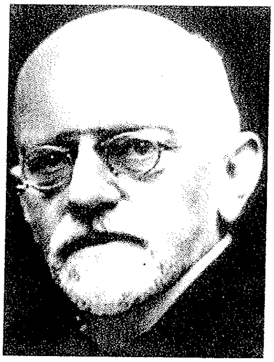

THE ORIGIN AND THE FURTHER DEVELOPMENT OF HILBERT "GRUNDLAGEN DER GEOMETRIE"

Michael Toepell

IL PENSIERO DI DAVIDE HILBERT

A CENTO ANNI DAI "GRUNDLAGEN DER GEOMETRIE"
E DAL CONGRESSO INTERNAZIONALE DI PARIGI

Università degli Studi di Catania – Dipartimento di Matematica – Catania 23-25 settembre 1999

LE MATEMATICHE
Vol. LV (2000) – Supplemento n. 1, pp. 207–226

THE ORIGINS AND THE FURTHER DEVELOPMENT OF HILBERT'S "GRUNDLAGEN DER GEOMETRIE"

MICHAEL TOEPELL

1. A new direction of mathematical thought.

The year 1999 stimulates to look back at the expiring century, especially to look back at the beginning of this century. In mathematics here we think at the decisive influential work, which led at the beginning of the new century – like the beat of a kettledrum – to a new thinking in all mathematical fields: David Hilbert's famous book *Grundlagen der Geometrie*. With it a new mathematical field was created, too. For nearly the whole century it should become the classical texbook for geometry in educating mathematicians and mathematics teachers.

In an admirable pregnant form Hilbert had contributed to a better understanding of the coherence of geometric and algebraic structures. Also the work impressed methodically by its conscious lack of intuition, of perception of space (*Anschauung*) and geometric experiments. So one aim of our conference is to trace the thoughts of Hilbert in this respect. What was it that happened hundred years ago?

As we could see in the preceding talks, towards the end of the 19th century a remarkable change came about in the field of the foundations of geometry. Whereas geometry had hitherto been based on empirical facts, it was now seen as a purely formal deductive system. Hilbert was not the first, but he perfected this method in his guiding book *Grundlagen der Geometrie.*

2. Fourteen editions in 100 years.

The work first appeared in June 1899 in a *Festschrift* commemorating the unveiling of the Gauss-Weber Monument in Göttingen. The 5th edition appeared in 1922 in the year of Hilbert's 60th birthday, the 9th edition in 1962 for the Hilbert centenary, the 12th edition for the bicentenary of Gauss's birth in 1977 and in summer 1999 the 14th edition appeared as a so-called *Jubiläumsausgabe* for the centenary of the book itself.

100 years "Grundlagen der Geometrie"

— Editions of David Hilbert's book –

Erste Auflage. Festchrift zur Feier der Enthüllung des Gauß-Weber-Denkmals in Göttingen. Herausgegeben von dem Fest-Comitee. Erster Teil. Leipzig, Verlag von B.G. Teubner 1899. 92 S.

Zweite, durch Zusätze vermehrte und mit fünf Anhängen versehene Auflage. Mit zahlreichen in den Text gedruckten Figuren. Leipzig, Druck und Verlag von B.G. Teubner 1903. [Erstmals:] Verzeichnis der Begriffsnamen. V + 175 S.

Dritte, durch Zusätze und Literaturhinweise von neuem vermehrte und mit sieben Anhängen versehene Auflage. Mit zahlreichen in den Text gedruckten Figuren. Leipzig und Berlin, Druck und Verlag von B.G. Teubner 1909. VI + 279 S.
[Dritte bis siebente Auflage:] Wissenschaft und Hypothese, Bd. VII.

Vierte,...[Untertitel wie 3. Aufl.] 1913. VI + 258 S.

Fünfte,...[Untertitel wie 3. Aufl.] 1922. [Erstmals, S. II:] Vorwort. [Zusätze:] Zu Anhang II: S. 259. Zu Anhang III: S. 263. Zu Kap. IV 21: S. 265. - VI + 265 S.

Sechste unveränderte Auflage. Anastatischer Nachdruck. Mit zahlreichen in den Text gedruckten Figuren. Leipzig und Berlin, Druck und Verlag von B.G. Teubner 1923. VI + 265 S.

Siebente umgearbeitete und vermehrte Auflage. Mit 100 in den Text gedruckten Figuren. [Jetzt zehn Anhänge.] Leipzig und Berlin, Druck und Verlag von B.G. Teubner 1930. VII + 326 S. [Die Paginierung von S. I bis S. 240 (Ende von Anh. V) wurde seitdem nicht mehr verändert.]

Achte Auflage, mit Revisionen und Ergänzungen von Dr. Paul Bernays. Mit 124 Abbildungen. B.G. Teubner Verlagsgesellschaft Stuttgart 1956. [Mit fünf Anhängen und drei Supplementen von P. Bernays] VII + 251 S.

Neunte Auflage, revidiert und ergänzt von Dr. Paul Bernays. Mit 129 Abbildungen.
B.G. Teubner Verlagsgesellschaft Stuttgart 1962. [Mit fünf Anhängen und acht Supplementen von P. Bernays] VII + 271 S.

Zehnte Auflage,... [wie 9. Aufl.] 1968.

11. Auflage. Mit Supplementen von Dr. Paul Bernays. B.G. Teubner Stuttgart 1972.

12. Auflage [wie 11. Aufl.] 1977

13. Auflage [wie 11. Aufl.] 1987

14. Auflage. Mit Suppl. v. Paul Bernays. Hrsg. u. m. Anhängen versehen v. Michael Toepell. B.G. Teubner Stuttgart. Leipzig 1999. XVI + VIII + 412 S. Teubner-Archiv zur Mathematik - Supplementbd. 6

The centenary suggests to revise and complete the edition. So the *Jubiläumsaugabe* contains contributions, which summarize the pre-history and the further development in the last hundred years, as well as documents and registers, which complete and organize the book. It contains as a highlight an interesting exercise-book on the foundations of projective geometry of Hilbert's time as a school-boy, an essay about the omitted projective geometry, a critical comparison of all the 13 editions, a register of literature and names, numerous photographs, title-pages and facsimiles.

3. Hilbert's papers.

Hilbert's method immediately gave a new direction to mathematical thought in the 20th century. Its impact on contemporaries has been studied and further developed in numerous publications. Let me remind to the works by Arnold Schmidt, Hans Freudenthal and Bartel L. Van Der Waerden cited in the foreword of the newer editions of the *Grundlagen.* However up to the 1980s little was known about the origins of Hilbert's *Grundlagen der Geometrie* or about the development that led him to it. The book itself offers nearly no information on this subject. Its concise style was typical for Hilbert's publications.

According to the biographies, Hilbert appeared to have worked almost exclusively with algebra and questions concerning the theory of numbers in the years prior to 1899. His publications on the theory of invariants suggest as much. With the *Grundlagen der Geometrie,* however, he presented to the public a thoroughly mature work about an entirely different subject.

In 1935 Otto Blumenthal wrote in his biographical sketch of Hilbert concerning the *Festschrift*: "It has brought up to its author a world-wide reputation, whereas up to that time he was appreciated only among experts. It is worth tracing the grounds for this success and the development of Hilbert's ideas" [Hilbert: GA 3, 402].

But this was not investigated in the following 50 years. One reason is that Hilbert's papers have been made accessible not before about 30 years after his death. The catalogue of his papers covers over 700 items, among them letters from roughly 500 correspondents and about 50 manuscripts of own lectures. From our viewpoint the principal correspondents are Felix Klein, Hermann Minkowski, Ferdinand Lindemann and Adolf Hurwitz. Lindemann is well known for his proof of the transcendence of π.

It may well be asked: How did Hilbert arrive at his creation? What kind of preparatory work did he find? What works did he study? To what extent had he dealt already with the foundations of geometry? Which problems had particularly stimulated him? A first complete set of axioms before Hilbert was constructed in 1882 by Moritz Pasch. Also he formulated the axioms of order which already Gauss had postulated.

The manuscripts of four Hilbert lectures form the *basis of the development*:

1. Hilbert's first geometric lectures were that about *Projektive Geometrie* (1891; **PG**). They study the properties, which are invariant under projections. Three examples of this manuscript can be found at the poster to this congress.
2. Then a manuscript to an axiomatic formulated non-euclidean geometry *Die Grundlagen der Geometrie* of 1894 (**GG**).
3. The manuscript to an Easter vacation-course in 1898 *Über den Begriff des Unendlichen* seems to be the very essence of the later book (**FK**).
4. From this emerged the detailed manuscript to euclidean geometry, Hilbert wrote in the winter-semester 1898/99: *Grundlagen der Euklidischen Geometrie* (**EG**).

The elaboration *Elemente der Euklidischen Geometrie* (**SG**) prepared by Hilbert's assistant von Schaper, appeared in March 1899 from the preceding lectures and from that Hilbert developed the *Festschrift* essay ([1]**GG**), published in June 1899.

The material makes it possible for us not only to demonstrate the origins of Hilbert's book but also to trace the steps that Hilbert had omitted from his publications. These concern, first of all, the role of intuition (perception), of experience and experiments, as well as questions of projective geometry, which we are missing in the book at all.

Thus the posthumous papers allow what hiherto was not possible: to see the master in his workshop. Besides we see the close connection between Hilbert's biography and the genesis of axiomatic thinking.

4. Foundations of projective geometry: Hilbert's exercise-book (1879).

Let us go back in his biography – as far as possible considering his papers. Of the time while he was still a schoolboy there are left two exercise-book. As if Hilbert would have anticipated it that it is interesting for us, one of them deals with projective geometry and the point of intersection theorems. This exercise-book from 1879 can be seen as the starting-point of his geometric studies. It is edited, covering 19 pages, as a supplement in the 14th edition [pp. 327–345] and is divided into four sections:

1. "Lerhsätze des Menelaos und Ceva mit ihren Umkehrungen und Erweiterungen."
2. "Die gesamte lehre von den harmonischen Punkten und Strahlen."
   It first of all deals with cross ratios, with the arithmetic, geometric und harmonic means, the complete quadrilateral and the polar theory.
3. "Die Lage der Fünf merkwürdigen Punkte im Dreiecke."
4. "Der Pascal'sche und Brianchon'sche Lehrsatz vom Sehnen - und Tangenten - sechseck."

Here is a construction by Hilbert with Euler's line and Feuerbach's nine-point circle. Unlike the later manuscripts this exercise-book is tidy. The diagrams are rather exactly drawn. The sections here about the harmonic points and rays and about Pascal's theorem have been elaborated by Hilbert in his later lectures on projective geometry in summer 1891.

5. Projective Geometry (1891).

In 1886 Hilbert habilitated at the age of 24, in 1889 he lectured on line-geometry, in 1890 on the theory of plane algebraic curves at the *Lehrstuhl* of his doctor-father Lindemann in Königsberg. In the summer 1891 the projective geometry has its turn. Hilbert manuscript comprises over 100 pages. He begins with a survey, dividing geometry into three parts, a division which he consistently followed in later years:

1. *Intuitive geometry.*

In this Hilbert included school geometry, projective geometry and the *Analysis Situs* (topology). In 1920 he delivered a lecture on *Anschauliche Geometrie* which he published together with Stefan Cohn-Vossen in 1932.

2. *Axioms of geometry.*

This part investigates, which axioms are used in the established facts in intuitive geometry and confronts these systematically with geometries in which some of these axioms are dropped. These considerations led Hilbert to his investigations of independence and to his own geometries in the manuscript of 1894.

3. *Analytical geometry,* in which from the outset a number is ascribed to the points in a line and thus reduces geometry to analysis.

The meaning of
1. ist ästhetisch und paedagogisch und praktisch.
2. erkenntnistheoretisch.

3. wissenschaftlich mathematisch.

How did Hilbert characterize "geometry" in 1891? He begins his lecture [Toepell 1986, 21]: "Geometry is the theory about the properties of space.... It is based on the simplest experiment that can be carried out, namely drawing." As a consequence Hilbert excludes computing and numbers.

He mentions the *Geometry der Lage* of von Staudt, which had appeared in 1847, following these pure methods in order to keep projective geometry free from axiomatic and analytic influences. In the first part Hilbert follows Reye's "Geometrie der Lage" and in the second Jakob Steiner's lectures on synthetic geometry.

We learn something about the audience of Hilbert's lectures from a letter of his to Klein. Hilbert wrote on the 30.6.1891: "Our audience consists in the main of two students, to whom may be added as a third man the directory of the royal art school - a painter interested in geometry - who also attends my lectures on projective geometry."

At the end of September 1891, Hilbert heard a lecture at the annual congress of natural scientists in Halle by Hermann Wiener *Über Grundlagen und Aufbau der Geometrie*. There Hilbert becomes acquainted with the general validity of the axiomatic method and in particular with the possibility of developing projective geometry from Pascal's and Desargues' point of intersection theorems taken as axioms.

According to Blumenthal, Hilbert is supposed to have uttered these famous word in a Berlin waiting-room on the return journey from Halle to Königsberg: "One should always be able to say, instead of points, lines and planes, tables, chairs and beer mugs."

If this oral tradition is reliable, already in 1891 Hilbert saw the intuitive part of geometrical concepts as being mathematically irrelevant. But only seven years later he did express that view in correspondingly radical written formulations.

6. Geometry as a system of axioms (1894).

In 1892 Hilbert married Käthe Jerosch. In April 1893 he wrote to Minkowski: "Now I made me acquainted with non-euclidean geometry, because I intend to lecture on this in the next semester." That would have been in 1893. But in a letter to Klein from 23.5.1893 we read: "concerning our audience in this semester it is worse than ever: I give two lectures for respectively one student ... I had no luck in delivering my third course of lectures, but I work it out for myself ..." [Toepell 1986, 44f].

A key-point for Hilbert was the construction of a system of *independent* axioms. After the study of the role of the axiom of parallels it was at least Grassmann, who demands in 1844 not to have any unnecessary axioms. As far as I could see, Peano was the first to speak of the concept of independence of axioms, he used the word "indipendenza" in his work of 1889 [Peano: Op. sc. 2, p. 57]: "Da quest'ordine nelle proposizioni risulta chiaro il valore degli assiomi, e si è moralmente certi della loro indipendenza."

Only in summer semester of 1894, the audience enabled Hilbert to give his lectures on non-euclidean geometry. His concept was: to produce the purest possible exact system of axiomatic, non-euclidean geometry concluding with euclidean geometry. Hilbert gave his lectures the title *Die Grundlagen der Geometrie.*

A large part of the manuscript was written by a second, easily legible hand, that of his wife Käthe, as can be concluded from one of the letters. Blumenthal writes about her: "She looks after her husband and son with understanding and courage. Many Hilbert manuscripts are written in her high strong handwriting."

Hilbert prefaced his manuscript with a *bibliography* of over 40 works, which are written in German. Amongst others he names Pasch, Helmholtz, Lobacevskij, Riemann, Klein, Peano, Killing, Lie, Clebsch, Lindemann, Möbius, von Staudt, Reye, Erdmann and Wiener. He mentions the Italian works as far as they are translated. So we find in this list the work of Giuseppe Peano: *Die Grundzüge des geometrischen Kalküls.* (1891). Original *Calcolo geometrico secondo l'Ausdehnungslehre di H. Grassmann.* (Torino 1888).

But Hilbert does not mention the important axiomatic studies by Peano *I principii di geometria logicamente esposti* (1889) or by Gino Fano *Sui postulati fondamentali della geometria proiettiva* (1892). Peano's *Sui fondamenti della geometria* (1894) had only just been appearing and Veronese's *Grundzüge der Geometrie von mehreren Dimensionen* (1894. Original: Padova 1891) just been translated. Since 1899 Hilbert mentions the Non-Archimedian geometry, Veronese tried to construct, and in the first to the sixth editions Hilbert gave a hint to Veronese's historical appendix.

How is Hilbert proceeding now in 1894? Contrary to 1891 he avoided any explicit definition of geometry like "Geometry is the theory about the properties of space." Otherwise phsical properties, like the falling principles, would belong to geometry, too. So Hilbert writes [Toepell 1986, 58]: "Among the phenomena, or facts of experience that we take into account observing nature, there is a particular group, namely the group of those facts which determine the external form of things. Geometry concerns itself with these facts."

He even regards the axioms as facts: " These unprovable facts have to be determined in advance and we term them axioms." In this Hilbert stood at the same level with Pasch, who likewise derived his axioms from "experience".

Hilbert, however, questions whether the axioms are *complete* or *independent*: "Our collegue's problem is this: which are the necessary and sufficient conditions, independent of each other, which one must posit for a system of things, so that every property of these things corresponds to a geometrical fact and vice versa, so that by means of such a system of things a complete description and ordering of all geometrical facts is possible."

Hilbert took up an idea of Heinrich Hertz, which in geometry leads to the use of space intuition (*Raumanschauung*) only in the sense of a possible intuitive analogy.

Whereas Pasch divided his axioms into eight axioms for lines and four for planes, Hilbert now arranged his axioms according to relations, an order he had already touched on in 1891. He first separates the axioms of connection and order.

Following von Staudt's and Möbius' practice he now ascribes rational numbers to the point of a line with the help of the construction of the fourth harmonic element. I order "to prevent a gap" in the transition to the real numbers, Hilbert states an axiom of *continuity*. From Pasch he adopted the formulation by Weierstrass.

Subsequently the fundamental theorems of projective geometry are derived. Then he specialized the projective system. This leads to the axioms of congruence, the determination of metrics to the hyperbolic and parabolic geometries. The second part is based on the non-axiomatically constructed "Vorlesungen über Geometrie" of Clebsch/Lindemann. A main problem for Hilbert was, to select and to formulate the suitable axioms.

In retrospect it is remarkable that Hilbert states the axiom of continuity straight after the first two groups of axioms. Together with the introduction of numbers this, for him, was a decisive process – in his words "it is of high epistemological significance". He had not anticipated from the outset the early introduction of numbers. In the second part of the lectures he often proceeded analytically - a logically, but for axiomatic system an impure method, which he wished to avoid in future. Remember his division of geometry in the different three parts. So Hilbert noticed in that regard [Toepell 1986, 104]: "If I lecture again, it will be on euclidean geometry." So he did in 1898.

And we understand, too, why Hilbert then added the axioms of continuity at the end and thus showed how dispensable they are. The intersection theorems of Desargues and Pascal enable Hilbert to establish a *segment arithmetic* without an axiom of continuity. In 1894 he had not even mentioned the intersection theorems and consequently had not examined their importance. Thus Wiener's contribution, which had been seen as decisive by Blumenthal, was applied in this way by Hilbert not before 1898!

In addition Hilbert viewed continuity as one of the assumptions of *projective geometry*. As he tried to avoid continuity, the projective studies thereby disappeared from his *Grundlagen der Geometrie*. Other reasons for the transition from projective to euclidean geometry are that the order relation of three elements had proved unsuitable in projective geometry and the principle of duality is not valid in the geometry Hilbert discussed in 1894. In the following decades projective geometry also gradually disappeared in school-geometry.

In August 1894 Hilbert closed his first geometrical publication *Über die gerade Linie als kürzeste Verbindung zweier Punkte*, where he shows that a generalized cayleyan determination of metrics fulfills the triangular inequality [Hilbert 1999, Appendix 1]. Eastern 1895 Hilbert accepted the chair at Göttingen. In the following years - until 1897 - he concerns himself principally with number theory.

7. A vacation-course for teachers – the very essence of the Festschrift (1898).

Hilbert's concern with the foundations of geometry rested for more than three years, until he was inspired to take up anew this field by a letter from Friedrich Schur to Felix Klein dated of the 30.1.1898. Hilbert wrote in March 1898 to Hurwitz [Toepell 1985, 641]: " This letter, which Schönflies introduced to us in a lecture to the mathematical society, has given me the inspiration to take up again my old ideas about the foundations of euclidean geometry. It is remarkable how many new things can be discovered in this field."

Also it just covers 27 pages, the manuscript of the Easter vacation- course of 1898 *Über den Begriff des Unendlichen* ("About the Concept of Infinity") forms the *kernel* of the whole development. As we see from the introduction to this course the contact to teachers and to school-mathematics was Hilbert's personal request. He addressed himself especially to the teachers as "the most competent collaborators". Perhaps it was the teachers who especially stimulated Hilbert to study the foundations of geometry. A reviewer even postulated, Hilbert's *Grundlagen der Geometrie* should be used as a textbook in school-geometry.

In the vacation-course Hilbert introduced his audience to the most up-to-date research questions. For the first time he constructs the axioms in what was subsequently to be their usual sequence. Then he directs the teachers to practicable problems: the geometrical constructions based on the theorems of congruence. The unrestricted constructability - for example the intersection of circles - requires an axiom of continuity, whose independence is subsequently examined. Also he asks for the first time, which axioms were dispensable if one premises Pascal's and Desargues' theorem in place of some axioms that were used to prove these theorems.

In this manuscript the arragement of the later *Festschrift* is already apparent: Axioms, proofs of independences, segment arithmetic, Desargues' theorem, Pascal's theorem and problems concerning constructability. We can also trace how Hilbert develops his ideas in two directions: to avoid assumptions of continuity and to construct plane geometry independent of spacial assumptions.

Once Hilbert's basic concept had been established, a number of individual problems came into focus on which he now worked intensively. That led him to the careful system of his decisive lecture for the *Festchrift*.

8. Lectures and an elaboration on Euclidean Geometry (WS 1898/99).

In the winter semester 1898/99 we read in the announcements of lectures in Göttingen: "Elemente der Euklidischen Geometrie: Prof. Hilbert, Montag und Dienstag 8 - 9 Uhr, privatim." Two hours per week.

Hilbert begins [Toepell 1986, 144]: "Concerning the content of the lectures, we shall study the theorems of elementary geometry, which we all learned at school: the theory of parallels, the theorems of congruence, the equality of polygons, the theorems about the circle etc. in the plane and the space."

The manuscript (EG) contains an exhaustive discussion of those areas that were mostly treated in brief in the vacation-course. The logical meaning of the axioms was studied by construction of arithmetical models. Amongst these were proofs of independence for axioms of the first two groups. In accordance to the theme of the lectures, Hilbert examined in detail the studies of congruence that were possible without using continuity. Much of this was omitted in the *Festschrift*, including a historical survey of the parallel axiom that follows, the detailed presentation of a non-euclidean geometry and the introduction of ideal elements.

Comparing this lecture with that of 1894, it is plausible when Klein remarks to the *Festschrift* that "... compared with earlier studies its main object is to state the importance of the axioms of continuity". The formulation of Freudenthal: " The so-called axioms of continuity are introduced by Hilbert to show that actually they are dispensable." Therefore they are introduced at the end.

In March 1899 Hilbert's assistant von Schaper had elaborated these lectures in the book *Elemente der Euklidischen Geometrie* (SG). It contains numerous remarks, motivations and examples, which were omitted in the concise presentation of the *Festschrift.*

Then Hilbert begins with the fundamental concepts. He doesn't explain like in the lectures "... es giebt ein System von Dingen, die wir Punkte nennen...", but formulates with abstract rigor: "Zum Aufbau der Geometrie denken wir uns drei Systeme von Dingen, die wir Punkte, Geraden und Ebenen nennen..." In the *Festschrift* he omitted even the words "zum Aufbau" and "wir" and writes: "Wir denken drei verschiedene System von Dingen..." Leo Unger translated (in the 2nd English edition 1971): "Consider three distinct sets of objects..."

"With these lion-claws", as Freudenthal describes it, "the navel-string between reality and geometry is cut through". Geometry seems to awake to an own existence, independent of any physical reality. Some months before, in the elaboration, Hilbert had seen in the axioms still "very simple... original facts", whose validity is experimentally provable in nature.

What Hilbert formulated may have been new in Germany, but it "was in the air". Already seven years earlier Gino Fano wrote in his dissertation *Sui postulati fondamentali della geometria proiettiva*: "A base del nostro studio noi mettiamo una verità qualsiasi di enti di qualunque natura; enti che chiameremo, per brevità, punti, indipendentemente però, ben intenso, dalla loro stessa natura" [Fano 1892; Toepell in: Hilbert 1999, 295]. Let me mention that in the following two years (1892-1894) Fano had been in Göttingen and just from 1899 to 1901 he was professor nearby in Messina.

Concerning the further development it is interesting that in the elaboration there appear for the first time studies, which Hilbert only published thirty years later in the 7th edition: the studies of the theorems of Legendre. Like several questions also this question is discussed in detail in the new edition [p. 298]. An eight-page introduction to projective geometry by means of ideal elements was also omitted in the *Festschrift.*

9. The algebraisation of geometry – the first edition in June 1899.

In spring 1899 Hilbert had once more revised his lecture - for the *Festschrift.* Now he concentrated his wide-spread investigation to question of independence and to Desargues' and Pascal's theorem in special chapters.

Hilbert *aim* from the outset seems to be the algebraisation of geometry. In 1894 he still was content with the introduction of coordinates by means of the

Möbius grid. At the end he established that it must also be possible to calculate with the numbers ascribed to geometrical objects. Hence the laws of fields (*Körpergesetze*) are required.

While Pasch speaks of primitive propositions "directly based on observation", from which he derived all the remaining theorems, for Hilbert after his manuscript of 1894 the relations between the objects of intuition provide the starting point. Having perceived both the starting-point and the aim, it remains only to find the way. Hilbert – like Euclid – proceeds axiomatically. Here the question arises, which axioms are required.

While the axioms of incidence are largely clear, the axioms of order are already somewhat problematical. Hence there are difficulties in projective geometry. Proceeding to euclidean geometry, the concept of congruence can be introduced without hesitation. But then appears the problem of the intersection theorems, of the axioms of parallels, of the archimedian axiom, of continuity - all the statements connected are not independent from one another. As our investigation demonstrates, it was not easy for Hilbert to find a suitable way through this maze of axioms.

10. The further development of Hilbert's "Grundlagen der Geometrie".

The development of Hilbert's *Grundlagen der Geometrie* was not finished with the first edition. The complete content was continuously revised - especially during Hilbert's lifetime. Hilbert considered new results, gave hints to articles and improved his own formulations. The most incisive changes went through the seventh edition - the last edition which appeared in Hilbert's lifetime. All the later editions kept its pagination. The comparison of all editions leads to a lot of remarkable differences, which are elaborated in an article (covering 40 pages) in the latest edition: "Zur Entstehung und Weiterentwicklung von David Hilberts *Grundlagen der Geometrie*".

Also by fine and small differences we state Hilbert's effort to have his words carefully chosen. Sometimes for example in the first editions he speaks of "Grundthatsachen", in later editions of "Grundsätzen". While he writes in the first edition that "none of the axioms can be deduced from the remaining ones" (10), he later on weakens this by stating that "no essential part of any one of theses groups of axioms can be deduced from the others". As Arnold Schmidt pronounced, Hilbert preferred in any case the conceptual understanding and intuition compared with the logical economy [Hilbert: GA 2, 407]. An excellent (but not categorical) system of geometrical axioms which are *absolutely* independent, was constructed by Oswald Veblen in his dissertation in 1904.

Some modifications of axioms and theorems led to further enlargements. So for example supplement I (p. 241 f.) goes back to a modification of the axiom of Pasch (II. 4) - suggested by Van Der Waerden. A further example is the theorem of four points by E.H. Moore, which in the first edition still was an axiom. The proof to theorem 9, which for Hilbert was a proof "ohne erhebliche Schwierigkeit", was delivered by Feigl 25 years later.

The axioms of congruence have been according to Hilbert "the most important and most difficult group" [Toepell 1986, 161]. So Hilbert had special interest in the functions of these axioms and less in the axiom of parallels. Poincaré in his review to the *Festschrift* concluded: "Lobachevsky and Riemann rejected the postulate of Euclid, but they preserved the metrical axioms; in the majority of his geometries, Professor Hilbert does the opposite" [Toepell in: Hilbert 1999, 297]. That means, he tried to reject the metrical axioms – as we also saw in the intersections theorems survey. Out of his study of axiom III. 5, the so-called *Umklappungssatz* (reflection theorem), emerged "Appendix II". This axiom plays an important role in the proof of Desargues' theorem in the plane (§ 22f).

It is remarkable that the whole theory of congruences was shortened from about 20 pages in the elaboration to 6 pages in the Festschrift. At the same time generalized the title from "Grundlagen der Euklidischen Geometrie" to "Grundlagen der Geometrie".

The coronation of Hilbert's axioms of *continuity* is the axiom of completeness. The first time Hilbert embedded it in his *Grundlagen* was in May 1900, in the french edition, after he had called attention to it already in his DMV-talk "Über den Zahlbegriff" at 12.10.1899. Out of this talk, which belonged to the *Grundlagen* as appendix VI up to the seventh edition, emerged supplement I.2 by Bernays in later editions. With this axiom of completeness the polymorphic system became categorical. This " axiom about axioms", whose "logical structure is complicated" (as Bernays said), is called by Freudenthal an "unlucky axiom", but Baldus thought it to be "the most original achievement by Hilbert in axiomatics". It was discussed exhaustively and repeatedly changed [Toepell in: Hilbert 1999, 299f]. The axioms of *continuity* conclude the system. Out of postulating them at the beginning emerged Hilbert's appendix IV in 1902.

That the *Grundlagen der Geometrie* contain much more than "the only insight in the nature of axiomatics" (Freudenthal) first of all becomes evident in the chapters three to seven. What he did in the third chapter, the theory of proportions, could have been - in the words of Freudenthal - "a Greek ideal: a pure geometric approach" by constructing a coordinate geometry with respect to a field which does not have to be archimedian - "an original idea with a powerful effect", removing the second stain in Euclid's "Elements". Preparing contributions are due to von Staudt and Schur. A simplification was delivered by Adolf Kneser and further ones were possible by the proofs of Hessenberg (1905) and Hjelmslev (1907).

The theory of plane area in chapter IV turns out to be – in Hilbert's words – "the supposedly most interesting application" of the axioms I to IV and "one of the most remarkable applications of Pascal's theorem in elementary geometry", because he does not need any axiom of continuity. In chapter V and VI Hilbert coordinates the affine plane by the affine form of Desargues' theorem. The role of this theorem came into focus.

Not long after the first edition Moulton (1902), Vahlen (1905) and Veblen/Wedderburn (1907) constructed further and also simpler non-desarguesian geometries. Hilbert took up the Moulton example in his 7th edition [p. 86f]. A systematic treatment of non-desarguesian geometries started in the early 1930s with the contributions by Ruth Moufang, who developed the theory of alternating fields.

THe so-called *new segment arithmetic* came out of the question how far Desargues' theorem is able to substitute the axioms of congruence. This new geometry, in which the commutative law of multiplication does not hold, opened - in the words of the reviewer Max Dehn in 1922 – "the view to a very large, still not investigated area", which in the following decades led to *non-commutative algebra.* The further development in the field *Grundlagen der Geometrie* – since about Hilbert's last editions during his lifetime - is discussed in the latest article "Das Forschungsgebiet 'Grundlagen der Geometrie' seit Hilbert" by Kiechle, Kreuzer and Wefelscheid in [Hilbert 1999, pp. 365–384].

For Friedrich Schur "the most important result" (1901) of the *Festschrift* was chapter VI, in which Hilbert shows that the proof of Pascal's theorem without the use of the axiom of congruence III. 5 is only possible with the aid of Archimedes' axiom.

The last chapter VII, the constructions with ruler and scale are a remarkable station in the history of elementary geometric constructions. The development can be seen in an impressive manner by means of the problem of constructing the missing centre of a circle. Euclid solves the problem by using a compass and a ruler, Abu Al-Wafa (10th c.) and Dürer (1525) reduce their tools to a ruler and a compass with a fixed opening. Mohr (1672) and Mascheroni (1797) use a compass alone, Lambert (1774) and Steiner (1833) use only a ruler and a fixed circle with its centre and Adler (1890) uses parallel- or angle-rulers. The development leads up straight to Hilbert and his student Feldblum, who showed the possibility to solve the problem with a ruler and a so-called "transferer of segments", in later editions with ruler and scale, a transferer of a single fixed segment.

11. A survey to the intersection theorems and the most important results.

The role and function of the intersection theorems are of highest importance for the whole book. In this preceding survey we have listed up Hilbert's decisive results (first steps in his manuscripts are set in brackets).

Besides this the following results of the historical investigations seem to be the *most important*:

1. A widely unknown fact is, that Hilbert studied the foundations of geometry as early as 1891 and perceived carefully the development in this field.

2. A key role is due to Friedrich Schur, who is responsible for the decisive stimulation of Hilbert, which led to an intensive period discussing the foundations of geometry.

3. The significance of intuition (*Anschauung*) for Hilbert was much more important than his publications suggest.

4. The projective geometry plays a remarkable role in the pre-history of the *Grundlagen der Geometrie*. We can trace it back until his time as a schoolboy. For Hilbert the projective geometry always belonged to the foundations of geometry, but there are different reasons why he omitted this field in the *Festschrift*.

12. Reactions and Conclusion.

The *Festschrift* led to the world-wide reputation of Hilbert. The first written congratulations came from Minkowski, Hurwitz and Aurel Voss. One of the first public reactions to the epistemological background can be found in Otto Hölder's Antrittsvorlesung on the 22nd of July in 1899. The philosophical

	Hilbert's stages 1898/99			
	FK	EG	SG	^{1}GG
Desargues (1648): L $\Rightarrow$ Des	FK 19			
Schur (1898): L III $\Rightarrow$ Pas	FK 19		SG 74	
Hilbert: L_2 III IV $\Rightarrow$ Pas	(FK 23)	(EG 92)	SG 108	III § 14
L_2 III IV $\Rightarrow$ Des	(FK 19)		SG 108	V § 22
L_2 $\nRightarrow$ Des		EG 30	SG 28	
L_2 IV V $\nRightarrow$ Des			SG 146	
L_2 III 1-4 IV* V $\nRightarrow$ Des				V § 23
L_2 IV Des $\Rightarrow$ new arith. seg.		EG 102	(SG 147)	V § 24
L_2 Des $\Rightarrow$ L		(EG 34)	(SG 32)	V § 30
L IV* V $\Rightarrow$ Pas			SG 146	VI § 31
L IV* $\nRightarrow$ Pas	FK 26		SG 147	VI § 31
L IV Pas $\Rightarrow$ every inters. th.	(FK 27)	EG 104	SG 167	VI § 31
Hessenberg (1905): L_2 IV* Pas $\Rightarrow$ Des				VI § 35 (ab^{3}GG)
Hjelmslev (1907): L_2 III $\Rightarrow$ Pas		(EG 106)		III § 14 (ab^{3}GG)
L=axioms I and II; L_2=axioms I and II for the plane; III axioms of congruence; IV: ^{14}GG 28; IV*: ^{14}GG 83; V: Archimede's axiom				

discussion about the nature of axioms with Gottlob Frege led to Hilbert's decisive article *Über die Grundlagen der Logik und Arithmetik*, a philosophical *Programmschrift*, which was appendix VII from the 3rd to the 7th edition. Important reviews go back to Sommer, Halsted, Veblen, Ziegler, Hedrick and Poincaré – they are discussed in the new edition (pp. 316–320).

It is not immediately obvious that the concept of the *Grundlangen der Geometrie* emerged from a vacation-course for teachers. The significance of intuition seems to be entirely subordinate. Also in his further publications Hilbert argues as a rule for the axiomatic method. Hence he was frequently seen as a formalist. On the other hand, Hilbert's manuscripts and letters show his intense concern with intuition and its significance for geometry. Regarding his attitude in later years we see how little Hilbert freed himself from intuition. He perceived that the consistency of his axiomatic system depends after all on what it means.

REFERENCES

[1] G. Fano, *Sui postulati fondamentali della geometria proiettiva,* Giornale di Matematiche, 30 (1892), pp. 106–132.

[2] D. Hilbert, *Grundlagen der Geometrie,* ([1]1899) Mit Suppl. v. Paul Bernays. Hrsg. u. m. Anhängen versehen v. Michael Toepell. B.G. Teubner Stuttgart · Leipzig [14]1999. XVI + VIII + 412 S. (Teubner-Archiv zur Mathematik - Supplementbd. 6).

[3] D. Hilbert, *Gesammelte Abhandlungen,* 3 Bde. Berlin Springer 1932, 1933, 1935. Reprint New York 1965.

[4] G. Peano, *Die Grundzüge des geometrischen Kalküls,* Deutsch v. A. Schepp Leipzig 1891. Original: Calcolo geometrico secondo l'Ausdehnungslehre di H. Grassmann. Torino 1888.

[5] G. Peano, *I principii di geometria logicamente esposti,* Torino 1889 - Opere scelte 2 (1958), pp. 56–91.

[6] G. Peano, *Sui fondamenti della geometria,* Rivista di Mathematica, 4 (1894), pp. 51–90 – Opere scelte 3 (1959), pp. 115–157.

[7] M. Toepell, *Zur Schlüsselrolle Friedrich Schurs bei der Entstehung von David Hilberts 'Grundlagen der Geometrie',* In: Mathemata. Festschrift für Helmuth Gericke. Hrsg. Menso Folkerts und Uta Lindgren. Franz Steiner, Stuttgart, 1985, pp. 637–649.

[8] M. Toepell, *Über die Entstehung von David Hilberts "Grundlagen der Geometrie",* (Dissertation München 1984). Vandenhoeck & Ruprecht Göttingen 1986. (Studien zur Wissenschafts, Sozial und Bildungsgeschichte der Mathematik Bd. 2).

[9] M. Toepell, *100 Jahre "Grundlagen der Geometrie" - David Hilberts entscheidender Beitrag zur Formalisierung der Mathematik,* In: Mitteilungen der Deutschen Mathematiker-Vereinigung (1999) Heft 1., pp. 10–15.

[10] M. Toepell, *Zur Entstehung und Weiterentwicklung von David Hilberts "Grundlagen der Geometrie",* In: David Hilbert: Grundlagen der Geometrie. [14]1999, pp. 283–324.

[11] M. Toepell [Hrsg.], *David Hilbert: Schulheft zur Neueren Geometrie. Wilhelmsgymnasium Königsberg i. Pr. 1879/80 [Cod. Ms. D. Hilbert 758]. Kommentiert und mit einer Einleitung versehen von M.T.,* In: David Hilbert: Grundlagen der Geometrie [14]1999, pp. 327–345.

[12] M. Toepell, *Die projektive Geometrie als Forschungsgrundlage David Hilberts,* In: David Hilbert: Grundlagen der Geometrie [14]1999, pp. 347–361.

[13] G. Veronese, *Grundzüge der Geometrie von mehreren Dimensionen,* Deutsch v. A. Schepp. Leipzig 1894. Original: Padova 1891.

Michael Toepell,
Universität Leipzig,
Karl-Heine-Str. 22 b,
D-04229 Leipzig (Germany)
e-mail: toepell@rz.uni-leipzig.de