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ON THE HILBERT SCHEME OF CURVES OF DEGREE d AND GENUS $\binom{d-3}{2} - 1$

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Dedicated to Silvio Greco in occasion of his 60th birthday.

In this paper we describe the components of the Hilbert scheme $H_{d,\tilde{g}}$ of locally Cohen–Macaulay curves of degree d and arithmetic genus $\tilde{g} = \binom{d-3}{2} - 1$. We show that $H_{d,\tilde{g}}$ is connected thanks to the irreducible component of extremal curves to which every curve can be connected.

1. Introduction.

The question of the connectedness of the Hilbert schemes $H_{d,g}$ of locally Cohen–Macaulay curves $\mathcal{C} \subset \mathbb{P}^3$ of degree d and arithmetic genus g arose naturally after Hartshorne proved in his PhD thesis that the Hilbert scheme of all one dimensional schemes with fixed Hilbert polynomial is connected. The answer to the question in case of locally Cohen–Macaulay curves is known, so far, only for low degrees or high genera. After the paper [6], it is well known that $H_{d,g}$ contains an irreducible component consisting of extremal curves (i.e. curves having the largest possible Rao function). This is the only component for $d \geq 5$ and $(d-3)(d-4)/2 + 1 < g \leq (d-2)(d-3)/2$ while in the cases $d \geq 5$, $g = (d-3)(d-4)/2 + 1$ and $d \geq 4$, $g = (d-3)(d-4)/2$ the Hilbert scheme is not irreducible, but it is connected (see [1], [9]). The connectedness has been proved also for $d \leq 4$ and any genus (see [9], [10] and the references therein). This note deals with the first unknown case for high genus, i.e. $\tilde{g} = (d-3)(d-4)/2 - 1$. The paper [2] has given a new light to the problem, in fact Hartshorne provides some methods to connect particular classes of curves to the irreducible component of extremal curves. The purpose of this note is to prove the connectedness of $H_{d,\tilde{g}}$, first by identifying its components for every d and then to connect them to extremal curves using [2] and its continuation [14].

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2. Preliminaries.

We will work on an algebraically closed field k of characteristic zero. A curve $\mathcal{C}$ in $\mathbb{P}^3 = \mathbb{P}^3_k$ is a closed subscheme of pure dimension one, locally Cohen–Macaulay. Its degree and arithmetic genus will be denoted by d and g respectively. The key tools we will use are the notion of *spectrum* of a curve (see [12], [13]) and the notion of *triangle diagram* (see [4]). We will recall here some of the basic results on these topics for sake of completeness.

Definition 2.1. *Let $\mathcal{C}$ be a curve in $\mathbb{P}^3$. The spectrum of $\mathcal{C}$ is the finitely supported function $h_{\mathcal{C}}(n) = \Delta^2 h^0(\mathcal{O}_{\mathcal{C}}(n))$.*

The following proposition shows how the spectrum is related to other invariants of the curve:

Proposition 2.2. *Let $\mathcal{C}$ be a curve with spectrum $h_{\mathcal{C}}$. Then*

$$d = \sum_n h_{\mathcal{C}}(n) \qquad g = 1 + \sum_n (n-1) h_{\mathcal{C}}(n).$$

Let now I be the saturated homogeneous ideal defining $\mathcal{C}$ in the homogeneous coordinate ring $k[X, Y, Z, T]$ of $\mathbb{P}^3$ and let $\operatorname{gin}(I)$ be its initial ideal. We define the lower triangle diagram by the function $\Delta_0 : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \cup \{\infty\}$ defined as follows: $\Delta_0(i, j) = \min\{k : X^i Y^j Z^k \in \operatorname{gin}(I)\}$ and $\Delta_0(i, j) = \infty$ if k does not exist. We can also define an upper triangle diagram $\Delta_1 : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \cup \{\infty\}$ by means of the lower triangle diagram of a curve algebraically linked to $\mathcal{C}$ in a suitable complete intersection. The pair of functions $\Delta = (\Delta_0, \Delta_1)$ is called the triangle diagram associated to the curve $\mathcal{C}$.
To our purposes, it is important to recall the following result, proved in [4], Prop. 3.6.3 that links the notion of triangle with the notion of spectrum:

$$A(n) = \sharp\{(i, j) : i + j - \Delta_1(i, j) = n\} = h_{\mathcal{C}}(n)$$

where $\sharp$ denotes the cardinality of a finite set. The following proposition describes some properties of the triangles that allow their construction (see [4]):

Proposition 2.3. *Let $\mathcal{C}$ be a curve in $\mathbb{P}^3$. Then Δ satisfies the following conditions:*

1. $\Delta_1(0, 0) = 0$,
2. $\Delta_1(i, j) = \infty \iff \Delta_0(i, j) \neq \infty$,
3. $\Delta_0(i, j + 1) < \Delta_0(i, j)$ *unless they are both* 0 *or* ∞,
4. $\Delta_0(i + 1, j - 1) \leq \Delta_0(i, j)$ *unless they are both* 0 *or* ∞,
5. $\Delta_1(i, j) < \Delta_1(i + 1, j)$ *unless they are both* 0 *or* ∞,
6. $\Delta_1(i, j) < \Delta_1(i, j + 1)$ *unless they are both* 0 *or* ∞.

Using the triangles one can compute the dimensions of the cohomology group of the ideal sheaf $\mathcal{I}_{\mathcal{C}}$ of the curve:

Proposition 2.4. *Let $\mathcal{C}$ be a curve with triangle diagram $\Delta = (\Delta_0, \Delta_1)$ and let*

$$B(n) = \sharp\{(i, j) \ : \ i + j + \Delta_0(i, j) = n\}.$$

Then we have

$$h^0(\mathcal{I}_{\mathcal{C}}(n)) = \sum_0^n (n - k + 1)B(k)$$

$$h^1(\mathcal{I}_{\mathcal{C}}(n)) = \sum_{0<\Delta_1(i,j)<+\infty} min\{\Delta_1(i, j), max(n + 1 - i - j + \Delta_1(i, j), 0)\} -$$

$$\sum_{0<\Delta_0(i,j)<+\infty} min\{\Delta_0(i, j), max(n + 1 - i - j, 0)\}.$$

3. The case of degree $d \geq 9$.

Let us consider the Hilbert scheme $H_{d,\tilde{g}}$ for $d \geq 9$ and $\tilde{g} = \binom{d-3}{2} - 1$. We will show that there are only two types of spectrum associated to the pair $(d, \tilde{g})$ and that $H_{d,\tilde{g}}$ does not contain ACM curves. We will classify the possible Rao modules and we will list the curves belonging to $H_{d,\tilde{g}}$.

Proposition 3.1. *Let $\mathcal{C}$ be a curve of degree $d \geq 9$ and genus $\tilde{g}$. Then its spectrum is either:*

$$sp_{\mathcal{C}} = \{-(d-3)\} \cup \{0, 1, 2, \ldots, d-2\} = \tag{1}$$

or

$$sp_{\mathcal{C}} = \{0^2, 1^2, 2, \ldots, d-3\}. \tag{2}$$

Proof. The spectrum (1) correspond to the case of an extremal curve having the given degree and genus and if $h_{\mathcal{C}}(d-2) \neq 0$ this is the only spectrum possible. If $h_{\mathcal{C}}(d-2) = 0$ and $h_{\mathcal{C}}(d-3) \neq 0$ the spectrum, that is connected in positive degrees, contains $\{0, 1, 2, \ldots, d-2\}$. It is now easy to verify, using the relations between the spectrum and d, $\tilde{g}$ in Prop. 2.2, that to get the desired degree and genus the only possibility is the one given in (2). If one supposes that $h_{\mathcal{C}}(d-3) = 0$ then there is no spectrum possible associated to the pair $(d, \tilde{g})$. This finishes the proof. $\square$

Proposition 3.2. *If $d \geq 9$, there are no ACM curves in $H_{d,\tilde{g}}$.*

Proof. Theorem 0.6 in [12] shows that there exists an ACM curve having spectrum $\{h_{\mathcal{C}}(n)\}$ if and only if $h_{\mathcal{C}}(n)$ is an s-sequence, where s is the least degree of a surface containing $\mathcal{C}$. In particular, one has $h_{\mathcal{C}}(n) = n + 1$ for $0 \leq n \leq s-1$. In our case it is $s \geq 2$ so that there exists an ACM curve $\mathcal{C}$ if and only if $h_{\mathcal{C}}(0) = 1$, $h_{\mathcal{C}}(1) = 2$. This is absurd being the spectrum of the type (1) or (2). $\square$

Theorem 3.3. *The Rao modules of a curve of degree $d \geq 6$ and genus $\tilde{g}$ associated to the spectra (1), (2) are (up to type):*

1. *the extremal module*

   $$M_1 = R(d-3)/(X, Y, f_3, f_4) \text{ where } \deg f_3 = d-2,\ \deg f_4 = 2(d-2),$$

   where (X, Y, f_3, f_4) form a regular sequence
2. *the subextremal module*

   $$M_2 = R/(X, Y, f_3, f_4) \text{ where } \deg f_3 = 2,\ \deg f_4 = d-2,$$

   where (X, Y, f_3, f_4) form a regular sequence
3. *the module*

   $$M_3 = R/(X, Y, Z^2, T^2),$$
4. *the module*

   $$M_4 = R/(X, Y, Z^2, ZT, T^{d-2}).$$

Proof. The spectrum (1) characterizes the extremal curves. From the general theory, see [6], the associated module is $M_1 = M(a-1)$ where $M = R/(f_1, f_2, f_3, f_4)$ is an extremal Koszul module with $degf_1 = degf_2 = 1$, $degf_3 = a$ and $degf_4 = a + l$. In our case, we have $a = \binom{d-2}{2} - \tilde{g} = d - 2$ and $l = d - 2$ and with a suitable choice of the coordinates we may suppose that the module is as above. In case the spectrum is (2), the curve $\mathcal{C}$ contains a plane curve $\mathcal{C}_{d-2}$ of degree $(d-2)$ (see [13], Theorem 3.2). Let now consider the integer $s(\mathcal{C}) = \min\{n \mid h^0(\mathcal{I}_{\mathcal{C}}(n)) \neq 0\}$. If $s(\mathcal{C}) = 2$ then the curve is subextremal. If $s(\mathcal{C}) \geq 3$ then $s(\mathcal{C}) = 3$ for reason of genus, being $g \leq \binom{d-3}{2} - \binom{s-1}{3}$ by Cor. 2.11 in [7], and the Rao function $\rho(n)$ is bounded above by the following inequalities given in [7]:

$$\rho(n) \leq \begin{cases} 0 & n < 0, \quad n > d-3 \\ 1 & n = 0 \\ 2 & n = 1 \\ 1 & 2 \leq n \leq d-3. \end{cases} \tag{3}$$

The Rao modules associated to the spectrum (2) are monogeneous, in fact if $\mu_{\mathcal{C}}(n)$ denotes the number of their minimal generators of in degree n, we have that $h_{\mathcal{C}}(n) \geq 1 + \mu_{\mathcal{C}}(n)$, for $0 \leq n \leq d-2$ and if the equality holds for $n = \ell$, $1 \leq \ell \leq d-2$ then $h_{\mathcal{C}}(n) = 1$ and $\mu_{\mathcal{C}}(n) = 0$ for $\ell \leq n \leq d-2$ (see [13], Theorem 3.2). So, for $n \geq 1$ the spectrum forces $\mu_{\mathcal{C}}(n) = 0$ and the only generator e appears in degree 0. Note that the spectrum fixes the Rao function for $n = 0, 1, 2$. In fact we have: $h^0(\mathcal{O}_{\mathcal{C}}(0)) = 2$, $h^0(\mathcal{O}_{\mathcal{C}}(1)) = 6$, $h^0(\mathcal{O}_{\mathcal{C}}(2)) = 11$ and $h^0(\mathcal{I}_{\mathcal{C}}(n)) = 0$ for $n = 0, 1, 2$. The Riemann–Roch theorem gives $\rho(n) = 1$ for $n = 0, 2$ and $\rho(1) = 2$. The relations that generate the module structure can only be of two types, in fact, up to a choice of the coordinates, the first relations are $Xe = Ye = 0$. The next relations can be either $Z^2e = T^2e = 0$ or $Z^2e = ZTe = 0$. In the first case the module is of the type M_3 and its Rao function vanishes for $n \geq 3$. In the second case, $T^ie = e_i$ is a basis of the i–th component $(M)_i$ of the Rao module for $i \geq 2$, so the module will be of the type $M_4 = R/(X, Y, Z^2, ZT, T^r)$ for some $r \geq 3$. Note that r cannot be 3 in fact, in this case, M_4 has the same Rao function as the module M_3. The Rao modules with $\rho(n) = 1$ for $n = 0, 2$ and $\rho(1) = 2$ are classified in [6] where the authors prove that M_4, for $r = 3$, is associated to a curve which is, up to a deformation, a curve of degree 5 and genus 0 obtained as the disjoint union of a line and a quartic of genus 1. Then $r > 3$ and we can look at the module M_4 as the module of a curve that is, up to a deformation, union the line $X = Y = 0$ and the curve $Z^2 = ZT = T^r$, so that, by reason of degree, $r = d - 2$ and the Rao function satisfies the equalities in (3). $\square$

Remark 3.4. If the curve $\mathcal{C}$ has the spectrum (2) and degree $d \geq 9$, we can consider the residual sequence with respect to the plane H containing the plane subcurve $\mathcal{C}_{d-2}$:

$$0 \longrightarrow \mathcal{I}_{\mathcal{C}'}(-1) \longrightarrow \mathcal{I}_{\mathcal{C}} \longrightarrow \mathcal{I}_{\mathcal{C}\cap H,H} \longrightarrow 0. \tag{4}$$

Following the proof of Theorem 3.1 of [11] we get that the genus of the residual curve $\mathcal{C}'$ is given by

$$g(\mathcal{C}') = r - 2$$

where $r = h^0(\mathcal{O}_{\mathcal{R}})$, $\mathcal{R}$ denotes the residual scheme coming from the sequence

$$0 \longrightarrow \mathcal{I}_{\mathcal{C}\cap H|H} \longrightarrow \mathcal{I}^{CM}_{\mathcal{C}\cap H|H} \longrightarrow \mathcal{O}_{\mathcal{R}} \longrightarrow 0, \tag{5}$$

and $\mathcal{I}^{CM}_{\mathcal{C}\cap H|H}$ is the largest locally Cohen–Macaulay curve contained in $\mathcal{C} \cap H$. This shows that $g(\mathcal{C}') \geq -2$. The genus $\tilde{g}$ allows only three cases. If $g(\mathcal{C}') = 0$ then $\mathcal{C}'$ is a conic, the curve $\mathcal{C}$ is a subextremal curve that is the disjoint union of two plane curves of degree $(d-2)$ and 2 respectively. Such a curve is obtained from an extremal curve by an elementary biliaison of height 1 on a quadric. Its Rao module is, modulo a shift, the Rao module of an extremal curve of parameters $a = 2$ and $l = d - 2$.
If $g(\mathcal{C}') = -1$ then the general curve $\mathcal{C}'$ is the union of two skew lines; one of them must intersect $\mathcal{C}_{d-2}$ giving rise to an ACM curve of degree $d-1$. The Rao module is M_4 by Cor. 3.5 I, [5]. The Rao function in this case is $\rho(n) = 1$ for $n = 0$ and $2 \leq n \leq d-3$, $\rho(1) = 2$ and $\rho(n) = 0$ otherwise. In fact, being the values of $\rho(n)$ fixed by the spectrum for $n = 0, 1, 2$ and since (3) holds, we only have to prove that $\rho(d-3) = 1$, since the module is monogeneous. This can be easily computed by using the residual sequence written above, in fact, from (4) we get $H^1(\mathcal{I}_{\mathcal{C}}(d-3)) \cong H^1(\mathcal{I}_{\mathcal{C}\cap H|H}(d-3))$ while from (5) it follows that

$$0 \longrightarrow H^0(\mathcal{I}^{CM}_{\mathcal{C}\cap H|H}(d-3)) \longrightarrow H^0(\mathcal{O}_{\mathcal{R}}(d-3)) \longrightarrow H^1(\mathcal{I}_{\mathcal{C}\cap H|H}(d-3)) \longrightarrow 0$$

so $h^1(\mathcal{I}_{\mathcal{C}\cap H|H}(d-3)) = r$ and the statement follows.
Finally, if $g(\mathcal{C}') = -2$ then $\mathcal{C}'$ is a double line of genus -2 that must intersect $\mathcal{C}_{d-2}$ in a 0–dimensional scheme of length 2. We wish to prove that the curve $\mathcal{C}$ is bilinked to $\mathcal{C}'$ on a surface of degree $d-2$. Note that there is a surface S containing $\mathcal{C}$ defined by $F \in H^0(\mathcal{I}_{\mathcal{C}}(d-2))$ and cutting properly the plane H. In fact, the surface S must contain H and $\mathcal{C}'$, so that $F = hF'$, where h is the linear form defining H and $F' \in H^0(\mathcal{I}_{\mathcal{C}'}(d-3))$. By the Riemann–Roch theorem one has $h^0(\mathcal{I}_{\mathcal{C}}(d-2)) \geq \binom{d+1}{3} - d(d-2) - 1 + \binom{d-3}{2} - 1$ while

$h^0(\mathcal{I}_{\mathcal{C}'}(d-3)) = \binom{d}{3} - d(d-3)+1$, so that $h^0(\mathcal{I}_{\mathcal{C}}(d-2)) > h^0(\mathcal{I}_{\mathcal{C}'}(d-3))$ and one can conclude that there are surfaces S as above. If we apply an elementary biliaison to $\mathcal{C}'$ on the surface S defined by F, we get the curve $\tilde{\mathcal{C}} = \mathcal{C}' \cup (S \cap H)$, but $S \cap H = \mathcal{C}_{d-2}$ for reason of degree, so $\tilde{\mathcal{C}} = \mathcal{C}$.

Note that, in case $\mathcal{C} \in H_{d,\tilde{g}}$ the Rao module is uniquely associated to the postulation of the curve and its Rao function and vice versa. So, in the proof of Theorem 3.3 it is contained also the proof of the following Corollary:

Corollary 3.5. *Let ψ_d the map that associates to each curve $\mathcal{C} \in H_{d,\tilde{g}}$ its postulation γ and its Rao function ρ. Then the image of ψ_d contains four pairs (γ_i, ρ_i).*

Remark 3.6. Let us set $H_{\gamma_i,\rho_i} = \psi_d^{-1}(\gamma_i, \rho_i)$; the subschemes H_{γ_i,ρ_i} for $i = 1, 2, 3$ are smooth and irreducible since the corresponding modules are Koszul. For $i = 4$, we have that H_{γ_4,ρ_4} contains only curves associated to the module M_4 so we can consider H_{γ_4,M_4}, instead of H_{γ_4,ρ_4}, which is smooth and irreducible by Corollaries 1.2 and 1.7, VII in [5]. Then it is possible to compute the dimensions t_{γ_i,ρ_i} of the schemes H_{γ_i,ρ_i}, $i = 2, 3, 4$; in the extremal case it is known that $t_{\gamma_1,\rho_1} = d(d+5)/2 - 1$, by Theorem 2.5, [6]. In the remaining cases $t_{\gamma_i,\rho_i} = \delta_{\gamma_i} + \epsilon_{\gamma_i,\rho_i} - h_{M_i} + ext^1(M_i, M_i)^0$, where all the quantities can be determined using the formulas in [5], IX and are based on the knowledge of the postulation γ_i, its first difference $\partial\gamma_i$ and the Rao function that are listed in the diagram.

n	0	1	2	3	4	5	6	7	$\dots$	$d-3$	$d-2$	$d-1$	d	$d+1$	$d+2$	$d+3$
γ_2	-1	-1	0	1	0	0	0	0	$\dots$	0	0	1	1	-1	0	0
$\partial\gamma_2$	-1	0	1	1	-1	0	0	0	$\dots$	0	0	1	0	-2	1	0
ρ_2	1	2	2	2	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	2	1	0	0	0	0	0
γ_3	-1	-1	-1	2	1	-1	0	0	$\dots$	0	1	0	0	0	0	0
$\partial\gamma_3$	-1	0	0	3	-1	-2	1	0	$\dots$	0	1	-1	0	$\dots$	0	0
ρ_3	1	2	1	0	0	0	0	0	$\dots$	0	0	0	0	0	0	0
γ_4	-1	-1	-1	3	-1	0	0	0	$\dots$	0	0	2	-1	0	0	0
$\partial\gamma_4$	-1	0	0	4	-4	1	0	0	$\dots$	0	0	2	-3	1	0	0
ρ_4	1	2	1	1	1	$\dots$	$\dots$	$\dots$	$\dots$	1	0	0	0	0	0	0

The dimensions are the following:

i	δ_{γ_i}	$\epsilon_{\gamma_i,\rho_i}$	h_{M_i}	$ext^1(M_i, M_i)^0$	t_{γ_i,ρ_i}
2	$d(d-1)/2+6$	-2	1	7	$d(d-1)/2+10$
3	$d(d-1)/2+8$	-4	1	6	$d(d-1)/2+9$
4	$d(d-1)/2+7$	-2	1	5	$d(d-1)/2+9$

We can now describe the general curve of each component of the Hilbert scheme:

Theorem 3.7. *The Hilbert scheme $H_{d,\tilde{g}}$ of curves of degree $d \geq 9$ and genus $\tilde{g}$ has four irreducible components (which are the closures of the subschemes $H_{\gamma,\rho}$):*

1. *The family H_1 of extremal curves, whose dimension is $\frac{d(d+5)}{2} - 1$.*
2. *The closure H_2 of the family of subextremal curves associated to the module M_2, whose general member is the disjoint union of two plane curves of degrees $d-2$ and 2. The dimension of H_2 is $\frac{d(d-1)}{2} + 10$.*
3. *The closure H_3 of the family of curves associated to the module M_3 whose general curve is obtained by a biliaison of height 1 on a surface of degree $d-2$ from a double line of genus -2 and corresponds to the union of a plane curve $\mathcal{C}_{d-2}$ of degree $d-2$ with a double line of genus -2 intersecting $\mathcal{C}_{d-2}$ in a zero–dimensional subscheme of length 2. The dimension of H_3 is $\frac{d(d-1)}{2} + 9$.*
4. *The closure H_4 of the family of curves associated to the module M_4 whose general member is the union of a plane curve $\mathcal{C}_{d-2}$ of degree $d-2$ with two skew lines, one of them intersecting transversally $\mathcal{C}_{d-2}$ in one point. The dimension of H_4 is $\frac{d(d-1)}{2} + 9$.*

Proof. The case of extremal curves is well known: they form an irreducible component of $H_{d,\tilde{g}}$ having the given dimension so we can consider non extremal curves. To prove the result we will show that the sets of curves described in *2, 3, 4* of the statement are contained in H_i, $i = 2, 3, 4$ respectively and that they form families with dimension equal the values already obtained in the diagram. If $s = 2$, we have already proved that the curve is associated to M_2 and it is contained in the union of two planes; if π is the plane containing $\mathcal{C}_{d-2}$ then the residual curve with respect to π is again a plane curve disjoint from $\mathcal{C}_{d-2}$.

Counting the irreducible choices we get t_{γ_2,ρ_2}. If $s = 3$ then the module is either M_3 or M_4. The module M_3 is, modulo a shift, the module of a double line $\mathcal{C}'$ of genus -2. There exists a surface of degree $d-2$ containing $\mathcal{C}'$ so that we can consider an elementary biliaison of height 1 on this surface that produces a curve having the given module. The general curve in H_3 can be found as in Remark 3.4 and the statement on the dimension follows from direct computation. Finally, M_4 is the module of a curve that specializes to the disjoint union of the line defined by the ideal (X, Y) and the ACM curve defined by (Z^2, ZT, T^{d-2}) that is contained in the double plane $Z^2 = 0$. The intersection with the $Z = 0$ gives the plane curve (Z, T^{d-2}) and the line (Z, T), so this ACM curve appears as the union of a plane curve of degree $d-2$ and a line meeting in one point. As before, if we count the choices made we get t_{γ_4,ρ_4}. The families of curves listed above are not contained one into another. First of all, note that the only inclusion allowed by the value of the dimensions and by the semicontinuity is $H_4 \subseteq H_3$. Now if $H_4 \subseteq H_3$ then H_3 must intersect the open subset of H_4 formed by the reduced curves but this is absurd, since the general curve in H_3 is non reduced. □

4. The case $d \leq 8$.

The Hilbert scheme $H_{d,g}$ with $d = 3$ was already studied for all the values of the genus in [8], moreover it is well known that all curves with $d \leq 2$ are planar or extremal. The case $(d, g) = (4, -1)$ has been treated in [10] while $(d, g) = (5, 0)$ was dealt by Liebling in his PhD thesis [4]. Then, the cases we have to consider are $(d, g) \in \{(6, 2), (7, 5), (8, 9)\}$.
Easy calculations prove the following:

Proposition 4.1. *Let $\mathcal{C}$ be a curve of degree $6 \leq d \leq 8$ and genus $\tilde{g}$. Then its spectrum is either (1) or (2) or*

$$sp_{\mathcal{C}} = \{0, 1^3, 2^2\} \qquad if \;\; d = 6, \tag{6}$$

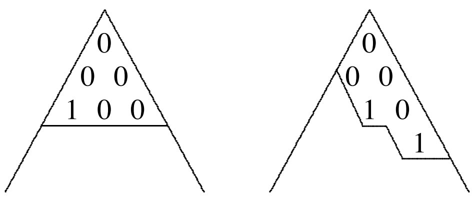
$$sp_{\mathcal{C}} = \{0, 1^2, 2^3, 3\} \qquad if \;\; d = 7, \tag{7}$$

$$sp_{\mathcal{C}} = \{0, 1^2, 2^2, 3^2, 4\} \quad if \;\; d = 8. \tag{8}$$

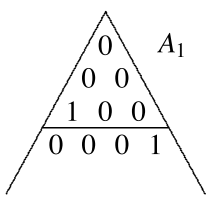
Remark 4.2. The Rao modules of the curves of degree $d \geq 6$ associated to spectra (1) and (2) can be classified as in Theorem 3.3 and the general curves are described in Theorem 3.7, in fact the results obtained in the previous section hold also for $d = 6, 7, 8$. In particular, the dimensions t_{γ_i,ρ_i}, $i = 1, \dots, 4$ can be computed as in Remark 3.6 giving the same results listed in the diagram. So we turn our attention to the spectra listed in the previous Proposition.

In this case, we will determine the cohomology and then the Rao modules of curves $\mathcal{C} \in H_{d,\tilde{g}}$, using the triangles introduced by Liebling in [4].

Curves with $(d, g) = (6, 2)$.
Following the notations in the introduction we consider $A(n) = \sharp\{\ (i, j)\ :\ i + j - \Delta_1(i, j) = n\}$. The knowledge of the spectrum allows the construction of the upper triangle diagram associated to $\mathcal{C}$, since $A(n) = h_{\mathcal{C}}(n)$. If the curve is in general coordinates, we have the following two types of triangles:



Since $\sum_{n\in\mathbb{Z}} A(n) = \sum_{n\in\mathbb{Z}} B(n)$, with $B(n) = \sharp\{(i, j) : i + j + \Delta_1(i, j) = n\}$, the complete triangle arising from the first type is



We can now prove the following classification:

Proposition 4.3. *The curves $\mathcal{C} \in H_{6,2}$ having spectrum (6) belong to the following families:*

1. *the closure H_5 of the family of curves associated to the module $R(-1)/(X, Y, Z, T^2)$ that can be obtained by applying a biliaison* $(3, 1)$ *to the disjoint union of a line and a conic. The dimension of the family H_5 is* 24.

while the second triangle gives rise to six different triangles that are listed below:

$$A_2:\ \begin{array}{c} 0 \\ 0\ \ 0 \\ 2\ \ 1\ \ 0 \\ 0\ \ 0\ \ 0\ \ 1 \\ 0\ \ 0\ \ 0\ \ 0\ \ 0 \end{array} \qquad A_3:\ \begin{array}{c} 0 \\ 0\ \ 0 \\ 0\ \ 1\ \ 0 \\ 0\ \ 0\ \ 2\ \ 1 \\ 0\ \ 0\ \ 0\ \ 0\ \ 0 \end{array} \qquad A_4:\ \begin{array}{c} 0 \\ 0\ \ 0 \\ 0\ \ 1\ \ 0 \\ 0\ \ 0\ \ 0\ \ 1 \\ 0\ \ 0\ \ 0\ \ 0\ \ 2 \end{array}$$

$$A_5:\ \begin{array}{c} 0 \\ 0\ \ 0 \\ 1\ \ 1\ \ 0 \\ 0\ \ 0\ \ 1\ \ 1 \\ 0\ \ 0\ \ 0\ \ 0\ \ 0 \end{array} \qquad A_6:\ \begin{array}{c} 0 \\ 0\ \ 0 \\ 0\ \ 1\ \ 0 \\ 0\ \ 0\ \ 1\ \ 1 \\ 0\ \ 0\ \ 0\ \ 0\ \ 1 \end{array} \qquad A_7:\ \begin{array}{c} 0 \\ 0\ \ 0 \\ 1\ \ 1\ \ 0 \\ 0\ \ 0\ \ 0\ \ 1 \\ 0\ \ 0\ \ 0\ \ 0\ \ 1 \end{array}$$

2. *the closure H_6 of the family of curves having Rao module $R(-1)/(X, Y, Z^2, T^2)$ that can be obtained by applying a biliaison $(2, 1)$ to the disjoint union of two conics. H_6 has dimension 23.*

Proof. The spectrum (6) implies that the Rao module is monogeneous and moreover that $\rho(0) = 0$, $\rho(1) = 1$. The triangles A_1, A_2 and A_5 give the same Rao function that is $\rho(2) = 1$, $\rho(n) = 0$ for $n \geq 3$ and $h^0(\mathcal{I}_{\mathcal{C}}(n)) = 0$ for $n = 0, 1, 2$, $h^0(\mathcal{I}_{\mathcal{C}}(n)) = \binom{n+3}{3} - 6n + 1$ for $n \geq 3$. When the Rao function of a connected module is bounded above by 1, Lemma 1.8 of [1] applies, so that the module in this case is $M(-h)/(X, Y, Z, T^2)$. By Prop. 0.5 in [6] the minimal curve $\mathcal{C}_0$ associated to this module has degree 3 and genus -1 and there is a sequence of curves $\{\mathcal{C}_i\}$ such that $\mathcal{C}_i$ is obtained from $\mathcal{C}_{i-1}$ by an elementary biliaison $(s_i, 1)$ and $\mathcal{C}_h = \mathcal{C}$. The only possibility is $h = 1$ and $s_1 = 3$, so that $\mathcal{C}_0$ is the disjoint union of a line and a conic and $\mathcal{C}$ is bilinked to the curve $\mathcal{C}_0$ on a cubic surface. The postulation is $\gamma_{\mathcal{C}}(n) = -1$, for $n = 0, 1, 2$, $\gamma_{\mathcal{C}}(n) = 2$, for $n = 3, 4$, $\gamma_{\mathcal{C}}(5) = -1$ and 0 otherwise; by the formulas in [5], IX it is possible to compute the dimension of the family; we have $\delta_\gamma = 24$, $\epsilon_{\gamma,\rho} = -2$, $h_M = 1$ and $ext^1(M_i, M_i)^0 = 3$.

The triangles A_3 and A_6 give the following cohomology: $h^0(\mathcal{I}_{\mathcal{C}}(n)) = 0$ for $n = 0, 1$, $h^0(\mathcal{I}_{\mathcal{C}}(2)) = 1$ so that the curve lies on a quadric, $h^0(\mathcal{I}_{\mathcal{C}}(3)) = 4$, $h^0(\mathcal{I}_{\mathcal{C}}(n)) = \binom{n+3}{3} - 6n + 1$ for $n \geq 4$, $\rho(n) = 1$ for $n = 1, 3$, $\rho(2) = 2$ and $\rho(n) = 0$ otherwise. A curve with this cohomology bilinks down on a quadric by Prop. 3.4.15 in [4], so that it lies in the biliaison class of the disjoint union of two conics. Its Rao module is of the type $R(-1)/(X, Y, Z^2, T^2)$. As above,

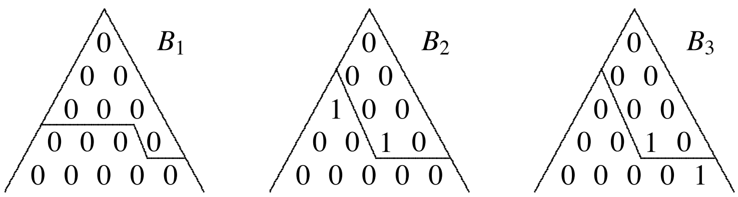
one can compute $\delta_\gamma = 22$, $\epsilon_{\gamma,\rho} = -4$, $h_M = 1$ and $ext^1(M_i, M_i)^0 = 6$ so that $\dim H_6 = 23$.
The triangle A_7 gives $h^0(\mathcal{I}_{\mathcal{C}}(n)) = 0$ for $n = 0, 1, 2$, $h^0(\mathcal{I}_{\mathcal{C}}(3)) = 4$, $h^0(\mathcal{I}_{\mathcal{C}}(n)) = \binom{n+3}{3} - 6n + 1$ for $n \geq 4$, $\rho(n) = 1$ for $n = 1, 2, 3$ and $\rho(n) = 0$ otherwise. Reasoning as above, we have that the Rao module can only be $R(-1)/(X, Y, Z, T^3)$ and the curve $\mathcal{C}$ would be in the biliaison class of the disjoint union of a line and a plane cubic. But such a curve would have genus 3 so this case cannot happen.
Finally, the triangle A_4 gives $\rho(n) = 1$ for $n = 1, 4$, $h^1(\mathcal{I}_{\mathcal{C}}(n)) = 2$ for $n = 2, 3$ and it is zero otherwise. If such a curve exists, it lies on a quadric so, by Prop. 3.4.15 in [4], it bilinks down to a quartic of genus -1 that, on the other hand, cannot have the given Rao function (modulo a shift) and this is absurd. □

Proposition 4.4. *The Hilbert scheme $H_{6,2}$ has five components:*

1. *the four components in Theorem 3.7,*
2. *the closure H_5 of the family of curves in the biliaison class of the disjoint union of a line and a conic.*

Proof. We have to consider only the curves having spectrum (6). The computations on the dimensions of the various families show that only H_5 can be a component, since, if X is a component, it is $\dim X \geq 24$. By the semicontinuity we have that the only inclusion allowed is $H_6 \subset H_5$, so $H_{6,2} = \bigcup_{i=1}^{5} H_i$. □

The case $(d, g) = (7, 5)$
The values of $A(n)$ and $B(n)$ introduced above, allow us to produce first the upper triangles (that can be only of two types) and then the complete triangles that are the following



The curves associated to these triangles are listed in the next Proposition.

Proposition 4.5. *The curves $\mathcal{C} \in H_{7,5}$ having spectrum (7) belong to the following families*

1. *the closure H_5 of family of ACM curves; H_5 has dimension* 28

2. *the closure H_6 of the family of curves having Rao module $R(-2)/(X, Y, Z, T^2)$, that are in the biliaison class of the disjoint union of a conic and a line; H_6 has dimension* 27.

Proof. First note that the Rao module must be monogeneous, by Theorem 3.2 in [13]. In the case B_1 the curve is ACM since all the entries of the triangles are zero, moreover $h^0(\mathcal{I}_{\mathcal{C}}(n)) = 0$ for $n = 0, 1, 2$, $h^0(\mathcal{I}_{\mathcal{C}}(3)) = 3$, $h^0(\mathcal{I}_{\mathcal{C}}(n)) = \binom{n+2}{3} + \binom{n-2}{2} - 3n + 2$, for $n \geq 4$. By direct computation it is possible to show that also a curve having the triangle B_2 is ACM and with the same cohomology as above. We have $\gamma_{\mathcal{C}}(n) = -1$ for $n = 0, 1, 2$, $\gamma_{\mathcal{C}}(3) = 2$, $\gamma_{\mathcal{C}}(4) = 1$ and 0 otherwise, $\delta_\gamma = 28$, $\epsilon_{\gamma,\rho} = h_M = ext^1(M_i, M_i)^0 = 0$ so that $\dim H_5 = 28$.
In the case B_3 we have $h^0(\mathcal{I}_{\mathcal{C}}(n)) = 0$ for $n = 0, 1$, $h^0(\mathcal{I}_{\mathcal{C}}(2)) = 1$, $h^0(\mathcal{I}_{\mathcal{C}}(3)) = 4$, $h^0(\mathcal{I}_{\mathcal{C}}(n)) = \binom{n+2}{3} + \binom{n-2}{2} - 3n + 2$, for $n \geq 4$. Moreover $\rho(n) = 1$ for $n = 2, 3$ and it vanishes otherwise. Reasoning as in the proof of Prop. 4.3, the Rao module is of the type $R(-h)/(X, Y, Z, T^2)$ and the curve $\mathcal{C}$ is not minimal. Its minimal model is given by a disjoint union of a line with a conic so that $h = 2$ and the curve $\mathcal{C}$ can be obtained, up to deformations, by two elementary biliaisons (2, 1). We also have $\delta_\gamma = 27$, $\epsilon_{\gamma,\rho} = -2$, $h_M = 1$, $ext^1(M_i, M_i)^0 = 3$ so that $\dim H_6 = 27$. □

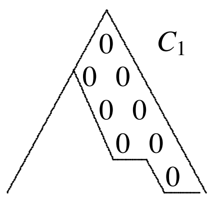
Proposition 4.6. *The Hilbert scheme $H_{7,5}$ has five irreducible components:*

1. *the four components listed in Theorem 3.7,*

2. *the closure of the family of ACM curves.*

Proof. We consider only the curves having spectrum (7): the dimensions of the two families show that only H_5 is a component and by semicontinuity, it is $H_6 \subset H_5$. □

The case $(d, g) = (8, 9)$
In this last case, the only upper triangle allowed by the spectrum (8) is



then it immediately follows that the curve $\mathcal{C} \in H_{8,9}$ having spectrum (8) is ACM. Computations as above show that $h^0(\mathcal{I}_{\mathcal{C}}(n) = 0$ for $n = 0, 1$ and $h^0(\mathcal{I}_{\mathcal{C}}(n) = \binom{n+2}{3} + \binom{n-3}{2} - 3n + 3$ for $n \geq 2$ so that $\gamma_{\mathcal{C}}(n) = -1$ for $n = 0, 1$, $\gamma_{\mathcal{C}}(4) = 1$, $\gamma_{\mathcal{C}}(5) = 1$ and 0 otherwise, $\delta_\gamma = 33 = t_{\gamma,\rho}$. Then we have the following

Proposition 4.7. *The Hilbert scheme $H_{8,9}$ has the four components listed in Theorem 3.7 and the closure H_5 of family of ACM curves that has dimension* 33.

Proof. Note that in this case, the family of ACM curves is a family that, by the semicontinuity, is not contained in any of the previous ones. The result follows. □

Now, we can state our main result:

Theorem 4.8. *The Hilbert scheme $H_{d,\tilde{g}}$ is connected for $d \geq 3$.*

Proof. Let $d \geq 9$: we have classified the curves $\mathcal{C}$ lying in $H_{d,\tilde{g}}$ in theorem 3.7. The subextremal curves can be connected to extremal curves by [3], Prop. 9.12, while curves in the family associated to the module M_3 are in the biliaison class of a double line of genus -2 so they can be connected to an extremal curve by [14], Lemma 2.1. Finally, a curve associated to M_4 can be connected to an extremal curve by [2], Propositions 2.2 and 3.4. Then, via the irreducible component of $H_{d,\tilde{g}}$ formed by extremal curves, we have shown that $H_{d,\tilde{g}}$ is connected. The cases of degrees $d = 3, 4, 5$ are studied in [9], [10], [4], so let $d = 6, 7, 8$. We already know that the four components listed in Theorem 3.7 contain curves that can be connected to an extremal curve. If $d = 6$, the curves in the component H_5 are in the biliaison class of the disjoint union of a line with a plane curve of degree two, so they can be connected to an extremal curve by Proposition 2.2 in [14]. If $d = 7$, again we only have to consider the component of ACM curves that can be connected to an extremal curve by Prop. 3.4 in [2]. The same argument shows also that the component H_5 of the case $H_{8,9}$ can be connected. This finishes the proof. □

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