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STABILITY OF CONSTANT EQUILIBRIA IN A DIFFUSIVE LOGISTIC SIR MODEL WITH SATURATED TREATMENT AND DEATH RATES

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This paper is concerned with the Neumann initial-boundary value problem for a diffusive logistic SIR model with saturated treatment and death rates. The purpose of this paper is to study stability of constant equilibria under some conditions on parameters. Moreover, several numerical simulations are given.

1. Introduction

Problem. Motivated by Nakiyama et al. [14] without the death rate θI, we consider the Neumann initial-boundary value problem including the death rate,

$$\begin{cases} \dfrac{\partial S}{\partial t} = d_S \Delta S - \beta S I + r S \left(1 - \dfrac{S}{K}\right), & x \in \Omega,\ t > 0, \\ \dfrac{\partial I}{\partial t} = d_I \Delta I + \beta S I - \dfrac{\lambda I}{1 + \varepsilon I} - \theta I, & x \in \Omega,\ t > 0, \\ \nabla S \cdot \vec{n} = \nabla I \cdot \vec{n} = 0, & x \in \partial\Omega,\ t > 0, \\ S(x,0) = S_0(x),\ I(x,0) = I_0(x), & x \in \Omega, \end{cases} \tag{1}$$

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where Ω is a bounded domain in $\mathbb{R}^N$ ($N \in \mathbb{N}$) with smooth boundary $\partial\Omega$, $d_S, d_I, r, K, \beta, \varepsilon, \theta$ and λ are positive constants and $\vec{n}$ denotes the outward normal vector on $\partial\Omega$. Moreover, $S_0, I_0 \in C(\overline{\Omega})$ fulfill $S_0, I_0 \geq 0$. The problem (1) is one of SIR models, which describes the infectious diseases dynamics. The functions $S(x,t)$ and $I(x,t)$ represent the number of susceptible and infected individuals at position x and time t, respectively. The constants d_S and d_I are the diffusion rates of susceptible and infected individuals, respectively; K represents the carrying capacity of susceptible population; r is the intrinsic growth rate of susceptible population; βSI idealizes the saturated incidence rate; $\frac{\lambda I}{1+\varepsilon I}$ means the saturated treatment rate; θI idealizes the total removal (including death) of infected individuals. One characteristic of the problem (1) is that it takes the death rate into account. This enables us to understand population trends with precision. The original SIR epidemic model ((1) with $d_S = d_I = r = \varepsilon = \theta = 0$) was proposed by Kermack and McKendrick [11]. Subsequently, many kinds of SIR epidemic models have been studied in e.g. [1, 3–5, 8, 9, 13–16, 18–24].

Stability of equilibria. When we investigate behavior of solutions to (1), it is basic to clarify constant equilibria, which are the solutions of the following system:

$$\begin{cases} -\beta SI + rS\left(1 - \dfrac{S}{K}\right) = 0, \\ \beta SI - \dfrac{\lambda I}{1+\varepsilon I} - \theta I = 0. \end{cases} \tag{2}$$

The system (2) possibly admits three types of the solutions $(0,0)$, $(K,0)$ and (S^*, I^*) with $S^*, I^* > 0$, where $(K,0)$ is called the disease-free equilibrium (DFE for short) and (S^*, I^*) is called the endemic equilibrium (EE for short). Also we define the basic reproduction number as

$$\mathcal{R}_0 := \frac{K\beta}{\theta + \lambda},$$

which represents the expectation of individuals directly infected when one infected person joins a population where no one has immunity. The specific expression of $\mathcal{R}_0$ is obtained in the analysis of the sign of the second solution component I of (2). We now mention known results about some problems related to (1). Avila-Vales et al. [1] studied the problem in which βSI in (1) is replaced by $\frac{\beta(x)SI}{1+\alpha I}$, where α is a positive constant and $\beta(x)$ is a Hölder continuous function, and showed that the DFE is globally asymptotically stable when $\theta > 0$ and $K\beta^* < \theta$, where $\beta^* := \max_{x\in\overline{\Omega}} \beta(x)$. Recently, the following system without the death rate θI was investigated by [14]:

$$\begin{cases} \dfrac{\partial S}{\partial t} = d_S \Delta S - \dfrac{\beta SI}{1+\alpha I} + rS\left(1 - \dfrac{S}{K}\right), & x \in \Omega,\ t > 0, \\ \dfrac{\partial I}{\partial t} = d_I \Delta I + \dfrac{\beta SI}{1+\alpha I} - \dfrac{\lambda I}{1+\varepsilon I}, & x \in \Omega,\ t > 0. \end{cases} \tag{3}$$

This system is obtained by replacing βSI by $\frac{\beta SI}{1+\alpha I}$ $(\alpha > 0)$ and setting $\theta = 0$ in (1). In [14, Theorem 1.1] global asymptotic stability of $(K, 0)$ was proved under the condition $\mathcal{R}_0^* := \frac{K\beta}{\lambda} \leq 1$ and one of the following conditions:

(I)* $\varepsilon \leq \alpha$;

(II)* $\varepsilon > \alpha$ and $K\beta \cdot \frac{\varepsilon}{\alpha} \leq \lambda$;

(III)* $\varepsilon > \alpha$, $\lambda < K\beta \cdot \frac{\varepsilon}{\alpha}$, $\|S_0\|_{L^\infty(\Omega)} \leq K$ and $\|I_0\|_{L^\infty(\Omega)} < \frac{\lambda - K\beta}{K\beta\varepsilon - \alpha\lambda}$.

Thus we wonder whether this result can be extended to the case that the second equation in (3) contains $-\theta I$ in both cases $K\beta \leq \theta$ and $K\beta > \theta$. As a first step in the extension, we study the case $\alpha = 0$ in this paper.

Main results. The first purpose of this paper is to establish that the DFE of the problem (1) is asymptotically stable in the case that $\mathcal{R}_0 = \frac{K\beta}{\theta+\lambda} < 1$. However, the number of solutions to (2) depends on the parameters (see Figure 1).

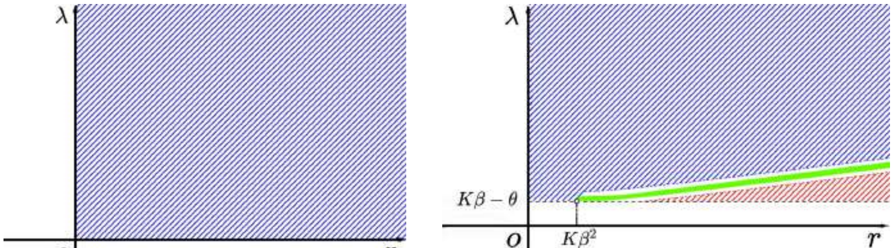

Figure 1: When $K\beta \leq \theta$, there are two constant equilibria of (1) with parameters chosen in the blue colored region as in the left figure. When $K\beta > \theta$, there are two, three and four constant equilibria of (1) with parameters chosen in the blue, green and red colored regions, respectively, as in the right figure.

In view of the right figure in Figure 1, when $K\beta > \theta$, solutions of (1) may converge to a point different from the DFE. In the first main theorem we give conditions such that solutions of (1) converge to the DFE when $\mathcal{R}_0 < 1$:

Theorem 1.1. *Let* $\mathcal{R}_0 = \frac{K\beta}{\theta+\lambda} < 1$. *Assume that* $S_0, I_0 \in C(\overline{\Omega})$, $S_0, I_0 \geq 0$, $S_0 \not\equiv 0$, *and that* $r, K, \beta, \varepsilon, \theta, \lambda, \|S_0\|_{L^\infty(\Omega)}, \|I_0\|_{L^\infty(\Omega)}$ *fulfill one of the following conditions*:

(I) $K\beta \leq \theta$;

(II) $K\beta > \theta, \quad \|S_0\|_{L^\infty(\Omega)} \leq K$ *and* $\|I_0\|_{L^\infty(\Omega)} < \frac{\theta+\lambda-K\beta}{(K\beta-\theta)\varepsilon}$;

(III) $K\beta > \theta, \quad K < \|S_0\|_{L^\infty(\Omega)} < \frac{\theta+\lambda}{\beta}$ *and* $\|I_0\|_{L^\infty(\Omega)} < \frac{\theta+\lambda-\beta\|S_0\|_{L^\infty(\Omega)}}{(\beta\|S_0\|_{L^\infty(\Omega)}-\theta)\varepsilon}$.

Then the global classical solution (S, I) *of* (1) *fulfills that as* $t \to \infty$,

$$(S(\cdot,t), I(\cdot,t)) \to (K, 0) \quad \text{in } L^\infty(\Omega) \times L^\infty(\Omega).$$

Remark 1.2. The condition $K\beta < \theta$ in **(I)** coincides with [1, (3.10) with $\beta(x) \equiv \beta$]. However, the case $K\beta = \theta$ was not covered in this literature. In addition, we can regard the condition **(II)** as an extension of **(III)*** in [14] in the case $\theta > 0$. Indeed, taking the limit $\alpha \to 0$ in **(III)***, we have

$$\varepsilon > 0, \quad \|S_0\|_{L^\infty(\Omega)} \leq K, \quad \|I_0\|_{L^\infty(\Omega)} \leq \frac{\lambda - K\beta}{K\beta\varepsilon},$$

which is also obtained by letting $\theta \to 0$ in **(II)**. Also, the condition **(III)** is the one for the case $\|S_0\|_{L^\infty(\Omega)} > K$, which is not presented in [14], and the restriction on $\|I_0\|_{L^\infty(\Omega)}$ changes depending on the value of $\|S_0\|_{L^\infty(\Omega)}$.

The second purpose of this paper is to discuss possibility of instability of the EE (S^*, I^*) when $\mathcal{R}_0 < 1$ and to give observations by numerical simulations.

Organization of the paper. This paper is organized as follows. In Section 2 we prove boundedness of solutions to (1). Section 3 is devoted to clarifying constant equilibria of (1). In Section 4 we show asymptotic stability of $(K, 0)$ by using a Lyapunov function. In Sections 5 and 6 we give a result on instability of (S^*, I^*) and several numerical simulations.

2. Global existence and boundedness

We first give a result on local existence of classical solutions to (1).

Lemma 2.1. *Let* $S_0, I_0 \in C(\overline{\Omega})$ *satisfy* $S_0, I_0 \geq 0$. *Then there exist* $T_{\max} \in (0, \infty]$ *and a unique pair* (S, I) *of nonnegative functions*

$$S, I \in C(\overline{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max})),$$

which solves (1) *in the classical sense. Moreover, if* $T_{\max} < \infty$, *then*

$$\lim_{t \nearrow T_{\max}} (\|S(\cdot,t)\|_{L^\infty(\Omega)} + \|I(\cdot,t)\|_{L^\infty(\Omega)}) = \infty.$$

Proof. The claim can be obtained by straightforward adaption of standard procedures based on arguments in appropriate fixed point frameworks e.g. as in [10] for closely related problems. □

We next prove global existence and boundedness of classical solutions obtained in Lemma 2.1.

Proposition 2.2. *Let* $S_0, I_0 \in C(\overline{\Omega})$ *satisfy* $S_0, I_0 \geq 0$. *Then there exists a unique pair* (S, I) *of nonnegative functions*

$$S, I \in C(\overline{\Omega} \times [0,\infty)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)), \tag{4}$$

which solves (1) *in the classical sense. Moreover, there exists* $M_0 > 0$ *such that*

$$\|S(\cdot,t)\|_{L^\infty(\Omega)} + \|I(\cdot,t)\|_{L^\infty(\Omega)} \leq M_0 \quad \textit{for all } t \geq 0. \tag{5}$$

Furthermore,

$$\begin{cases} \textit{if } S_0 \not\equiv 0, \quad \textit{then} \quad S(x,t) > 0 \quad \textit{for all } x \in \overline{\Omega} \textit{ and } t > 0, \\ \textit{if } I_0 \not\equiv 0, \quad \textit{then} \quad I(x,t) > 0 \quad \textit{for all } x \in \overline{\Omega} \textit{ and } t > 0. \end{cases} \tag{6}$$

In particular, if the condition **(II)** *or* **(III)** *in Theorem* 1.1 *are satisfied, then*

$$\|I(\cdot,t)\|_{L^\infty(\Omega)} \leq \|I_0\|_{L^\infty(\Omega)} \quad \textit{for all } t \geq 0. \tag{7}$$

Proof. Let $S_0, I_0 \in C(\overline{\Omega})$ fulfill $S_0, I_0 \geq 0$. From Lemma 2.1 there exist $T_{\max} \in (0,\infty]$ and a unique pair (S, I) of nonnegative functions

$$S, I \in C(\overline{\Omega} \times [0,T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0,T_{\max})),$$

which solves (1) in the classical sense. We shall show boundedness of $S(\cdot,t)$ by using the comparison principle. Let $\hat{S}$ be a solution of the initial value problem

$$\begin{cases} \dfrac{d\hat{S}}{dt} = r\hat{S}\Big(1 - \dfrac{\hat{S}}{K}\Big), \quad t > 0, \\ \hat{S}(0) = \|S_0\|_{L^\infty(\Omega)}. \end{cases}$$

Solving this problem, we see that

$$0 \leq \hat{S}(t) = \frac{K\hat{S}(0)}{\hat{S}(0) + (K - \hat{S}(0))e^{-rt}} \leq \begin{cases} \|S_0\|_{L^\infty(\Omega)} & (\|S_0\|_{L^\infty(\Omega)} > K), \\ K & (\|S_0\|_{L^\infty(\Omega)} \leq K). \end{cases}$$

Setting $M_0' := \max\{\|S_0\|_{L^\infty(\Omega)}, K\}$, we deduce that $\hat{S}(t) \leq M_0'$ for all $t > 0$. Hence the comparison principle tells us that

$$0 \leq S(\cdot,t) \leq \hat{S}(t) \leq M_0' \quad \text{for all } t \in (0,T_{\max}), \tag{8}$$

which means boundedness of $S(\cdot,t)$. We turn our eyes to uniform boundedness of $I(\cdot,t)$. To see this we first verify that there exists $M_1 > 0$ satisfying

$$\|I(\cdot,t)\|_{L^1(\Omega)} \le M_1 \quad \text{for all } t \in (0,T_{\max}). \tag{9}$$

Adding the first and second equations in (1) and integrating it over Ω, we infer from the boundary conditions in (1) and nonnegativity of I that

$$\begin{aligned} \frac{d}{dt}\int_\Omega (S+I) &= d_S \int_\Omega \Delta S + d_I \int_\Omega \Delta I + \int_\Omega \left[rS\left(1-\frac{S}{K}\right) - \frac{\lambda I}{1+\varepsilon I} - \theta I \right] \\ &\le \int_\Omega \left[-\frac{r}{K}S^2 + (r+\theta)S - \theta(S+I) \right]. \end{aligned}$$

Noticing that $-\frac{r}{K}S^2 + (r+\theta)S = -\frac{r}{K}\left[S - \frac{K(r+\theta)}{2r}\right]^2 + \frac{K(r+\theta)^2}{4r} \le \frac{K(r+\theta)^2}{4r}$, we obtain

$$\frac{d}{dt}\int_\Omega (S+I) \le \frac{K(r+\theta)^2}{4r}|\Omega| - \theta \int_\Omega (S+I),$$

which provides

$$\int_\Omega (S+I) \le M_1,$$

where $M_1 := \max\{\int_\Omega (S_0+I_0), \frac{K(r+\theta)^2}{4r\theta}|\Omega|\}$. This along with nonnegativity of S, I derives the estimate (9). We shift to proving uniform boundedness of $I(\cdot,t)$. Set a function f as $f(x,t,I) := \beta SI - \frac{\lambda I}{1+\varepsilon I} - \theta I$. Then we deduce from (8) and nonnegativity of I that

$$|f(x,t,I)| \le \beta SI + \frac{\lambda I}{1+\varepsilon I} + \theta I \le (\beta M_0' + \lambda + \theta) I.$$

Noting this together with (9) and applying [6, Lemma 2.1 with $p_0 = q = 1$], we make sure that there exists $M_0'' > 0$ satisfying

$$\|I(\cdot,t)\|_{L^\infty(\Omega)} \le M_0'' \quad \text{for all } t \in [0,T_{\max}).$$

Setting $M_0 := M_0' + M_0''$, we see from this and (8) that

$$\|S(\cdot,t)\|_{L^\infty(\Omega)} + \|I(\cdot,t)\|_{L^\infty(\Omega)} \le M_0 \quad \text{for all } t \in [0,T_{\max}),$$

which leads to $T_{\max} = \infty$ by Lemma 2.1. Thus we conclude that (S,I) has the properties (4) and (5). Also, the positivity property (6) is guaranteed by means of the strong maximum principle for parabolic equations as in [17]. We next move on to proving (7). Let $\hat{I}$ be a nonnegative solution of the initial value problem

$$\begin{cases} \dfrac{d\hat{I}}{dt} = \beta \hat{S}\hat{I} - \dfrac{\lambda \hat{I}}{1+\varepsilon \hat{I}} - \theta \hat{I}, & t > 0, \\ \hat{I}(0) = \|I_0\|_{L^\infty(\Omega)}. \end{cases} \tag{10}$$

Thanks to the comparison principle, we know that

$$0 \le I(\cdot,t) \le \hat{I}(t) \quad \text{for all } t>0. \tag{11}$$

Under the condition **(II)** or **(III)** in Theorem 1.1, we shall show that

$$\hat{I}(t) \le \hat{I}(0) = \|I_0\|_{L^\infty(\Omega)} \quad \text{for all } t>0. \tag{12}$$

We first prove (12) in the case **(II)**. Since $M_0' = \max\{\|S_0\|_{L^\infty(\Omega)}, K\} = K$ in this case, combining (8) with (10) derives

$$\begin{aligned}\frac{d\hat{I}}{dt} &\le \frac{(K\beta-\theta-\lambda)\hat{I}+(K\beta-\theta)\varepsilon\hat{I}^2}{1+\varepsilon\hat{I}} \\ &= -(K\beta-\theta)\varepsilon\cdot\frac{\left(\frac{\theta+\lambda-K\beta}{(K\beta-\theta)\varepsilon}-\hat{I}\right)\hat{I}}{1+\varepsilon\hat{I}}.\end{aligned}$$

Here, by the condition **(II)**, we note that

$$K\beta-\theta>0$$

and the comparison principle implies that

$$\hat{I}(t) \le \frac{\theta+\lambda-K\beta}{(K\beta-\theta)\varepsilon} \quad \text{for all } t>0,$$

so that $\frac{d\hat{I}}{dt}(t) \le 0$ for all $t>0$. Consequently, the estimate (12) holds in the case **(II)**. Next we show (12) in the case **(III)**. Since $M_0' = \|S_0\|_{L^\infty(\Omega)}$ in this case, (10) together with (8) leads to

$$\begin{aligned}\frac{d\hat{I}}{dt} &\le \frac{(\beta\|S_0\|_{L^\infty(\Omega)}-\theta-\lambda)\hat{I}+(\beta\|S_0\|_{L^\infty(\Omega)}-\theta)\varepsilon\hat{I}^2}{1+\varepsilon\hat{I}} \\ &= -(\beta\|S_0\|_{L^\infty(\Omega)}-\theta)\varepsilon\cdot\frac{\left(\frac{\theta+\lambda-\beta\|S_0\|_{L^\infty(\Omega)}}{(\beta\|S_0\|_{L^\infty(\Omega)}-\theta)\varepsilon}-\hat{I}\right)\hat{I}}{1+\varepsilon\hat{I}}.\end{aligned}$$

Here, in view of the condition **(III)**, we observe that

$$\beta\|S_0\|_{L^\infty(\Omega)}-\theta > K\beta-\theta>0$$

and the comparison principle yields

$$\hat{I}(t) \le \frac{\theta+\lambda-\beta\|S_0\|_{L^\infty(\Omega)}}{(\beta\|S_0\|_{L^\infty(\Omega)}-\theta)\varepsilon} \quad \text{for all } t>0,$$

so that $\frac{d\hat{I}}{dt}(t) \le 0$ for all $t>0$, and hence (12) holds. Therefore the estimate (12) holds under the condition **(III)**. Thus under the condition **(II)** or **(III)** we infer from (11) and (12) that

$$0 \le I(\cdot,t) \le \hat{I}(t) \le \|I_0\|_{L^\infty(\Omega)} \quad \text{for all } t>0,$$

which proves (7). □

3. Constant equilibria in a small reproduction number

We define P, S_0^*, I_0^*, $S_\pm^*$ and $I_\pm^*$ as follows:

$$P := \frac{[r\varepsilon(K\beta-\theta)+K\beta^2]^2}{4rK\beta^2\varepsilon}, \tag{13}$$

$$S_0^* := \frac{r\varepsilon(K\beta+\theta)+K\beta^2}{2r\beta\varepsilon}, \tag{14}$$

$$I_0^* := \frac{r\varepsilon(K\beta-\theta)-K\beta^2}{2K\beta^2\varepsilon}, \tag{15}$$

$$S_\pm^* := \frac{r\varepsilon(K\beta+\theta)+K\beta^2 \pm \sqrt{D}}{2r\beta\varepsilon}, \tag{16}$$

$$I_\pm^* := \frac{r\varepsilon(K\beta-\theta)-K\beta^2 \pm \sqrt{D}}{2K\beta^2\varepsilon}, \tag{17}$$

where

$$D := [r\varepsilon(K\beta-\theta)+K\beta^2]^2 - 4rK\beta^2\varepsilon\lambda. \tag{18}$$

Our goal in this section is to prove the following proposition.

Proposition 3.1. *Suppose that*

$$\mathcal{R}_0 = \frac{K\beta}{\theta+\lambda} < 1.$$

Then constant equilibria of (1) *are given below*:

(I) *If*

$$K\beta \le \theta \quad or \quad \begin{cases} K\beta > \theta,\ r < \dfrac{K\beta^2}{(K\beta-\theta)\varepsilon}, \\ K\beta > \theta,\ r \ge \dfrac{K\beta^2}{(K\beta-\theta)\varepsilon} \text{ and } \lambda > P, \end{cases}$$

then constant equilibria of (1) *are* $(0,0)$ *and* $(K,0)$.

(II) *If*

$$K\beta > \theta,\ r > \frac{K\beta^2}{(K\beta-\theta)\varepsilon} \text{ and } \lambda = P,$$

then constant equilibria of (1) *are* $(0,0)$, $(K,0)$ *and* (S_0^*, I_0^*).

(III) *If*

$$K\beta > \theta,\ r > \frac{K\beta^2}{(K\beta-\theta)\varepsilon} \text{ and } \lambda < P,$$

then constant equilibria of (1) *are* $(0,0)$, $(K,0)$, (S_-^*, I_+^*) *and* (S_+^*, I_-^*).

In order to show Proposition 3.1, for the moment we find all solutions (S,I) of (2) with $S,I \in \mathbb{R}$.

Lemma 3.2. *Let* $\mathcal{R}_0 = \frac{K\beta}{\theta+\lambda} < 1$. *Then solutions of* (2) *are given as follows*:

- **(i)** *If* $\lambda > P$, *then solutions of* (2) *are* $(0,0)$, $(K,0)$.
- **(ii)** *If* $\lambda = P$, *then solutions of* (2) *are* $(0,0)$, $(K,0)$ *and* (S_0^*, I_0^*).
- **(iii)** *If* $\lambda < P$, *then solutions of* (2) *are* $(0,0)$, $(K,0)$, (S_-^*, I_+^*) *and* (S_+^*, I_-^*).

Proof. In view of the first equation in (2), we see that $S = 0$ or

$$-\beta I + r\left(1 - \frac{S}{K}\right) = 0 \tag{19}$$

holds. Also we turn out from the second equation in (2) that $I = 0$ or

$$\beta S - \frac{\lambda}{1+\varepsilon I} - \theta = 0 \tag{20}$$

holds. If $I = 0$, it follows from the first equation in (2) that $S = 0$ or $S = K$ holds. On the other hand, if (20) holds, then we have the identity

$$S = \frac{\lambda}{\beta(1+\varepsilon I)} + \frac{\theta}{\beta} = \frac{\lambda + \theta(1+\varepsilon I)}{\beta(1+\varepsilon I)}. \tag{21}$$

This in conjunction with (19) guarantees that

$$-\beta I + r\left(1 - \frac{\lambda + \theta(1+\varepsilon I)}{K\beta(1+\varepsilon I)}\right) = 0,$$

which is equivalent to

$$K\beta^2\varepsilon I^2 - [r\varepsilon(K\beta - \theta) - K\beta^2]I - r(K\beta - \theta - \lambda) = 0. \tag{22}$$

We compute the discriminant D of (22). Then we observe that D is given by (18):

$$\begin{aligned} D &= [r\varepsilon(K\beta - \theta) - K\beta^2]^2 + 4rK\beta^2\varepsilon(K\beta - \theta - \lambda) \\ &= K^2\beta^4 + 2rK\beta^2\varepsilon(K\beta - \theta) + r^2\varepsilon^2(K\beta - \theta)^2 - 4rK\beta^2\varepsilon\lambda \\ &= [r\varepsilon(K\beta - \theta) + K\beta^2]^2 - 4rK\beta^2\varepsilon\lambda. \end{aligned}$$

The rest of the proof is divided into the three cases $D < 0$, $D = 0$ and $D > 0$.

We first deal with the case that $D < 0$ i.e.

$$\lambda > \frac{[r\varepsilon(K\beta - \theta) + K\beta^2]^2}{4rK\beta^2\varepsilon} = P,$$

where P is defined by (13). Then the equation (22) in $I \in \mathbb{R}$ has no solution, which means that the solutions of (2) are $(0,0)$ and $(K,0)$. Hence we obtain the conclusion in the case **(i)**.

Next we move on to consideration about the case that $D = 0$ i.e.

$$\lambda = \frac{[r\varepsilon(K\beta - \theta) + K\beta^2]^2}{4rK\beta^2\varepsilon} = P.$$

In this case there exists a unique solution of (22) such that

$$I = \frac{r\varepsilon(K\beta - \theta) - K\beta^2}{2K\beta^2\varepsilon} = I_0^*. \tag{23}$$

Substituting $I = \frac{r\varepsilon(K\beta-\theta)-K\beta^2}{2K\beta^2\varepsilon}$ and $\lambda = \frac{[r\varepsilon(K\beta-\theta)+K\beta^2]^2}{4rK\beta^2\varepsilon}$ into (21), we obtain

$$\begin{aligned} S &= \frac{\frac{[r\varepsilon(K\beta-\theta)+K\beta^2]^2}{4rK\beta^2\varepsilon}}{\beta\left(1+\varepsilon\cdot\frac{r\varepsilon(K\beta-\theta)-K\beta^2}{2K\beta^2\varepsilon}\right)} + \frac{\theta}{\beta} \\ &= \frac{K\beta^2 + rK\beta\varepsilon - r\varepsilon\theta}{2r\beta\varepsilon} + \frac{\theta}{\beta} \\ &= \frac{r\varepsilon(K\beta+\theta) + K\beta^2}{2r\beta\varepsilon} = S_0^*. \end{aligned} \tag{24}$$

Therefore we conclude in the case **(ii)** that the solutions of (2) are $(0,0)$, $(K,0)$ and (S_0^*, I_0^*), where S_0^* and I_0^* are defined in (14) and (15), respectively.

Finally we consider the case that $D > 0$ i.e.

$$\lambda < \frac{[r\varepsilon(K\beta - \theta) + K\beta^2]^2}{4rK\beta^2\varepsilon} = P.$$

Then (22) admits the two solutions given by

$$I = \frac{r\varepsilon(K\beta - \theta) - K\beta^2 \pm \sqrt{D}}{2K\beta^2\varepsilon} = I_\pm^*. \tag{25}$$

Hence, (21) along with this yields

$$\begin{aligned} S &= \frac{\lambda}{\beta\left(1+\varepsilon\cdot\frac{r\varepsilon(K\beta-\theta)-K\beta^2\pm\sqrt{D}}{2K\beta^2\varepsilon}\right)} + \frac{\theta}{\beta} \\ &= \frac{2K\beta\lambda}{r\varepsilon(K\beta-\theta)+K\beta^2\pm\sqrt{D}} + \frac{\theta}{\beta} \\ &= \frac{2K\beta\lambda\left[r\varepsilon(K\beta-\theta)+K\beta^2\mp\sqrt{D}\right]}{[r\varepsilon(K\beta-\theta)+K\beta^2]^2 - D} + \frac{\theta}{\beta}. \end{aligned}$$

Recalling that $D = [r\varepsilon(K\beta - \theta) + K\beta^2]^2 - 4rK\beta^2\varepsilon\lambda$ by (18), we have

$$\begin{aligned} S &= \frac{r\varepsilon(K\beta - \theta) + K\beta^2 \mp \sqrt{D}}{2r\beta\varepsilon} + \frac{\theta}{\beta} \\ &= \frac{r\varepsilon(K\beta + \theta) + K\beta^2 \mp \sqrt{D}}{2r\beta\varepsilon} = S^*_{\mp}. \end{aligned} \tag{26}$$

Consequently, we see that the solutions of (2) are $(0,0)$, $(K,0)$, (S^*_-, I^*_+) and (S^*_+, I^*_-) in the case **(iii)**, where $S^*_\pm$ and $I^*_\pm$ are defined in (16) and (17), respectively. ☐

In Lemma 3.2 we found all solutions of (2) in $\mathbb{R}^2$. Denoting by (S^*, I^*) any solution of (2) except $(0,0)$ and $(K,0)$, we focus on the signs of S^* and I^*. We first verify positivity of S^*.

Lemma 3.3. *Let* $\mathcal{R}_0 = \frac{K\beta}{\theta+\lambda} < 1$ *and let* $\lambda \le P$*. Then*

$$S^* > 0.$$

Proof. In view of the proof of Lemma 3.2 we see from the condition $\lambda \le P$ that $D \ge 0$, and we observe from (24) and (26) that S^* is written as

$$\begin{aligned} S^*_0 &= \frac{r\varepsilon(K\beta + \theta) + K\beta^2}{2r\beta\varepsilon}, \\ S^*_+ &= \frac{r\varepsilon(K\beta + \theta) + K\beta^2 + \sqrt{D}}{2r\beta\varepsilon}, \\ S^*_- &= \frac{r\varepsilon(K\beta + \theta) + K\beta^2 - \sqrt{D}}{2r\beta\varepsilon}. \end{aligned}$$

Recalling that $r, K, \beta, \varepsilon, \theta > 0$, we have

$$r\varepsilon(K\beta + \theta) + K\beta^2 > 0, \quad r\beta\varepsilon > 0. \tag{27}$$

This warrants that $S^*_0, S^*_+ > 0$, so that we have only to show that $S^*_- > 0$. Noting that $D = [r\varepsilon(K\beta - \theta) + K\beta^2]^2 - 4rK\beta^2\varepsilon\lambda$ by (18), we infer that

$$[r\varepsilon(K\beta + \theta) + K\beta^2]^2 - D = 4r\varepsilon\theta(K\beta^2 + rK\beta\varepsilon) + 4rK\beta^2\varepsilon\lambda > 0.$$

This implies that

$$0 \le \sqrt{D} < r\varepsilon(K\beta + \theta) + K\beta^2,$$

which together with (27) yields $S^*_- > 0$. ☐

We next give conditions to reveal the sign of I^*.

Lemma 3.4. *Let* $\mathcal{R}_0 = \frac{K\beta}{\theta+\lambda} < 1$ *and let* $\lambda \le P$. *Then the following holds*:

(i) $r\varepsilon(K\beta-\theta)-K\beta^2 > 0$ *is equivalent to* $I^* > 0$.

(ii) $r\varepsilon(K\beta-\theta)-K\beta^2 < 0$ *is equivalent to* $I^* < 0$.

Proof. Recalling the proof of Lemma 3.2, we deduce from the condition $\lambda \le P$ that $D \ge 0$, and we infer from (23) and (25) that I^* is written as

$$I_0^* = \frac{r\varepsilon(K\beta-\theta)-K\beta^2}{2K\beta^2\varepsilon}, \tag{28}$$

$$I_+^* = \frac{r\varepsilon(K\beta-\theta)-K\beta^2+\sqrt{D}}{2K\beta^2\varepsilon}, \tag{29}$$

$$I_-^* = \frac{r\varepsilon(K\beta-\theta)-K\beta^2-\sqrt{D}}{2K\beta^2\varepsilon}. \tag{30}$$

Focusing on the denominators in (28), (29) and (30), we note that

$$K\beta^2\varepsilon > 0, \tag{31}$$

which implies that the signs of I_0^* and $I_\pm^*$ depend on those of $r\varepsilon(K\beta-\theta)-K\beta^2$ and $r\varepsilon(K\beta-\theta)-K\beta^2 \pm \sqrt{D}$, respectively. Thus it suffices to show that the signs of I_+^* and I_-^* coincide with those of $r\varepsilon(K\beta-\theta)-K\beta^2$. Noting that $D = [r\varepsilon(K\beta-\theta)+K\beta^2]^2 - 4rK\beta^2\varepsilon\lambda$, we have

$$\begin{aligned}[r\varepsilon(K\beta-\theta)-K\beta^2]^2 - D &= -4rK\beta^2\varepsilon(K\beta-\theta)+4rK\beta^2\varepsilon\lambda \\ &= 4rK\beta^2\varepsilon(\theta+\lambda-K\beta).\end{aligned}$$

Since the condition $\mathcal{R}_0 = \frac{K\beta}{\theta+\lambda} < 1$ provides $\theta+\lambda-K\beta > 0$, we see that

$$[r\varepsilon(K\beta-\theta)-K\beta^2]^2 - D = 4rK\beta^2\varepsilon(\theta+\lambda-K\beta) > 0.$$

This is equivalent to

$$[r\varepsilon(K\beta-\theta)-K\beta^2]^2 > D,$$

which is rewritten as

$$|r\varepsilon(K\beta-\theta)-K\beta^2| > \sqrt{D}.$$

Therefore, in view of (29) and (30) together with (31) we conclude that

$$\begin{cases} I_+^* > 0 \text{ and } I_-^* > 0 & \text{is equivalent to} \quad r\varepsilon(K\beta-\theta)-K\beta^2 > 0, \\ I_+^* < 0 \text{ and } I_-^* < 0 & \text{is equivalent to} \quad r\varepsilon(K\beta-\theta)-K\beta^2 < 0. \end{cases}$$

Thus the proof is complete. □

Proof of Proposition 3.1. We first deal with the case **(I)**. In view of Lemma 3.2 **(i)** we know that solutions of (2) are $(0,0)$, $(K,0)$ if $\lambda > P$. Hence we focus on the case $\lambda \le P$. In this case it follows from Lemma 3.2 **(ii)** and **(iii)** that solutions of (2) are given by $[(0,0)$, $(K,0)$ and $(S_0^*, I_0^*)]$ or $[(0,0)$, $(K,0)$, (S_+^*, I_-^*) and $(S_-^*, I_+^*)]$. Here, the condition $[K\beta \le \theta]$ or $\left[K\beta > \theta \text{ and } r < \frac{K\beta^2}{(K\beta-\theta)\varepsilon}\right]$ leads to $r\varepsilon(K\beta-\theta) - K\beta^2 < 0$, which implies from Lemma 3.4 **(ii)** that $I_0^*, I_\pm^* < 0$. Thus we arrive at the conclusion in the case **(I)**. We next consider the case **(II)**. Lemma 3.2 **(ii)** along with $\lambda = P$ guarantees that solutions of (2) are $(0,0)$, $(K,0)$ and (S_0^*, I_0^*). Assuming that $K\beta > \theta$ and $r > \frac{K\beta^2}{(K\beta-\theta)\varepsilon}$, we deduce that $r\varepsilon(K\beta-\theta) - K\beta^2 > 0$, which means that $I_0^* > 0$ by Lemma 3.4 **(i)**. Noting that $S_0^* > 0$ when $\lambda = P$, we arrive at the conclusion in the case **(II)**. Finally we consider the case **(III)**. The condition $\lambda < P$ together with Lemma 3.2 **(iii)** warrants that solutions of (2) are $(0,0)$, $(K,0)$, (S_+^*, I_-^*) and (S_-^*, I_+^*). Noting that the conditions $K\beta > \theta$ and $r > \frac{K\beta^2}{(K\beta-\theta)\varepsilon}$ are same as those in the case **(II)**, we obtain the conclusion in the case **(III)**. $\square$

4. Proof of Theorem 1.1

Given any initial data $S_0, I_0 \in C(\overline{\Omega})$ satisfying $S_0, I_0 \ge 0$ and $S_0 \not\equiv 0$, let (S,I) be a unique global classical solution of (1) provided by Proposition 2.2. In the same way as in [4, Proof of Proposition 2.4], we use a Lyapunov function $V(S,I)$ as

$$V(S,I) := \int_\Omega K\left(\frac{S}{K} - \log\frac{S}{K} - 1\right) + \int_\Omega I \ge 0. \tag{32}$$

By the first and second equations in (1) and integration by parts, we observe that

$$\begin{aligned}
\frac{d}{dt}V(S(\cdot,t),I(\cdot,t)) &= \int_\Omega \left(1-\frac{K}{S}\right)\frac{\partial S}{\partial t} + \int_\Omega \frac{\partial I}{\partial t} \\
&= \int_\Omega \left(1-\frac{K}{S}\right)\left[d_S\Delta S - \beta SI + rS\left(1-\frac{S}{K}\right)\right] \\
&\quad + \int_\Omega \left(d_I\Delta I + \beta SI - \frac{\lambda I}{1+\varepsilon I} - \theta I\right) \\
&= -d_S K\int_\Omega \frac{|\nabla S|^2}{S^2} - \int_\Omega \left(1-\frac{K}{S}\right)\left[\beta SI - rS\left(1-\frac{S}{K}\right)\right] \\
&\quad + \int_\Omega \left(\beta SI - \frac{\lambda I}{1+\varepsilon I} - \theta I\right) \\
&\le -\frac{r}{K}\int_\Omega (S-K)^2 - \int_\Omega \frac{\lambda I}{1+\varepsilon I} + \int_\Omega (K\beta-\theta)I
\end{aligned} \tag{33}$$

for all $t > 0$.

We first consider the case **(I)**. In this case the inequality (33) together with the condition $K\beta \le \theta$ yields

$$\frac{d}{dt}V(S(\cdot,t),I(\cdot,t)) \le -\frac{r}{K}\int_\Omega (S-K)^2 - \int_\Omega \frac{\lambda I}{1+\varepsilon I} \le 0 \quad \text{for all } t > 0.$$

Let $t > 1$. Integrating this over $[1,t]$, we see that

$$\begin{aligned} &V(S(\cdot,t),I(\cdot,t)) + \frac{r}{K}\int_1^t \int_\Omega (S-K)^2 + \int_1^t \int_\Omega \frac{\lambda I}{1+\varepsilon I} \\ &\le V(S(\cdot,1),I(\cdot,1)) \quad \text{for all } t > 1. \end{aligned}$$

This in conjunction with (5) and (32) guarantees

$$0 < \frac{r}{K}\int_1^t \int_\Omega (S-K)^2 + \frac{\lambda}{1+\varepsilon M_0}\int_1^t \int_\Omega I \le V(S(\cdot,1),I(\cdot,1)) \quad \text{for all } t > 1,$$

which leads to

$$\int_1^t \int_\Omega (S-K)^2 + \int_1^t \int_\Omega I \le \frac{1}{\min\left\{\frac{r}{K}, \frac{\lambda}{1+\varepsilon M_0}\right\}} V(S(\cdot,1),I(\cdot,1)) \quad \text{for all } t > 1.$$

Letting $t \to \infty$, we have

$$\int_1^\infty \int_\Omega (S-K)^2 + \int_1^\infty \int_\Omega I < \infty. \tag{34}$$

Here we set functions $\varphi_1(t), \varphi_2(t)$ as

$$\begin{aligned} \varphi_1(t) &:= \|S(\cdot,t) - K\|^2_{L^2(\Omega)}, \\ \varphi_2(t) &:= \|I(\cdot,t)\|_{L^1(\Omega)} \end{aligned}$$

for $t \ge 1$. In view of (5), we know that $\|S(\cdot,t)\|_{L^\infty(\Omega)}, \|I(\cdot,t)\|_{L^\infty(\Omega)} \le M_0$ for all $t \ge 1$. Thus, applying standard parabolic regularity theory in [12], we can confirm that there exist $\eta_1, \eta_2 \in (0,1)$ and $c_1, c_2 > 0$ such that

$$\begin{aligned} \|S\|_{C^{2+\eta_1, 1+\frac{\eta_1}{2}}(\overline{\Omega}\times[1,\infty))} &\le c_1, \\ \|I\|_{C^{2+\eta_2, 1+\frac{\eta_2}{2}}(\overline{\Omega}\times[1,\infty))} &\le c_2. \end{aligned}$$

These imply that φ_1, φ_2 are uniformly continuous on $[1,\infty)$. Since (34) means that

$$\begin{aligned} \int_1^\infty \varphi_1(t)\,dt &< \infty, \\ \int_1^\infty \varphi_2(t)\,dt &< \infty, \end{aligned}$$

we deduce from the Barbalat lemma in [2, Lemma 3.1] that

$$\varphi_1(t) \to 0, \quad \varphi_2(t) \to 0$$

as $t \to \infty$. In other words,

$$S(\cdot,t) \to K \text{ in } L^2(\Omega), \quad I(\cdot,t) \to 0 \text{ in } L^1(\Omega) \tag{35}$$

as $t \to \infty$. The Gagliardo–Nirenberg inequality in [7, Theorem 10.1] claims that there exist $c_3, c_4 > 0$ such that

$$\begin{aligned} \|S(\cdot,t)-K\|_{L^\infty(\Omega)} &\le c_3 \|S(\cdot,t)-K\|_{W^{1,\infty}(\Omega)}^{\frac{N}{N+2}} \|S(\cdot,t)-K\|_{L^2(\Omega)}^{\frac{2}{N+2}}, \\ \|I(\cdot,t)\|_{L^\infty(\Omega)} &\le c_4 \|I(\cdot,t)\|_{W^{1,\infty}(\Omega)}^{\frac{N}{N+1}} \|I(\cdot,t)\|_{L^1(\Omega)}^{\frac{1}{N+1}}. \end{aligned}$$

Setting $c_5 := \|S\|_{C^{1,0}(\overline{\Omega}\times[1,\infty))} + K$ and $c_6 := \|I\|_{C^{1,0}(\overline{\Omega}\times[1,\infty))}$, we have

$$\begin{aligned} \|S(\cdot,t)-K\|_{W^{1,\infty}(\Omega)}^{\frac{N}{N+2}} &\le c_5^{\frac{N}{N+2}}, \\ \|I(\cdot,t)\|_{W^{1,\infty}(\Omega)}^{\frac{N}{N+1}} &\le c_6^{\frac{N}{N+1}}, \end{aligned}$$

and hence

$$\begin{aligned} \|S(\cdot,t)-K\|_{L^\infty(\Omega)} &\le c_3 c_5^{\frac{N}{N+2}} \|S(\cdot,t)-K\|_{L^2(\Omega)}^{\frac{2}{N+2}} \to 0, \\ \|I(\cdot,t)\|_{L^\infty(\Omega)} &\le c_4 c_6^{\frac{N}{N+1}} \|I(\cdot,t)\|_{L^1(\Omega)}^{\frac{1}{N+1}} \to 0 \end{aligned}$$

as $t \to \infty$ by (35). Therefore we arrive at the conclusion in the case **(I)**.

We next consider the case **(II)**. In this case, since $\|I_0\|_{L^\infty(\Omega)} < \frac{\theta+\lambda-K\beta}{(K\beta-\theta)\varepsilon}$, we can take $\delta > 0$ satisfying

$$0 < \delta < \frac{\theta+\lambda-K\beta}{(K\beta-\theta)\varepsilon} - \|I_0\|_{L^\infty(\Omega)},$$

which in conjunction with (7) leads to

$$0 < \delta < \frac{\theta+\lambda-K\beta}{(K\beta-\theta)\varepsilon} - \|I(\cdot,t)\|_{L^\infty(\Omega)}$$

for all $t > 0$. Hence the inequality (33) entails that

$$\begin{aligned} \frac{d}{dt}V(S(\cdot,t),I(\cdot,t)) &\le -\frac{r}{K}\int_\Omega (S-K)^2 - \int_\Omega \frac{\lambda I}{1+\varepsilon I} + \int_\Omega (K\beta-\theta)I \\ &= -\frac{r}{K}\int_\Omega (S-K)^2 - \int_\Omega \frac{(\theta+\lambda-K\beta)I - (K\beta-\theta)\varepsilon I^2}{1+\varepsilon I} \\ &= -\frac{r}{K}\int_\Omega (S-K)^2 - (K\beta-\theta)\varepsilon \int_\Omega \frac{\left(\frac{\theta+\lambda-K\beta}{(K\beta-\theta)\varepsilon} - I\right)I}{1+\varepsilon I} \\ &\le -\frac{r}{K}\int_\Omega (S-K)^2 - (K\beta-\theta)\varepsilon\cdot\delta \int_\Omega \frac{I}{1+\varepsilon I} < 0 \end{aligned} \tag{36}$$

for all $t > 0$. Integrating (36) over $[1,t]$ ($t > 1$), we have

$$\begin{aligned} &V(S(\cdot,t),I(\cdot,t)) + \frac{r}{K}\int_1^t\int_\Omega (S-K)^2 + (K\beta-\theta)\varepsilon\cdot\delta\int_1^t\int_\Omega \frac{I}{1+\varepsilon I} \\ &\le V(S(\cdot,1),I(\cdot,1)) \end{aligned}$$

for all $t > 1$. Hence we see from (7) and (32) that

$$\begin{aligned} 0 &< \frac{r}{K}\int_1^t\int_\Omega (S-K)^2 + \frac{(K\beta-\theta)\delta\varepsilon}{1+\varepsilon\|I_0\|_{L^\infty(\Omega)}}\int_1^t\int_\Omega I \\ &\le V(S(\cdot,1),I(\cdot,1)), \end{aligned}$$

which is rewritten as

$$\int_1^t\int_\Omega (S-K)^2 + \int_1^t\int_\Omega I \le \frac{1}{\min\left\{\frac{r}{K}, \frac{(K\beta-\theta)\delta\varepsilon}{1+\varepsilon\|I_0\|_{L^\infty(\Omega)}}\right\}} V(S(\cdot,1),I(\cdot,1))$$

for all $t > 1$. Letting $t \to \infty$, we arrive at the same property as in (34). Thus the claim can be derived similarly to the case **(I)**.

Finally we deal with the case **(III)**. In this case, the condition $K < \|S_0\|_{L^\infty(\Omega)}$ yields

$$\begin{aligned} \|I_0\|_{L^\infty(\Omega)} &\le \frac{\theta+\lambda-\beta\|S_0\|_{L^\infty(\Omega)}}{(\beta\|S_0\|_{L^\infty(\Omega)}-\theta)\varepsilon} \\ &< \frac{\theta+\lambda-K\beta}{(K\beta-\theta)\varepsilon} \end{aligned}$$

and (7) still holds. Therefore the conclusion can be proved similarly to the case **(II)**. □

5. Possibility of instability of (S^*,I^*)

The purpose of this section is to show instability of the EE (S^*,I^*) defined before Lemma 3.3. To see this we focus on the positive solutions of (1) which are constants with respect to $x \in \Omega$, that is, the positive solutions of the system of ordinary differential equations

$$\begin{cases} \dfrac{dS}{dt}(t) = -\beta S(t)I(t) + rS(t)\left(1-\dfrac{S(t)}{K}\right), & t > 0, \\ \dfrac{dI}{dt}(t) = \beta S(t)I(t) - \dfrac{\lambda I(t)}{1+\varepsilon I(t)} - \theta I(t), & t > 0. \end{cases}$$

Sufficient conditions for instability and local asymptotic stability are given as follows:

Proposition 5.1. *Let* $\mathcal{R}_0 = \frac{K\beta}{\theta+\lambda} < 1$. *For* (S_+^*, I_-^*) *and* (S_-^*, I_+^*) *given in Proposition 3.1, the following hold*:

(i) *If*

$$K\beta > \theta, \quad \frac{K\beta^2}{(K\beta-\theta)\varepsilon} < r \quad \text{and} \quad \lambda < P = \frac{[r\varepsilon(K\beta-\theta)+K\beta^2]^2}{4rK\beta^2\varepsilon},$$

then (S_+^*, I_-^*) *is unstable.*

(ii) *If*

$$\begin{aligned} &K\beta > \theta, \quad \max\left\{\frac{K\beta^2}{(K\beta-\theta)\varepsilon}, K\beta\right\} < r \quad \text{and} \\ &\lambda < P = \frac{[r\varepsilon(K\beta-\theta)+K\beta^2]^2}{4rK\beta^2\varepsilon}, \end{aligned}$$

then (S_-^*, I_+^*) *is locally asymptotically stable.*

Proof. We will use the Routh–Hurwitz criteria. We consider the function from $\mathbb{R}^2$ to $\mathbb{R}^2$ defined by

$$\begin{pmatrix} S \\ I \end{pmatrix} \longmapsto \begin{pmatrix} -\beta SI + rS\left(1-\frac{S}{K}\right) \\ \beta SI - \frac{\lambda I}{1+\varepsilon I} - \theta I \end{pmatrix},$$

and focus on the real parts of the eigenvalues of the Jacobian matrix of this function at the point $(S^*, I^*) := (S_\pm^*, I_\mp^*)$ given by

$$J(S^*, I^*) := \begin{pmatrix} -\beta I^* + r - \frac{2r}{K}S^* & -\beta S^* \\ \beta I^* & \beta S^* - \frac{\lambda}{(1+\varepsilon I^*)^2} - \theta \end{pmatrix}.$$

Since the characteristic equation of $J(S^*, I^*)$ is

$$\xi^2 - \operatorname{tr}(J(S^*, I^*))\xi + \det(J(S^*, I^*)) = 0, \tag{37}$$

the signs of the real parts of the eigenvalues are determined by the signs of $\operatorname{tr}(J(S^*, I^*))$ and $\det(J(S^*, I^*))$. Here, $\operatorname{tr}(J(S^*, I^*))$ and $\det(J(S^*, I^*))$ are written as

$$\begin{aligned} \operatorname{tr}(J(S^*, I^*)) &= \left(-\beta I^* + r - \frac{2r}{K}S^*\right) + \left(\beta S^* - \frac{\lambda}{(1+\varepsilon I^*)^2} - \theta\right), \\ \det(J(S^*, I^*)) &= \left(-\beta I^* + r - \frac{2r}{K}S^*\right)\left(\beta S^* - \frac{\lambda}{(1+\varepsilon I^*)^2} - \theta\right) + \beta^2 S^* I^*. \end{aligned}$$

Recalling that (S^*, I^*) satisfies (19) and (20), we have

$$\begin{aligned}
\operatorname{tr}(J(S^*,I^*)) &= -\frac{r}{K}S^* + \frac{\varepsilon I^*}{1+\varepsilon I^*}(\beta S^* - \theta)\\
&= \frac{1}{K(1+\varepsilon I^*)}\{-rS^* - \varepsilon S^* I^*(r-K\beta) - K\varepsilon\theta I^*\},\\
\det(J(S^*,I^*)) &= -\frac{r}{K}S^* \cdot \frac{\varepsilon I^*}{1+\varepsilon I^*}(\beta S^* - \theta) + \beta^2 S^* I^*\\
&= \frac{S^* I^*}{K(1+\varepsilon I^*)}\{-r\varepsilon(\beta S^* - \theta) + K\beta^2(1+\varepsilon I^*)\}.
\end{aligned}$$

To verify the signs of $\operatorname{tr}(J(S^*,I^*))$ and $\det(J(S^*,I^*))$ we set

$$\begin{aligned}
\sigma(S^*,I^*) &:= -rS^* - \varepsilon S^* I^*(r-K\beta) - K\varepsilon\theta I^*,\\
\varphi(S^*,I^*) &:= -r\varepsilon(\beta S^* - \theta) + K\beta^2(1+\varepsilon I^*),
\end{aligned}$$

and show the conclusions **(i)** and **(ii)**.

We first consider the conclusion **(i)**. Under the conditions in **(i)**, Proposition 3.1 gives the existence of the constant equilibrium (S_+^*, I_-^*) of (1). Here we recall that

$$(S_\pm^*, I_\mp^*) = \left(\frac{r\varepsilon(K\beta+\theta) + K\beta^2 \pm \sqrt{D}}{2r\beta\varepsilon}, \frac{r\varepsilon(K\beta-\theta) - K\beta^2 \mp \sqrt{D}}{2K\beta^2\varepsilon}\right),$$

where D is defined by (18). Inserting $(S^*,I^*) = (S_+^*, I_-^*)$ into $\varphi(S^*,I^*)$, we have

$$\begin{aligned}
\varphi(S_+^*, I_-^*) &= -r\varepsilon\left\{\beta \cdot \frac{r\varepsilon(K\beta+\theta) + K\beta^2 + \sqrt{D}}{2r\beta\varepsilon} - \theta\right\}\\
&\quad + K\beta^2\left\{1 + \varepsilon \cdot \frac{r\varepsilon(K\beta-\theta) - K\beta^2 - \sqrt{D}}{2K\beta^2\varepsilon}\right\}\\
&= -\frac{r\varepsilon(K\beta-\theta) + K\beta^2 + \sqrt{D}}{2} + \frac{r\varepsilon(K\beta-\theta) + K\beta^2 - \sqrt{D}}{2}\\
&= -\sqrt{D} < 0, \qquad (38)
\end{aligned}$$

Noticing that $K, S_+^*, I_-^* > 0$, we see that

$$\det(J(S_+^*, I_-^*)) = \frac{S_+^* I_-^*}{K(1+\varepsilon I_-^*)} \cdot \varphi(S_+^*, I_-^*) < 0,$$

which combined with (37) shows that the Jacobian matrix $J(S_+^*, I_-^*)$ has one positive and one negative eigenvalues. This means that (S_+^*, I_-^*) is a saddle point by the Routh–Hurwitz criteria, and hence (S_+^*, I_-^*) is unstable.

We next consider the conclusion **(ii)**. The conditions in **(ii)** are more restrictive than the one in **(i)**, which warrants that there exists (S_-^*, I_+^*) by applying Proposition 3.1. Substituting $(S^*, I^*) = (S_-^*, I_+^*)$ into $\varphi(S^*, I^*)$ and proceeding similarly to (38), we have $\varphi(S_-^*, I_+^*) = \sqrt{D} > 0$. Hence, it follows that $\det(J(S_-^*, I_+^*)) = \frac{S_-^* I_+^*}{K(1+\varepsilon I_+^*)} \cdot \varphi(S_-^*, I_+^*) > 0$. Thus we need to verify the sign of $\operatorname{tr}(J(S_-^*, I_+^*))$. Recalling that $r > K\beta$ and $S_-^*, I_+^* > 0$, we have $\sigma(S_-^*, I_+^*) = -rS_-^* - \varepsilon S_-^* I_+^*(r - K\beta) - K\varepsilon\theta I_+^* < 0$. This yields $\operatorname{tr}(J(S_-^*, I_+^*)) = \frac{1}{K(1+\varepsilon I_+^*)} \cdot \sigma(S_-^*, I_+^*) < 0$, which together with the fact that $\det(J(S_-^*, I_+^*)) > 0$ and (37) implies that the real parts of all eigenvalues of $J(S_-^*, I_+^*)$ are negative. Thus we conclude that (S_-^*, I_+^*) is locally asymptotically stable. ☐

6. Numerical simulations

In this section we verify the mathematical results in Sections 1–5 through numerical simulations. For simplicity, we assume that $N = 1$ and $\Omega = (0, 1)$.

(a) Convergence to the DFE $(K, 0)$**.** We choose the parameters as

$$d_S = d_I = 1,\ r = 5,\ K = 1,\ \beta = 10,\ \varepsilon = 100,\ \theta = 12,\ \lambda = 4,$$

and then we have

$$\mathcal{R}_0 = \frac{K\beta}{\theta + \lambda} = \frac{5}{8} < 1, \quad K\beta - \theta = -2 < 0,$$

so that these parameters satisfy the conditions in Theorem 1.1 **(I)**. Also we put

$$S_0(x) = K + 1 = 2, \quad I_0(x) = e^{10}.$$

Then we see that the solution (S, I) of (1) converges to $(K, 0)$ as in Figure 2.

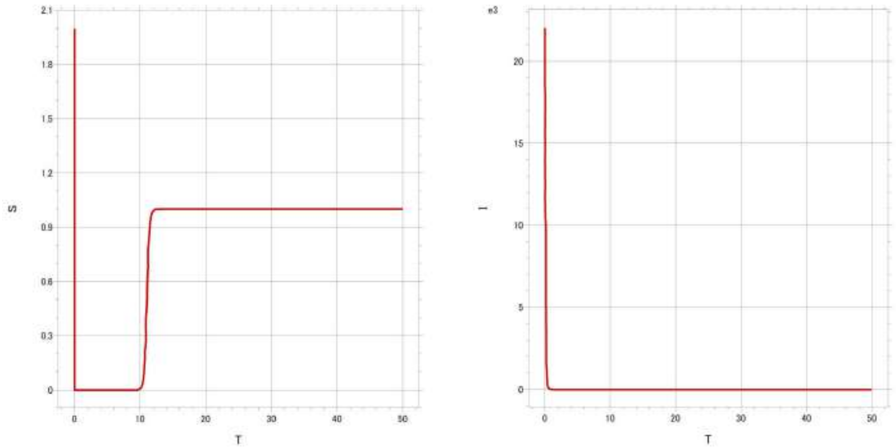

Figure 2: Numerical simulation at $x = \frac{1}{2}$ in Theorem 1.1 **(I)**.

(b) Instability of (S_+^*, I_-^*). We give the parameters as follows:

$$d_S = d_I = 1,\ r = 5,\ K = 1,\ \beta = 10,\ \varepsilon = 100,\ \theta = 8,\ \lambda = 4. \tag{39}$$

This warrants that

$$\begin{aligned}
&\mathcal{R}_0 = \frac{K\beta}{\theta+\lambda} = \frac{5}{6} < 1,\\
&K\beta - \theta = 2 > 0,\\
&r - \frac{K\beta^2}{(K\beta-\theta)\varepsilon} = \frac{9}{2} > 0,\\
&\frac{[r\varepsilon(K\beta-\theta)+K\beta^2]^2}{4rK\beta^2\varepsilon} - \lambda = \frac{41}{20} > 0,
\end{aligned}$$

and hence these parameters fulfill the conditions in Proposition 5.1 **(i)**. None of the solutions to (1) converge to (S_+^*, I_-^*), even if the initial value is sufficiently close to this point. For example, we take the initial data as

$$\begin{aligned}
S_0(x) &= S_+^* + 0.01 = \frac{91+\sqrt{41}}{100} + 0.01 \approx 0.9840,\\
I_0(x) &= e^{0.01} I_-^* = e^{0.01} \cdot \frac{9-\sqrt{41}}{200} \approx 0.0131.
\end{aligned}$$

Then we infer that the solution (S, I) of (1) converges to (S_-^*, I_+^*) as in Figure 3.

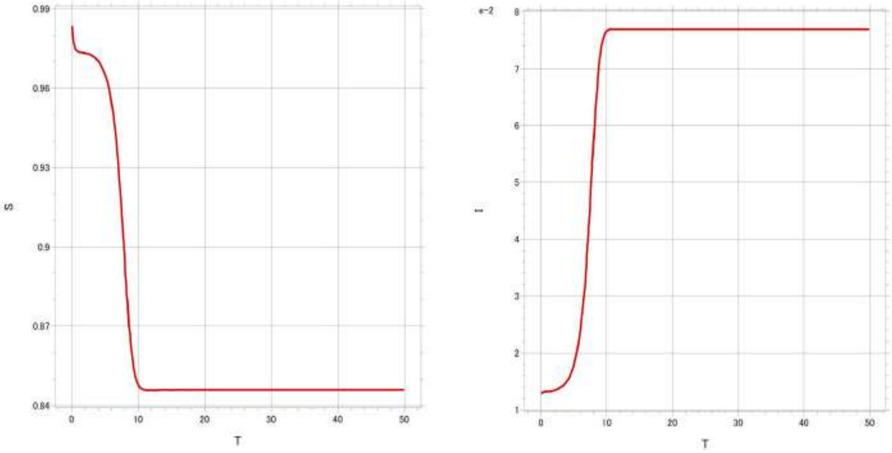

Figure 3: Numerical simulation at $x = \frac{1}{2}$ in Proposition 5.1 **(i)**. A point in a neighborhood of (S_+^*, I_-^*) converges to $(S_-^*, I_+^*) \approx (0.8460, 0.0770)$.

(c) Local asymptotic stability of (S_-^*, I_+^*)**.** We verify that the conditions in Proposition 5.1 **(ii)** provide that if the initial data is sufficiently close to (S_-^*, I_+^*), the solution (S, I) of (1) converges to this equilibrium. Setting the parameters as

$$d_S = d_I = 1,\ r = 12,\ K = 1,\ \beta = 10,\ \varepsilon = 100,\ \theta = 8,\ \lambda = 4, \tag{40}$$

we have

$$\begin{aligned}
&\mathcal{R}_0 = \frac{K\beta}{\theta+\lambda} = \frac{5}{6} < 1,\\
&K\beta - \theta = 2 > 0,\\
&r - \max\left\{\frac{K\beta^2}{(K\beta-\theta)\varepsilon}, K\beta\right\} = 2 > 0,\\
&\frac{[r\varepsilon(K\beta-\theta)+K\beta^2]^2}{4rK\beta^2\varepsilon} - \lambda = \frac{433}{48} > 0.
\end{aligned}$$

Then we see from (40) that the conditions in Proposition 5.1 **(ii)** are satisfied. Thus if the initial data is sufficiently close to (S_-^*, I_+^*), e.g.,

$$\begin{aligned}
S_0(x) &= S_-^* + 0.01 = \frac{217-\sqrt{433}}{240} + 0.01 \approx 0.8275,\\
I_0(x) &= e^{0.01} I_+^* = e^{0.01} \cdot \frac{23+\sqrt{433}}{200} \approx 0.2212,
\end{aligned}$$

then the solution (S, I) converges to (S_-^*, I_+^*) as in Figure 4.

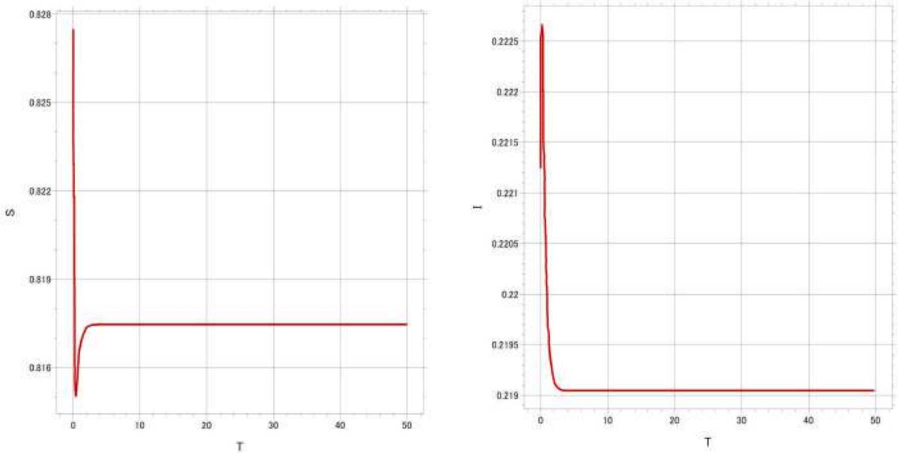

Figure 4: Numerical simulation at $x = \frac{1}{2}$ in Proposition 5.1 **(ii)**. A point in a neighborhood of (S_-^*, I_+^*) converges to $(S_-^*, I_+^*) \approx (0.8175, 0.2190)$.

(d) Stability and instability of (S_-^*, I_+^*). We take the parameters as in (39). Then we have $K\beta - r = 5 > 0$ so that $r < K\beta$, which implies that if the initial data is given as

$$S_0(x) = S_-^* + 0.01 = \frac{91 - \sqrt{41}}{100} + 0.01 \approx 0.8560,$$

$$I_0(x) = e^{0.01} I_+^* = e^{0.01} \cdot \frac{9 + \sqrt{41}}{200} \approx 0.0778,$$

then the solution (S, I) converges to (S_-^*, I_+^*) as in Figure 5.

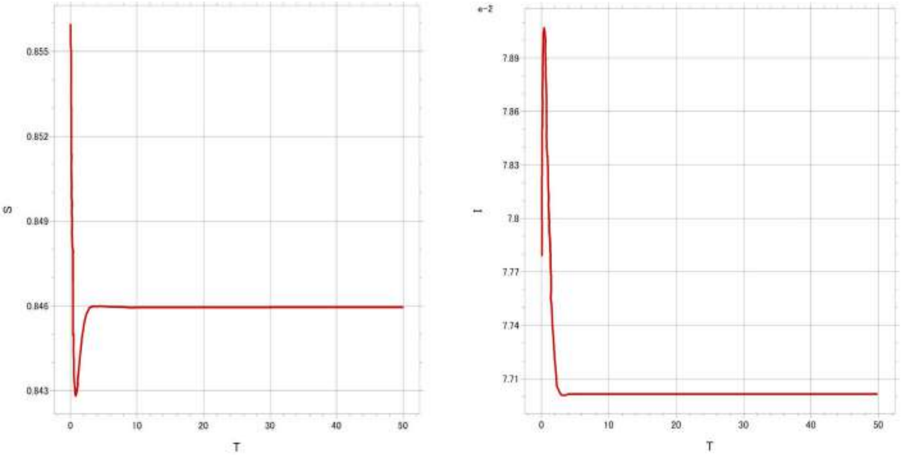

Figure 5: A point with $x = \frac{1}{2}$ near (S_-^*, I_+^*) converges to $(S_-^*, I_+^*) \approx (0.8460, 0.0770)$.

We see from numerical simulations that any points in a neighborhood of either (S_+^*, I_-^*) or (S_-^*, I_+^*) converge to (S_-^*, I_+^*) except for the case that (S_-^*, I_+^*) is unstable. Also, the condition $r < K\beta$ is necessary for instability of (S_-^*, I_+^*). However we cannot derive the necessary and sufficient conditions for instability of (S_-^*, I_+^*). To determine this condition is an open problem.

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