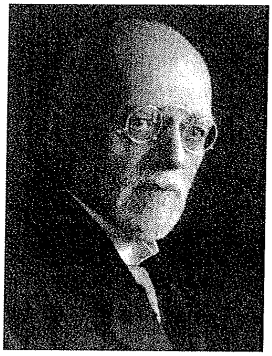

CURRENT PROBLEMS IN MATHEMATICAL PHYSICS: THE ORIGINAL CONTRIBUTION OF DAVID HILBERT

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IL PENSIERO DI DAVIDE HILBERT

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CURRENT PROBLEM IN MATHEMATICAL PHYSICS: THE ORIGINAL CONTRIBUTION OF DAVID HILBERT

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The significance of Hilbert's seminal work in the kinetic theory of gases for modern research in mathematical physics is investigated. In particular it is shown that the heuristic idea of the Hilbert expansion has been very fruitful in several application areas of great importance for plasma physics, astrophysics and solid state physics. Furthermore questions regarding the well posedeness and convergence of the Hilbert expansion have spurred investigations which lately have achieved significant results utilizing the most advanced functional analytical techniques.

1. Introduction.

The attitude of Hilbert towards mathematical physics is succintly expressed by the following sentence which is attributed to him:

Physics is obviously far too difficult to be left to the physicists!

meaning that the developments of modern physics (of which he was extremely aware, to the point of employing an assistant in order to keep him abreast of the most recent results in the fields of quantum mechanics, radiation theory, relativity, etc.) warrants more sophisticated and deeper mathematical concepts in order to achieve consistent theories. Also for several years he taught physics courses ranging from mechanics to electromagnetism, quantum theory, etc. There is no doubt that he tried to apply, at least as a matter of principle, an axiomatization paradigm also to physical theories: a satisfactory physical theory should be expressed in terms of (usually few and physically motivated) axioms where from all consequences could be derived by mathematical reasoning. This attitude he tried to apply to the several areas of physics he was concerned with.

Usually the impact of Hilbert's work on mathematical physics is restricted to the problem of the priority of the discovery of the field equations of general relativity. Many have claimed that in 1915 Hilbert discovered the correct field equations for general relativity before but never claimed priority. Recent articles however, cast some doubts on this view (for a review see T. Sauer,1999, [1]).

Regardless of the priority issue it must be said that: Einstein's derivation is based on clear physical principles (the principle of general covariance and the equivalence principle), whereas Hilbert's derivation is based on a formal mathematical argument. However in modern theoretical physics, in theories which generalize general relativity (gauge theories, string theories, etc.) one adopts Hilbert's methods because, once on the basis of symmetry arguments a lagrangian density has been selected, Hilbert's approach yields directly the field equations through a variational principle. In this seminar I shall not dwell on the well studied question of the field equations of General Relativity but I shall concentrate on another important contribution of Hilbert to mathematical physics, the Hilbert expansion, which is still at the core of recent active research in kinetic theory.

The plan of these lecture notes is the following. In sec. 2 I shall comment on Hilbert's view of mathematical physics.

In sec. 3 I shall introduce the modern framework of multiscale science in which Hilbert's expansion nowadays set and understood. In sec. 4 I shall introduce the Hilbert expansion in general terms. In sec. 5 I shall briefly describe the Hilbert expansion and its modern developments in the area of rarefied gas dynamics. In sec. 6 I shall outline a sketch of the Hilbert expansion in the modern area of plasma physics (both astrophysical and solid state plasmas). In sec. 7 I shall comment on very recent results concerning the Hilbert expansion for radiation transfer of interest to astrophysics.

2. Hilbert's view of mathematical physics.

Another glimpse of Hilbert's attitude towards mathematical physics can be obtained from the following excerpt of his lecture: *Mathematical Problems*, lecture delivered before the International Congress of Mathematicians at Paris in 1900 by Professor David Hilbert:

"While insisting on rigor in the proof as a requirement for a perfect solution of a problem, I should like, on the other hand, to oppose the opinion that only the concepts of analysis, or even those of arithmetic alone, are susceptible of a fully rigorous treatment. This opinion, occasionally advocated by eminent men, I consider entirely erroneous. Such a one-sided interpretation of the requirement of rigor would soon lead to the ignoring of all concepts arising from geometry, mechanics and physics, to a stoppage of the flow of new material from the outside world, and finally, indeed, as a last consequence, to the rejection of the ideas of the continuum and of the irrational number. But what an important nerve, vital to mathematical science, would be cut by the extirpation of geometry and mathematical physics! On the contrary I think that wherever, from the side of the theory of knowledge or in geometry, or from the theories of natural or physical science, mathematical ideas come up, the problem arises for mathematical science to investigate the principles underlying these ideas and so to establish them upon a simple and complete system of axioms, that the exactness of the new ideas and their applicability to deduction shall be in no respect inferior to those of the old arithmetical concepts."

At the beginning of the century a philosophical question of great interest was that of *reductionism*, i.e. the attempt to reduce all physical phenomena to simple mechanical atomistic laws. If this attempt could have been successfull, Hilbert's idea of a satisfactory theory of Physics in the sense of a (at least partially) axiomatized system would have been realized. Therefore it is no wonder that one of his main concerns turned out to be *how to reduce the macroscopic laws of continuum physics to the mechanics of the single atoms and molecules of which they are constituted.* Stated as such this problem was of enormous difficulty (also because of the lack of an accepted mathematical model at the atomic or molecular level in general) and therefore he chose to tackle it in the case in which there was some hope of achieving a solution, the case of rarefied gas dynamics. In the latter case, Boltzmann, at the turn of the century, had introduced his celebrated Boltzmann Transport Equation, describing the evolution of the probability density (distribution function) of molecules. Therefore it was possible to considere two mathematical models of the same reality:

– the continuum description of a gas in terms of the fluid equations

– the molecular description of a gas in terms of the Boltzmann Transport Equation.

The problem which Hilbert tackled was then *how to reduce the fluid equations describing a rarefied gas to the Boltzmann equation for the molecular dis-*

tribution function. The method which he invented for this purpose is now called the *Hilbert expansion*, and is based on the idea of exploiting the various time and length scales implicitly appearing in the Boltzmann Transport Equation. In modern terms we would say that Hilbert resorts to concepts of what nowadays is called *multiscale science*, (whose origin in fact can be traced back to Hilbert's ideas as well as to Poincaré's work on celestial mechanics, as well as to others).

3. Multiscale Science.

The importance of multiscale approach to many problems in the applied sciences cannot be overstressed. In this respect, an idea of the importance of multiscale science in the present scientific environment can be obtained from the following excerpt by two eminent applied mathematicians [2], J.Glimm and D.Sharp. "Multiscale science is the study of phenomena that couple distinct length or time scales. It would be difficult to overstate the scope of multiscale science, which is central to such diverse fields as fluid dynamics, materials science, biology, environmental science, chemistry, geology, meteorology, and high-energy physics.... Because closure introduces new physical assumptions and is not derivable at a formal mathematical level, the range of validity of closure will be less than for the original primitive equations. Determination of the range of validity introduces a new range of scientific questions that are an essential aspect of multiscale science. ... Just as nonlinear and stochastic phenomena have been recognized as crosscutting approaches, we see that multiscale science lies at the heart of many of the most challenging problems of contemporary science."

As said before, one of the earliest contributions to Multiscale Science for fluid mechanics and kinetic theory is that of Hilbert (paralleled by that of Poincaré for celestial mechanics). I shall introduce the original Hilbert expansion for rarefied gas dynamics and shall highlight the relevance of this approach for modern research in kinetic theory. Then, as said already in the Introduction, I shall present extensions of the Hilbert expansion method to very modern and active areas of research which are important for applications: Plasma physics (Astrophysics, Space Science, Geophysics, Semiconductors and other technology applications as free electron lasers, etc.); Radiative transfer (Astrophysics, Space science, High temperature industrial processes as in glass manifacturing, etc.). I will highlight the mathematical problems arising in these areas, stemming from Hilbert's original contribution. It is remarkable that Hilbert's idea of the Hilbert expansion has given rise not only to challenging mathematical problems but also to very useful computational methods for solving practical problems arising in technology, as electron transport in semiconductor devices, or radiation thermal heating in glass manufacturing processes. This shows that to draw a sharp division between Applied and Pure Mathematics is misleading: a theory developed on the basis of philosophical considerations (like the reductionism which inspired Hilbert's expansion) can give rise to deep mathematical problems *as well* as to practical computational methods.

4. The Hilbert Expansion.

Hilbert's idea of the Hilbert expansion is tied up with his previous work on integral equations. Hilbert's work in integral equations in about 1909 led directly to 20th-century research in functional analysis and also established the basis for his work on infinite-dimensional space, later called Hilbert space, a concept that is essential in mathematical analysis and quantum mechanics. Making systematic use of his results on integral equations, Hilbert contributed to the development of mathematical physics by his important memoirs on kinetic gas theory and the theory of radiations.

D. Hilbert, in an article published in 1912 [3], motivated by the reductionistic principle of reducing all the physical reality to the mechanics of atoms (later he modified drastically this principle, adopting instead an electromagnetic reductionistic principle) introduces the basic formalism of the Hilbert expansion. Hilbert stated this program very clearly and it is enlightning to quote directly from: D. Hilbert, *Begrundung der elementaren Strahlungstheorie*, Gött. Nac. 1912, pp. 773–789.

"In my treatise on the Foundations of the kinetic theory of gases I have showed, using the theory of linear integral equations, that starting alone from the Maxwell-Boltzmann fundamental formula – the so called collision formula – it is possible to construct the kinetic theory of gases systematically. This construction is such that it only requires a consistent implementation of the methods of certain mathematical operations prescribed in advance, in order to obtain the proof of the second law of thermodynamics, of Boltzmann's expression for the entropy of a gas, of the equations of motion that take into account both internal friction and heat conduction."

More on the historical and philosophical background can be found in two recent articles by Corry [4],[5].

5. Gas Dynamics.

As stated before Hilbert's expansion is rooted in Hilbert's work on integral equations, published in the book: D. Hilbert, *Grundzuge einer allgemeinen Theorie der linearen Integralgleichungen*, Teubner, Leipzig, 1912. The application to the Boltzmann Transport Equation for rarefied gases is presented in an article in Mathematische Annalen, 72, p. 562, 1912. The presentation is clear, following Hilbert's own suggestion:

I have tried to avoid long numerical computations, thereby following Riemann's postulate that proofs should be given through ideas and not voluminous computations. Report on Number Theory, 1897.

Now we state the fundamental problem. The macroscopic description of ideal gases is given by the Euler equations. These are essentially balance equations and are:
mass conservation equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u^j}{\partial x^j} = 0. \tag{1}$$

momentum conservation equation

$$\frac{\partial \rho u^i}{\partial t} + \frac{\partial (\rho u^i u^j + p\delta^{ij})}{\partial x^j} = 0 \tag{2}$$

energy conservation equation

$$\frac{\partial \rho(e + u^2/2)}{\partial t} + \frac{\partial((\rho u^j (e + u^2/2) + pu^j)}{\partial x^j} = 0 \tag{3}$$

where ρ is the mass density, u^i is the macroscopic velocity, e is the internal energy per unit mass and p is the pressure.

The microscopic description of a rarefied gas is given by the Boltzmann Transport Equation

$$\frac{\partial f}{\partial t} + \underline{v} \cdot \nabla f = Q \tag{4}$$

for the 1-particle probability density $f(t, x, v)$, also called *distribution function.* $Q(f)$ is the collision operator (in general a non linear integral operator), where a quantity appears which is related to the mean free path, i.e. the average distance traversed by a particle between two collisions. In terms of the 1-particle probability density one can identify the macroscopic quantities appearing in the Euler equations in terms of the moments of the distribution function,

$$\rho = \int f\,dv \tag{5}$$

$$\rho u^j = \int f v^j\,dv \tag{6}$$

and likewise for the pressure as the isotropic part of the deviatoric second order moment.

Now let $\alpha > 0$ and choose scaled space-time variables $x' = x/\alpha$ and $t' = t/\alpha$. When $\alpha \to 0$ this corresponds to considering *length and time scales much larger than the mean free path.*

With these new variables, the Boltzmann Transport Equation writes

$$\frac{\partial f}{\partial t} + \underline{v}\cdot \nabla f = \frac{Q}{\alpha} \tag{7}$$

The Hilbert expansion consists in assuming a formal expansion for the distribution function f, in the form

$$f = f_0 + \alpha f_1 + \alpha^2 f_2 + \cdots \tag{8}$$

Proceeding formally, one obtains to the zeroth order

$$Q(f) = 0 \tag{9}$$

The solutions to this equation correspond to those distribution functions representing the equilibrium solutions, local maxwellians:

$$M(t, v, x) = \frac{\rho(t, x, v)}{2\pi T(t, x, v)^{3/2}} \exp -((v-u)^2/(2T(t, x)) \tag{10}$$

where $T(t, x)$ is the local temperature.

The *solvability condition for* f_1, arising from

$$\frac{\partial f_0}{\partial t} + \underline{v}\cdot \nabla f_0 = 2Q(f_0, f_1) \tag{11}$$

(where $Q(f_0, f_1)$ is the linearized collision operator around f_0) *consists just of the Euler equations (1),(2),(3).*

This remarkable result of Hilbert is however at the formal level. Many years elapsed before a rigorous mathematical theory could be constructed. In fact the earliest convergence results are due to Nishida, in 1978; to Caflish, in 1980; Asano and Ukai in 1983. For an exhaustive treatment see the beautiful book by Cercignani, Illner and Pulvirenti [6] and references therein.

6. Plasma Physics (Astrophysics, Semiconductors).

After the introduction of the Hilbert expansion, as a formal mathematical tool for the Boltzmann Transport Equation for rarefied gases, it was natural to extend the technique to similar problems arising in other areas of physics. This is still an active area of research, for all those problems where a kinetic-like transport equation can be formulated.

In the case of a plasma, either for astrophysical or solid state applications, the fundamental description is in terms of the Vlasov- Boltzmann equation for electrons moving under an electrostatic field (self consistent) of potential U. We shall write it in the sequel in a form which is suitable for electron transport in a semiconductor. With slight modifications it can cover also more general situations. The Vlasov-Boltzmann equation is then [7] [8]

$$\frac{\partial f}{\partial t} + \underline{v}(\underline{k})\cdot \nabla f - qE\cdot \nabla_{\underline{k}} f = Q \tag{12}$$

for the one-particle distribution function [4] $f(\underline{x}, t, \underline{k})$, where q is the absolute value of the electron charge, $\underline{k}$ is the crystal momentum, $\underline{v}(\underline{k})$ the electron velocity given by

$$\underline{v}(\underline{k}) = \nabla_{\underline{k}} \mathcal{E} \tag{13}$$

with $\mathcal{E}(\underline{k})$ defined by the band structure of the crystal and Q the collision term

$$Q = \int d\underline{k}'[w(\underline{k}, \underline{k}')(1-f)f' - w(\underline{k}', \underline{k})f(1-f')] \tag{14}$$

with $w(\underline{k}, \underline{k}')$ the scattering rate

$$w(\underline{k}, \underline{k}') = 2\pi |m_{\underline{k}-\underline{k}'}|^2 [n_B(\omega_{\underline{k}-\underline{k}'})\delta(\mathcal{E}_{\underline{k}} - \mathcal{E}_{\underline{k}'} - \omega_{\underline{k}-\underline{k}'}) + \tag{15}$$

$$+ (n_B(\omega_{\underline{k}-\underline{k}'}) + 1)\delta(\mathcal{E}_{\underline{k}} - \mathcal{E}_{\underline{k}'} - \omega_{\underline{k}+\underline{k}'})] + 2\pi |V_{\underline{k}-\underline{k}'}|^2 \delta(\mathcal{E}_{\underline{k}} - \mathcal{E}_{\underline{k}'})$$

Here the first term represents scattering with phonons (acoustic and optical) and the second with impurities.

In eq. (14) f' stands for $f(\underline{x}, t, \underline{k}')$ and the integral is assumed to be over $\mathcal{B}$, the first Brillouin zone. The electric field $\underline{E}$ is related to the potential U by the usual relationship $E_j = -\frac{\partial U}{\partial x^j}$. We shall assume the effective mass approximation (parabolic band)

$$\mathcal{E}(\underline{k}) = \frac{\underline{k}^2}{2m^*} \tag{16}$$

(17) $$\underline{v}(\underline{k}) = \frac{\underline{k}}{m^*}$$

with m^* the effective electron mass and this implies that $\mathcal{B}$ expands to cover all $\mathcal{R}^3$. Furthermore we restrict k-space to a single band (neglecting interband collisions) in order to clarify the basic concepts in the simplest case. We remark that the theory can be extended to cover the non parabolic band case.

Let us define the particle density $n(\underline{x}, t)$

(18) $$n(\underline{x}, t) = \int d\underline{k} f(\underline{x}, t, \underline{k})$$

and the mean velocity $\underline{u}(\underline{x}, t)$

(19) $$\underline{u}(\underline{x}, t) = \frac{\int d\underline{k}\underline{v}(\underline{k}) f(\underline{x}, t, \underline{k})}{n}$$

By integrating (1) in $\underline{k}$-space we obtain, assuming as usual that $f(\underline{x}, t, \underline{k})$ vanishes sufficiently fast at infinity, the particle continuity equation is obtained

(20) $$\frac{\partial n}{\partial t} + \nabla \cdot (n\underline{u}) = 0$$

First of all we scale the equations according to the following units : velocity with the thermal velocity $v0$, $v0 = K_B T/m(K_B$ being the Boltzmann constant) of order of the sound speed; time with τ, the mean collision time (the typical time between two collisions of an electron with either another electron, or an ion, or a phonon, depending on the physical situation). We have that $Q(f)$ is of order $1/\tau$. As unit of length we take $l = v0\tau$ the mean free path, (average distance travelled between collisions). Also, the potential U is scaled according to $U0 = K_B T/q$, the thermal potential.

With these units the Vlasov-Boltzmann Transport Equation writes

(21) $$\frac{\partial f}{\partial t} + \underline{v}(\underline{k}) \cdot \nabla f - E \cdot \nabla_{\underline{k}} f = Q$$

where now all variables are dimensionless. Now let $\alpha > 0$ and choose scaled space-time variables $x' = x/\alpha$ and $t' = t/\alpha^2$. When $\alpha \to 0$ this corresponds to considering length scales much larger than the mean free path and time scales quadratically larger (diffusion limit). With these new variables, the BTE writes

(22) $$\alpha^2 \frac{\partial f}{\partial t} + \alpha(\underline{v}(\underline{k}) \cdot \nabla f - E \cdot \nabla_{\underline{k}} f) = Q$$

The Hilbert expansion consists in assuming a formal expansion for the distribution function f, in the form [8], [9]

$$f = f_0 + \alpha f_1 + \alpha^2 f_2 + \cdots \tag{23}$$

inserting it in the BTE and equating the various powers of α.

To the zeroth order one has $Q(f_0) = 0$. The solutions to this equation correspond to those distribution functions representing the equilibrium solutions. Physically one expects that, asymptotically in time, the solution to a given IVP will tend to such a solution. If we assume that $w(\underline{k}, \underline{k}')$ is measurable and bounded from below and above, together with other minor restrictions then the solution is of the kind [10]

$$f_0(t, x, v) = n(x, t)M(v) \tag{24}$$

where $n(x, t)$ is the density field, $n = \int f_0 dv$ and $M(v)$ is the scaled maxwellian

$$M(v) = \frac{1}{2\pi^{3/2}} \exp(-v^2/2) \tag{25}$$

In order to proceed it is necessary to resort to the following lemma:

Lemma. [10].

A) *A necessary and sufficient condition for the solvability of an equation of the form*

$$Q(f) = g \tag{26}$$

is

$$\int g \, dv = 0 \tag{27}$$

Under such condition the equation $Q(f) = g$ admits a one dimensional linear space of solutions of the form $f = f*(v) + aM(v)$, where $f*(v)$ is a particular solution and a a parameter.

B) *The equation*

$$Q(h) = M(v)v \tag{28}$$

has a vector solution $h(x, v)$ which satisfies

$$\int v_i h_j \, dv = -\mu(x)\delta_{ij} \tag{29}$$

with $\mu(x)$ an appropriate function, called mobility.

By applying this lemma, one obtains, to the order α, the following drift diffusion equations (or Van Roosbroek equations), which in terms of the original variables read:

$$\frac{\partial n}{\partial t} - \frac{\partial J^j}{\partial x^j} = 0 \tag{30}$$

which represents the conservation of electric charge, J being the electric current $J = -q \int f v dv$ and the drift-diffusion constitutive equation

$$J_j = q\mu(x)(U_0 \frac{\partial n}{\partial x^j} - n \frac{\partial U}{\partial x^j}) \,. \tag{31}$$

In the absence of density gradients we have

$$J_j = q\mu(x) n E_j \tag{32}$$

where $E_j = -\frac{\partial U}{\partial x^j}$ is the electric field, which is a pure drift equation. This justifies calling $\mu(x)$ the mobility.

In the absence of electric fields we obtain a form of Fick's law for diffusion

$$J_j = q\mu(x) U_0 \frac{\partial n}{\partial x^j} \tag{33}$$

Which allows to identify the diffusion coefficient $D = \mu(x)U_0$, which corresponds to the Einstein relation between diffusivity and mobility (obtained by Einstein through statistical mechanics).

The drift diffusion system is an approximation to the Boltzmann-Vlasov Transport Equation. Computationally is much more advantageous, compared to the effort of solving the full Boltzmann-Vlasov Transport Equation either by finite difference methods (in 6 space-velocity dimensions) or with Monte Carlo simulations. For these reasons the drift-diffusion equations, obtained by applying Hilbert's expansion to the Boltzmann-Vlasov Transport Equation, form the core of standard computer programs for describing charge carrier transport in microelectronics devices. We witness, after several decades, the enormous practical importance of a mathematical idea, the Hilbert expansion, whose motivation was philosophical and mathematical.

7. The null space of the collision operator in the general case.

Crucial to the Hilbert expansion is the knowledge of the null space of the collision operator.

The collision operator for electron-phonon collision (which is the dominant one at medium and high energy in semiconductors) cannot be put in the simple form above, i.e. with $w(\underline{k}, \underline{k'})$ measurable and bounded because the full scattering rate is the sum of Dirac measures. In order to determine, then the correct mobility and diffusion coefficients which appear in the drift-diffusion equations, one must generalise the previous approach. This is the subject of current research. The problem of determining the null space of $Q(f)$ in this general case was tackled and solved by A. Majorana [11], [12]. He proved that the equilibrium solutions are an infinite sequence of functions of the kind

$$f(k) = \frac{1}{(1 + h(\epsilon(k)) \exp(\epsilon(k)/T))} \tag{34}$$

where $h(\epsilon) = h(\epsilon + \epsilon_{ph})$ is a periodic function of period ϵ_{ph}/n, n integer.

The derivation of the generalized drift-diffusion equations, starting from these results, is a matter of current research. We mention the recent articles by Abdallah, Degond, Markowich and Schmeiser [13] and by Majorana and Liotta [14].

8. Radiation Transfer.

Another area of great interest where the Hilbert expansion plays a very important role, is the theory of the radiation transfer equation, a subject which is crucial for astrophysics and also for advanced technological manifacturing (like the glass industry). The radiative transfer equation is

$$(1/c)\frac{\partial I}{\partial t} + \omega^j \frac{\partial I}{\partial x^j} + \sigma I = (\sigma)/4(\pi)acT^4 \tag{35}$$

where I is the radiation intensity, c is the speed of light, ω the angle unit vector, σ the absorption coefficient, a Boltzmann's constant and T the matter temperature. One must also consider the heat equation in the matter

$$C_V \frac{\partial T}{\partial t} = \Delta T + \sigma(\Phi - acT^4) \tag{36}$$

where C_V is the constant volume specific heat and

$$\Phi = \int I \, d\Omega \tag{37}$$

the radiation intensity integrated over angle. Introduce the diffusion scaling x', t' given by $x' = x/\alpha$ and $t' = t/\alpha^2$. The system becomes, in appropriate units

$$(1/c)\alpha^2 \frac{\partial I}{\partial t} + \alpha\omega^j \frac{\partial I}{\partial x^j} + \sigma I = (\sigma)/4(\pi)acT^4 \tag{38}$$

$$\alpha^2 C_V \frac{\partial T}{\partial t} = \alpha^2 \Delta T + \sigma(\Phi - acT^4) \tag{39}$$

The Hilbert expansion is then:

$$I = I_0 + \alpha I_1 + \alpha^2 I_2 + \cdots \tag{40}$$

$$T = T_0 + \alpha T_1 + \alpha^2 T_2 + \cdots \tag{41}$$

In the limit, one obtains the radiation diffusion equation

$$\frac{\partial(B(T_0) + T_0)}{\partial t} = \nabla\nabla T_0 + D\nabla B(T_0) \tag{42}$$

where $B(T_0) = acT^4$ the Planck function.

This reduced equation is also currently used in astrophysics and in high temperature technology (and it is computationally much more advantageous than solving the full radiative transfer equation). As with the case of gas dynamics, only recently rigorous existence and convergence results have been obtained by Klar and Schmeiser [14].

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