C(\mathbb{R})$.

*Proof of Theorem 1.* Define a map

$$f : X \longrightarrow \mathbb{P}^2(\mathbb{R})^{\curlyvee}$$

as follows. Let $D \in X$. We distinguish two cases: either $D = P + Q$, where $P$ and $Q$ are closed points of $C$ of degree 1, or $D = P$, where $P$ is a closed point of $C$ of degree 2. In the latter case, let $f(D)$ be the unique real line in $\mathbb{P}^2$ passing through $P$. In the first case, let $f(D)$ be the unique real line in $\mathbb{P}^2$ passing through $P$ and $Q$. If $P = Q$ then $f(D)$ is to be the tangent line to $C$ at $P = Q$.

Now, the object is to show that $f$ is a topological covering map, using the fact that $C$ is unramified. It is clear that $f$ is continuous. Since $X$ is compact and since $\mathbb{P}^2(\mathbb{R})^{\curlyvee}$ is Hausdorff, it suffices to show that $f$ is locally injective. Let $D \in X$ and put $L = f(D)$. There are three cases to consider: $D = P$, where $P$ is a closed point of $C$ of degree 2, $D = 2P$, where $P$ is a closed point of $C$ of degree 1, and $D = P + Q$, where $P$ and $Q$ are distinct closed points of $C$ of degree 1.

Suppose that $D = P$, where $P$ is a closed point of $C$ of degree 2. Then, the multiplicity of $P$ in the intersection product $L \cdot C$ is equal to 1. Indeed, let $m$ be this multiplicity. Since $P \in L$, one has $m \geq 1$. Since $C$ is unramified,

$$2(m-1) = (m-1) \cdot \deg(P) \leq \deg(L \cdot C) - \deg(L \cdot C)_{\mathrm{red}} \leq 1.$$

Hence, $m \leq \frac{3}{2}$. Therefore, $m = 1$. It follows that there is an open neighborhood $U$ of $D$ in $X$ such that the restriction of $f$ to $U$ is injective.

Suppose that $D = 2P$, where $P$ is a closed point of $C$ of degree 1. Then, the multiplicity of $P$ in the intersection product $L \cdot C$ is equal to 2. Indeed, let $m$ be this multiplicity. Since $L$ is the tangent line to $C$ at $P$, one has $m \geq 2$. Since $C$ is unramified,

$$m - 1 \leq \deg(L \cdot C) - \deg(L \cdot C)_{\mathrm{red}} \leq 1.$$

Hence, $m \leq 2$. Therefore, $m = 2$. It follows that there is an open neighborhood $U$ of $D$ in $X$ such that the restriction of $f$ to $U$ is injective.

Suppose that $D = P + Q$, where $P$ and $Q$ are distinct closed points of $C$ of degree 1. It is here where we use the preceding lemmas. Indeed, by Lemma 5, $L$ is not tangent to $P$ or $Q$. Therefore, the multiplicities of $P$ and $Q$ in $L \cdot C$ are equal to 1. It follows that there is an open neighborhood $U$ of $D$ in $X$ such that the restriction of $f$ to $U$ is injective.

We have proven that $f$ is a topological covering map. The surface $\mathbb{P}^2(\mathbb{R})$ admits only two connected coverings, the trivial one, and the covering by the 2-sphere $S^2$. Since $\mathbb{P}^2(\mathbb{R})^\curlyvee$ is homeomorphic to $\mathbb{P}^2(\mathbb{R})$ and since $X$ is connected, $X$ is either homeomorphic to $\mathbb{P}^2(\mathbb{R})$ or to $S^2$. In particular, the Euler characteristic $\chi(X)$ of $X$ is positive.

Now, we show that $C$ is a conic. Let $g$ be the genus of $C$. Then, $\chi(C_{\mathrm{top}}) = \frac{1}{2}(2 - 2g) = 1 - g$. Since the Euler characteristic of a Möbius strip vanishes, $\chi(X) = \chi(C_{\mathrm{top}}) = 1 - g$. Since $\chi(X)$ is positive, $g = 0$. Let $d$ be the degree of $C$. Since $C$ is smooth, $g = \frac{1}{2}(d-1)(d-2)$. Hence, $d = 1$ or 2. Since $C$ is unramified, $C$ is not contained in a real projective line, i.e., $d \neq 1$. Therefore, $d = 2$ and $C$ is a conic. $\square$

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