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WEAK SOLUTIONS FOR NONLOCAL PROBLEMS IN FRACTIONAL ORLICZ-SOBOLEV SPACES

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We study a class of nonlocal boundary value problems involving the fractional $a(\cdot)$-Laplacian operator in fractional Orlicz–Sobolev spaces. The main equation features nonlinear source terms with a nonnegative parameter μ and is posed on a bounded domain with homogeneous Dirichlet conditions in the nonlocal sense. Using variational approaches, we establish the existence of at least one nontrivial, nonnegative weak solution when the parameter μ is sufficiently small. Furthermore, we identify conditions under which the solution is strictly positive. Two examples are included to demonstrate the applicability of the main result.

1. Introduction

In recent years, the mathematical community has shown a growing interest in investigating nonlinear phenomena governed by nonlocal operators within modular-type function spaces. Notably, considerable effort has been devoted to the analysis in fractional Orlicz-Sobolev spaces; see, for example, [2–4, 9–12, 14, 21]. These spaces accommodate nonpolynomial growth conditions and arise naturally in several applied contexts. A typical example is the modeling of electrorheological fluids, where the viscosity depends on electromagnetic fields [16, 17]. In image processing, nonlocal diffusion with Orlicz growth effectively

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preserves edges while denoising [6]. In mathematical finance, this framework supports models involving jump processes and nonlinear utility functions [7]. It also finds applications in biological models, such as those describing anomalous diffusion [1].

The present paper extends the variational approach in [18] to the significantly broader class of fractional $a(\cdot)$-Laplacian operators within fractional Orlicz–Sobolev spaces. While [18] relies on fixed-exponent growth and standard Sobolev embeddings, we employ the modular function space framework developed in [21], which accommodates N-functions A with variable growth rates. The abstract variational tools from [19, 20] (Ricceri's three-critical-points theorem and related principles) are adapted to this setting by establishing coercivity, weak lower semicontinuity, and nontriviality in the modular topology. This synthesis allows us to treat nonlocal problems with nonpolynomial diffusion, such as those arising in electrorheological fluids with field-dependent viscosity or anomalous diffusion.

In this work, we establish the existence of at least one weak solution to the following problem in an appropriate fractional Orlicz-Sobolev space

$$\begin{cases} (-\Delta)^s_{a(\cdot)} u = h(x)f(u) + \mu w(x)g(u) & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subseteq \mathbb{R}^N$, with $N \geq 1$, is a bounded domain with Lipschitz boundary $\partial\Omega$, $0 < s < 1$, $h, w \in L^\infty(\Omega) \setminus \{0\}$ with $h, w \geq 0$, $\mu \geq 0$ is a parameter, $f, g : \mathbb{R}^+ \to \mathbb{R}^+$ are continuous functions, and $(-\Delta)^s_{a(\cdot)}$ is the fractional a-Laplacian operator defined by

$$\begin{aligned} &(-\Delta)^s_{a(\cdot)} u(x) \\ &= 2 \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} a\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right) \frac{u(x)-u(y)}{|u(x)-u(y)|} \frac{dy}{|x-y|^{N+s}}. \end{aligned} \tag{1.2}$$

Here, $a : \mathbb{R}^+ \to \mathbb{R}^+$ is nondecreasing and right-continuous, with $a(0) = 0$, $a(t) > 0$ for all $t > 0$, and $a(t) \to \infty$ as $t \to \infty$. Define a function $A : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$A(t) = \int_0^t a(\tau)d\tau. \tag{1.3}$$

Then, A is an N-function; that is, $A : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous, convex, and increasing function such that

$$\frac{A(t)}{t} \to 0 \quad \text{as } t \to 0, \quad \text{and} \quad \frac{A(t)}{t} \to \infty \quad \text{as } t \to \infty.$$

Throughout this paper, we assume that

$$1 < p^- := \inf_{t>0} \frac{ta(t)}{A(t)} \le p^+ := \sup_{t>0} \frac{ta(t)}{A(t)} < \infty. \tag{1.4}$$

As established in [15, Theorem 3.34], the right-hand side inequality in (1.4) is equivalent to the condition that A satisfies the Δ_2 condition; that is, there exists a constant $C_1 > 0$ such that $A(2x) \le C_1 A(x)$ for all $x \in \mathbb{R}$.

The operator $(-\Delta)^s_{a(\cdot)}$ was introduced in [9] in 2019,. With $a(t) = t^{p-1}$, $(-\Delta)^s_{a\cdot}$ reduces to the well known fractional p-Laplacian operator, defined by

$$(-\Delta)^s_p u(x) = 2 \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}}\, dy.$$

Due to its versatility in capturing a broad class of nonlocal phenomena, the operator $(-\Delta)^s_a$ has been widely adopted in recent research on nonlocal equations. Some recent contributions include [2, 3, 5, 10, 21].

This paper is devoted to proving the existence of nontrivial, nonnegative weak solutions to problem (1.1), with a particular emphasis on the case where the parameter $\mu \ge 0$ is sufficiently small. In addition, we identify sufficient conditions that ensure the existence of positive solutions to problem (1.1). The analysis is based on variational principles developed by Ricceri, particularly those presented in [18, 19] and formalized in Lemmas 2.1 and 2.2 of Section 2.

The rest of the paper is organized as follows. Section 2 presents a review of key results on fractional Orlicz–Sobolev spaces that are essential to our analysis. In Section 3, we state and prove the main results and include two illustrative examples.

2. Fractional Orlicz-Sobolev spaces

In this section, we recall essential results on fractional Orlicz–Sobolev spaces needed for this work; see, e.g., [2, 3, 9, 10].

For simplicity of notation, we define

$$D_{x,y}(u) := \frac{u(x)-u(y)}{|x-y|^s} \quad \text{and} \quad dm(x,y) := \frac{dx\,dy}{|x-y|^N}.$$

Let Ω be an open subset of $\mathbb{R}^N$, and let A be the N-function defined by (1.3). We define the corresponding Orlicz space by

$$L^A(\Omega) = \Big\{ u : \Omega \to \mathbb{R} \,\Big|\, u \text{ is measurable and} \\ \int_\Omega A(\lambda |u(x)|)\, dx < \infty \text{ for some } \lambda > 0 \Big\},$$

which is a Banach space equipped with the Luxemburg norm

$$\|u\|_A = \inf\left\{\lambda > 0 : \int_\Omega A\left(\frac{|u(x)|}{\lambda}\right) dx \leq 1\right\}.$$

Next, we define the fractional Orlicz-Sobolev space $W^{s,A}(\Omega)$ as

$$W^{s,A}(\Omega) = \Bigg\{ u \in L^A(\Omega) : \int_\Omega \int_\Omega A\Big(\frac{|D_{x,y}(u)|}{\lambda}\Big)\, dm(x,y) < \infty \text{ for some } \lambda > 0 \Bigg\}.$$

This space is endowed with the norm

$$\|u\|_{s,A} = \|u\|_A + [u]_{s,A},$$

where $[u]_{s,A}$ is the Gagliardo-type seminorm defined by

$$[u]_{s,A} = \inf\left\{\lambda > 0 : \int_\Omega \int_\Omega A\left(\frac{|D_{x,y}(u)|}{\lambda}\right) dm(x,y) \leq 1\right\}.$$

By [9, Proposition 2.10], $W^{s,A}(X)$ is a separable and reflexive Banach space.

Lemmas 2.1 and 2.2 below are taken from [3, Proposition 2] and [3, Corollary 2], respectively.

Lemma 2.1. *The following inequalities hold:*

$$\|u\|^{p^-} \leq \phi(u) \leq \|u\|^{p^+} \quad \textit{for all } u \in W^{s,A}(\Omega) \textit{ with } \|u\| > 1, \tag{2.1}$$

$$\|u\|^{p^+} \leq \phi(u) \leq \|u\|^{p^-} \quad \textit{for all } u \in W^{s,A}(\Omega) \textit{ with } \|u\| < 1, \tag{2.2}$$

where

$$\phi(u) := \int_\Omega \int_\Omega A\left(|D_{x,y}(u)|\right) dm(x,y).$$

Lemma 2.2. *Let Ω be a bounded open subset of $\mathbb{R}^N$ with $C^{0,1}$ regularity and bounded boundary. Let $0 < s' < s < 1$ and define*

$$p^*_{s'} = \begin{cases} \frac{Np^-}{N-s'p^-}, & \textit{if } N > s'p^-, \\ \infty, & \textit{if } N \leq s'p^-. \end{cases}$$

Then, we have

(a) *If $s'p^- < N$, then $W^{s,A}(\Omega) \hookrightarrow L^q(\Omega)$ for all $q \in [1, p^*_{s'}]$, and the embedding is compact for all $q \in [1, p^*_{s'})$.*

(b) *If* $s'p^- = N$*, then* $W^{s,A}(\Omega) \hookrightarrow L^q(\Omega)$ *for all* $q \in [1,\infty]$*, and the embedding is compact for all* $q \in [1,\infty)$*.*

(c) *If* $s'p^- > N$*, then the embedding* $W^{s,A}(\Omega) \hookrightarrow L^\infty(\Omega)$ *is compact.*

In Lemma 2.2, the use of $s' < s$ ensures compact embedding up to but not including the critical exponent when $N > s'p^-$.

We consider solutions to problem (1.1) in the space

$$W_0^{s,A}(\Omega) = \left\{u \in W^{s,A}(\mathbb{R}^n) \, : \, u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\right\}.$$

According to [3, Corollary 1], there exists a constant $C_2 > 0$ such that

$$\|u\|_A \le C_2[u]_{s,A} \quad \text{for all } u \in W_0^{s,A}(\Omega).$$

Therefore, if we let $\|\cdot\| := [\cdot]_{s,A}$, then $\|\cdot\|$ and $\|\cdot\|_{s,A}$ are equivalent norms on $W_0^{s,A}(\Omega)$. In this paper, for simplicity, we will use the norm $\|\cdot\|$ for the space $W_0^{s,A}(\Omega)$.

3. Main results

Let q be a real number such that

$$q \in \left[1, \frac{Np^-}{N - sp^-}\right) \quad \text{if } N > sp^-, \quad \text{or} \quad q \in [1,\infty) \quad \text{if } N \le sp^-.$$

By Lemma 2.2, the embedding $W^{s,A}(\Omega) \hookrightarrow L^q(\Omega)$ is compact. Therefore, if we define

$$c_q = \sup_{u \in W^{s,A}(\Omega \setminus \{0\})} \frac{\|u\|_q^q}{\|u\|^q}, \tag{3.1}$$

where $\|u\|_q^q = \int_\Omega |u(x)|^q dx$, then $c_q \in (0,\infty)$.

We extend f and g to $\mathbb{R}$ by setting

$$f(t) = \begin{cases} f(t) & \text{if } t \ge 0, \\ f(0) & \text{if } t < 0, \end{cases} \quad \text{and} \quad g(t) = \begin{cases} g(t) & \text{if } t \ge 0, \\ g(0) & \text{if } t < 0. \end{cases}$$

Then, $f, g \ge 0$ on $\mathbb{R}$. Let

$$F(t) = \int_0^t f(s)ds \quad \text{and} \quad G(t) = \int_0^t g(s)ds \quad \text{for any } t \in \mathbb{R}.$$

We make the following assumptions:

(H1) $F+G$ has no global maximum in $\mathbb{R}$.

(H2) The function $t \mapsto \frac{f(t)+g(t)}{t^{q-1}}$ is strictly decreasing on $(0,\infty)$ and satisfies

$$\lim_{t\to\infty} \frac{f(t)+g(t)}{t^{q-1}} = 0. \tag{3.2}$$

(H3) There exist $r>0$ and $\varepsilon \in (0,1)$ such that

$$\begin{aligned} &F\left(r^{1/q}\right)+G\left(r^{1/q}\right) \\ <\ & \frac{1}{\int_\Omega h(x)dx} \min\left\{ \left(\frac{\varepsilon r \int_\Omega h(x)dx}{c_q \operatorname{ess\,sup}_\Omega h}\right)^{p^-/q}, \left(\frac{\varepsilon r \int_\Omega h(x)dx}{c_q \operatorname{ess\,sup}_\Omega h}\right)^{p^+/q} \right\}, \end{aligned}$$

where c_q is defined by (3.1).

(H4) Either $f(0)>0$ or $sp^- > N$ and $\lim_{t\to 0^+} \frac{F(t)}{t^{p^-}} = \infty$.

Remark 3.1. (a) If

$$\lim_{r\to\infty} \frac{F\left(r^{1/q}\right)+G\left(r^{1/q}\right)}{r^{p^-/q}} = 0, \tag{3.3}$$

then for sufficiently large r, the left side of (H3) is small enough to satisfy the inequality, as the denominator grows faster.

(b) The case $f(0)>0$ directly provides positive source terms near zero, ensuring nontrivial solutions. When $f(0)=0$, the alternative $sp^- > N$ together with $\lim_{t\to 0^+} F(t)/t^{p^-} = \infty$ allows detection of small-amplitude solutions via growth rates.

Definition 3.1. A function $u \in W_0^{s,A}(\Omega)$ is called a weak solution to problem (1.1) if

$$\begin{aligned} &\int_\Omega\int_\Omega a(|D_{x,y}(u)|)\frac{D_{x,y}(u)}{|D_{x,y}(u)|} D_{x,y}(v)\,dm(x,y) \\ &\qquad - \int_\Omega h(x) f(u(x))v(x)\,dx - \mu\int_\Omega w(x)g(u(x))v(x)\,dx = 0 \end{aligned}$$

for all $v \in W_0^{s,A}(\Omega)$.

Define two functionals $I,J : W_0^{s,A}(\Omega) \to \mathbb{R}$ by

$$I(u) = \int_\Omega\int_\Omega A(|D_{x,y}(u)|)dm(x,y) \tag{3.4}$$

and

$$J(u) = \int_\Omega h(x)F(u(x))dx + \mu\int_\Omega w(x)G(u(x))dx.$$

Then $I, J \in C^1(W_0^{s,A}(\Omega), \mathbb{R})$ with Gâteaux derivatives given by

$$\langle I'(u), v\rangle = \int_\Omega \int_\Omega a(|D_{x,y}(u)|) \frac{D_{x,y}(u)}{|D_{x,y}(u)|} D_{x,y}(v)\, dm(x,y) \tag{3.5}$$

and

$$\langle J'(u), v\rangle = \int_\Omega h(x) f(u(x)) v(x) dx + \mu \int_\Omega w(x) g(u(x)) v(x)\, dx \tag{3.6}$$

for all $u, v \in W_0^{s,A}(\Omega)$. The proofs of (3.5) and (3.6) are provided in [3, Lemmas 6 and 7], or alternatively in [2, Lemmas 3 and 4]. See also [21, Proposition 4.1] for a related result. Clearly, any weak solution of problem (1.1) corresponds to a critical point of the energy functional $I - J$ in the space $W_0^{s,A}(\Omega)$.

We now state the main theorem of the paper.

Theorem 3.1. Assume that (1.4) and conditions (H1)–(H4) hold. Define

$$\mu^* := \frac{\operatorname{ess\,inf}_\Omega h}{\operatorname{ess\,sup}_\Omega w} \geq 0. \tag{3.7}$$

Then, for every $\mu \in [0, \mu^*]$, problem (1.1) admits a nontrivial, nonnegative weak solution u. Moreover, u satisfies

$$\begin{aligned} &\int_\Omega \int_\Omega A\big(|D_{x,y} u|\big)\, dm(x,y) \\ &\leq \max\left\{ \left(\frac{r \int_\Omega h(x)\, dx}{c_q \operatorname{ess\,sup}_\Omega h} \right)^{p^-/q}, \left(\frac{r \int_\Omega h(x)\, dx}{c_q \operatorname{ess\,sup}_\Omega h} \right)^{p^+/q} \right\}. \end{aligned} \tag{3.8}$$

Remark 3.2. Note that if $\operatorname{essinf}_\Omega h = 0$, then $\mu^* = 0$, yielding existence solely for the unperturbed problem; to ensure $\mu^* > 0$, one may assume $\operatorname{essinf}_\Omega h > 0$, as is common in such settings.

With some additional assumptions, the solution u is positive, as stated below.

Corollary 3.1. In addition to all the assumptions in Theorem 3.1, suppose further that

$$p^- > \frac{N}{s(1-s)}. \tag{3.9}$$

Then the solution u of problem (1.1), as given in Theorem 3.1, is positive in Ω.

Before proving Theorem 3.1, we provide two examples to illustrate its applicability.

Example 3.1. In problem (1.1), let $N = 2$, $s = \frac{1}{2}$, $h(x) = 2 + \sin x$, $w(x) = \pi/2 + \arctan x$, $a(t) = t^9 \ln(b + ct)$, $f(t) = \tau_1 t^{\tau_1 - 1}$, and $g(t) = \tau_2 t^{\tau_2 - 1}$, where $b, c > 0$ and $\tau_1, \tau_2 \geq 1$.

We claim that, for all $\mu \in [0, 1/\pi]$*, problem* (1.1) *has a positive weak solution if* $\tau_1 < p^-$.

In fact, it is obvious that $F(t) = t^{\tau_1}$, $G(t) = t^{\tau_2}$, $\frac{N}{s(1-s)} = 8$, $\operatorname{ess\,inf}_\Omega h = 1$, and $\operatorname{ess\,sup}_\Omega w = \pi$. Moreover, (1.4) holds with $p^- = 10$ and $p^+ = 11$ (see [21, Example 2.4 (2)]). Thus, we have:

- (H1) and (3.9) clearly hold.
- For any $q > \max\{\tau_1, \tau_2\}$, (H2) and (3.3) hold. Then, in view of Remark 3.1, (H3) holds.
- $sp^- > N$, and $\tau_1 < p^-$ implies that $\lim_{t \to 0^+} \frac{F(t)}{t^{p^-}} = \infty$, so (H4) holds.

Therefore, all the conditions of Corollary 3.1 are satisfied, and the claim follows directly from Corollary 3.1.

Example 3.2. In problem (1.1), let $N = 4$, $s = \frac{1}{3}$, and $h(x) = w(x) = 1$. Suppose a satisfies

$$a \in C(\mathbb{R}^+), \quad a(t) = c_1 t^{a_1} \text{ for } t \leq t_0, \quad \text{and} \quad a(t) = c_2 t^{a_2} + d \text{ for } t \geq t_0,$$

where $t_0 > 0$ and $a_1, a_2, c_1, c_2 > 0$ are such that $a \in C(\mathbb{R}^+)$. Let

$$f(t) = \frac{t^{\gamma - 1}}{1 + t^{2\gamma}} \quad \text{and} \quad g(t) = e^{-t}, \quad \text{with } \gamma > 1.$$

We claim that, for all $\mu \in [0, 1]$*, problem* (1.1) *has a positive weak solution if*

$$\min\{a_1, a_2\} > \max\{17, \gamma - 1\}. \tag{3.10}$$

Indeed, we have $\operatorname{ess\,inf}_\Omega h = \operatorname{ess\,sup}_\Omega w = 1$ and

$$\frac{N}{s(1-s)} = \frac{4}{\frac{1}{3}(1 - \frac{1}{3})} = 18.$$

For $t \in \mathbb{R}^+$, we have

$$F(t) = \int_0^t \frac{s^{\gamma - 1}}{1 + s^{2\gamma}} ds = \frac{1}{\gamma} \arctan(t^\gamma),$$

and

$$G(t) = \int_0^t e^{-s}\,ds = -e^{-t} + 1.$$

Moreover, (1.4) holds with (see [21, Example 2.4 (3)])

$$p^- = 1 + \min\{a_1, a_2\}, \quad p^+ = 1 + \min\{a_1, a_2\}.$$

Hence, from (3.10), it follows that (3.9) is satisfied.

We now verify the assumptions of Corollary 3.1:

- (H1) clearly holds.
- For any $q > 1$, both (H2) and (3.3) hold. Then, by Remark 3.1, condition (H3) also holds.
- Clearly, $sp^- > N$, and from (3.10), we obtain

  $$\begin{aligned} \lim_{t\to 0^+} \frac{F(t)}{t^{p^-}} &= \frac{1}{\gamma} \lim_{t\to 0^+} \frac{\arctan(t^\gamma)}{t^{1+\min\{a_1,a_2\}}} \\ &= \lim_{t\to 0^+} \frac{t^{\gamma-1}}{(1+t^{2\gamma})(1+\min\{a_1,a_2\})t^{\min\{a_1,a_2\}}} = \infty, \end{aligned}$$

  so (H4) is also satisfied.

Therefore, all the assumptions of Corollary 3.1 are satisfied, and the claim follows directly from Corollary 3.1.

These examples cover broad classes of admissible functions: $a(\cdot)$ nondecreasing, right-continuous, with $a(0) = 0$ and satisfying (1.4) and Δ_2; f, g continuous, nonnegative, subcritical at infinity ((H2)), and satisfying (H1)–(H3). To show (H4) is sharp, if $f(t) = 0$ near 0 and $sp^- \le N$, then only the trivial solution may exist, even if (H1)–(H3) hold.

Next, we prove Theorem 3.1. Lemma 3.1 below is essential in the proof and is taken from [18, Theorem A]. For some other related results and applications, see [8, 13, 20].

Lemma 3.1. *Let X be a reflexive real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two sequentially weakly lower semicontinuous functionals with Ψ also coercive and $\Phi(0) = \Psi(0) = 0$. Then, for each $\sigma > \inf_X \Psi$ and each λ satisfying*

$$\lambda > -\frac{\inf_{\Psi^{-1}((-\infty,\sigma])} \Phi}{\sigma},$$

the restriction of $\lambda\Psi + \Phi$ to $\Psi^{-1}((-\infty, \sigma])$ has a global minimum.

Let $0 \leq a < b \leq \infty$. For a generic pair of functions $\phi, \psi : \mathbb{R} \to \mathbb{R}$, if $\lambda \in [a,b]$, we denote by $M(\phi, \psi, \lambda)$ the set of all global minimum of the function $\lambda \psi - \phi$ or the empty set according to whether $\lambda < \infty$ or $\lambda = \infty$. We adopt the conventions $\sup \emptyset = -\infty$ and $\inf \emptyset = \infty$. Let

$$\alpha(\phi, \psi, b) = \max \left\{ \inf_{\mathbb{R}} \psi, \sup_{M(\phi,\psi,b)} \psi \right\}$$

and

$$\beta(\phi, \psi, a) = \min \left\{ \sup_{\mathbb{R}} \psi, \inf_{M(\phi,\psi,a)} \psi \right\}.$$

The following minimax result can be found in [18, Propostion A], which is a special case of [19, Theorem 1].

Lemma 3.2. *Let $\phi, \psi : \mathbb{R} \to \mathbb{R}$ be two functions such that, for each $\lambda \in (a,b)$, the function $\lambda \psi - \phi$ is lower semicontinuous, coercive, and has a unique global minimum in $\mathbb{R}$. Assume that*

$$\alpha(\phi, \psi, b) < \beta(\phi, \psi, a).$$

Then, for each $s \in (\alpha(\phi, \psi, b), \beta(\phi, \psi, a))$, there exists $\lambda_s \in (a,b)$ such that the global minimum of the function $\lambda_s \psi - \phi$ lies in $\psi^{-1}(s)$.

Below, Lemma 3.3 is taken from [21, Proposition 5.1] and Lemma 3.4 can be found in [21, Lemma 2.1].

Lemma 3.3. *For each $\tau > 0$, the minimization problem*

$$\alpha_\tau = \inf_{u \in K_\tau} \frac{\int_\Omega \int_\Omega A(|D_{x,y}(u)|) dm(x,y)}{\int_\Omega A(|u|) dx} \tag{3.11}$$

has a solution $\alpha_{1,\tau}$, where

$$K_\tau = \left\{ u \in W_0^{s,A} : \int_\Omega A(|u|) dx = \tau \right\}.$$

Lemma 3.4. *Let $y, z > 0$. Then*

$$\max \left\{ y^{p^-}, y^{p^+} \right\} A(z) \geq A(yz).$$

We are now ready to prove our main theorem. The proof proceeds variationally: Lemma 3.1 ensures the functional's geometry, Lemma 3.2 verifies compactness and boundedness, and Lemma 3.3 guarantees nontriviality via the test function construction.

Proof of Theorem 3.1. By Lemma 2.1 and (3.4), we have

$$I(u) \geq \min\{\|u\|^{p^-}, \|u\|^{p^+}\}. \tag{3.12}$$

Below, we always assume that $\mu \in [0, \mu^*]$, where μ^* is defined by (3.7). Let $r > 0$ and $\varepsilon \in (0,1)$ be given in (H4). Choose $\sigma > 0$ so that

$$\begin{aligned} \frac{\varepsilon r \int_\Omega h(x)\,dx}{c_q \operatorname{ess\,sup}_\Omega h} &\leq \min\left\{\sigma^{q/p^-}, \sigma^{q/p^+}\right\} \\ &\leq \max\left\{\sigma^{q/p^-}, \sigma^{q/p^+}\right\} \leq \frac{r \int_\Omega h(x)\,dx}{c_q \operatorname{ess\,sup}_\Omega h}. \end{aligned} \tag{3.13}$$

Then, we have

$$c_q \operatorname{ess\,sup}_\Omega h \max\left\{\sigma^{q/p^-}, \sigma^{q/p^+}\right\} \leq r \int_\Omega h(x)dx. \tag{3.14}$$

From (3.1) and (3.12), it follows that

$$\begin{aligned} &\left\{u \in W_0^{s,A}(\Omega) : I(u) \leq \sigma\right\} \\ \subseteq &\left\{u \in W_0^{s,A}(\Omega) : \|u\| \leq \max\left\{\sigma^{1/p^-}, \sigma^{1/p^+}\right\}\right\} \\ \subseteq &\left\{u \in L^q(\Omega) : \|u\|_q^q \leq c_q \max\left\{\sigma^{q/p^-}, \sigma^{q/p^+}\right\}\right\}. \end{aligned}$$

This, together with (3.14), implies that

$$\begin{aligned} &\left\{u \in W_0^{s,A}(\Omega) : I(u) \leq \sigma\right\} \\ \subseteq &\left\{u \in L^q(\Omega) : \int_\Omega h(x)|u(x)|^q dx \leq c_q \operatorname{ess\,sup}_\Omega h \max\left\{\sigma^{q/p^-}, \sigma^{q/p^+}\right\}\right\} \\ = &\left\{u \in L^q(\Omega) : \int_\Omega h(x)|u(x)|^q dx \leq r \int_\Omega h(x)dx\right\}. \end{aligned} \tag{3.15}$$

For any $\lambda > 0$, the equation (3.2) in (H2) implies that the function

$$\ell_\lambda(t) := \lambda |t|^q - F(t) - G(t)$$

is lower semicontinuous and coercive; that is, $\ell_\lambda(t) \to \infty$ as $|t| \to \infty$.

We now prove that $\ell_\lambda(t)$ has a unique global minimum in $\mathbb{R}$. Suppose, for contradiction, that $\ell_\lambda(t)$ admits two distinct global minimizers $t_1, t_2 \in \mathbb{R}$ with $t_1 < t_2$. Since $\ell_\lambda(t) > 0$ for all $t < 0$, it follows that $t_1 \geq 0$. By Rolle's theorem, there exists $t_3 \in (t_1, t_2)$ such that $\ell'_\lambda(t_3) = 0$. Also, note that $\ell'_\lambda(t_2) = 0$. Therefore,

$$\lambda q t_2^{q-1} - f(t_2) - g(t_2) = \lambda q t_3^{q-1} - f(t_3) - g(t_3) = 0.$$

Thus,

$$\frac{f(t_2)+g(t_2)}{t_2^{q-1}} = \frac{f(t_3)+g(t_3)}{t_3^{q-1}} = \lambda q,$$

which contradicts the assumption in (H2) that the function $t \mapsto \frac{f(t)+g(t)}{t^{q-1}}$ is strictly decreasing on $(0,\infty)$. Hence, we conclude that for each $\lambda \in (0,\infty)$, the function $\ell_\lambda(t)$ has a unique global minimum in $\mathbb{R}$. Below, let θ_λ denote the unique global minimizer of $l_\lambda(t)$ in $\mathbb{R}$.

Obviously, $\alpha(F+G,|\cdot|^q,\infty)=0$, and by (H1), $\beta(F+G,|\cdot|^q,0)=\infty$. Therefore, setting $\phi(t)=F(t)+G(t)$, $\psi(t)=|t|^q$, $a=0$, $b=\infty$, and choosing $\lambda \in (a,b)$, all the assumptions of Lemma 3.2 are satisfied. Note that

$$r \in (\alpha(F+G,|\cdot|^q,\infty), \beta(F+G,|\cdot|^q,0)).$$

Then Lemma 3.2 ensures the existence of $\lambda_r \in (0,\infty)$ such that

$$\psi(\theta_\lambda) = |\theta_\lambda|^q = r$$

and

$$\lambda_r r - F(\theta_{\lambda_r}) - G(\theta_{\lambda_r}) \le \lambda_r \psi(t) - \phi(t) = \lambda_r |t|^q - F(t) - G(t) \quad \text{for all } t \in \mathbb{R}.$$

Therefore, using the fact that $f,g \ge 0$ on $\mathbb{R}$, we conclude that

$$F(\theta_{\lambda_r}) + G(\theta_{\lambda_r}) = \sup_{|s|^q=r} (F(s)+G(s)) = F\left(r^{1/q}\right) + G\left(-r^{1/q}\right). \tag{3.16}$$

Moreover, for each $u \in L^q(\Omega)$, we have

$$\begin{aligned}
&(\lambda_r r - F(\theta_{\lambda_r}) - G(\theta_{\lambda_r})) \int_\Omega h(x)dx \\
\le{}& \int_\Omega h(x)(\lambda_r |u(x)|^q - F(u(x)) - G(u(x)))dx \\
\le{}& \lambda_r \int_\Omega h(x)|u(x)|^q dx - \int_\Omega h(x)F(u(x))dx - \operatorname{ess\,inf}_\Omega h \int_\Omega G(u(x))dx \\
\le{}& \lambda_r \int_\Omega h(x)|u(x)|^q dx - \int_\Omega h(x)F(u(x))dx - \mu \operatorname{ess\,sup}_\Omega w \int_\Omega G(u(x))dx \\
\le{}& \lambda_r \int_\Omega h(x)|u(x)|^q dx - \int_\Omega h(x)F(u(x))dx - \mu \int_\Omega w(x)G(u(x))dx.
\end{aligned} \tag{3.17}$$

The derivation of (3.17) uses the assumption that $\mu \in [0,\mu^*]$. It is easy to check that (3.16) and (3.17) further imply that, for each $u \in L^q(\Omega)$ satisfying

$$\int_\Omega h(x)|u(x)|^q dx < r \int_\Omega h(x)dx,$$

we have

$$\begin{aligned} & \int_\Omega h(x)F(u(x))dx + \mu \int_\Omega w(x)G(u(x))dx \\ \leq \quad & \left(F(\theta_{\lambda_r}) + G(\theta_{\lambda_r})\right) \int_\Omega h(x)dx \\ = \quad & \left(F\left(r^{1/q}\right) + G\left(r^{1/q}\right)\right) \int_\Omega h(x)dx. \end{aligned}$$

Then, in view of (3.15), we have

$$\begin{aligned} & \sup_{I^{-1}((-\infty,\sigma])} \left(\int_\Omega h(x)F(u(x))dx + \mu \int_\Omega w(x)G(u(x))dx\right) \\ \leq \quad & \left(F\left(r^{1/q}\right) + G\left(r^{1/q}\right)\right) \int_\Omega h(x)dx. \end{aligned} \tag{3.18}$$

Using (H4), equations (3.13) and (3.18), we obtain

$$\sup_{I^{-1}((-\infty,\sigma])} \left(\int_\Omega h(x)F(u(x))\,dx + \mu \int_\Omega w(x)G(u(x))\,dx\right) < \sigma,$$

which implies that

$$-\frac{\inf_{I^{-1}((-\infty,\sigma])}(-J(u))}{\sigma} < 1.$$

By setting $X = W_0^{s,A}(\Omega)$, $\Phi = -J$, $\Psi = I$, and $\lambda = 1$, we can apply Lemma 3.1 to deduce the existence of a weak solution $u \in W_0^{s,A}(\Omega)$ to problem (1.1). This solution is a local minimizer of $I - J$; indeed, it is a global minimizer of $I - J$ when restricted to the set $I^{-1}((-\infty,\sigma])$. Furthermore, u satisfies (3.8), as a consequence of (3.12) and (3.13). In addition, the weak maximum principle in fractional Orlicz–Sobolev spaces (see [21, Proposition 3.7]) implies that $u \geq 0$ in Ω.

Finally, we prove that 0 is not a local minimizer of $I - J$, which implies that the solution u obtained above is not identically zero. If the first part of (H4) holds, that is, $f(0) > 0$, then u is clearly nontrivial. Now assume the second part of (H4) holds, namely, $sp^- > N$ and $\lim_{t\to 0^+} \frac{F(t)}{t^{p^-}} = \infty$. Fix $\tau = 1$ in Lemma 3.3 and choose a small $\varepsilon > 0$. Then, Lemma 3.3 implies that there exists a nonnegative function $v \in W_0^{s,A}(\Omega) \setminus \{0\}$ such that

$$\int_\Omega A(|v|)dx = 1 \quad \text{and} \quad \int_\Omega \int_\Omega A(|D_{x,y}(v)|)dm(x,y) \leq \alpha_1 + \varepsilon. \tag{3.19}$$

Now, if $\operatorname{ess\,inf}_\Omega h = 0$, we let δ be a positive number such that the set $h^{-1}([\delta,\infty))$ has a positive measure. On the other hand, if $\operatorname{ess\,inf}_\Omega h > 0$, we set $\delta = \operatorname{ess\,inf}_\Omega h$.

Since $sp^- > N$, we have $v \in L^\infty(\Omega)$ by Lemma 2.2. Now, since $\lim_{t\to 0^+} \frac{F(t)}{t^{p^-}} = \infty$, there exists $\eta > 0$ small enough such that $\frac{\eta}{\operatorname{ess\,sup}_\Omega v} < 1$, and

$$F(t) > \frac{\alpha_1 + \varepsilon}{\delta \int_{h^{-1}([\delta,\infty))} |v(x)|^{p^-}\, dx} t^{p^-} \quad \text{for all } t \in (0, \eta]. \tag{3.20}$$

Then, for each $\rho \in \left(0, \frac{\eta}{\operatorname{ess\,sup}_\Omega v}\right)$, using (3.19) and (3.20), we have

$$\begin{aligned}
J(\rho v) &= \int_\Omega h(x) F(\rho v(x))\, dx + \mu \int_\Omega w(x) G(\rho v(x))\, dx \\
&\geq \int_{h^{-1}([\delta,\infty))} h(x) F(\rho v(x))\, dx \\
&> \frac{\alpha_1 + \varepsilon}{\delta \int_{h^{-1}([\delta,\infty))} |v(x)|^{p^-}\, dx} \int_{h^{-1}([\delta,\infty))} h(x) |\rho v(x)|^{p^-}\, dx \\
&= \frac{\rho^{p^-}(\alpha_1 + \varepsilon)}{\delta \int_{h^{-1}([\delta,\infty))} |\rho v(x)|^{p^-}\, dx} \int_{h^{-1}([\delta,\infty))} h(x) |\rho v(x)|^{p^-}\, dx \\
&\geq \frac{\rho^{p^-} \int_\Omega \int_\Omega A(|D_{x,y}(v)|)\, dm(x,y)}{\delta \int_{h^{-1}([\delta,\infty))} |\rho v(x)|^{p^-}\, dx} \cdot \delta \int_{h^{-1}([\delta,\infty))} |\rho v(x)|^{p^-}\, dx \\
&= \rho^{p^-} \int_\Omega \int_\Omega A(|D_{x,y}(v)|)\, dm(x,y).
\end{aligned}$$

In view of the fact that $0 < \rho < 1$, from Lemma 3.4, we have

$$\rho^{p^-} A\big(|D_{x,y}(v)|\big) \geq A\big(\rho\, |D_{x,y}(v)|\big) = A\big(|D_{x,y}(\rho v)|\big).$$

Thus,

$$J(\rho v) > \int_\Omega \int_\Omega A\big(|D_{x,y}(\rho v)|\big)\, dm(x,y) = I(\rho v).$$

This shows that $I - J$ takes negative values in every ball of $W_0^{s,A}(\Omega)$ centered at 0. Therefore, 0 is not a local minimum of $I - J$. This concludes the proof of the theorem. □

Proof of Corollary 3.1. All conditions of [21, Proposition 3.8]) hold: h, f, g are continuous and nonnegative, the weak solution u is nonnegative by Theorem 3.1, and the strong maximum principle applies due to the positivity of the source terms and the interior regularity in fractional Orlicz–Sobolev spaces. Therefore, the conclusion follows. □

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