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ON QUADRISECANT LINES OF THREEFOLDS IN $\mathbb{P}^5$

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Dedicated to Silvio Greco in occasion of his 60-th birthday.

We study smooth threefolds of $\mathbb{P}^5$ whose quadrisecant lines don't fill up the space. We give a complete classification of those threefolds X whose only quadrisecant lines are the lines contained in X. Then we prove that, if X admits "true" quadrisecant lines, but they don't fill up $\mathbb{P}^5$, then either X is contained in a cubic hypersurface, or it contains a family of dimension at least two of plane curves of degree at least four.

Introduction.

The classical theorem of general projection for surfaces says that a general projection in $\mathbb{P}^3$ of a smooth complex projective surface S of $\mathbb{P}^5$ is a surface F with *ordinary singularities* i.e. its singular locus is either empty or is a curve γ such that:

(i) γ is either non singular or has at most a finite number of ordinary triple points;
(ii) every smooth point of γ is either a nodal point or a pinch-point of F;
(iii) the general point of γ is a nodal point for F;
(iv) every triple point of γ is an ordinary triple point of F.

(see [6], [11])

Moreover γ is empty if and only if S is already contained in a $\mathbb{P}^3$.

Note that the projection to $\mathbb{P}^3$ can be split in two steps: in the first step from $\mathbb{P}^5$ to $\mathbb{P}^4$ S acquires only double points, while triple points appear only in the second step from $\mathbb{P}^4$ to $\mathbb{P}^3$.

The problem of classifying the surfaces S such that F does not have any triple point is equivalent to the problem of classifying the intermediate surfaces S' of $\mathbb{P}^4$ whose trisecant lines don't fill up $\mathbb{P}^4$, or "without apparent triple points" in the old fashioned terminology. This problem had been tackled by Severi in [17]. His approach was based on the description of hypersurfaces of $\mathbb{P}^4$ containing a 3–dimensional family of lines: they are quadrics and hypersurfaces birationally fibered by planes. By consequence his theorem says that a surface S' without apparent triple points either is contained in a quadric or is birationally fibered by plane curves of degree at least 3. Recently Aure ([1]) made this result precise under smoothness assumption, proving that, if a surface S' as above is not contained in a quadric, then it is an elliptic normal scroll.

In the study of threefolds, several analogous questions appear, not all completely answered yet. Here we are concerned mainly with smooth threefolds of $\mathbb{P}^5$ and their projections to $\mathbb{P}^4$. We want to study their 4-secant lines, trying in particular to describe threefolds whose 4-secant lines don't fill up the space. We first study threefolds X whose only 4-secant lines are the lines contained in X: we give a complete description of them (Theorem 2.1). Then we consider the threefolds with a 5-dimensional family of 4-secant lines (or more generally k-secant lines, with $k \geq 4$): we find that these lines cannot fill up $\mathbb{P}^4$ and that X is birationally ruled by surfaces of $\mathbb{P}^3$ of degree k (Theorem 2.3). There are no examples of this situation and it seems sensible to guess that in fact it cannot happen.

The general situation is that of 3-folds whose 4-secant lines form a family of dimension four, i.e. a congruence of lines. To understand the case of a congruence of order 0, i.e. of lines not filling up $\mathbb{P}^5$, we imitate the approach of Severi: we have to look at hypersurfaces Y of $\mathbb{P}^4$ covered by a 4-dimensional family of lines. We find that a priori there are many possibilities for such hypersurfaces. More precisely, if we consider a general hyperplane section V of such a Y, this is a threefold of $\mathbb{P}^4$ covered by a 2-dimensional family of lines. The threefolds like that are studied in [12], where the following result is proved:

Theorem 0.1. *Let $V \subset \mathbb{P}^4$ be a projective, integral hypersurface over an algebraically closed field of characteristic zero, covered by lines. Let $\Sigma \subset \mathbb{G}(1,4)$ denote the Fano scheme of the lines on V. Assume that Σ is generically reduced of dimension 2. Let μ denote the number of lines of Σ passing through a general point of V and g the sectional genus of V, i.e. the geometric genus of a plane section of V. Then $\mu \leq 6$ and one of the following happens:*

(i) $\mu = 1$, *i.e.* V *is birationally a scroll over a surface;*

(ii) V *is birationally ruled by smooth quadric surfaces over a curve* ($\mu = 2$)*;*

(iii) V *is a cubic hypersurface with singular locus of dimension at most one; if* V *is smooth, then* Σ *is irreducible and* $\mu = 6$*;*

(iv) V *has degree* $d \leq 6$, $g = 1$, $2 \leq \mu \leq 4$ *and* V *is a projection in* $\mathbb{P}^4$ *of one of the following:*

- *a complete intersection of two hyperquadrics in* $\mathbb{P}^5$, $d = 4$*;*
- *a section of* $\mathbb{G}(1, 4)$ *with a* $\mathbb{P}^6$, $d = 5$*;*
- *a hyperplane section of* $\mathbb{P}^2 \times \mathbb{P}^2$, $d = 6$*;*
- $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, $d \leq 6$.

To apply this result to a fourfold Y generated by the 4-secant lines of a threefold, it is necessary first of all to understand the meaning of the assumption of generic reducedness on Σ. We prove that this hypothesis is equivalent to the non-existence of a fixed tangent plane to V along a general line of Σ. Threefolds V not satisfying this assumption are then described in Proposition 3.4.

So it is possible to perform an analysis of the possible cases for the fourfold Y. This leads to a result very similar to the theorem of Severi for surfaces quoted above:

Theorem 0.2. *Let* X *be a smooth non-degenerate threefold of* $\mathbb{P}^5$ *not contained in a quadric. Let* Σ *be an irreducible component of dimension* 4 *of* $\Sigma_4(X)$ *such that a general line of* Σ *is* k*-secant* X ($k \geq 4$)*. Assume that the union of the lines of* Σ *is a hypersurface* Y*. Then either* Y *is a cubic or* Y *contains a family of planes of dimension* 2 *which cut on* X *a family of plane curves of degree* k*.*

Recently, a different approach to the study of multisecant lines of smooth threefolds of $\mathbb{P}^5$ has been considered by Sijong Kwak ([8]). It is based on the well-known monoidal construction. He proves that, if the 4-secant lines of X don't fill up $\mathbb{P}^5$, then either $h^2(\mathcal{O}_X) \neq 0$ or $h^1(\mathcal{O}_X(1)) \neq 0$. Moreover he gives an explicit formula for $q_4(X)$, the number of 4-secant lines through a general point of $\mathbb{P}^5$, depending on $\deg X$, on the sectional genus and on the two Euler characteristics $\chi(\mathcal{O}_X)$ and $\chi(\mathcal{O}_S)$, where S is a general hyperplane section. It is interesting to note that, testing this formula on all known smooth threefolds of $\mathbb{P}^5$, one gets $q_4(X) = 0$ only for those contained in a cubic hypersurface.

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1. Multisecants lines of threefolds in $\mathbb{P}^5$.

Let X be an integral smooth threefold of $\mathbb{P}^5$ not contained in a hyperplane. To define the multisecant lines of X, we follow the approach of Le Barz ([9]). Let $k \geq 2$ be an integer number. Let $Hilb^k\mathbb{P}^5$ be the Hilbert scheme of subschemes of length k of $\mathbb{P}^5$, and $Hilb^k_c\mathbb{P}^5$ be its smooth open subvariety parametrizing curvilinear subschemes, i.e. subschemes which are contained in a smooth curve. Let $Al^k\mathbb{P}^5$ denote the subscheme of $Hilb^k_c\mathbb{P}^5$ of length k subschemes lying on a line and $Hilb^k_cX$ that of subschemes contained in X. The following cartesian diagram defines Al^kX, the scheme of aligned k-tuples of points of X:

$$\begin{array}{ccc} Al^kX & \longrightarrow & Al^k\mathbb{P}^5 \\ \downarrow & & \downarrow \\ Hilb^k_cX & \longrightarrow & Hilb^k_c\mathbb{P}^5. \end{array}$$

We have: $\dim Hilb^k_c\mathbb{P}^5 = 5k$, $\dim Al^k\mathbb{P}^5 = 10 + (k-2) = 8+k$, $\dim Hilb^k_cX = 3k$; so, if Al^kX is non–empty, then any irreducible component of its has dimension at least $(8+k)+(3k)-(5k) = 8-k$.

Let now

$$a : Al^k\mathbb{P}^5 \longrightarrow \mathbb{G}(1,5)$$

be the natural map (axe) to the Grassmannian of lines of $\mathbb{P}^5$. Note that all fibers of a have dimension k.

The image scheme $\Sigma_k(X) := a(Al^k(X))$ is by definition the family of k-secant lines of X. Clearly all lines contained in X belong to $\Sigma_k(X)$. If $Al^kX = \emptyset$, then obviously also $\Sigma_k(X) = \emptyset$: in this case no line cuts X in at least k points or is contained in X.

Let us consider now the restriction $\bar{a}$ of a to an irreducible component Σ of Al^kX:

$$\bar{a} : \Sigma \longrightarrow \bar{a}(\Sigma) \subset \mathbb{G}(1,5).$$

We have : $\dim \bar{a}(\Sigma) = \dim \Sigma - \dim \Sigma_l$, where $\Sigma_l := \bar{a}^{-1}(l)$ is the fibre over l, a general line of $\bar{a}(\Sigma)$. There are two possibilities, i.e. either $\dim \Sigma_l = k$ if $l \subset X$, or $\dim \Sigma_l = 0$ if $l \cap X$ is a scheme of finite length. By consequence, either $\dim \bar{a}(\Sigma) = \dim \Sigma - k$, if any line of $\bar{a}(\Sigma)$ is contained in X, or else $\dim \bar{a}(\Sigma) = \dim \Sigma$ if a general line of $\bar{a}(\Sigma)$ is not contained in X.

Some rather precise information on the families of k-secant lines of threefolds for particular k come from the classical theorems of "general projection". For smooth curves in $\mathbb{P}^3$ and smooth surfaces in $\mathbb{P}^4$ there are very precise theorems, describing the singular locus of the projected variety (see [7], [11], [1]).

From these results, passing to general sections with linear spaces of dimension 3 and 4, it follows that a general projection X' in $\mathbb{P}^4$ of a smooth threefold X of $\mathbb{P}^5$ acquires a double surface D, i.e. a surface whose points have multiplicity at least two on X', and a triple curve $T \subset D$, i.e. a curve whose points have multiplicity at least three on X'. Moreover, D is non-empty unless X is degenerate and T is non-empty unless X is contained in a quadric. In terms of multisecant lines, this means that, through a general point P of $\mathbb{P}^5$, there passes a 2-dimensional family of 2-secant lines of X: we have that $\Sigma_2(X)$ is irreducible of dimension 6. If moreover X is not contained in a quadric, then the trisecant lines through P form a family of dimension 1, so $\Sigma_3(X)$ has dimension 5 and its lines fill up $\mathbb{P}^5$. I would like to emphasize that D is truly double and T is truly triple for X', or, in other words, a general secant line of X is not trisecant and a general trisecant line is not quadrisecant.

On the other hand, it has been proved that X' does not have any point of multiplicity 5 or more (see [15], [10]). Hence the 5-secant lines of X never fill up $\mathbb{P}^5$.

Remark 1.1. It is interesting to note that no smooth threefold X in $\mathbb{P}^5$ has $Al^3(X) = \emptyset$. Indeed, if so, a general curve section C of X would be a smooth curve of $\mathbb{P}^3$ without trisecant lines. It is well known that such a curve C is either a skew cubic or an elliptic quartic. So X could be either $\mathbb{P}^1 \times \mathbb{P}^2$ or a complete intersection of two quadrics: in both cases, X is an intersection of quadrics, so the trisecant lines are necessarily contained in X. But both threefolds contain lines: they form a family of dimension 3 in the first case and of dimension 2 in the second one.

2. Quadrisecant lines: special cases.

The first case we consider is that of threefolds without "true" quadrisecant lines.

Theorem 2.1. *Let X be a smooth threefold of $\mathbb{P}^5$. Then $Al^4(X) \neq \emptyset$. If its quadrisecant lines are all contained in X, then $\sigma_4 := \dim \Sigma_4(X) \leq 4$ and one of the following possibilities occurs:*

$\sigma_4 = 4$, *X is a $\mathbb{P}^3$;*

$\sigma_4 = 3$, X is a quadric hypersurface (contained in a hyperplane of $\mathbb{P}^5$), or $\mathbb{P}^1 \times \mathbb{P}^2$;

$\sigma_4 = 2$, X is a cubic hypersurface (contained in a hyperplane of $\mathbb{P}^5$), or a complete intersection of type $(2, 2)$, or a Castelnuovo threefold, or a Bordiga scroll;

$\sigma_4 = 1$, X is a complete intersection of type $(2, 3)$, or an inner projection of a complete intersection of type $(2, 2, 2)$ in $\mathbb{P}^6$;

$\sigma_4 = 0$, X is a complete intersection of type $(3, 3)$.

Proof. By [16], the maximal dimension of a family of lines contained in a threefold X is 4, and the maximum is attained only by linear spaces. Moreover, if the dimension is 3, then either X is a quadric or it is birationally a scroll over a curve. Being X smooth, in the last case X is $\mathbb{P}^1 \times \mathbb{P}^2$ (see [14]).

If $\sigma_4 \leq 2$, then a general hyperplane section S of X contains only a finite number of lines and does not possess any other 4-secant line. In [3] one proves that there is a finite explicit list of such surfaces S. They have all degree at most 9 and are all arithmetically Cohen-Macaulay, except the elliptic scroll. The smooth threefolds X having them as general hyperplane sections are all described (see for instance [5]) and are precisely those appearing in the list above. More precisely, a Castelnuovo threefold has degree 5, its ideal is generated by the maximal minors of a 2×3 matrix of forms: in the first two columns the entries are linear while in the third one they are quadratic, X is fibered by quadrics over $\mathbb{P}^1$. The Bordiga scroll has degree 6, its ideal is generated by the maximal minors of a 3×4 matrix of linear forms, it is a scroll over $\mathbb{P}^2$. Finally, the computation of the dimension of the family of lines contained in a smooth complete intersection as above is classical. Note that in all cases X contains lines, so $Al^4(X) \neq \emptyset$.

Remark 2.2. Note that all threefolds whose only quadrisecant lines are the lines contained in them are cut out by quadrics and cubics.

From now on we will consider only smooth non-degenerate threefolds in $\mathbb{P}^5$ such that the general line of at least one irreducible component of $\Sigma_4(X)$ is not contained in X. Hence the dimension of such a component Σ is at least 4. On the other hand $\dim \Sigma < 6$, otherwise every secant line would be quadrisecant, which is excluded by general projection theorems. If the dimension of such a component is 5, then we have the following result.

Theorem 2.3. *Let X be a smooth non-degenerate threefold of $\mathbb{P}^5$, let Σ be an irreducible component of $\Sigma_4(X)$ of dimension 5. Then the lines of Σ don't fill up $\mathbb{P}^5$. More precisely either their union is a quadric or it is a hypersurface birationally ruled by $\mathbb{P}^3$'s over a curve.*

Proof. Let H be a general hyperplane and let $S := X \cap H$, $\Sigma' := \Sigma \cap \mathbb{G}(1, H)$. S is a smooth surface of $\mathbb{P}^4$ and Σ' is a family of dimension 3 of quadrisecant lines of S. From the general projection result for surfaces, it follows that the lines of Σ' don't fill up H, so their union is a hypersurface V in H. By [16], either V is a quadric or it is birationally fibered by planes. In the first case, V lifts to a quadric containg X and all its quadrisecant lines.

In the second case, the planes of V cut on S a one-dimensional family of plane curves of degree, say, a: since the lines of these planes have to be 4-secant S, then $a \geq 4$. Coming back to $\mathbb{P}^5$, X contains a family of dimension at least 4 of plane curves of degree at least 4. Let W be the subvariety of $\mathbb{G}(2, 5)$ parametrizing those planes. We consider the focal locus of the family W on a fixed plane π (see [4] for generalities about the theory of foci): it must contain the plane curve of X lying on π. But the matrix representing the characteristic map of W restricted to π is a 3×4 matrix of linear forms on π, so it cannot degenerate along a curve of degree strictly bigger than 3, unless it degenerates everywhere on π. So all planes of the family are focal planes. Let f be the projection from the incidence correspondence of W to $\mathbb{P}^5$: the differential of f has always a kernel of dimension two and image of dimension 4. By the analogous of Sard's theorem, it follows that the union of the planes of W is a variety Y of dimension 4. By [16], we conclude that Y is birationally ruled by $\mathbb{P}^3$'s over a curve.

Remark 2.4. Under the assumption of Theorem 2.3, if X is not contained in a quadric, then it is covered by a one-dimensional family of surfaces of $\mathbb{P}^3$ of degree at least 4, whose hyperplane sections are the plane curves covering S. So the plane curves on X are cut by the planes of the $\mathbb{P}^3$'s of Y.

3. Quadrisecant lines not filling up the spaces.

We assume now that X is a non-degenerate smooth threefold in $\mathbb{P}^5$, such that all irreducible components of $\Sigma_4(X)$, corresponding to lines not all contained in X, have dimension 4. A subscheme of dimension 4 of $\mathbb{G}(1, 5)$ is called a congruence of lines. To a congruence of lines Σ one associates an integer number, its *order*: the number of lines of Σ passing through a general point of $\mathbb{P}^5$. More formally, it is the intersection number of Σ with the Schubert cycle of lines through a point. The order of $\Sigma_4(X)$ will be denoted by $q_4(X)$. It is clear that if X is contained in a quadric or in a cubic hypersurface, then this hypersurface contains also the quadrisecant lines of X, hence $q_4(X) = 0$. It is natural to try to reverse this implication, so one can consider the following

Question . *Do there exist smooth threefolds X in $\mathbb{P}^5$, not contained in a cubic, but such that the 4-secant lines of X form a congruence with $q_4(X) = 0$?*

From now on, we assume that $H^0(\mathcal{I}_X(3)) = (0)$, $\dim \Sigma_4(X) = 4$ and $q_4(X) = 0$. Let Y be the hypersurface of $\mathbb{P}^5$ union of the 4-secant lines of X. Let Σ be the Fano scheme of lines contained in Y: $\Sigma_4(X)$ is a union of one or more irreducible components of Σ. Let now H be a general hyperplane, $S := X \cap H$ and $V := Y \cap H$. So $\Sigma' := \Sigma \cap \mathbb{G}(1, H)$ is the Fano scheme of lines contained in V and $\Sigma_4(S) = \Sigma_4(X) \cap \mathbb{G}(1, H)$ parametrizes 4-secant lines of S.

In order to apply Theorem 0.1 to our situation, we want to give some characterization of threefolds covered by lines with non-reduced associated Fano scheme. First of all we recall a result from [12].

Let V be a threefold of $\mathbb{P}^4$ covered by a two dimensional family of lines and let $\bar{\Sigma}$ be an irreducible component of dimension two of its Fano scheme of lines. Let r be a line on V which is a general point of $\bar{\Sigma}$, let P be a general point of r and let $\mathbb{P}(T_P V)$ be the projective plane obtained by projectivization from the tangent space to V at P, its points correspond to tangent lines to V at P. Choose homogeneous coordinates in $\mathbb{P}^4$ such that $P = [1, 0, \dots, 0]$ and $T_P V$ has equation $x_4 = 0$. In the affine chart $x_0 \neq 0$ with non-homogeneous coordinates $y_i = x_i/x_0$, $i = 1, \dots, 4$, V has an equation $G = G_1 + G_2 + G_3 + \dots + G_d = 0$, where the G_i are the homogeneous components of G and $G_1 = x_4$. It is convenient to write $G_i = F_i + y_4 H_i$, where the F_i are polynomials in y_1, y_2, y_3. The equations $y_4 = F_2 = 0$ (resp. $y_4 = F_2 = F_3 = 0$) represent lines in $\mathbb{P}(T_P V)$ which are at least 3-tangent (resp. 4-tangent) to V at P.

Proposition 3.1. *With the notations just introduced, $\bar{\Sigma}$ is reduced at r if and only if in $\mathbb{P}(T_P V)$ the intersection of the conic $F_2 = 0$ with the cubic $F_3 = 0$ is reduced at the point corresponding to r.*

Proof. [12], Proposition 1.3.

In the following characterization, we need again the notion of focal scheme of a family of lines (see [4]).

Proposition 3.2. *Let V be a threefold of $\mathbb{P}^4$ covered by a two dimensional family of lines. Let $\bar{\Sigma}$ be an irreducible component of dimension two of the Fano scheme of lines on V. Then the following are equivalent:*

(1) $\bar{\Sigma}$ is non-reduced;
(2) V has a fixed tangent space of dimension at least two along a general line of $\bar{\Sigma}$;
(3) on each general line of the family $\bar{\Sigma}$ there is at least one focal point.

Proof. (1) $\Leftrightarrow$ (2). One implication is Proposition 1.5 of [12]. This implication and the inverse one, which is similar, follow from a local computation and from Proposition 3.1.

(2) $\Leftrightarrow$ (3). Let the line r be a smooth, general point of $\bar{\Sigma}$ and let $[x_0, \ldots, x_4]$ be homogeneous coordinates in $\mathbb{P}^4$ such that r has equations $x_2 = x_3 = x_4 = 0$. We consider the restriction to r of the global characteristic map relative to the family of lines $\bar{\Sigma}$:

$$\chi(r) : T_r\bar{\Sigma} \otimes \mathcal{O}_r \to \mathcal{N}_{r/\mathbb{P}^4}.$$

Since $T_r\bar{\Sigma} \otimes \mathcal{O}_r \simeq \mathcal{O}_r^2$ and $\mathcal{N}_{r/\mathbb{P}^4} \simeq \mathcal{O}_r(1)^3$, the map $\chi(r)$ can be represented by a suitable 3×2 matrix $\mathcal{M}$, with linear entries $l_{ij}(x_0, x_1)$. If there is a fixed tangent plane M_r to V along r, it gives a (fixed) normal direction to r in $\mathbb{P}^4$. If $\Lambda \subset K^5$ is the vector space of dimension two corresponding to r, this normal direction can be represented by a vector $v \in K^5/\Lambda$, with $v \neq 0$. Moreover, for any $P \in r$, the columns of $\mathcal{M}$ evaluated at P are elements of K^5/Λ.

With this set-up we can rephrase the condition that the tangent spaces to V at the points of r all contain the plane M_r as follows, where $v = (v_1, v_2, v_3)$:

$$(*) \qquad det \begin{pmatrix} v_1 & l_{11}(P) & l_{12}(P) \\ v_2 & l_{21}(P) & l_{22}(P) \\ v_3 & l_{31}(P) & l_{32}(P) \end{pmatrix} = 0$$

for every $P \in r$. The development of the above determinant is a quadratic form in x_0, x_1, whose three coefficients linearly depend on v_1, v_2, v_3. Since the determinant vanishes for each choice of x_0, x_1, these coefficients have to be identically zero. This can be interpreted as a homogeneous linear system of three equations which admits the non-trivial solution (v_1, v_2, v_3). The determinant of the matrix of the coefficients of the system is therefore zero. It is a polynomial G, homogeneous of degree 6 in the coefficients of the linear forms l_{ij}, which can be explicitly written. If $\varphi_{12}, \varphi_{13}, \varphi_{23}$ are the quadratic forms given by the 2×2 minors of $\mathcal{M}$, it is possible to verify that the resultant of any two of them is a multiple of G. Being $G = 0$, it follows that the polynomials φ_{ij}'s have a common linear factor. Hence on a general $r \in \Sigma_1$ there exists a focal point.

The inverse implication is similar: if the polynomials φ_{ij} have a common linear factor L, such that $\varphi_{ij} = L\psi_{ij}$, for all i, j, then the $(*)$ takes the form $v_1\psi_{23} - v_2\psi_{13} + v_3\psi_{12} = 0$: this is an equation in v_1, v_2, v_3 which certainly admits a non-zero solution. This gives a vector $v \in K^5/\Lambda$, hence a normal direction to r that generates the required plane M_r.

Proposition 3.3. *If the equivalent conditions of Proposition 3.2 are satisfied, let F be the focal scheme on V. Then F is a point or a curve or a surface. In the first case V is a cone, in the second case F is a fundamental curve for the lines of $\bar{\Sigma}$ and V is a union of cones with vertex on F, in the third case all lines of $\bar{\Sigma}$ are tangent to F.*

Proof. Let $I \subset \bar{\Sigma} \times \mathbb{P}^4$ be the incidence correspondence, and let $f : I \to V$ and $q : I \to \bar{\Sigma}$ be the projections. The focal scheme on V can be seen as the branch locus of the map f, i. e. the image of the ramification locus $\mathcal{F}$, which is a surface. So $\dim F \leq 2$.

The first two cases are clear. We have to show that, if F is a surface, then all lines of $\bar{\Sigma}$ are tangent to V. Let P be a focal point on r and assume that P is a smooth point for F. Let $s \subset I$ be the fibre of q over the point representing r. The tangent space to I at (P, r) contains the tangent space to $\mathcal{F}$ at (P, r), the line s and the kernel of the differential map df of f at (P, r). Since F is smooth at P, this latter space is transversal to $T_{(P,r)}\mathcal{F}$, and the image of df is $df(T_{(P,r)}\mathcal{F}) = T_P F$. But also s is transversal to $ker(df)$, hence $r = df(s) \subset T_P F$.

Remark 3.4.

1. One can prove that, if on each line r of $\bar{\Sigma}$ there is also a second focal point, possibly coinciding with the first one, then the tangent space to V is fixed along r and $\bar{\Sigma}$ is the family of the fibres of the Gauss map of V (see [13]). In this case, clearly, only one line of $\bar{\Sigma}$ passes through a general point of V.

2. Also in the last case of Proposition 3.3, i.e. if the focal locus on V is a surface F and on a general line r of $\bar{\Sigma}$ there is only one simple focus, we can conclude that only one line of $\bar{\Sigma}$ passes through a general point of V. Indeed, first of all let us exclude that there are two lines r and r' of $\bar{\Sigma}$ which are both tangent to F at a general point P. Otherwise r and r' are both contained in $T_P F$ and the hyperplanes which are tangent to V along r vary in the pencil containing the fixed plane M_r, which coincides with $T_P F$ in this case. So the pencil would be the same for r and r', and every hyperplane in the pencil would be tangent to V at two points, one on r and the other on r', which is impossible. So only one line of $\bar{\Sigma}$ passes through a general focal point on V. But then a fortiori the same conclusion holds true also for a general non-focal point of V.

We are now able to prove Theorem 0.2 stated in the Introduction.

Proof of Theorem 0.2. Let $V = Y \cap H$, where H is a general hyperplane. Hence V is a hypersurface of $\mathbb{P}^4$ covered by a 2-dimensional family of lines: this is the situation of Theorem 0.1. If one irreducible component $\bar{\Sigma}$ of the Fano scheme of lines on V is non-reduced, then it follows from Proposition 3.3 and the subsequent Remark 3.4 that V is a cone, or a union of cones with vertices on a curve C, or a union of lines all tangent to a surface F: in this last case only one line of $\bar{\Sigma}$ passes through a general point of V. It is easy to check that, in the first two cases, to have such a V as general hyperplane section, Y has to be a cone over V. In the third case, the lines through a general point of Y form a surface which intersects the general hyperplane H in one line (Remark 3.4), so this surface is necessarily a plane. In any event Y contains a 2-dimensional family of planes, cutting plane curves on X.

Now we assume that all irreducible components $\bar{\Sigma}$ of the Fano scheme of lines on V are reduced. If V is as in case (i) of Theorem 0.1, i.e. if $\mu = 1$, then the lines of Y through a general point form a plane, and we are done.

We consider now case (ii): we prove first that Y cannot be birationally fibered by smooth quadric surfaces. Assume, by contradiction, that Y contains such a family of quadrics and let P be a fixed general point of Y. Then only one quadric F_P of the family passes through P, so the lines contained in Y and passing through P form a quadric cone Q_P, the intersection of F_P with its tangent space at P. The linear span $\mathbb{P}^3_P := < Q_P >$ is the tangent space to F_P at P. We consider the curve $C_P := X \cap Q_P$: it is a k-secant curve on the cone Q_P, so $\deg C_P = 2k$ and $p_a(C_P) = (k-1)^2$. On the other hand $X \cap \mathbb{P}^3_P$ is a connected curve of degree $d = \deg X$. If it contains also another curve C'_P different from C_P, then every point of $C_P \cap C'_P$ is singular for $X \cap \mathbb{P}^3_P$, so, being X smooth, $\mathbb{P}^3_P$ has to be tangent to X at each point of $C_P \cap C'_P$. But $\{\mathbb{P}^3_P\}_{P \in Y}$ is a family of dimension 4 of 3-spaces and the tangent spaces to X form a family of dimension 3. Therefore every $\mathbb{P}^3_P$ should be tangent to infinitely many quadrics of Y, i.e. to all quadrics of Y, which is impossible. So $X \cap \mathbb{P}^3_P = C_P$, $d = 2k$ and the sectional genus of X is $(k-1)^2 = (\frac{d}{2}-1)^2$. But this is the Castelnuovo bound, so every curve section of X with a 3-space is contained in a quadric, which implies that also X is contained in a quadric hypersurface: this gives the required contradiction. As a consequence, if V is as in (ii) of Theorem 0.1, then Y is birationally fibered by quadrics of rank at most 3. So the k-secant lines of X are necessarily cut by the planes contained in these quadrics.

It remains to analyze the four cases of (iv) in Theorem 0.1, with $g = 1$. If V is a projection of a complete intersection of type $(2, 2)$, then also Y is a projection of a fourfold Z of degree 4 in $\mathbb{P}^6$, complete intersection of two quadrics. We have the following diagram:

$$\begin{array}{ccccc} & & Z & \subset & \mathbb{P}^6 \\ & & \pi \downarrow & & \\ X & \hookrightarrow & Y & \subset & \mathbb{P}^5 \end{array}$$

where π is the projection from a suitable point P. $P \notin Z$, because $d = 4$, hence the singular locus of Y is a threefold D of degree 2, according to the formula $\deg D = (d-1)(d-2)/2 - g$, where $d = \deg Z$ and g is the sectional genus, so D does not contain X. Therefore the restriction of $\pi : \pi^{-1}(X) \to X$ is regular and birational: but X, being smooth, is linearly normal, so $\pi^{-1}(X)$ is already contained in a $\mathbb{P}^5$ and the projection is an isomorphism. In this case $\deg X < \deg Z = 4$, but the smooth threefolds of low degree in $\mathbb{P}^5$ are all completely described (see for instance [2]) and this possibility is excluded.

The second possibility for V is being a projection of $\mathbb{G}(1, 4) \cap \mathbb{P}^6$ of degree 5. So Y is a projection from a line Λ of a fourfold Z of degree 5 in $\mathbb{P}^7$. Arguing as in the previous case, we get that $\Lambda \cap Z = \emptyset$, then either X is contained in the double locus of Y, which has degree 5, or $\pi^{-1}(X)$ is contained in a $\mathbb{P}^5$ and again $\deg X < 5$. Both possibilities are excluded as before.

The last case is when V is a projection of a threefold of degree 6 and sectional genus one of $\mathbb{P}^7$. If $\Lambda \cap Z = \emptyset$, it can be treated in the same way, observing that in this case the degree of the double locus of Y is 9. So $\Lambda \cap Z \neq \emptyset$ and the intersection should contain the whole centre of projection. But then $\deg Y = 3$.

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