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PLURICANONICAL MAPS OF A THREEFOLD OF GENERAL TYPE

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Dedicated to Silvio Greco in occasion of his 60-th birthday.

We construct a non-singular threefold X of general type as a desingularization of a hypersurface of degree six in $\mathbb{P}^4$, having the birational invariants $q_1 = q_2 = p_g = 0$, $P_2 = 1$, $P_3 = 2$, $P_4 = P_5 = 3$, $P_6 = 5$. Moreover, we prove that the m-*canonical* map $\varphi_{|mK|}$, where K is a *canonical* divisor on X, has fibers that are generically finite sets if and only if $m \geq 6$ and it is birational if and only if $m \geq 11$.

Introduction.

In this paper we summarize the results of a paper with the same title which will be published elsewhere. The results are presented here without complete proofs, but the idea giving the birationality of the m-canonical map $\varphi_{|mK|}$ if and only if $m \geq 11$ is written in some detail.

Let V be a reduced, irreducible algebraic hypersurface of degree 6 in the projective space $\mathbb{P}^4 = \mathbb{P}^4_k$, where k is an algebraically closed field of characteristic zero, which we may assume to be the field of complex numbers.

We impose five triple points on V at the five vertices A_0, A_1, A_2, A_3, A_4 of the fundamental tetrahedron. We impose a double surface S_i infinitely near each point A_i, $i = 0, 1, 2, 3, 4$. Other unimposed singularities appear on V, close to these imposed singularities; they are actual and infinitely near singularities. As usual, we call *actual* a singularity on V to distinguish it from the infinitely near singularities that are (actual) singularities on strict transforms of V belonging to exceptional divisors. By calling $\sigma : X \longrightarrow V$ a desingularization of V, we obtain that the unimposed singularities do not affect the birational invariants of X.

The birational invariants we find for X are: $q_1 = q_2 = p_g = 0$, $P_2 = 1$, $P_3 = 2$, $P_4 = P_5 = 3$, $P_6 = 5$, $P_7 = 6$, $P_8 = 8$, $P_9 = 10$, $P_{10} = 13$, where $q_i = \dim_k H^i(X, \mathcal{O}_X)$, $P_m = \dim_k H^0(X, \mathcal{O}_X(mK))$, K denotes a canonical divisor on X; the Kodaira dimension of X is three, i.e. X is of general type, and the canonical divisors K on X do not satisfy the two properties (simultaneously): the highest self-intersection number (K^3) is positive (i.e. $(K^3) > 0$) and $(K \cdot C) \geq 0$, for any curve C on X. If K enjoys the latter property, it is called *numerically effective*, abbreviated as *nef*. Furthermore, regarding the m-canonical map $\varphi_{|mK|} : X - - \rightarrow \mathbb{P}^{P_m - 1}$, we have the following results: $\varphi_{|mK|}$ has fibers that are generically given by two points if and only if $6 \leq m \leq 10$, and $\varphi_{|mK|}$ is birational if and only if $m \geq 11$.

We prefer from now on to call $\varphi_{|mK|}$ an m-canonical transformation, rather than an m-canonical map, in order to emphasize that $\varphi_{|mK|}$ is not, strictly speaking, a map: it does not need to be defined on all of X. Moreover, in the following pages a rational transformation having the generic fiber given by a finite set of n points will be called a rational transformation $n : 1$.

In the literature, the following results are given for the m-canonical transformation of a non-singular threefold of general type X having canonical divisors K satisfying the two properties: $(K^3) > 0$ and K nef. The m-canonical transformation $\varphi_{|mK|}$ is a birational transformation providing

$m \geq 9$, X. Benveniste 1984, [1];

$m \geq 8$, K. Matsuki 1986, [7];

$m \geq 6$, M. Chen and S. Lee (independently) 1998-1999, [2], [6].

In the case of a non-singular threefold of general type Y, a lemma proved by A. Sommese (cf. [4], p. 44) states that if the canonical divisors K_Y on Y are nef, then $(K_Y^3) > 0$. So, from Sommese's result, we deduce that the canonical divisors K of our example X are not nef.

As far as I know, the results for the birationality of $\varphi_{|mK|}$ in the case of non-singular projective threefolds of general type X without the two hypotheses for K ($(K^3) > 0$ and K nef) are as follows.

– J. Kollàr [5] has proved that if $P_r \geq 2$, then $\varphi_{|(7r+3)K|}$ is generically finite and $\varphi_{|(11r+5)K|}$ is birational;

– S. Chiaruttini and R. Gattazzo [3] constructed a non-singular threefold of general type such that the m-canonical transformation is birational if and only if $m \geq 6$.

Many problems regarding the birationality of $\varphi_{|mK|}$, and the fact that it is a rational transformation $n:1$, are therefore still open if we abandon the hypothesis of K nef. The example constructed in the present paper is a contribution in the direction of these problems.

1. Imposing singularities on a degree six hypersurface V in $\mathbb{P}^4$.

Let us indicate as $f_6(X_0, X_1, X_2, X_3, X_4)$ a form (homogeneous polynomial) defining a hypersurface of degree six $V \subset \mathbb{P}^4$ with a triple point at each of the five vertices $A_0 = (1, 0, 0, 0, 0)$, $A_1 = (0, 1, 0, 0, 0)$, $A_2 = (0, 0, 1, 0, 0)$, $A_3 = (0, 0, 0, 1, 0,)$, $A_4 = (0, 0, 0, 0, 1)$ of the fundamantal tetrahedron. The equation of V is given by

$$V : f_6(X_0, X_1, X_2, X_3, X_4) =$$
$$X_0^3(a_{33000}X_1^3 + \ldots) + X_1^3(a_{23100}X_0^2X_2 + \ldots) + X_2^3(...) + X_3^3(...) +$$
$$+ X_4^3(...)a_{22200}X_0^2X_1^2X_2^2 + a_{22110}X_0^2X_1^2X_2X_3 + \ldots + a_{00222}X_2^2X_3^2X_4^2 = 0,$$

where $a_{ijklh} \in k$ denotes the coefficient of the monomial $X_0^iX_1^jX_2^kX_3^lX_4^h$.

We impose an infinitely near double surface $\mathcal{S}_i$ at the point A_i, $i = 0, 1, 2, 3, 4$, in the first neighbourhood. The surface $\mathcal{S}_i$ is locally isomorphic to a plane, according to our hypothesis on the singularities in [8], the Introduction and section 1.

We follow the same method as we used in $[St]$, section 5 and impose a double surface $\mathcal{S}_0$ infinitely near A_0, then - by means of a permutation of indices and variables - we impose the same singularity at A_1, A_2, A_3 and A_4. We also use the same permutations of indices and variables as in [8].

We give the final equation for our hypersurface V, after imposing all the above-said singularities.

$$V : f_6(X_0, X_1, X_2, X_3, X_4) =$$
$$X_0^3(a_{33000}X_1^3 + a_{32100}X_1^2X_2 + a_{32010}X_1^2X_3 + a_{32001}X_1^2X_4)+$$
$$X_1^3(\qquad a_{23100}X_0^2X_2 + a_{23010}X_0^2X_3 + a_{23001}X_0^2X_4)+$$
$$X_2^3(a_{21300}X_0^2X_1)+$$
$$X_3^3(a_{12030}X_0X_1^2)+$$
$$X_4^3(a_{10203}X_0X_2^2)+$$
$$a_{22200}X_0^2X_1^2X_2^2 + a_{22110}X_0^2X_1^2X_2X_3 + a_{22101}X_0^2X_1^2X_2X_4+$$
$$a_{22020}X_0^2X_1^2X_3^2 + a_{22011}X_0^2X_1^2X_3X_4 + a_{21210}X_0^2X_1X_2^2X_3+$$
$$a_{21201}X_0^2X_1X_2^2X_4 + a_{21120}X_0^2X_1X_2X_3^2 + a_{21111}X_0^2X_1X_2X_3X_4+$$

$$a_{21102}X_0^2X_1X_2X_4^2+a_{21021}X_0^2X_1X_3^2X_4+a_{12210}X_0X_1^2X_2^2X_3+$$
$$a_{12201}X_0X_1^2X_2^2X_4+a_{12120}X_0X_1^2X_2X_3^2+a_{12111}X_0X_1^2X_2X_3X_4+$$
$$a_{12102}X_0X_1^2X_2X_4^2+a_{12021}X_0X_1^2X_3^2X_4+a_{11220}X_0X_1X_2^2X_3^2+$$
$$a_{11211}X_0X_1X_2^2X_3X_4+a_{11202}X_0X_1X_2^2X_4^2+a_{11121}X_0X_1X_2X_3^2X_4+$$
$$a_{11112}X_0X_1X_2X_3X_4^2+a_{10212}X_0X_2^2X_3X_4^2+a_{01122}X_1X_2X_3^2X_4^2=0.$$

In the sequel, V denotes this final hypersurface defined by the above final form $f_6(X_0, X_1, X_2, X_3, X_4)$ for a generic choice of the parameters a_{ijklh}.

2. Global and non-global m-canonical adjoints to $\mathbf{V} \subset \mathbb{P}^4$.

Let

$$\mathbb{P}_r \xrightarrow{\pi_r} \dots \xrightarrow{\pi_3} \mathbb{P}_2 \xrightarrow{\pi_2} \mathbb{P}_1 \xrightarrow{\pi_1} \mathbb{P}_0 = \mathbb{P}^4$$

be a sequence of blow-ups resolving the singularities on V.

If we call $V_i \subset \mathbb{P}_i$ the *strict transform* of V_{i-1} with respect to π_i, then, from the above sequence, we obtain

$$X = V_r \xrightarrow{\pi'_r} \dots \xrightarrow{\pi'_3} V_2 \xrightarrow{\pi'_2} V_1 \xrightarrow{\pi'_1} V_0 = V,$$

where $\pi'_i = \pi_{i_{|V_i}} : V_i \longrightarrow V_{i-1}$ and X is a desingularization of $V \subset \mathbb{P}^4$.

Let us assume that π_i is a blow-up along a subvariety Z_{i-1} of $\mathbb{P}_{i-1}$, of dimension j_{i-1}, which can be either singular or non-singular subvariety of $V_{i-1} \subset \mathbb{P}_{i-1}$ (i.e. Z_{i-1} is the locus of singular or simple points of V_{i-1}). Let m_{i-1} be the multiplicity of the variety Z_{i-1} on V_{i-1}.

Let us set $n_{i-1} = -3 + j_{i-1} + m_{i-1}$, for $i = 1, ..., r$ and $\deg(V) = d$.

A hypersurface $\Phi_{m(d-5)}$ of degree $m(d-5)$ in $\mathbb{P}^4$ is an *m-canonical adjoint* to V (with respect to the sequence of the blow-ups $\pi_1, \dots, \pi_r$) if the restriction to X of the divisor

$$D_m = \pi_r^*\{\pi_{r-1}^*[\dots\pi_1^*(\Phi_{m(d-5)}) - mn_0E_1 \dots] - mn_{r-2}E_{r-1}\} - mn_{r-1}E_r$$

is effective, i.e. $D_{m|_X} \geq 0$, where $E_i = \pi^{-1}(Z_{i-1})$ is the exceptional divisor of π_i and $\pi_i^* : Div(\mathbb{P}_{i-1}) \longrightarrow Div(\mathbb{P}_i)$ is the homomorphism of the Cartier (or locally principal) divisor groups (cf. [8], sections 1,2).

A hypersurface $\Phi_{m(d-5)}$ of degree $m(d-5)$ in $\mathbb{P}^4$ is a *global* m-canonical adjoint to V (with respect to $\pi_1, \dots, \pi_r$) if the divisor D_m is effective on $\mathbb{P}_r$, i.e. $D_m \geq 0$ (loc. cit.).

As usual, if $D_{m|_X} \geq 0$ but $D_m \not\geq 0$, then the hypersurface $\Phi_{m(d-5)}$ will be called a *non-global* m-canonical adjoint to V.

Note that if $\Phi_{m(d-5)}$ is an m-canonical adjoint to V, then $D_{m|_X} \equiv mK$, where $\equiv$ denotes linear equivalence and K denotes a canonical divisor on X.

In our example, the blow-up π_1 is the blow-up at A_0 and π_2 is the blow-up along the surface $\mathcal{S}_0$ infinitely near A_0, π_3 is the blow-up at A_1 and π_4 is the blow-up along the surface $\mathcal{S}_1$ infinitely near $A_1, \ldots$; i.e. π_{2j+1} is the blow-up at A_j and π_{2j+2} is the blow-up along the surface $\mathcal{S}_j$ infinitely near A_j, $j = 0, 1, 2, 3, 4$. Moreover, we have $d = 6$ and D_m is given by:

$$(*) \quad D_m = \pi_r^* \ldots \{\pi_2^*[\pi_1^*(\Phi_m)] - mE_2\} - mE_4 - mE_6 - mE_8 - mE_{10} + mE,$$

where E_2 is the exceptional divisor of the blow-up π_2 along the surface $\mathcal{S}_0$ infinitely near A_0, E_4 is the exceptional divisor of the blow-up π_4 along the surface $\mathcal{S}_1$ infinitely near $A_1, \ldots, E_{10}$ is the exceptional divisor of the blow-up π_{10} along the surface $\mathcal{S}_4$ infinitely near A_4; the divisor E is the exceptional divisor of the blow-up at an unimposed double isolate point.

No other exceptional divisors appear in $(*)$, because the unimposed singularities, which differ from the above isolated double point, are either actual or infinitely near double singular curves on our (generic) V. So, the exceptional divisors of the blow-ups along these curves appear with coefficient $n_h = 0$ in the above expression of D_m.

Warning. For the sake of brevity, from now on we omit the divisor E, since it is not essential for our purposes.

3. In search of non-global m-canonical adjoints to V.

If Φ_m is a *non-global* m-canonical adjoint to our V, it may be that a *global* m-canonical adjoint Φ'_m to V exists such that

$$\Phi_{m|_V} = \Phi'_{m|_V}.$$

We note that the equality $\Phi_{m|_V} = \Phi'_{m|_V}$ is equivalent to the equality $D_{m|_X} = D'_{m|_X}$, where $D_m = \pi_r^* \ldots \{\pi_2^*[\pi_1^*(\Phi_m)] - mE_2\} - mE_4 - mE_6 - mE_8 - mE_{10}$ and $D'_m = \pi_r^* \ldots \{\pi_2^*[\pi_1^*(\Phi'_m)] - mE_2\} - mE_4 - mE_6 - mE_8 - mE_{10}$ (cf. also $(*)$, section 2).

Theorem 1. *Let Φ_m be a* non-global *m-canonical adjoint to V. A* global *m-canonical adjoint Φ'_m to V exists such that $D_{m|_X} = D'_{m|_X}$ if and only if $m \leq 10$.*

As in [8], we denote by W'_m the vector space of degree m forms F_m defining global m-canonical adjoints to V; we denote as $\overline{W'}_m$ the vector space of the elements $\overline{F}_m \in k[V]$ = homogeneous coordinate ring of V, where F_m is a degree m form defining a global m-canonical adjoint to V; we denote as $\mathcal{W}'_m$ the vector space of the m degree forms $\mathcal{F}_m$ defining m-canonical adjoints to V; we denote as $\overline{\mathcal{W}'}_m$ the vector space of the elements $\overline{\mathcal{F}}_m \in k[V]$, where $\mathcal{F}_m$ is a degree m form defining an m-canonical adjoint to V. There is the inclusion $\overline{W'}_m \subseteq \overline{\mathcal{W}'}_m$

With these notations, the above Theorem 1 states

Theorem 1′. *We have* $\overline{W'}_m = \overline{\mathcal{W}'}_m$ *if and only if* $m \leq 10$.

Proof. (Sketch only.) The long and tedious proof of theorem 1 (and 1′) consists in proving that if $m \leq 10$ and Φ_m is a non-global m-canonical adjoint to V, then a global m-canonical adjoint Ψ_m to V exists such that

$$\Phi_{m|_V} = \Psi_{m|_V}.$$

Omitting this part, let us consider $m = 11$. The form defining V can be written in the following way:

$f_6(X_0, X_1, X_2, X_3, X_4) = \xi_0(X_0, X_1, X_2, X_3, X_4)X_0 + f'_6(X_0, X_1, X_2, X_3, X_4)$, where $\xi_0 = a_{32100}X_0^2X_1^2X_2 + a_{32010}X_0^2X_1^2X_3 + a_{32001}X_0^2X_1^2X_4 +$ $a_{21300}X_0X_1X_2^3 + a_{10203}X_2^2X_4^3 + a_{21210}X_0X_1X_2^2X_3 + a_{21201}X_0X_1X_2^2X_4 +$ $a_{21120}X_0X_1X_2X_3^2 + a_{21111}X_0X_1X_2X_3X_4 + a_{21102}X_0X_1X_2X_3X_5^2 +$ $a_{21021}X_0X_1X_3^2X_4 + a_{10212}X_2^2X_3X_4^2$ is a form such that $F_{11} = \xi_0X_0^3X_1^2X_2$ defines a non-global 11-canonical adjoint Φ_{11} and a global 11-canonical adjoint Ψ_{11} does not exist with $\Phi_{11|_V} = \Psi_{11|_V}$.

Now, we consider $m = 12$. As in the case of $m = 11$, the form $F_{12} = \xi_0X_0^3X_1^2X_2X_3$ defines a non-global 12-canonical adjoint Φ_{12} and a global 12-canonical adjoint Ψ_{12} such that $\Phi_{12|_V} = \Psi_{12|_V}$ does not exist.

Next, in the case $m \geq 12$, we argue as follows: the form $F_2 = X_0X_1$ defines a global bicanonical adjoint to V, then the forms $F_{11}F_2^q$ and $F_{12}F_2^q$ define the desired non-global m-canonical adjoints to V.

This proves the theorem.

4. Computing the m-genus P_m of X for $m \leq 10$.

Lemma. *If F_m is a form of degree $m \geq 1$ defining a global m-canonical adjoint to V, then any monomial $\mathcal{M}$ in F_m is given by*

$$\mathcal{M} = cX_0^sX_1^sX_2^uX_3^vX_4^w, c \in k \text{ and } s > 0.$$

We note that X_0 and X_1 have the same exponent s.

Corollary 1. *Let $A \neq 0$ be a form of degree $m - 6 \geq 0$. If F_m is a form, of degree m, defining a global $m-$canonical adjoint to V, then the form:*

$$F_m + Af_6$$

cannot define a global $m-$canonical adjoint (where the form f_6 defines V).

Next corollary follows from Corollary 1.

Corollary 2. *If F_m and F'_m are two forms, of degree $m \geq 6$, defining two global m-canonical adjoints Φ_m and Φ'_m, respectively, then*

$$F'_m = F_m + Af_6 \Longrightarrow A = 0 \ \ and \ \ F'_m = F_m,$$
$$equivalently$$
$$\Phi_{m_{|V}} = \Phi'_{m_{|V}} \Longrightarrow \Phi_m = \Phi'_m.$$

With the notations at the beginning of section 3, from Corollary 2, we have $W'_m = \overline{W'}_m$, for $m \geq 6$, and clearly $W'_m = \overline{W'}_m$ for if $m < 6$. It follows from Theorem $1'$, section 3, that $\overline{W'}_m = \overline{W}'_m$, for $m \leq 10$. Next, from [8], Lemma 4 and Corollary 8, section 3, $P_m = \dim_k(\overline{W}'_m)$. Therefore, we obtain

$$P_m = \dim_k(W'_m), \ \text{ for } m \leq 10,$$

and now it is possible to calculate the vector space W'_m from the above Lemma. Let us write $W'_m = \{F_m\}$, varying F_m in the set of forms, of degree m, defining global m-canonical adjoints; computing $\{F_m\}$, we obtain:

$P_1 = p_g = 0$;
$P_2 = 1$, because $W'_2 = \{aX_0X_1\}$, $a \in k$;
$P_3 = 2$, because $W'_3 = \{X_0X_1(a_1X_2 + a_2X_3)\}$, $a_i \in k$;
$P_4 = 3$, because $W'_4 = \{b_1X_0^2X_1^2 + X_0X_1(b_2X_2X_3 + b_3X_2X_4)\}$, $b_i \in k$;
$P_5 = 3$, because $W'_5 = \{X_0^2X_1^2(c_1X_2 + c_2X_3) + c_3X_0X_1X_2X_3X_4\}$, $c_i \in k$;
$P_6 = 5$, because $W'_6 = \{d_1X_0^3X_1^3 + X_0^2X_1^2(d_2X_2^2 + d_3X_2X_3 + d_4X_3^2 + d_5X_2X_4)\}$, $d_i \in k$;
$P_7 = 6$, because $W'_7 = \{X_0^3X_1^3(e_1X_2 + e_2X_3) + X_0^2X_1^2X_2(e_3X_2X_3 + e_4X_2X_4 + e_5X_3^2 + e_6X_3X_4)\}$, $e_i \in k$;
$P_8 = 8$; $P_9 = 10$ and $P_{10} = 13$.

5. The m-canonical transformation $\varphi_{|mK|}$.

Let us consider the following commutative triangle

$$\begin{array}{ccc} X & \overset{\varphi_{|mK|}}{\dashrightarrow} & \mathbb{P}^{P_m-1} \\ & {\scriptstyle\sigma_{|X}}\searrow & \uparrow {\scriptstyle\varphi_{L_m}} \\ & & V \end{array}$$

where $\sigma_{|_X} : X \longrightarrow V$, with $\sigma = \pi_r \circ \ldots \circ \pi_1$, is our desingularization of V and where $L_m = \mathbb{P}(\overline{\mathcal{W}'}_m) = (\overline{\mathcal{W}'}_m)^*/k^*$ = (incomplete) linear system of m-canonical adjoints to V restricted to V [with $(\ldots)^*$ indicating $(\ldots) \setminus \{0\}$], and φ_{L_m} is the rational transformation defined by L_m.

Proposition 1. *The rational transformation φ_{L_m} (equivalently $\varphi_{|mK|}$) is a rational transformation $n : 1$, with $n \leq 2$, if and only if $m \geq 6$.*

Corollary 3. *The rational transformation φ_{L_m} (equivalently $\varphi_{|mK|}$) is a rational transformation* $2 : 1$ *for* $6 \leq m \leq 10$.

Proof. Let F_m be a form, of degree m, defining a global m-canonical adjoint to V. It follows from the Lemma in section 4 that any monomial $\mathcal{M}$ in F_m is given by $cX_0^sX_1^sX_2^uX_3^vX_4^w$. Then, in an affine open set $U \subset \mathbb{P}^4$, the intersection of the hyperbola $X_0X_1 = const$ with $V \cap U$ gives exactly two points; such points go to one and the same point in the image of φ_{L_m}. So, the thesis follows from Proposition 1 and from the equality $\overline{W'}_m = \overline{\mathcal{W}'}_m$ if $m \leq 10$ (Theorem 1′, section 3).

Proposition 2. *The m-canonical transformation $\varphi_{|mK|}$ is birational if and only if $m \geq 11$.*

Proof. The "only if" part follows from Corollary 3 and from $P_m \leq 3$ for $m < 6$ (section 4).

It remains for us to prove that $\varphi_{|mK|}$ is birational if $m \geq 11$. First we prove that $\varphi_{|11K|}$ is birational. To do so, we consider the six forms of the vector space $\mathcal{W}'_{11}$ given by the five generators of W'_6 multiplied by $X_0^2X_1^2X_2$, plus the form $\xi_0X_0^3X_1^2X_2$. We must remember that $X_0^2X_1^2X_2$ defines a global 5-canonical adjoint, and that the product with elements of W'_6 gives elements in $W'_{11} \subset \mathcal{W}'_{11}$. The form $F_{11} = \xi_0X_0^3X_1^2X_2$ defines the non-global 11-canonical adjoint that we considered in the proof of Theorem 1, section 3. If we prove that the six forms considered in $\mathcal{W}'_{11}$ define a birational transformation on V, then $\varphi_{|11K|}$ is also birational. The six forms define a rational transformation that we denote as $\psi : \mathbb{P}^4 - - \to \mathbb{P}^5$ given by

$$\begin{cases} Y_0 = X_0^5 X_1^5 X_2 \\ Y_1 = X_0^4 X_1^4 X_2^3 \\ Y_2 = X_0^4 X_1^4 X_2^2 X_3 \\ Y_3 = X_0^4 X_1^4 X_2 X_3^2 \\ Y_4 = X_0^4 X_1^4 X_2^2 X_4 \\ Y_5 = \xi_0 X_0^3 X_1^2 X_2 \end{cases}.$$

In the affine coordinates

$$X = \frac{Y_0}{Y_1},\ Y = \frac{Y_2}{Y_1},\ Z = \frac{Y_3}{Y_1},\ T = \frac{Y_4}{Y_1},\ W = \frac{Y_5}{Y_1},$$

$$x = \frac{X_0}{X_2},\ y = \frac{X_1}{X_2},\ z = \frac{X_3}{X_2},\ t = \frac{X_4}{X_2},$$

we obtain that the restriction of ψ to k^4, of affine coordinates (x, y, z, t), is given by

$$\begin{cases} X = xy \\ Y = z \\ Z = z^2 \\ T = t \\ W = \dfrac{\xi_0(x, y, 1, z, t)}{xy^2} \end{cases}$$

We need to prove that $\psi_{|V}$ is birational, but a more important fact is true, i.e. that ψ is birational. Now, let us prove that ψ is birational. This follows from the equality $\xi_0(x, y, 1, z, t) = x^2y^2A + xyB + C$, with A, B, C polynomials in z, t (see the definition of ξ_0 in section 3). In fact, let us consider $P = (x_0, y_0, z_0, t_0)$ and $Q = (x_1, y_1, z_1, t_1)$ such that $\psi(P) = \psi(Q)$, i.e. such that $x_0y_0 = x_1y_1, z_0 = z_1, t_0 = t_1$ and $\frac{\xi_0(x_0,y_0,1,z_0,t_0)}{x_0y_0^2} = \frac{\xi_0(x_1,y_1,1,z_1,t_1)}{x_1y_1^2}$. Considering the monomials in the polynomial $\xi_0(x, y, 1, z, t)$, we deduce that $\xi_0(x_0, y_0, 1, z_0, t_0) = \xi_0(x_1, y_1, 1, z_1, t_1)$. Thus, we obtain that $x_0y_0^2 = x_1y_1^2$ and then $y_0 = y_1$. So $P = Q$, proving that ψ is a rational transformation $1 : 1$, and this is the same as saying that ψ is birational. (It is also not difficult to find ψ^{-1} directly). This proves that $\varphi_{|11K|}$ is birational.

Similarly, if we consider the form $F_{12} = \xi_0 X_0^3 X_1^2 X_2 X_3$ defining a non-global 12-canonical adjoint, it can be demonstrated that $\varphi_{|12K|}$ is birational. Therefore, multiplying by the form $F_2^q = X_0^q X_1^q$ as usual, we find that $\varphi_{|mK|}$ is birational, for $m \geq 11$, proving the proposition.

6. Computing the irregularities of X .

There remains for us to prove that $q_i = \dim_k H^i(X, \mathcal{O}_X) = 0$, for $i = 1, 2$. We know that $q_1 = \dim_k H^1(X, \mathcal{O}_X) = q(S_r) = \dim_k H^1(S_r, \mathcal{O}_{S_r})$, where $S_r \subset X$ is the strict transform of a generic hyperplane section S of V (cf., for instance, [8], section 4). S has finite many isolated (actual or infinitely near) double points and no other singularities. So, we obtain $q_1 = 0$.

To prove that $q_2 = 0$ we use the formula (36), section 4 in [8], which states that:

$$q_2 = p_g(X) + p_g(S_r) - \dim_k(W_2),$$

where W_2 is the vector space of the degree 2 forms defining global adjoints Φ_2 to V of degree 2, i.e. defining hyperquadrics Φ_2 such that

$$\pi_r^* \dots \pi_2^*[\pi_1^*(\Phi_2)] - E_2 - E_4 - E_6 - E_8 - E_{10} \geq 0,$$

that is hyperquadrics passing through the points A_0, A_1, A_2, A_3 and A_4. Thus, we have: $\dim_k(W_2) = 15 - 5 = 10$.

It follows from $p_g(S_r) = 10$ and from $p_g(X) = 0$, section 4, that $q_2 = 0$.

REFERENCES

[1] X. Benveniste, *Sur les applications pluricanoniques des variétés de type très général en dimension 3,* Amer. J. Math., 108 (1986), pp. 433–449.

[2] M. Chen, *On pluricanonical maps for threefolds of general type,* J. Math. Soc. Japan, 50 (1998), pp. 615–621.

[3] S. Chiaruttini - R. Gattazzo, *Examples of birationality of pluricanonical maps,* Rend. Sem. Mat. Univ. Padova, 107 (2002), to appear.

[4] Y. Kawamata, *A generalization of Kodaira-Ramanujam's vanishing theorem,* Math. Ann., 261 (1982), pp. 43–46.

[5] J. Kollár, *Higher direct images of dualizing sheaves I,* Ann. of Math., 123 (1986), pp. 11–42.

[6] S. Lee, *Remarks on the pluricanonical and the adjoint linear series on projective threefolds,* Comm. in Algebra, 27 (1999), pp. 4459–4476.

[7] K. Matsuki, *On pluricanonical maps for 3-folds of general type,* J. Math. Soc. Japan, 38 (1986), pp. 339–359.

[8] E. Stagnaro, *Adjoints and pluricanonical adjoints to an algebraic hypersurface,* Annali di Mat. Pura ed Appl., 180, (2001), pp. 147–201.

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