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EVOLUTION PROBLEMS IN MATERIALS WITH FADING MEMORY

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Evolution problems in materials with memory are here considered. Thus, linear integro-differential equations with Volterra type kernel are investigated. Specifically, initial boundary value problems are studied; physical properties of the material under investigation are shown to induce the choice of a suitable function space, where solutions are looked for. Then, combination with the application of Fourier transforms, allows to prove existence and uniquenes of the solution. Indeed, the original evolution problem is related to an elliptic one: existence and uniqueness results are proved for the latter and, thus, for the original problem. Two different evolution initial boundary value problems with memory which arise, in turn, in the framework of linear heat conduction and of linear viscoelasticity are compared.

1. Introduction

The model of a rigid heat conductor with memory we refer to is the well known one proposed by Gurtin and Pipkin [11], subsequently, widely investigated by many authors among them Fabrizio, Gentili and Reynolds [8] studied the thermodynamics connected with such a model. Indeed, they assumed the thermodynamic state of the material to be determined when the absolute temperature θ, its history θ^t and the integrated history of the temperature gradient are known.

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Here, the internal energy and the relative temperature are assumed linearly related; that is the internal energy e reads:

$$e(\mathbf{x},t) = \alpha(\mathbf{x})u(\mathbf{x},t) \ , \tag{1}$$

where $\mathbf{x} \in \Omega \subset \mathrm{I\!R}^3$ denotes the position, $t \in \mathrm{I\!R}^+$, the time variable and $u = \theta - \theta_0$ the temperature difference where θ_0 is a fixed reference temperature[1] and $\alpha > 0$ the specific heat. The constitutive equation

$$\mathbf{q}(\mathbf{x},t) = -\int_0^\infty k(\mathbf{x},\tau)\nabla u(\mathbf{x},t-\tau)\, d\tau \tag{2}$$

relates the heat flux $\mathbf{q} \in \mathrm{I\!R}^3$ with the temperature gradient, denoted as $\nabla u = \nabla(\theta - \theta_0)$, while $k(\mathbf{x},\tau)$ represents the heat flux relaxation function [8]. The latter, when the heat conductor is isotropic, is independent on the spatial variable $\mathbf{x}$, thus omitted, letting $k(\mathbf{x},\tau) = k(\tau)$, assumed of the form:

$$k(t) = k_0 + \int_0^t \dot{k}(s)\, ds \ , \tag{3}$$

where $k_0 \equiv k(0)$ denotes the initial value of the heat flux relaxation function, termed *initial heat flux relaxation coefficient* and superscript dot, according to the notation in [8], here throughout, derivation with respect to time. Indeed, crucial in this study is the dependence with respect to the time variable which is through both the present as well as the past time. It is further required that

$$\dot{k} \in L^1(\mathrm{I\!R}^+) \cap L^2(\mathrm{I\!R}^+) \quad \textit{and} \quad k \in L^1(\mathrm{I\!R}^+) \tag{4}$$

which imply $k(\infty) := \lim_{t\to\infty} k(t) = 0$. These assumptions can be physically interpreted recalling that there is no heat flux when, at infinity, the thermal equilibrium is reached.

The constitutive equations (1) and (2) can be re-written, in turn, as follows

$$e(\mathbf{x},t) = \alpha_0\,[\theta(\mathbf{x},t) - \theta_0] \tag{5}$$

and, when the heat flux relaxation function k satisfies both conditions (3) and (4), the heat flux $\mathbf{q}$ can be written in the following two equivalent forms:

$$\mathbf{q}(\mathbf{x},t) = -\int_0^\infty k(\tau)\mathbf{g}(\mathbf{x},t-\tau)\, d\tau \tag{6}$$

or

$$\mathbf{q}(\mathbf{x},t) = \int_0^\infty \dot{k}(\tau)\bar{\mathbf{g}}^t(\mathbf{x},\tau)\, d\tau; \tag{7}$$

[1] According to [6], θ_0 represents the temperature of a surrounding environment which is supposed not to be effected by the presence of the conductor.

where $\mathbf{g}(\mathbf{x},t) := \nabla\theta(\mathbf{x},t)$ denotes the temperature-gradient, and

$$\bar{\mathbf{g}}^t(\mathbf{x},\tau) = \int_{t-\tau}^{t} \mathbf{g}(\mathbf{x},s)\, ds \tag{8}$$

represents the integrated history of the temperature-gradient. Hence, the thermodynamical state of the conductor is assigned when, according to Fabrizio, Gentili and Reynolds[8] and McCarthy [12], the *thermodynamic state function* $\sigma : \Omega \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}^3$ which associates $(\mathbf{x},t) \mapsto \sigma(\mathbf{x},t) \equiv (\theta(\mathbf{x},t), \bar{\mathbf{g}}^t)$ is introduced.

The material is organized as follows. The opening Section 2 is devoted to introduce the evolution problem under investigation. The subsequent Section 3 is concerned about existence and uniqueness results. In the next Section 4 a partial differential boundary value problem which describes the evolution of the stress tensor within a viscoelastic body with memory is briefly considered to point out similarities between the two problems. A closing Appendix collects some background definitions to help the reader.

2. Evolution Problem 1: Rigid Thermodynamics

This Section is devoted to state the evolution problem which describes the temperature evolution within a rigid heat conductor with memory; that is, write an appropriate evolution equation together with prescribed initial and boundary conditions. Specifically, the attention is focussed on the particular case when the initial temperature distribution within the conductor is given and, in addition, the temperature on the boundary is assigned.

The space-time domain wherein the unknown function u, which represents the temperature, is defined is $Q_T = \Omega \times (0,T) \subset \mathbb{R}^3 \times \mathbb{R}$, where $\Omega \subset \mathbb{R}^3$ denotes the heat conductor configuration domain. The evolution problem reads:

$$u_t = -\nabla \cdot q(\mathbf{x},t) + r(\mathbf{x},t) \tag{9}$$

$$\mathbf{q}(\mathbf{x},\mathrm{t}) = +\int_0^\infty \dot{k}(s)\overline{\nabla u}^t(\mathbf{x},s)\, ds \tag{10}$$

where there is no heat supply, namely $r(\mathbf{x},t) \equiv 0$. Let

$$\bar{I}^0(\mathbf{x},\tau) := \int_0^\infty \dot{k}(t+\tau)\overline{\nabla u}^0(\mathbf{x},\tau)\, d\tau \tag{11}$$

which is assigned when the thermal history, up to the initial time $t = 0$ is known, then the right hand side of (10) can be written

$$\int_0^\infty k(s)\overline{\nabla u}^t(\mathbf{x},s)\, ds = \int_0^t k(s)\overline{\nabla u}^t(\mathbf{x},s)\, ds + \bar{\mathbf{I}}^0(\mathbf{x},t). \tag{12}$$

Then, the evolution problem (9)-(10), reads

$$u_t = -\nabla \cdot \left(\int_0^t k(s) \overline{\nabla u}^t (\mathbf{x}, s)\, ds - \bar{\mathbf{I}}^0(\mathbf{x}, t) \right) . \tag{13}$$

where $\bar{\mathbf{I}}^0(\mathbf{x},t)$, defined by (12), characterizes the given initial status. Note that a more general case[2] can be considered within this framework letting a non trivial sorce term, which plays the role of a forcing term, appear on the right hand side of (13). The initial and boundary conditions here imposed read

$$\bar{\mathbf{I}}^{t=0}(\mathbf{x},t) = \bar{\mathbf{I}}^0(\mathbf{x},t), \quad t \in (0,T), \tag{14}$$

$$u(\mathbf{x},0) = u_0(\mathbf{x}) \tag{15}$$

$$u(\mathbf{x},t)|_{\partial\Omega} = 0 \ , \qquad \forall t \geq 0 \tag{16}$$

The weak formulation of the evolution problem, together with the initial boundary conditions (14)-(16), gives:

$$\begin{aligned} &\int_0^\infty \int_\Omega \Big\{ \alpha_0 u_t(\mathbf{x},t) \varphi(\mathbf{x},t) + \Big[k(t) \overline{\nabla u}^t (\mathbf{x},t) + \\ &- \int_0^t \dot{k}(\tau) \overline{\nabla u}^t (\mathbf{x},\tau)\, d\tau \Big] \nabla\varphi(\mathbf{x},t) \Big\} d\mathbf{x}dt = \int_0^\infty \int_\Omega \nabla \bar{\mathbf{I}}^0(\mathbf{x},t) \nabla\varphi(\mathbf{x},t) d\mathbf{x}dt \ . \\ &\bar{\mathbf{I}}^0(\mathbf{x},t) := \int_0^\infty \dot{k}(t+\tau) \overline{\nabla u}^0 (\mathbf{x},\tau)\, d\tau \end{aligned} \tag{17}$$

where a subscript t denotes partial derivative with respect to the time variable and $\varphi : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ is any *test* function, while

$$\overline{\nabla u}^0 (\mathbf{x},t) := \int_{-t}^0 \nabla u(\mathbf{x},s)\ ds \ . \tag{18}$$

First, rewrite (17) in the equivalent form

$$\begin{aligned} &\int_0^\infty \int_\Omega \left[u_t(\mathbf{x},t)\varphi(\mathbf{x},t) - \int_0^t k(t-\tau) \nabla u(\mathbf{x},\tau) \cdot \nabla\varphi(\mathbf{x},t) d\tau \right] d\mathbf{x}dt \\ &= - \int_0^\infty \int_\Omega \mathbf{I}^0(\mathbf{x},t) \nabla\varphi(\mathbf{x},t) d\mathbf{x}dt \ ; \end{aligned} \tag{19}$$

where the test function φ can be arbitrarily chosen in a suitable function space. Notably, the suitable functional space wherein the test functions φ can be chosen arbitrarily is defined via the set of all finite work processes. Indeed, the set

[2]the case when the source term is not zero can be incorporated in this framework provided it is in gradient form, that is $r(\mathbf{x},t) = -\nabla \cdot R(\mathbf{x},t)$.

of admissible states comprises all those states which correspond to physically admissible processes, namely, those ones which are associated to a finite thermal work. The *set of all finite thermal work states* is represented by the functional space

$$\mathscr{H}_G(\mathbb{R}^+;H^1_0(\Omega)) := \Big\{ u \in L^2_{loc}(\mathbb{R}^+;H^1_0(\Omega)) : \\ \int_0^\infty \int_0^\infty \int_\Omega k(|\tau-\tau'|)\nabla u(\mathbf{x},\tau')\cdot\nabla u(\mathbf{x},\tau)d\mathbf{x}d\tau'd\tau \Big\} . \tag{20}$$

On recalling Plancharel's theorem, since

$$\int_0^\infty \int_0^\infty \int_\Omega k(|\tau-\tau'|)\nabla u(\mathbf{x},\tau')\cdot\nabla u(\mathbf{x},\tau)d\mathbf{x}d\tau'd\tau = \\ = \frac{1}{\pi}\int_{-\infty}^{\infty}\int_\Omega k_c(\omega)\nabla u(\mathbf{x},\omega)\cdot\nabla u(\mathbf{x},\omega)d\mathbf{x}d\omega , \tag{21}$$

then, the function space $\mathscr{H}_G(\mathbb{R}^+;H^1_0(\Omega))$ can be characterized as

$$\mathscr{H}_G(\mathbb{R}^+;H^1_0(\Omega)) := \Big\{ \ u \in L^2_{loc}(\mathbb{R};H^1_0(\Omega)) : \\ \frac{1}{\pi}\int_{-\infty}^{\infty}\int_\Omega k_c(\omega)\nabla u(\mathbf{x},\omega)\cdot\nabla u(\mathbf{x},\omega)d\mathbf{x}d\omega < \infty \Big\} \tag{22}$$

Precisely in such functional space the evolution equation (13), together with the initial boundary conditions (14)-(16), admits a unique solution according to the results comprised in the following Section. Furthermore, the function space $\mathscr{H}_G$ is a Hilbert space when equipped with, in turn, the inner product and the norm, as shown in [2], see formulae (3.20) and (3.21) therein. Thus, when the dependence on the spatial **x**-variable is recovered and the function space which comprises all admissible thermal states is defined in terms of the temperature distribution [3], then the inner product given by (3.20) in [2], reads

$$\langle f,\varphi\rangle_{\mathscr{H}_G} := \int_{-\infty}^{+\infty}\int_\Omega k_c(\omega)\, f_+(\mathbf{x},\omega)\cdot\overline{\nabla\varphi_+(\mathbf{x},\omega)}\; d\mathbf{x}d\omega \ . \tag{23}$$

and the corresponding norm, (3.21) in [2], now is

$$\|\varphi\|_{\mathscr{H}_G} := \langle\varphi,\varphi\rangle_{\mathscr{H}_G} = \int_{-\infty}^{+\infty}\int_\Omega k_c(\omega)\,\nabla\varphi_+(\mathbf{x},\omega)\cdot\overline{\nabla\varphi_+(\mathbf{x},\omega)}\; d\mathbf{x}d\omega \ . \tag{24}$$

Notably the evolution problem under investigation admits a unique solution, according to the results comprised in [5].

The proof follows on showing that:

[3] note that the unknown in the evolution problem under investigation is the temperature distribution $u:\Omega\times(0,T)\subset\mathbb{R}^3\times\mathbb{R}\to\mathbb{R}$ and, hence, here the function space which comprises all admissible thermal states is defined in terms of the temperature distribution, while in [2] such space is related to the temperature gradient $g=\nabla u$ and its integrated history.

- the evolution problem is related, via duality under Fourier transform, to an equivalent elliptic problem;
- such an elliptic problem enjoys coercivity property;
- known results imply existence and uniqueness of the solution of the original problem.

Key tool is represented by the arbitrariness of the test function φ in the weak formulation of the evolution problem (13): it can be equivalently written under the form

$$u_t(\mathbf{x},t) = \nabla \cdot \int_0^t k(t-\tau)\nabla u(\mathbf{x},\tau)d\tau - \nabla \cdot \mathbf{I}^0(\mathbf{x},t) \tag{25}$$

together with the initial and boundary conditions which are now expressed as

$$u(\mathbf{x},0) = u_0(\mathbf{x}), \quad u(\mathbf{x},t)|_{\partial\Omega} = 0\ , \tag{26}$$

where the unknown temperature is now denoted by $u : \Omega \times \mathbb{R}^+ \to \mathbb{R}$.

Problem (25), subject to the initial and boundary conditions (26), on application of Fourier Transform, it is related via duality with the following one

$$-i\omega u_+(\mathbf{x},\omega) + \nabla\left[k_+(\omega)\nabla u_+(\mathbf{x},\omega)\right] = \nabla I^0_+(\mathbf{x},\omega) \tag{27}$$

with the initial and boundary conditions (26)

$$u(\mathbf{x},0) = 0, \;\; u_+(\mathbf{x},\omega)|_{\partial\Omega} = 0\ , \tag{28}$$

where

$$\mathbf{I}^0_+(\mathbf{x},\omega) := \int_0^\infty \mathbf{I}^0(\mathbf{x},\tau)\, e^{-i\omega\tau} d\tau \;\;,\;\;\; u_+(\omega) := \int_0^\infty u(\mathbf{x},\tau)\, e^{-i\omega\tau} d\tau\ , \tag{29}$$

denote, in turn, the half-line Fourier Transforms of $\mathbf{I}^0(\mathbf{x},\tau)$ and $u(\mathbf{x},\tau)$.

Hence, as soon as a solution of the problem (27)- (28) is known, a solution of the original problem (13), with the related initial and boundary conditions (14)-(16), is obtained. Accordingly, existence and uniqueness results can be stated as far as the problem (27)- (28) is concerned.

Proposition 2.1. *If $k_c(\omega) > 0 \;\; \forall \omega \in \mathbb{R}$, then the initial boundary value problem (27), subject to the initial and boundary conditions (28), admits a unique solution.*

Proof. The proposition is proved as soon as it is shown that (27) enjoys coercivity property. Thus, consider the bilinear form

$$a\left(u_+(\omega), u_+(\omega)\right) := \int_\Omega \Big(i\omega u_+(\mathbf{x},\omega) \cdot \overline{u_+(\mathbf{x},\omega)} + k_c(\omega)\nabla u_+(\mathbf{x},\omega) \cdot \overline{\nabla u_+(\mathbf{x},\omega)} \Big) d\mathbf{x}. \tag{30}$$

Indeed, fixed any $\mathbf{x}\in\Omega$, the bilinear form $a\left(u_{+}(\omega),u_{+}(\omega)\right)$ is bounded whenever a solution in the function space of finite thermal work states $\mathscr{H}_G(\mathbb{R}^+;H_0^1(\Omega))$, (22), is looked for. Coercivity follows since, $\forall\omega\in\mathbb{R}$, definition (30) implies that, $\forall\mathbf{x}\in\Omega$, $\forall u_+(\mathbf{x},\omega)\in H_0^1(\Omega)$, the following inequality is satisfied

$$|a(u_+(\omega),u_+(\omega))|\geq k(\omega)\,\|u_+(\mathbf{x},\omega)\|^2_{H_0^1} \tag{31}$$

where the adopted thermodynamical assumptions, (3) and (4), imply $k(\omega)$ is a positive constant and

$$\|\varphi\|_{H_0^1}:=\left(\int_{-\infty}^{+\infty}\int_\Omega|\nabla\varphi_+(\mathbf{x},\omega)|^2+|\varphi_+(\mathbf{x},\omega)|^2\,d\mathbf{x}d\omega\right)^{\frac{1}{2}}\quad. \tag{32}$$

□

Hence, for any fixed $\omega\neq 0$, the problem (27)-(28) admits a solution $u_+\in H_0^1(\Omega)$, whenever $\nabla\cdot\mathbf{I}_+^0(\mathbf{x},\omega)\in H^{-1}(\Omega)$.

Let

$$\mathscr{L}(\mathbf{I}^0,\nabla u):=\int_0^\infty\int_\Omega\mathbf{I}^0(\mathbf{x},\tau)\cdot\nabla u(\mathbf{x},\tau)d\mathbf{x}d\tau \tag{33}$$

define a linear functional on $\mathscr{H}_G(\mathbb{R}^+;H_0^1(\Omega))$, then Riesz theorem, see, for instace, [13], implies the existence of an element $\nabla u^{\mathbf{I}^0}\in\mathscr{H}'_G(\mathbb{R}^+;H_0^1(\Omega))$ such that

$$\forall\nabla u(\mathbf{x},\tau)\in\mathscr{H}_G(\mathbb{R}^+;H_0^1(\Omega))$$

$$\mathscr{L}(\mathbf{I}^0,\nabla u)=\int_0^\infty\int_0^\infty\int_\Omega(k(\mathbf{x},|\tau-\tau'|))\nabla u^{\mathbf{I}^0}(\mathbf{x},\tau')\cdot\nabla u(\mathbf{x},\tau)d\mathbf{x}d\tau'd\tau\ , \tag{34}$$

Hence, the functional

$$\mathbf{F}^0(\mathbf{x},\tau):=\int_0^\infty(k(\mathbf{x},|\tau-\tau'|))\nabla u^{\mathbf{I}^0}(\mathbf{x},\tau')d\tau'\ ,\ \tau\in\mathbb{R}, \tag{35}$$

can be defined; it, by definition, is unique when $\mathbf{I}^0$ has been arbitrarily chosen; that is, there is only one function $\mathbf{F}^0(\mathbf{x},\tau)$, defined by (35), such that, fixed $\tau\in\mathbb{R}^+$, $\mathbf{I}^0(\mathbf{x},\tau)=\mathbf{F}^0(\mathbf{x},\tau)$. Hence, on integration over both time and space variables, it follows

$$\mathscr{L}(\mathbf{I}^0,\nabla u)=\int_0^\infty\int_\Omega\mathbf{I}^0(\mathbf{x},\tau)\cdot\nabla u(\mathbf{x},\tau)d\mathbf{x}d\tau=\int_0^\infty\int_\Omega\mathbf{F}^0(\mathbf{x},\tau)\cdot\nabla u(\mathbf{x},\tau)d\mathbf{x}d\tau \tag{36}$$

for all $u(\mathbf{x},\tau)\in\mathscr{H}_G(\mathbb{R}^+;H_0^1(\Omega))$.

3. Evolution Problem 2: Isothermal Viscoelasticity

The evolution problem which describes the evolution of an isothermal viscoelastic material with memory. Indeed, such a model is well known: an overview concerning isothermal viscoelasticity is comprised in [10] and in [7] as well as in references therein. The first is devoted to minimum free energy and its connection to maximum recoverable work, while a partial differential equation is studied in [7]. That is, a linear viscoelastic materials with memory is characterized by the partial differential equations describing the behaviour of bodies constituted by such materials; let $Q_T = \Omega \times (0,T) \subset \mathbb{R}^3 \times \mathbb{R}$:

$$u_{tt} = \nabla \cdot \mathbf{T}(\mathbf{x},t) \tag{37}$$

where $\mathbf{T}(\mathbf{x},t)$ denotes the stress tensor given by:

$$\mathbf{T}(\mathbf{x},t) = \int_0^\infty \mathbb{G}(\tau)\, \dot{\mathbf{E}}(\mathbf{x},t-\tau)\, d\tau \tag{38}$$

or

$$\mathbf{T}(\mathbf{x},t) = \mathbb{G}_0 E(t) + \int_0^\infty \dot{\mathbb{G}}(\tau)\mathbf{E}(\mathbf{x},t-\tau)\, d\tau \tag{39}$$

where, according to the notation in [7], superscript dot denotes derivation with respect to the time variable, and, in turn, the fourth order tensor $\mathbb{G}(t)$, the elastic modulus, $\mathbf{E}(\mathbf{x},t)$, the value of the strain at the time t and $\mathbf{E}^t$ the past history defined by

$$\begin{aligned} \mathbf{E}^t : \Omega \times (0,\infty) & \longrightarrow Sym \\ (\mathbf{x},t) & \longmapsto \mathbf{E}^t(\mathbf{x},s) := \mathbf{E}(\mathbf{x},t-s)\,. \end{aligned} \tag{40}$$

The elastic modulus is assumed such that $\dot{\mathbb{G}} \in L^1(\mathbb{R}^+, Lin(Sym))$, that is, its time derivative satisfies, for all positive t,

$$\mathbb{G}(t) = \mathbb{G}_0 + \int_0^t \dot{\mathbb{G}}(s)ds \quad , \quad \mathbb{G}_0 := \mathbb{G}(0) \tag{41}$$

where the initial value of the elastic modulus $\mathbb{G}_0$ is termed *instantaneous elastic modulus* [10]; hence

$$\mathbb{G}(\infty) := \lim_{t\to\infty} \mathbb{G}(t) \in Lin(Sym) \tag{42}$$

which represents the *equilibrium elastic modulus*. Now, the analogy with the rigid thermodynamic problem can be better pointed out introduction of the symmetric tensor defined via

$$\breve{\mathbb{G}}(t) := \mathbb{G}(t) - \mathbb{G}(\infty) \implies \lim_{t\to\infty} \breve{\mathbb{G}}(t) = 0, \tag{43}$$

thus, $\breve{\mathbb{G}}^T(t) = \breve{\mathbb{G}}(t)$ and $|\breve{\mathbb{G}}(t)| \in L^1$; which shows that such a tensor plays, in the isothermal viscoelastic model the same role played by $k(t)$ in rigid thermodynamics. The state and the strain history of a viscoelastic body is characterized, again in analogy with rigid thermodynamics, according to [10], [9] and [7], by a *viscoelstic state function* $\sigma : \Omega \times \mathbb{R} \to Sym \times Sym$, which associates $(\mathbf{x},t) \mapsto \sigma(t) :\equiv (\mathbf{E}(\mathbf{x},t), \mathbf{E}^t)$. Hence, the viscoelastic state function is known when the strain tensor $\mathbf{E}(\mathbf{x},t)$ and the strain past history, $\mathbf{E}^t$ which belong to a suitable Hilbert space, are assigned. Physically meaningful viscoelastic phenomena, are characterized by a finite stress tensor $\mathbf{T}(\mathbf{x},t)$ for all points in the space-time definition set $\Omega \times \mathbb{R}^+$, that is they belong to the vectorial space

$$\Gamma := \left\{ \mathbf{E}^t : \Omega \times (0,\infty) \to Sym : \left| \int_\Omega \int_0^\infty \dot{\mathbb{G}}(s+\tau)\, \mathbf{E}^t(\mathbf{x},s) ds d\mathbf{x} \right| < \infty, \forall \tau \geq 0 \right\}. \tag{44}$$

According to [10], the property $\dot{\mathbb{G}} \in L^1(\mathbb{R}^)$ is termed *fading memory* and it implies that corresponding to any arbitrary $\varepsilon > 0$ there exists a positive constant $\tilde{a} = a(\varepsilon, \mathbf{E}^t)$ s.t.

$$\left| \int_\Omega \int_0^\infty \dot{\mathbb{G}}(s+\tau)\, \mathbf{E}^t(\mathbf{x},s) ds d\mathbf{x} \right| < \varepsilon \,, \;\; \forall a > \tilde{a}. \tag{45}$$

The space-time domain wherein the unknown function u, which represents the temperature, is defined in $Q_T = \Omega \times (0,T) \subset \mathbb{R}^3 \times \mathbb{R}$, where $\Omega \subset \mathbb{R}^3$ denotes the heat conductor configuration domain. The evolution problem reads:

$$u_{tt} = \nabla \cdot \mathbf{T}(\mathbf{x},t) \tag{46}$$

$$\mathbf{T}(\mathbf{x},t) = \int_0^\infty \mathbb{G}(\tau) \dot{\mathbf{E}}(\mathbf{x},t-\tau)\, d\tau \tag{47}$$

where $\mathbf{T}$ can, equivalently, be written under the form

$$\mathbf{T}(\mathbf{x},t) = \mathbb{G}_0 E(\mathbf{x},t) + \int_0^\infty \dot{\mathbb{G}}(\tau) \mathbf{E}(\mathbf{x},t-\tau)\, d\tau\,. \tag{48}$$

The initial and boundary conditions are

$$u(\mathbf{x},0) = u_0(\mathbf{x}), \quad u_t(\mathbf{x},0) = u_{0t}(\mathbf{x}), \quad u(\mathbf{x},t)|_{\partial\Omega} = 0 \tag{49}$$

in addition, the past history is assigned, that is

$$u^{t=0}(\mathbf{x},s) = u_0(\mathbf{x},s), \;\; (\mathbf{x},s) \in \Omega \times \mathbb{R}^+, \quad \forall t \geq 0 \tag{50}$$

are given. According to [7], the initial boundary value problem (46)-(50), the initial status as well as the source term can be, more conveniently, be assigned on fixing

$$\bar{\mathbf{I}}^0(\mathbf{x},\tau) := \int_0^\infty \dot{\mathbb{G}}(t+\tau) \overline{\nabla u}^0(\mathbf{x},\tau)\, d\tau \tag{51}$$

which is the analog of (11) in the case of thermodynamics, and the initial and boundary conditions are now given by

$$\begin{aligned}\bar{\mathbf{I}}^{t=0}(\mathbf{x},t) &= \bar{\mathbf{I}}^0(\mathbf{x},t), \quad t \in (0,T), &(52)\\ u(\mathbf{x},0) &= u_0(\mathbf{x}) &(53)\\ u_t(\mathbf{x},0) &= u_{0t}(\mathbf{x}) &(54)\\ u(\mathbf{x},t)|_{\partial\Omega} &= 0\ , \qquad \forall t \geq 0 &(55)\end{aligned}$$

The weak formulation of the initial and boundary value problem, together with the introduction of the new dependent variable

$$v(\mathbf{x},t) := u_t(\mathbf{x},t) \tag{56}$$

according to the detailed computations by Deseri, Fabrizio and Golden [7], allows to prove that the isothermal viscoelastic is equivalent to the following one:

$$v_t = -\nabla \cdot \int_0^t \dot{\mathbb{G}}(t-s)\overline{\nabla v}(\mathbf{x},s)\,ds - \nabla\bar{\mathbf{I}}^0(\mathbf{x},t), \tag{57}$$

where the initial and boundary conditions are now given by

$$\begin{aligned}v(\mathbf{x},0) &= v_0(\mathbf{x}) &(58)\\ v(\mathbf{x},t)|_{\partial\Omega} &= 0\ , \quad \forall t \geq 0\ . &(59)\end{aligned}$$

Remarkably, the latter represents an evolution problem which exhibits interesting analogies with the initial boundary value problem (17). Again, the functional space wherein the test functions φ can be choosen arbitrarily is defined via the set of all finite work processes. Indeed, the set of admissible states comprises all those states which correspond to physically admissible processes, namely, those ones which are associated to a finite viscoelastic work. The *set of all finite thermal work states* is represented by the functional space

$$\begin{aligned}\mathscr{H}_G(\mathbb{R}^+;H_0^1(\Omega)) := \Big\{ v \in L^2_{loc}(\mathbb{R}^+;H_0^1(\Omega)) :\\ \int_0^\infty\!\!\int_0^\infty\!\!\int_\Omega \mathbb{G}(|\tau-\tau'|)\nabla v(\mathbf{x},\tau')\cdot\nabla v(\mathbf{x},\tau)d\mathbf{x}d\tau' d\tau \Big\}\ .\end{aligned} \tag{60}$$

When Plancharel's theorem is applied

$$\begin{aligned}\int_0^\infty\!\!\int_0^\infty\!\!\int_\Omega \mathbb{G}(|\tau-\tau'|)\nabla v(\mathbf{x},\tau')\cdot\nabla v(\mathbf{x},\tau)d\mathbf{x}d\tau' d\tau =\\ \frac{1}{\pi}\int_{-\infty}^\infty\!\!\int_\Omega \mathbb{G}_c(\omega)\nabla v(\mathbf{x},\omega)\cdot\nabla v(\mathbf{x},\omega)d\mathbf{x}d\omega\ ,\end{aligned} \tag{61}$$

then, the function space $\mathscr{H}_G(\mathbb{R}^+; H_0^1(\Omega))$ can be characterized as

$$\begin{aligned}\mathscr{H}_G(\mathbb{R}^+; H_0^1(\Omega)) := \Big\{ \; u \in L^2_{loc}(\mathbb{R}; H_0^1(\Omega)) : \\ = \frac{1}{\pi}\int_{-\infty}^{\infty}\int_\Omega \mathbb{G}_c(\omega)\nabla v(\mathbf{x},\omega)\cdot\nabla v(\mathbf{x},\omega)d\mathbf{x}d\omega < \infty \Big\}\end{aligned} \tag{62}$$

Precisely in such functional space the evolution equation (25), together with the initial boundary conditions (26), admits a unique solution according to the results comprised obtained by Desero, Fabrizio and Golden [7]. The function space $\mathscr{H}_G$ is a Hilbert space when equipped with, in turn, the following inner product

$$\langle f, \varphi \rangle_{\mathscr{H}_G} := \int_{-\infty}^{+\infty}\int_\Omega \mathbb{G}_c(\omega) f_+(\mathbf{x},\omega)\cdot \overline{\nabla\varphi_+(\mathbf{x},\omega)}\; d\mathbf{x}d\omega \;\; . \tag{63}$$

and the norm

$$\|\varphi\|_{\mathscr{H}_G} := \langle \varphi, \varphi \rangle_{\mathscr{H}_G} = \int_{-\infty}^{+\infty}\int_\Omega \mathbb{G}_c(\omega)\,\nabla\varphi_+(\mathbf{x},\omega)\cdot \overline{\nabla\varphi_+(\mathbf{x},\omega)}\; d\mathbf{x}d\omega \;\; . \tag{64}$$

The remarkable analogy in the analytic form the two different models of materials with memory, leads to think that other phenomena which are characterized by an evolution which involves an integro-differential with memory, then, existence and uniqueness of the solution can be stated provided the suitable function space is considered. Furthermore, the dependence with respect to the space variable $\mathbf{x} \in \Omega$ in both rigid thermodynamics, as well as isothermal viscoelasticity, of, in turn, the heat flux relaxation function and the elastic modulus has been here assumed only for sake of simplicity. Indeed, in both frameworks, according to [5] and [7], all the results remain true also when such a simplifying assumption is removed.

4. Appendix

In this Section some basic definitions concerning Fourier transforms are briefly recalled; thus, according to the classical notation, (see for instance [1]), the following Fourier transforms are defined. Given a function $f : [0,\infty) \to \mathbb{R}^n$, the formal half range Fourier sine and cosine transforms are defined, in turn, by

$$f_s(\omega) = \int_0^\infty f(\tau)\sin\; \omega\tau\; d\tau \tag{65}$$

and

$$f_c(\omega) = \int_0^\infty f(\tau)\cos\; \omega\tau\; d\tau \;\; . \tag{66}$$

In the case when its trivial prolongation over the whole $\mathbb{R}$, denoted again as f,

$$f(\tau) := \begin{cases} f(\tau) & \tau \geq 0 \\ 0 & \tau < 0\,, \end{cases} \tag{67}$$

is considered, then, the corresponding Fourier transform is given by

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} f(\tau) e^{-i\omega\tau}\, d\tau \ , \tag{68}$$

which is related to Fourier sine and cosine transforms via

$$\tilde{f}(\omega) = f_c(\omega) - i f_s(\omega) \ . \tag{69}$$

In addition, when the even prolongation of f over the whole $\mathbb{R}$, denoted as $f(|\tau|)$, is considered

$$f(|\tau|) := \begin{cases} f(\tau) & \tau \geq 0 \\ f(-\tau) & \tau < 0\,, \end{cases} \tag{70}$$

then, since $f(|\tau|)$, even by definition (70), is real valued, then the corresponding Fourier transform, given by (68), is an even real valued function, i.e. $\Im(\omega) = 0$ implies $\Im(\tilde{f}(\omega)) = 0$ (see for instance [1]), related to Fourier cosine transforms via

$$\tilde{f}(\omega) = 2 f_c(\omega) \ . \tag{71}$$

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