EUCLID'S AND HILBERT'S FOUNDATIONS OF GEOMETRY

Benno Artmann

IL PENSIERO DI DAVIDE HILBERT

A CENTO ANNI DAI "GRUNDLAGEN DER GEOMETRIE"
E DAL CONGRESSO INTERNAZIONALE DI PARIGI

Università degli Studi di Catania – Dipartimento di Matematica – Catania 23-25 settembre 1999

LE MATEMATICHE
Vol. LV (2000) – Supplemento n. 1, pp. 143–160

EUCLID'S AND HILBERT'S FOUNDATIONS OF GEOMETRY

BENNO ARTMANN

A. Introduction.

The purpose of this note is to substantiate and explain the following quote from Hilbert. In fact, we will see that the correspondence between Hilbert's and Euclid's foundations of geometry is even greater than Hilbert himself observed. Hilbert writes about Euclid in the Lecture Notes (p. 44/45) from 1898/99, after he has discussed the axioms:

The sequence of our theorems will differ greatly from what one usually finds in text books on elementary geometry. It will however frequently be the same as in Euclid's Elements. Thus we will be led by our most modern investigations to appreciate the acute insight of this ancient mathematician and to admire him in the highest degree.

(... So führen uns diese ganz modernen Untersuchungen dazu, den Scharfsinn dieses alten Mathematikers recht zu würdigen und aufs höchste zu bewundern.)

As the basis of our discussion I am using the English translation of Euclid's *Elements* by Th. L. Heath.

We have to keep in mind that the name "Euclid" is ambiguous: (a) it stands for the author/editor of the *Elements*, and (b) it is a collective name of the many Greek mathematicians 450 - 300 BC who contributed to the *Elements*.

From Hilbert I am using not the polished text of the official editions of the *Foundations of Geometry*, but rather the much more lively lecture notes (by

H. v. Schaper) of his course *Elemente der Euklidischen Geometrie* during the winter term 1898/99, which precedes the first edition of the *Foundations* on 17 June 1899. All my quotations are from these lecture notes (my translations) unless stated explicitely otherwise.

B. Congruence Geometry.

We compare the axiomatics of Euclid and Hilbert in a schematic way:

Euclid	**Hilbert**
"Common notions" (Axioms) like (1) equality is transitive. (4) Things wich coincide wich each other are equal. (5) The whole is greater than the part.	—
Postulates (1) Existence of segment PQ, but no axioms of 3-dim. geometry. .... (5) Parallel postulate.	Axioms of incidence, including 3-space The parallel axiom.
Ordering used implicitely (segments instead of lines), except "archimedean postulate"	Axioms of ordering
Congruence "Theorem" side-angle-side, SAS (prop. I, 4) (connects to Common Notion (4))	Axioms of congruence, esp. SAS (Hilbert p. 75: *essentially we have Euclid's axioms*)
Area: Book I, 32–48 and more (Implicit definitions and assumptions are listed by Hartshorne.)	**Area**: incorporated into the theory of proportions, Euclid's Comm. Notions (4,5) very important.
Proportion: General theory in Book V for general magnitudes, Similarity based on V	Proportion and similarity geometry based on congruence axioms

Example 1. The congruence axiom SAS.
Euclid "proves" the congruence property side-angle-side for triangles, SAS, in his Prop. I,4 by superposition, an undefined concept, so that in fact he uses SAS as an axiom. Hilbert starts with congruence axioms for segments etc. and states SAS as his most important axiom. Others, like F. Klein, M. Pieri 1899 and F. Schur 1909 have preferred to put the group of congruence mappings into the foreground.

Example 2. The exterior angle of a triangle.

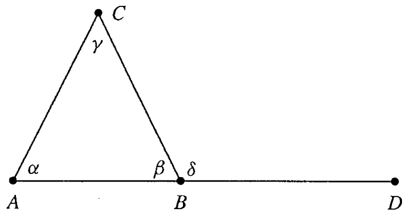

Fig. 1: The exterior angle of a triangle

Euclid states in Prop. I, 16: *In any triangle the exterior angle δ is greater than either of the interior and opposite angles α or γ.*

In the proof Euclid uses implicitely properties of ordering. Hilbert p. 46–51 has the same proposition and uses ordering explicitely. Note that after the introduction of the parallel postulate in I, 29 the result of I, 16 would be a trivial consequence of the theorem about the sum of angles in any triangle. Both Euclid and Hilbert deduce important consequences from I, 16:

(i) The triangle inequality (I, 20)
(ii) The existence of non-intersecting lines (parallels) in I, 27
At this occasion Hilbert praises Euclid once more (p. 76):

The fact that Euclid proves I, 16 before the introduction of the parallel postulate indicates how deeply he understood the mutual dependencies of the theorems of geometry.

C. Similarity geometry.

Hilbert (p. 77) quotes the Englishman Henry Savile, who wrote in 1621: There are

duo macula in pulcherrimo geometriae corpore,

namely

(i) the theory of parallels
(ii) the theory of proportion.

We leave aside the much debated theory of parallels and concern ourselves with proportion and similarity geometry.

When Hilbert starts his treatment of proportion, he writes p. 111/112:

The fundamental importance of the theorem just proved [i.e. Pappus-Pascal] *consists in enabling us to develop the theory of proportion without any new axiom. This shows that in this case – as before – Euclid is fundamentally right. We have however to add: the particular way that Euclid introduces proportion is completely misguided.*

In order to develop an opinion about this seemingly contradictory statement we have to look more closely at Euclid's and Hilbert's theories of proportion. Our result will be: Most likely, Hilbert did not find the time to study Euclid's *Elements* carefully, because otherwise, he would have been even more enthusiastic about how similar Euclid's theories are to his own ones. This concerns two points: (i) the definition of proportion and (ii) the use of archimedean ordering in order to prove the theorem of Pappus-Pascal.

Before going into a detailed discussion of points (i) and (ii), we state the two most important theorems and present Euclid's and Hilbert's theories of proportion in a schematic way

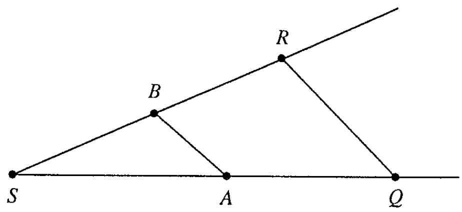

Fig. 2: The theorem of proportional segments.

With the notation of Fig. 2, the theorem of proportional segments (Thalete, Strahlensatz) says

$$AB \parallel QR \Longleftrightarrow SA : SB = SQ : SR.$$

For both Euclid and Hilbert this theorem is the basis of similarity geometry. Thus the foundation of similarity geometry consists in defining proportion and proving the theorem of proportional segments.

Our second theorem is the affine theorem of Pappus, or, as Hilbert says, "Pascal's theorem of lines". It is (after Hilbert and many other authors) well known that the theorem of Pappus is the strongest configuration theorem in the incidence-theory of affine and projective planes. Its validity implies that the plane may be coordinatized by a field. Hence it is of great interest to see how it can be deduced from other axioms (Hilbert) or what its implicit equivalents are (Euclid).

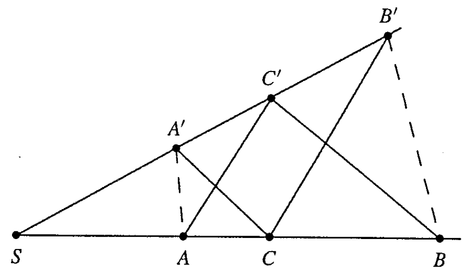

Fig. 3: The theorem of Pappus.

With the notation of Fig. 3, the theorem of Pappus says

$$AC' \parallel CB' \text{ and } A'C \parallel C'B \Longleftrightarrow AA' \parallel BB'$$

(By the way, Pappus himself proves the projective variant of the theorem in a series of propositions in his Collectio, Book VII, 207 in the edition of A. Jones).

Just as a reminder, let us state a third theorem from elementary geometry, which plays an essential role in both Euclid's and Hilbert's theories (see Fig. 4).

Hilbert's Way to Similarity Geometry.

Axioms of incidence, ordering, congruence and the parallel axiom
$\Rightarrow$ (p. 108) Theorem about the invariance of angles in segments of circles
$\Rightarrow$ (p. 110) Theorem of Pappus ("Pascal for lines")

p. 113: **Definition** of the multiplication of segments.
Important: The product ab is a segment.

Pappus $\Rightarrow$ Field axioms for the arithmetic of segments

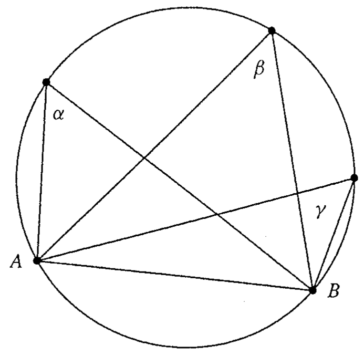

Fig. 4: It is the theorem about the invariance of angles in segments of circules, $\alpha = \beta = \gamma$ etc.

p. 116: **Definition** $a : b = c : d$ means nothing else than $ad = bc$ for segments a, b, c, d, ad, bc.

$\Rightarrow$ (p. 117/118) Theorem of proportional segments (similar triangles) p. 122
Now we could do all of similarity geometry, coordinatization
(p. 123-138) Theory of areas of polygons, measurement of areas and finally
$a : b = c : d \Leftrightarrow$ areas of rectangles $ad = bc$ (product measure)
(p. 163) Archimedean ordering $\Rightarrow$ Pappus

Euclid's Way to Similarity Geometry.

Axioms of incidence, (ordering tacitely assumed), congruence, parallel axiom

Book I, 33-45 Parallelograms and comparison of areas (no measurement!) of parallelograms and triangles
Book I, 46-48 Theorem of Pythagoras ("equal areas")
Book III, 21 Theorem about the invariance of angles in segments of circles
Book V Theory of proportion for general magnitudes (esp. length, area). Def. 4 assumes archimedean ordering of the magnitudes. Ratio remains undefined, but the equality of ratios (i.e. proportion) is defined by a procedure equivalent to Dedekind cuts.
Book V, 16 Archimedean ordering implies alternation (enallax), that is

$$a : b = c : d \qquad \Leftrightarrow \qquad a : c = b : d$$

Book VI, 1 In effect a product measure for areas is constructed. (But never any formulas!)

Book VI, 2 Theorem of proportional segments via areas of triangles
Book VI, 16 $a : b = c : d \Leftrightarrow$ equality of rectangle (a, d) and rectangle (b, c).

The Definition of Proportion.

At a closer look, one finds in Euclid's *Elements*, Book I–IV, several instances where Euclid proves theorems typical for similarity geometry by congruence methods. In order to emphasize the difficulties involved, let us state the problem in modern terms.
We have the congruence group (say of $\mathbb{R}^2$) of mappings of the type

$$x \to Qx + t, \quad \text{where} \quad Q \in O_2(\mathbb{R}) \quad \text{and} \quad t \in \mathbb{R}^2.$$

Especially we have $|\det Q| = 1$. Is it possible to construct with the help of these mappings a similarity mapping

$$x \to \lambda x, \quad \text{where} \quad \lambda \in \mathbb{R} \quad \text{and} \quad |\lambda| \neq 1?$$

The answer is

yes: Hilbert 1898/99, the essential tool being Pappus-Pascal and the arithmetic of segments.
yes: Hjelmslev $\approx$ 1905-1910, theory of reflections and "half turns", the direct way. (A modern presentation of Hjelmslev's work can be found in the book by F. Bachmann 1959. One basic tool is the so-called "Lotensatz", a special case of the theorem about the invariance of angles in segments of circles, which was introduced by Hilbert p. 109 in order to prove Pappus' theorem.)
almost yes: Euclid 300 BC.

Because Euclid's "almost yes" can be found in his first four Books before he introduces proportion in Book V, we will deal with this issue first.

When Euclid (or one of his predecessors not long before him) wrote the Books I-IV, he knew quite well that in Book VI he was going to establish the equivalence for segments a, b, c, d (see Fig. 5)

$$\text{Prop. VI, 16:} \qquad a : b = c : d \Longleftrightarrow \text{rectangle}(a, d) = \text{rectangle}(b, c)$$

In the first Books Euclid uses the area-part of this equivalence at several instances. We present one typical example, an exhaustive discussion can be found in the paper Artmann 1985.

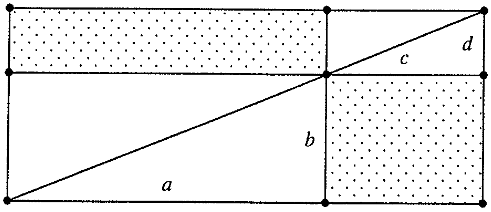

Fig. 5: Euclid's Prop. VI, 16

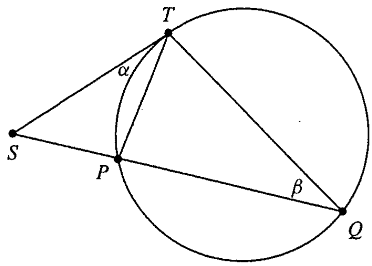

Fig. 6: Hilbert's proof

The theorern about tangents and secants (in modern terms the power of a point with respect to a circle), (see Fig. 6).

With the notation of Fig. 6 and $ST = t$, $SP = p$, $SQ = q$ we have
Euclid III, 36: $pq = t^2$.

Hilbert's proof (p. 122): Use similar triangles. By Euclid's III, 32 we have $\alpha = \beta$, hence ΔSPT is similar to ΔSTQ. This gives us $p : t = t : q$ and finally $pq = t^2$.

Euclid's proof: In spite of the just preceding proposition III, 32, Euclid cannot speak of similar triangles. We present his proof for the case that the center M of the circle is on SQ. (The general case is a little more complicated but uses the same idea, see Euclid).

Let r be the radius of the circle and $SM = a$, hence $SP = p = a - r$ and $SQ = q = a + r$.
We get

$$\begin{aligned} pq &= (a - r)\,(a + r) \\ &= a^2 - r^2 && \text{by Euclid II, 5/6} \\ &= t^2 && \text{by Pythagoras I, 47.} \end{aligned}$$

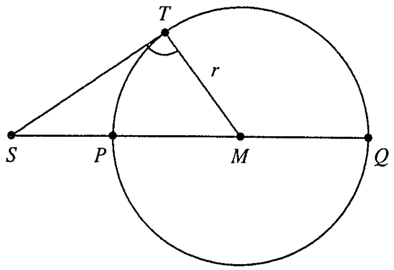

Fig. 7: Euclid's proof

Using similar procedures, always combining II 5/6 with the theorem of Pythagoras, Euclid proves II, 11, II, 14, III, 35 and ultimately manages to construct the regular pentagon in IV, 11 without ever mentioning proportion. In the end we see that Euclid uses the same trick as Hilbert in his definition of proportion. The essential difference lies in Hilbert's interpretation of *ad* and *bc* as segments, so that he avoids area, or at least the comparison of areas, which is indispensable for Euclid. But I think that Euclid has shown that he is as clever as Hilbert in this respect. We will get the same impression in our next section.

Hilbert uses the theorem of Pappus as the cornerstone of his theory of coordinatization and the subsequent definition of proportion plus the proof of the theorem of proportional segments. He deduces the theorem of Pappus from the invariance of angles in segments of circles, that is ultimately from the axioms of congruence. This substantiates his claim that he does not need any new axioms for the theory of proportion, but it still takes considerable time to arrive at the state where he says (p. 122) *Now we could do all of similarity geometry*

Hilbert is critical about Euclid's introduction of proportion because Euclid in his Book V develops his theory for general magnitudes and not just for segments. Euclid, on the other hand, is proud to have found *the unifying bond of the mathematical sciences* (Eratosthenes) in his overarching theory of proportion for general magnitudes. Aristotle (Post. Analytics I, 5; 74a 18-24) explicitely praises the mathematicians of his time because they now have a general concept of magnitude and can prove things like alternation (to be described below) at one stroke, whereas before they had to give separate proofs in the cases of numbers, length, times and solids.

Even if Aristotle, Eratosthenes and certainly all the Greek mathematicians saw the principal merits of Euclid's Book V in its generality, we will for the present purposes restrict our attention to the case of segments (and areas). This is the case that concerns the foundations of geometry.

Euclid's Book V is completely independent of the preceding Books. We quote the important definitions at the beginning of Book V in modernized language. “Magnitude” and “measuring” are undefined concepts in Euclid’s theory.

V Def 3: A **ratio** is a sort of relation in respect of size between two magnitudes of the same kind.

V Def 4: Magnitudes are said to **have a ratio** to one another if they are capable, when multiplied, of exceeding one another.

In other words, magnitudes a, b have a ratio if there exist natural numbers r, s such that

$$b < ra \quad \text{and} \quad a < sb.$$

This, clearly, is the condition of archimedean ordering for the “magnitudes of the same kind”.

V Def 5: Magnitudes a, b, c, d are said to be **in the same ratio**,

$$a : b = c : d,$$

if for all natural numbers r, s

$$sa > rb \Leftrightarrow sc > rd$$

$$sa = rb \Leftrightarrow sc = rd$$

$$sa < rb \Leftrightarrow sc < rd.$$

If, for the moment, we interpret a and b as positive real numbers, then a short calculation derives from the first line

$$\text{for all} \quad rs: \qquad \frac{r}{s} < ab^{-1} \quad \Leftrightarrow \quad \frac{r}{s} < cd^{-1}.$$

We see that the equality of the real numbers/ratios ab^{-1} and cd^{-1} is defined via the equality of sets of rational numbers just as it is done in the theory of Dedekind cuts. We should however keep in mind that Euclid does not define or create any new numbers. In his case of lines (or segments), ab^{-1} will be a constructible algebraic number. Further, in the case of segments Def. 5 defines an equivalence relation for pairs of segments (transitivity is established by Euclid in Prop. V, 11 directly from the definition). Then, again with a language not available to Euclid, the ratio $a : b$ will be the equivalence class of the pair (a, b) of lines determined by Def. 5.

Returning to Hilbert's objections, it seems that he does not like the (hidden) intrusion of (a subset of) the real numbers into geometry. Hilbert is able to avoid Euclid's undefined magnitudes. If, on the other hand, one would restrict the attention to segments constructible by Euclidean means, Euclid's procedure would finally result in the same field of constructible algebraic numbers as the one defined by Hilbert's segment arithmetic. But even if we have quite analoguous results in the case of segments, Euclid has to transgress this theory in order to prove the theorem of proportional segments where he needs the equality of ratios of segments a, b and of rectangles with sides a, b and common height h. (This is shown in Prop. VI, 1 with a proof using nothing but V Def. 5). Because Euclid does not have an explicit theory of area, this gives Hilbert another reason to believe that Euclid is misguided.

D. Archimedean Ordering and the Theorem of Pappus.

In projective or affine planes without any further structure, one has to take the configuration theorem of Pappus as an axiom in order to be able to coordinatize the plane by a field. Hilbert has shown that one can prove the theorem of Pappus from the congruence axioms. At the end of his lecture (p. 139–166, esp. 163–166) he discusses the axioms of ordering in more detail, showing that without the theorem/axiom of Pappus one may coordinatize the plane by means of a skew field, and that axiom of Archimedes implies commutativity, which in turn is equivalent to the theorem of Pappus. Below we will indicate that hidden in Euclid's Book V one can find very similar ideas about archimedean ordering and commutativity. Before sketching the proof we will explain the relation of Euclid's Prop. V, 16 (alternation) and the theorem of Pappus.

Euclid Prop. V, 16 (alternation, enallax) is:

$$a : b = c : d \Leftrightarrow a : c = b : d$$

If we use the equivalence of proportion and products from VI, 16 and look at this carefully in an algebraic way, we see

$$a : b = c : d \quad \Longleftrightarrow \quad ad = bc$$

$$a : c = b : d \quad \Longleftrightarrow \quad ad = cb$$

that is, enallax is equivalent to the commutativity of multiplication, and hence to the theorem of Pappus.

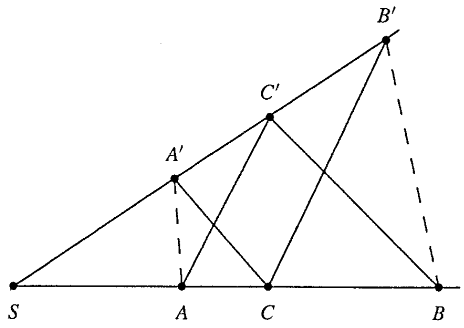

Fig. 8: Alternation implies Pappus

A more direct geometric proof runs as follows. In addition to alternation it uses the theorem of proportional segments VI, 2, which Euclid derives via VI, 1 directly from V, Def. 5 (see Fig. 8).

As the hypothesis of Pappus' theorem we have

$$AC' \parallel CB' \quad \text{and} \quad A'C \parallel C'B.$$

We have to show $AA' \parallel BB'$.

From the hypothesis we get

(1) $SA : SC' = SC : SB'$

(2) $SC : SA' = SB : SC'$.

Alternation gives us

(3) $SA : SC = SC' : SB'$

(4) $SC : SB = SA' : SC'$

Now we may use Euclid's "ex aequali" Prop. V, 22 to derive

(5) $SA : SB = SA' : SB'$

and again by alternation

(6) $SA : SA' = SB : SB'$,

resulting in

$$AA' \parallel BB'$$

as desired. (A similar derivation has been, proposed by Freudenthal 1957, who uses congruent segments in order to prove the reverse implication. Because this is clear enough from the algebraic interpretation, we leave it aside.)

After this the point of main interest to us is Euclid's derivation of alternation from the archimedean postulate. We will present the main outlines of Euclid's arguments in modern notation, for the details the interested reader should consult Euclid's *Elements* or Freudenthal 1957, Beckmann 1967 or Mueller 1981. As above we will restrict our attention to segments and interpret ratios as equivalence classes of pairs of segments. In a series of propositions in Book V Euclid shows how the concept of ratio behaves with respect of certain operations with segments, for instance

$$\text{Prop. V, 12:} \quad a : b = c : d \rightarrow (a + c) : (b + d) = a : b$$

We are concerned with ordering. Euclid tacitely assumes the linear ordering of his magnitudes and defines for ratios:

V Def 7: If, for magnitudes a, b, c, d, there exist natural numbers r, s such that $ra \geq sb$ and $rc < sd$, then we will say

$$a : b > c : d.$$

(A short calculation shows $cd^{-1} < \frac{s}{r} \leq ab^{-1}$, hence the ratios are separated by a rational number. But again we stress that Euclid does not know rational numbers, he always works with multiples as in the definition. Taking into account the linear ordering of the magnitudes, Def. 7 amounts to nothing else than the negation of Def. 5).

The key for the proof of alternation is the proposition V, 8, which carries the ordering of segments over to ratios.

Prop. V, 8 : $a > c \Rightarrow a : b > c : b$.
Proof: By the archimedean postulate in V, Def. 4 there exist $r, s \in \mathbb{N}$ such that

(1) $r(a - c) > b$

(2) $sb > rc \geq (s - 1)b$.

This implies

$$ra = r(a - c) + rc > b + (s - 1)b = sb,$$

hence we have

$$ra > sb \quad \text{and} \quad rc < sb,$$

that is, according to the definition,

$$a : b > c : b.$$

At this point an aside concerning stylistic observations may be of interest. Euclid uses segments as variables for magnitudes, but he does not know variables for numbers. Our relation (2) reads in Euclid's words: "let L be taken double of b, M triple of it, and successive multiples increasing by one, until what is taken is a multiple of b and the first one greater than rc. Let it be taken, and let it be N which is quadruple of b and the first multiple of it that is greater than $rc, \ldots$ and let M be triple of b". Hence in fact he uses 4 and 3 as variables for s and s - 1, but from the text it is clear what he means. Another point is the whole presentation of the proof by Euclid. Heath and other authors have noted Euclid's long windes arguments and superfluous parts of the proof. My personal interpretation of this fact is that we have in our hands an original proof that has been written down and never re-worked so as to give a smoother presentation. Because Eudoxos is said to have been the original author of Book V, this ingenious definition and proof may well bave been preserved as a piece of Eudoxos' personal writing.

With the aid of Prop. V, 8 Euclid proves Prop V, 14: $a : b = c : d$ and $a > c \Rightarrow b > d$. This, together with some other simple propositions, leads to the proof of alternation.

Prop. V, 16: $a : b = c : d \Rightarrow a : c = b : d$. We have to check that for all $r, s \in \mathbb{N}$

$$sa > rc \Rightarrow sb > rd \quad \text{etc.}$$

starting from $a : b = c : d$ Euclid has for all r, s:

$$sa : sb = rc : rd$$

Now V, 14 implies

$$sa > rc \Rightarrow sb > rd \quad \text{etc.}$$

and hence Def. 5 is verified.
(Once more, the reader interested in the details should look up Euclid and the comments by Beckmann 1967 and Mueller 1981)

E. Conclusion.

Our last two sections make obvious that the analogy of contents between the *Elements* and the *Foundations* is even closer than Hilbert himself had observed. With respect to the axioms Hilbert says on p. 75: *essentially our axioms are the same as Euclid's ones*, obviously including the axioms of order. Typically the most important archimedean property is stated by Euclid, whereas he keeps quiet about the more obvious aspects of ordering. Seen that way, Hilbert completes and refines Euclid's axioms.

On the other hand, Hilbert is radically different from Euclid in his conception of the objects of geometry. The very first words of the printed version of the *Foundations* are: "We think certain objects..." instead of "We imagine..., the German is: "Vir denken drei verschiedene Systeme..." instead of the colloquial: "Wir denken uns..." as the lecture notes p. 4 have it. This is quite exacting: he creates in bis mind the very objects he is talking about!

Moreover we find in the *Foundations* the rightly famous algebra of segments, which has had its forerunners, but certainly not Euclid. Even if I have stressed the analogy between enallax and Pappus, we cannot say that Euclid has recognized its fundamental importance.

Finally the *Foundations* have opened up the way to the structural understanding of mathematics, which via Emmy Noether and B. L. van der Waerden has culminated in Bourbaki's work.

REFERENCES

[1] B. Artmann, *Über voreuklidische Elemente, deren Autor Proportionen vermied,* Archive for History of Exact Sciences, 33 (1985), pp. 291–306.

[2] B. Artmann, *Euclid – The Creation of Mathematics,* Springer, New York, 1999.

[3] F. Bachmann, *Aufbau der Geometrie aus dem Spiegelungsbegriff,* Springer, Heidelberg, 1959.

[4] F. Beckmann, *Neue Gesichtspunkte zum 5. Buch Euklids,* Archive for History of Exact Sciences, 4 (1967), pp. 1–44.

[5] Euclid - T.L. Heath, *The Thirteen Books of Euclid's Elements,* Cambridge University Press, Cambridge, 1926.

[6] Euclid - B. Vitrac, *Euclide: Les Éléments,* Presses Universitaires de France, Paris. (Vol. I (1990), Vol. II (1994), Vol. III (1998), Vol. IV to appear).

[7] R. Hartshorne, *Companion to Euclid,* (A course of geometry, based on Euclid's Elements and its modern descendants) AMS, Berkeley Center of Pure and Applied Mathematics, 1997.

[8] D. Hilbert - H.v. Schaper, *Elemente der Euklidischen Geometrie,* Lecture Notes of the course in the winterterm 1898/1899 by H.v. Schaper, 1899.

[9] D. Hilbert - M. Toepell, *Grundlagen der Geometrie,* 14. ed. by M. Toepell, B.G. Teubner, Stuttgart und Leipzig, 1999.

[10] I. Mueller, *Philosophy of Mathematics and Deductive Structure in Euclid's Elements,* MIT Press, Cambridge, Mass., 1981.

[11] I. Mueller, *On the Notion of a Mathematical Starting Point in Plato, Aristoteles, and Euclid,* In: A.C. Bowen, ad.: Science and Philosophy in Classical Greece. New York: Garland, 1991, pp. 59–97.

[12] R. Netz , *The shaping of Deduction in Greek Mathematics,* Cambridge University Press, Cambridge, UK, 1999.

[13] Pappus - A. Jones, *Pappus of Alexandria. Book 7 of the Collection,* Edited with translation and commentary by A. Jones, Springer, New York, 1986.

[14] M. Pieri, *I principi della geometria di positione, compositi in sistema logico deduttivo,* Memorie della Reale Accademia delle Science di Torino, (2) 48 (1899), pp. 1–62.

[15] F. Schur, *Grundlagen der Geometrie,* Teubner, Leipzig, 1909.

[16] B. L. van der Waerden , *Science Awakening,* Groningen, Nordhoft, 1954.

Benno Artmann,
Mathematisches Institut,
Bunsenstr. 3-5
D-37073 Göttingen (Germany)
e-mail: artmann@uni-math.gwdg.de