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STOCHASTIC GAMES WITH RISK SENSITIVE PAY OFFS FOR N-PLAYERS

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Dedicated to Professor Sergio Campanato on his 70th birthday

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1. Introduction.

In this paper we extend some of the results of the authors on stochastic differential games ([1]) to the case of risk sensitive payoffs. We consider that the stochastic process describing the system is stopped at the exit of a domain $\mathcal{O}$ of $\mathbb{R}^n$. Like in the non stopped case (finite time horizon), the risk sensitive parameter cannot be arbitrary, see H. Nagai [10], A. Bensoussan - J. Frehse - H. Nagai [5] for the finite horizon case. On the other hand, the fact that we consider risk sensitive payoffs prevents to make use of discount factors in the cost functions. The system of Bellman equations related to the value functions has Dirichlet boundary conditions, and does not contain zero order terms. This complicates obtaining L^∞ bounds, since maximum principle arguments cannot be obtained easily. Other methods, already used by the authors to solve ergodic control problems apply conveniently to the present case.

Our approach to solve the stochastic differential games, taken as usual in the sense of Nash, is to prove regularity results for the solution of the system of Bellman equations. Then a standard verification argument can be used.

2. Setting of the Problem.

Let

(2.1) $$\Omega = C^0([0,\infty);\mathbb{R}^n)\,, \quad \mathcal{A} = \text{Borel } \sigma - \text{algebra on } \Omega.$$

The elements of Ω are denoted by $\omega = \omega(t)$, and we equip Ω with a probability measure P such that

(2.2) $\omega(t)$ *is a standardized n dimensional Wiener process.*

([1]) A. Bensoussan - J. Frehse [2], A. Bensoussan - J. Frehse [3].

We then set

(2.3) $$\mathcal{F}^t = \sigma\{\omega(s)\,, s \le t\,,\ \omega \in \Omega\}.$$

A trajectory starting at x, is simply

(2.4) $$x(t;\omega) = x + \omega(t).$$

We now consider N players, each of them acting through a control $v_\nu(t)$, $\nu = 1, \dots, N$. We assume that

(2.5) $v_\nu(t) = (v_1(t), \dots, v_N(t))$, adapted process with bounded values in $\mathbb{R}^{nN}$,

which we call an admissible control. Let also

(2.6) $g(x)$ be a measurable bounded function with values in $\mathbb{R}^n$.

To a pair $x, v(t)$, where $v(t)$ is a control vector as above, we associate the process

(2.7) $$\beta_{x,v}(t) = g(x(t)) + \sum_\mu v_\nu(t)$$

and the probability $P_{x,v}$ such that

(2.8) $$\frac{dP_{x,v}}{dP}|_{\mathcal{F}^t} = \exp\left\{\int_0^t \beta_{x,v}(s)\,d\omega(s) - \frac{1}{2}\int_0^t |\beta_{x,v}(s)|^2\,ds\right\}.$$

From the Girsanov theorem, if we introduce the process

(2.9) $$w_{x,v}(t) = \omega(t) - \int_0^t \beta_{x,v}(s)\,ds,$$

then the system $\Omega, \mathcal{A}, \mathcal{F}^t P_{x,v} w_{x,v}(t)$ forms a probability system in which $w_{x,v}(t)$ is an $\mathcal{F}^t$ standardized Wiener process. Note that from (2.9) one has

(2.10) $$dx = (g(x(t)) + \sum_\mu v_\nu(t))\,dt + dw_{x,v}(t)\,,\ x(0) = x\,.$$

Let now

(2.11) $\mathcal{O}$ = open smooth bounded domain of $\mathbb{R}^n$, and let

(2.12) $$\tau_x = \inf\{t \mid x(t) \notin \mathcal{O}\}.$$

We shall stop the process $x(t)$ at the exit of the domain $\mathcal{O}$, and to save notation, we shall denote by $x(t)$ also the stopped process. Let also

(2.13) $f_\nu(x)$ be a scalar measurable bounded function.

In the sequel, we shall use the notation

(2.14) $$v(t) = (v_\nu(t)\,,\ v^\nu(t))$$

when $v^\nu(t)$ represents all components which are different from v_ν. We also shall use the notation

(2.15) $$\bar{v}_\nu(t) = \sum_{\mu \neq \nu} v_\nu(t).$$

The payoff of player ν is given by

(2.16) $$Ju(x, \nu(.)) = Ju(x, v_\nu(.), v^\nu(.)) = \\ = \frac{1}{\delta} \log E_{x,v} \exp \delta \Big[\int_0^{\tau_x} (f_\nu(x(t)) + \frac{1}{2}|v_\nu(t)|^2 + \theta v_\nu(t) \cdot \bar{v}_\nu(t))\, dt \Big]$$

where δ stands for the risk parameter. Note that as δ tends to 0, this reduces to

$$E_{x,v}\Big[\int_0^{\tau_x} \Big(f_\nu(x(t)) + \frac{1}{2}|v_\nu(t)|^2 + \theta v_\nu(t) \cdot \bar{v}_\nu(t)\Big)\, dt \Big]$$

which is the payoff considered in [4], except for the fact that we do not have a discount factor any more. If there is only one player, then the parameter σ is irrelevant.

A Nash point for the games defined by the functionals (2.16) is a control $\hat{v}(.)$ such that

(2.17) $$J_\nu(x, \hat{v}_\nu, \hat{v}^\nu) \leq J_\nu(x, v_\nu, \hat{v}^\nu)\,,\ \forall \nu$$

for any admissible control $v(.)$. See J. Nash [11], J. P. Aubin [1] for the concept of Nash point.

As indicated in the introduction, the method to prove the existence of a Nash point, will be to consider a system of Bellman equations for the value functions of the game. This means, that, for a convenient control $\hat{v}$, possibly depending on x (the initial state), the functions

$$u_\nu(x) = J_\nu(x, \hat{v}) \tag{2.18}$$

are the solutions of a system of partial differential equations. This system will allow to characterize optimal feedbacks for the N players. The proof of optimality will be performed by a verification argument. A key point is to obtain sufficient regularity properties for the value functions, otherwise, it is not possible to obtain feedbacks. Techniques of partial differential equation are instrumental in obtaining these necessary regularity properties.

3. Preliminaries.

3.1. *Lagrangians.*

We introduce the Lagrangians by the following definition:

$$L_\nu(v, p) = \frac{1}{2}|v_\nu|^2 + \theta v_\nu \cdot \bar{v}_\nu + p_\nu \cdot \sum_\mu v_\nu \,, \tag{3.1}$$

where

$$p = (p_1, \dots, p_N) \in \mathbb{R}^{nN},$$

$$v = (v_1, \dots, v_N) \in \mathbb{R}^{nN}$$

and, consistently, with the notation (2.15), we denote

$$\bar{v}_\nu = \sum_{\mu \neq \nu} v_\mu \,, \tag{3.2}$$

$$\bar{p}_\nu = \sum_{\mu \neq \nu} p_\mu \,. \tag{3.3}$$

The first step is to consider, for a given p, a Nash point in v for the functions $L_\nu(v, p)$. Clearly, the following conditions must hold (by differentiation) for such a Nash point $v(p)$:

$$v_\nu(p) + \theta \bar{v}_\nu(p) + p_\nu = 0 \,. \tag{3.4}$$

Provided

$$\theta \neq 1\,, \qquad \theta \neq -\frac{1}{N-1}\,, \tag{3.5}$$

it is easy to check that the system (3.4) has a unique solution given by the formulas

$$v_\nu(p) = \frac{\theta \sum_\mu p_\mu}{(1-\theta)(1+(N-1)\theta)} - \frac{p_\nu}{1-\theta}\,. \tag{3.6}$$

We note also the complementary formulas

$$\bar{v}_\nu(p) = \frac{-\sum_\mu p_\mu}{(1-\theta)(1+(N-1)\theta)} + \frac{p_\nu}{1-\theta}\,. \tag{3.7}$$

Then, we can define the quantities

$$L_\nu(p) = L_\nu(v(p), p). \tag{3.8}$$

It is useful to express also, from (3.6) and (3.7), the vectors p_ν in terms of $v(p)$ as follows:

$$p_\nu = -v_\nu(p) - \theta \bar{v}_\nu(p) \tag{3.9}$$

and also

$$\bar{p}_\nu = -(N-1)\theta v_\nu(p) + (-N\theta + 2\theta - 1)\bar{v}_\nu(p). \tag{3.10}$$

In particular, we can write

$$L_\nu(p) = -\frac{1}{2}|v_\nu(p)|^2 + p_\nu \cdot \bar{v}_\nu(p) \tag{3.11}$$

and also using (3.6), (3.7) in (3.11) after easy calculations we obtain

$$\begin{aligned} L_\nu(p) = &-\frac{\theta^2}{2(1-\theta)^2(1+(N-1)\theta)^2}\Big|\sum_\mu p_\mu\Big|^2 + \\ &+ \frac{1-2\theta}{2(1-\theta)^2}|p_\nu|^2 + \frac{2\theta-1}{(1-\theta)^2(1+(N-1)\theta)} p_\nu \cdot \sum_\mu p_\mu\,. \end{aligned} \tag{3.12}$$

3.2. *More developments on Lagrangians.*

We continue some useful developments on Lagrangians. We first note by summing up (3.12)

$$\sum_\nu L_\nu(p) = \frac{\theta^2(3N-4) - 2\theta(N-3) - 2}{2(1-\theta)^2(1+(N-1)\theta)^2}|\sum_\nu p_\nu|^2 + \tag{3.13}$$

$$+ \frac{1-2\theta}{2(1-\theta)^2}\sum_\nu |p_\nu|^2 =$$

$$= \frac{-2(N-1)^2\theta^3 + (N^2-3N+1)\theta^2 + 2\theta - 1}{2(1-\theta)^2(1+(N-1)\theta)^2}\sum_\nu |p_\nu|^2 +$$

$$+ 2\frac{(3N-4)\theta^2 - 2(N-3)\theta - 2}{2(1-\theta)^2(1+(N-1)\theta)^2}\sum_{\substack{\mu,\nu \\ \mu\neq\nu}} p_\mu \cdot p_\nu \,.$$

We shall be interested in guaranteeing the property

$$\sum_\nu L_\nu(p) \geq c_0|p|^2, \quad \forall p, \; c_0 > 0, \tag{3.14}$$

or, alternatively,

$$\sum_\nu L_\nu(p) \leq -c_1|p|^2, \quad \forall p, \; c_1 > 0\,. \tag{3.15}$$

From conditions on quadratic forms, in order to obtain (3.14) it will be enough to have

$$F(\theta) = -(N-1)^2\theta^3 + (N^2-3N+1)\frac{\theta^2}{2} + \theta - \frac{1}{2} - \tag{3.16}$$

$$-(N-1)\left|\left(\frac{3N}{2}-2\right)\theta^2 - (N-3)\theta - 1\right| > 0$$

and

$$c_0 = \frac{F(\theta)}{(1-\theta)^2(1+(N-1)\theta)^2}\,.$$

Similarly to obtain (3.15), it will be sufficient to assert that

$$\hat{F}(\theta) = -(N-1)^2\theta^3 + (N^2-3N+1)\frac{\theta^2}{2} + \theta - \frac{1}{2} + \tag{3.17}$$

$$+(N-1)\left|\left(\frac{3N}{2}-2\right)\theta^2-(N-3)\theta-1\right|<0$$

and

$$c_1=\frac{-\hat{F}(\theta)}{(1-\theta)^2(1+(N-1)\theta)^2}\,.$$

We shall characterize the values of θ for which (3.16) and (3.17) hold. Considering the two possibilities for the absolute values, we are led to introducing the following two functions (after reduction of terms)

$$F_1(\theta)=-(N-1)^2\theta^3+\frac{-2N^2+4N-3}{2}\theta^2+\theta(N^2-4N+4)+N-\frac{3}{2}\,, \tag{3.18}$$

$$F_2(\theta)=-(N-1)^2\theta^3+\frac{4N^2-10N+5}{2}\theta^2-\theta(N^2-4N+2)-N+\frac{1}{2}\,. \tag{3.19}$$

To make explicit the values of $F(\theta)$ and $\hat{F}(\theta)$, we consider the two roots of

$$(3N-4)\frac{\theta^2}{2}-(N-3)\theta-1,$$

namely

$$\theta_0=\frac{N-3-\sqrt{N^2+1}}{3N-4}\,,\quad \theta_0>-1(\text{for } N\geq 3),\, \theta_0<-1(\text{for } N=2) \tag{3.20}$$

$$\theta_0'=\frac{N-3+\sqrt{N^2+1}}{3N-4}\,,\qquad \theta_0'<1, \tag{3.21}$$

and we have

$$\begin{aligned} F(\theta)&=F_1(\theta)\,, &&\text{if } \theta\leq\theta_0 \text{ or } \theta\geq\theta_0'\,,\\ F(\theta)&=F_2(\theta), &&\text{if } \theta_0\leq\theta\leq\theta_0'\,, \end{aligned} \tag{3.22}$$

$$\begin{aligned} \hat{F}(\theta)&=F_1(\theta), &&\text{if } \theta_0\leq\theta\leq\theta_0'\,,\\ \hat{F}(\theta)&=F_2(\theta), &&\text{if } \theta\leq\theta_0 \text{ or } \theta\geq\theta_0'\,. \end{aligned} \tag{3.23}$$

Fortunately $F_2(\theta)$ is quite simple. Indeed, from (3.19) it is easy to check that

$$F_2(\theta)=(\theta-1)^2\left(-(N-1)^2\theta-N+\frac{1}{2}\right). \tag{3.24}$$

Let then

$$\bar{\theta} = -\frac{N - \frac{1}{2}}{(N-1)^2}, \tag{3.25}$$

and note that

$$\theta_0 < \bar{\theta} < 0 \quad (-1 < \theta_0 < \bar{\theta} \text{ for } N \geq 3), (\theta_0 < \bar{\theta} < -1 \text{ for } N = 2). \tag{3.26}$$

Clearly

$$\begin{aligned} F_2(\theta) &> 0 \quad \text{if } \theta \leq \bar{\theta}, \\ F_2(\theta) &\leq 0 \quad \text{if } \theta \geq \bar{\theta}, \end{aligned} \tag{3.27}$$

From the identity (3.22) for $F(\theta)$, we can immediately assert that

$$\begin{aligned} F(\theta) &> 0 \quad \text{for } \theta_0 \leq \theta < \bar{\theta}, \\ F(\theta) &\leq 0 \quad \text{for } \bar{\theta} \leq \theta \leq \theta_0', \end{aligned} \tag{3.28}$$

and similarly from (3.23)

$$\begin{aligned} \hat{F}(\theta) &< 0 \quad \text{for } \theta \geq \theta_0', \ \theta \neq 1, \\ \hat{F}(\theta) &> 0 \quad \text{for } \theta \leq \theta_0. \end{aligned} \tag{3.29}$$

To proceed, we must study the sign of $F_1(\theta)$, which is less simple.
However the case $N = 2$ is very simple and particular, since in this case one has

$$F_1(\theta) = (\theta + 1)^2\Big(-\theta + \frac{1}{2}\Big) \tag{3.30}$$

and note that for $N = 2$ we have

$$\theta_0 = \frac{-1 - 1\sqrt{5}}{2}, \quad \theta_0' = \frac{-1 + \sqrt{5}}{2}, \quad \bar{\theta} = -\frac{3}{2}, \tag{3.31}$$

and thus we can complete (3.28), (3.29) for $N = 2$ easily

$$\begin{aligned} F(\theta) = F_1(\theta) &> 0 \quad \text{for } \theta \leq \theta_0, \\ F(\theta) = F_1(\theta) &< 0 \quad \text{for } \theta \geq \theta_0', \end{aligned} \tag{3.32}$$

$$\hat{F}(\theta) = F_1(\theta) < 0 \quad \text{for } \frac{1}{2} < \theta \leq \theta_0', \tag{3.33}$$
$$\hat{F}(\theta) = F_1(\theta) \geq 0 \quad \text{for } \theta_0 \leq \theta \leq \frac{1}{2}.$$

Therefore we can state

$$\text{for } N = 2,\ F(\theta) > 0 \text{ is equivalent to } \theta < -\frac{3}{2} = \bar{\theta}, \tag{3.34}$$
$$\hat{F}(\theta) < 0 \text{ is equivalent to } \theta > \frac{1}{2} = \theta'',\ \theta \neq 1.$$

To study $F_1(\theta)$, for $N \geq 3$, we consider

$$F_1'(\theta) = -3(N-1)^2\theta^2 + (-2N^2 + 4N - 3)\theta + N^2 - 4N + 4$$

whose roots are θ_1, θ_1' given by the formulas

$$\theta_1 = \frac{-2N^2 + 4N - 3 - \sqrt{16N^4 - 88N^3 + 184N^2 - 168N + 57}}{6(N-1)^2} \tag{3.35}$$

$$\theta_1' = \frac{-2N^2 + 4N - 3 + \sqrt{16N^4 - 88N^3 + 184N^2 - 168N + 57}}{6(N-1)^2} \tag{3.36}$$

and we have the configuration

$$-1 < \theta_1 < \theta_0 < \bar{\theta} < 0 < \theta_1' < \theta_0' < 1 \tag{3.37}$$

and $F_1'(\theta) < 0$ for $\theta < \theta_1$, and $\theta > \theta_1'$, whereas $F_1'(\theta) \geq 0$ for $\theta_1 \leq \theta \leq \theta_1'$. Note also that

$$F_1(-\infty) = +\infty, \quad F_1(-1) \leq 0, \quad F_1(\theta_0) = F_2(\theta_0) > 0, \tag{3.38}$$
$$F_1(\theta_0') = F_2(\theta_0') < 0, \quad F_1(+\infty) = -\infty.$$

From the sign of $F_1'(\theta)$, it also follows that

$$F_1(\theta_1) < 0, \quad \text{minimum of } F_1(\theta), \tag{3.39}$$
$$F_1(\theta_1') > 0, \quad \text{maximum of } F_1(\theta).$$

Therefore $F_1(\theta)$ has three roots, two being negative, θ', θ'', and one positive θ''' with the following location:

$$\theta' \leq -1, \qquad \theta_1 < \theta'' < \theta_0, \qquad \theta_1' < \frac{1}{2} < \theta''' < \theta_0' \tag{3.40}$$

and we can conclude easily that

$$F(\theta) > 0 \text{ for } \theta < \theta' \text{ or } \theta'' < \theta < \bar{\theta}, \tag{3.41}$$

$$\hat{F}(\theta) < 0 \text{ for } \theta > \theta''', \qquad \theta \neq 1. \tag{3.42}$$

We then state the

Lemma 3.1. *For* $N = 2$, (3.14) *holds whenever* $\theta < -\frac{3}{2}$, *and* (3.15) *holds when* $\theta > \frac{1}{2}, \theta \neq 1$. *For* $N \geq 3$, *considering the numbers* $\theta', \theta'', \theta''', \bar{\theta}$, *where* θ', θ'' *are the two negative roots of* $F_1(\theta)$, θ''' *the positive root and* $\bar{\theta}$ *is given by* (3.25), *the property* (3.14) *holds when* $\theta < \theta'$ *or* $\theta'' < \theta < \bar{\theta}$, *and the property* (3.15) *holds when* $\theta > \theta''', \theta \neq 1$.

Remark 3.1. In (3.5) we had excluded the values $\theta = 1$, and $\theta = -\frac{1}{N-1}$. Since $\bar{\theta} < -\frac{1}{N-1} < 0$, the value $-\frac{1}{N-1}$ is out of the validity intervals defined in Lemma 3.1. The value $\theta = 1$, valid for (3.15) but not for (3.14) has to be excluded.

Remark 3.2. For $N = 3$, we have $\theta' = -1, \theta'' = \frac{-1-\sqrt{97}}{16}, \theta''' = \frac{-1+\sqrt{97}}{16}$, $\theta_1 = \frac{-9-\sqrt{129}}{24}, \theta_1' = \frac{-9+\sqrt{129}}{24}, \theta_0 = -\frac{\sqrt{10}}{5}, \theta_0' = \frac{\sqrt{10}}{5}, \bar{\theta} = -\frac{5}{8}$.

3.3. *Other properties.*

From formula (3.12) we can deduce by using Young's inequality

$$L_\nu(p) \leq \frac{1-2\theta}{2\theta^2}|p_\nu|^2. \tag{3.43}$$

We proceed now with a different estimate. Note first that from (3.7) we can write

$$\bar{v}_\nu = \bar{v}_\nu(p) = \frac{(N-1)\theta p_\nu - \bar{p}_\nu}{(1-\theta)(1+(N-1)\theta)}. \tag{3.44}$$

By analogy with the formula (3.44) we consider a similar combination of the Lagrangians, namely

$$\tilde{L}_\nu(p) = (N-1)\theta Ł_\nu(p) - \sum_{\mu \neq \nu} L_\mu(p). \tag{3.45}$$

Consider (3.9) and (3.11) which yields

$$L_\nu(p) = -\frac{1}{2}|v_\nu|^2 - v_\nu\bar{v}_\nu - \theta|\bar{v}_\nu|^2 = -\frac{1}{2}|v_\nu + \bar{v}_\nu|^2 + \left(\frac{1}{2} - \theta\right)|\bar{v}_\nu|^2 \tag{3.46}$$

hence as easily seen

$$\tilde{L}_\nu(p) = (N-1)\theta\left(\frac{1}{2} - \theta\right)|\bar{v}_\nu|^2 + \frac{1}{2}(N-1)(1-\theta)|v_\nu + \bar{v}_\nu|^2 - \tag{3.47}$$

$$-\left(\frac{1}{2}-\theta\right)\sum_{\mu\neq\nu}|\bar{v}_\mu|^2 = (N-1)\theta\left(\frac{1}{2}-\theta\right)|\bar{v}_\nu|^2 + \frac{1-\theta}{2(N-1)}|\sum_{\mu}\bar{v}_\mu|^2 -$$

$$-\left(\frac{1}{2}-\theta\right)\sum_{\mu\neq\nu}|\bar{v}_\mu|^2 = \left[(N-1)\theta\left(\frac{1}{2}-\theta\right)+\frac{1}{2}\frac{1-\theta}{N-1}\right]|\bar{v}_\nu|^2 +$$

$$+\frac{1}{2}\frac{1-\theta}{N-1}|\sum_{\mu\neq\nu}\bar{v}_\mu|^2 + \frac{1-\theta}{N-1}\bar{v}_\nu\sum_{\mu\neq\nu}\bar{v}_\mu + \left(\theta-\frac{1}{2}\right)\sum_{\mu\neq\nu}|\bar{v}_\mu|^2.$$

Using

$$|\sum_{\mu\neq\nu}\bar{v}_\mu|^2 \leq (N-1)\sum_{\mu\neq\nu}|\bar{v}_\mu|^2 \tag{3.48}$$

and assuming

$$\theta \geq \frac{1}{2} \tag{3.49}$$

we deduce

$$\tilde{L}_\nu(p) \geq \left[(N-1)\theta\left(\frac{1}{2}-\theta\right)+\frac{1}{2}\frac{1-\theta}{N-1}\right]|\bar{v}_\nu|^2+\frac{\theta}{2(N-1)}|\sum_{\mu\neq\nu}\bar{v}_\mu|^2+ \tag{3.50}$$

$$+\frac{1-\theta}{N-1}\bar{v}_\nu\sum_{\mu\neq\nu}\bar{v}_\mu \geq -\frac{2\theta-1}{2\theta(N-1)}(\theta^2(N-1)^2+\theta-1)|\bar{v}_\nu|^2.$$

So we state the

Lemma 3.2. *When $\theta \geq \frac{1}{2}$ one has the property*

$$\tilde{L}_\nu(p) \geq -k|\bar{v}_\nu|^2. \tag{3.51}$$

In the sequel we shall use (3.43) and (3.14) together when $\theta < 0$ (θ satisfying the conditions of Lemma 3.1) and (3.15), (3.51) together when $\theta > 0$ (in fact $\theta > \theta''' > \frac{1}{2}, \theta \neq 1$).

3.4. *Taking account of the risk factor.*

The risk factor will imply a perturbation of the Lagrangian, namely $L_\nu(p)$ has to be replaced for

$$M_\nu(p) = L_\nu(p) + \frac{\delta}{2}|p_\nu|^2. \tag{3.52}$$

The property (3.43) is clearly unchanged

$$M_\nu(p) \leq \left(\frac{1-2\theta}{2\theta^2} + \frac{\delta}{2}\right)|p_\nu|^2. \tag{3.53}$$

The property (3.14) is improved, since

$$\sum_\nu M_\nu(p) = \sum_\nu L_\nu(p) + \frac{\delta}{2}|p|^2 \tag{3.54}$$

and if (3.14) holds, a fortiori

$$\sum_\nu M_\nu(p) \geq c_0|p|^2. \tag{3.55}$$

Since the risk factor here helps, it modifies the discussion on θ. From formula (3.13), what matters now is to have

$$F_\delta(\theta) = \frac{\delta}{2}(1-\theta)^2(1+(N-1)\theta)^2 - (N-1)^2\theta^3 + \tag{3.56}$$

$$+(N^2-3N+1)\frac{\theta^2}{2} + \theta - \frac{1}{2} - (N-1)\left|\left(\frac{3N}{2}-2\right)\theta^2 - (N-3)\theta - 1\right| > 0.$$

Define

$$F_{1\delta}(\theta) = F_1(\theta) + \frac{\delta}{2}(1-\theta)^2(1+(N-1)\theta)^2, \tag{3.57}$$

$$F_{2\delta}(\theta) = F_2(\theta) + \frac{\delta}{2}(1-\theta)^2(1+(N-1)\theta)^2. \tag{3.58}$$

We have

$$F_\delta(\theta) = F_{1\delta}(\theta) \qquad \text{if } \theta \leq \theta_0 \text{ or } \theta \geq \theta_0', \tag{3.59}$$

$$F_\delta(\theta) = F_{2\delta}(\theta) \quad \text{if } \theta_0 \le \theta \le \theta_0'.$$

The function $F_{2\delta}(\theta)$ is easy to compute (using (3.24))

(3.60) $$F_{2\delta}(\theta) = \frac{(\theta-1)^2}{2}[\delta(N-1)^2\theta^2 + 2(N-1)\theta(\delta-(N-1)) - 2N + 1 + \delta]$$

hence the two roots not equal to 1 are

(3.61) $$\bar\theta_\delta = \frac{1 - \frac{\delta}{N-1} - \sqrt{1 + \frac{\delta}{(N-1)^2}}}{\delta}\, \bar\theta_\delta' = \frac{1 - \frac{\delta}{N-1} + \sqrt{1 + \frac{\delta}{(N-1)^2}}}{\delta}$$

and $\bar\theta < \bar\theta_\delta$. Therefore

(3.62) $$\begin{aligned} &F_{2\delta}(\theta) > 0 \quad \text{if } \theta < \bar\theta_\delta \text{ or } \theta > \bar\theta_\delta', \quad \theta \neq 1, \\ &F_{2\delta}(\theta) \le 0 \quad \text{if } \bar\theta_\delta \le \theta \le \bar\delta_\delta'. \end{aligned}$$

We need next to study $F_{1\delta}(\theta)$. We assume $N \ge 3$. Note that

(3.63) $$F_{1\delta}'(\theta) = F_1'(\theta) + \delta(\theta-1)(1+(N-1)\theta)(2\theta(N-1)-(N-2)).$$

For $\theta < -\frac{1}{N-1}$, we have $F_{1\delta}'(\theta) < F_1'(\theta)$. Hence

$$F_{1\delta}'(\theta) < 0 \qquad \text{for } \theta \le \theta_1.$$

Noting that $\theta_1' < \frac{N-2}{2(N-1)} < \theta_0'$, we have also

$$F_{1\delta}'(\theta) < 0 \qquad \text{for} \qquad \frac{N-2}{2(N-1)} \le \theta \le 1.$$

Similarly, since $\bar\theta < -\frac{1}{N-1}$, one has also

$$F_{1\delta}'(\theta) > 0 \qquad \text{for} \qquad -\frac{1}{N-1} \le \theta \le \theta_1'.$$

Furthermore $F_{1\delta}'(\theta) > 0$ for θ sufficiently larger > 1. Therefore $F_{1\delta}'(\theta)$ has necessarily three roots, which we denote $\theta_{1\delta}, \theta_{1\delta}', \theta_{1\delta}''$. We can also assert that

(3.64) $$\begin{aligned} &\theta_1 < \theta_{1\delta} < -\frac{1}{N-1} \\ &\theta_1' < \theta_{1\delta}' < \frac{N-2}{2(N-1)}, \qquad \theta_{1\delta}'' > 1. \end{aligned}$$

Moreover as $\delta \to 0$, $\theta_{1\delta} \to \theta_1$, $\theta'_{1\delta} \to \theta'_1$, $\theta''_{1\delta} \to +\infty$, and as $\delta \to \infty$, $\theta_{1\delta} \to -\frac{1}{N-1}$, $\theta'_{1\delta} \to \frac{N-2}{2(N-1)}$, $\theta''_{1\delta} \to 1$. Hence $F_{1\delta}(\theta)$ has two local minima, $\theta_{1\delta}$ and $\theta''_{1\delta}$, and one local maximum $\theta'_{1\delta}$. Note the following properties

$$
\begin{aligned}
&F_{1\delta}(\theta) > 0, \quad \theta \leq \theta' \quad \text{and} \quad \theta'' \leq \theta \leq \theta''', \\
&F_{1\delta}(1) = F_1(1) = -N^2 + N < 0, \\
&F_{1\delta}(-\infty) = F_{1\delta}(+\infty) = +\infty.
\end{aligned} \tag{3.65}
$$

Therefore the value at the local minimum $\theta''_{1\delta}$ is strictly negative, and thus $F_{1\delta}$ has two positive roots $\theta'''_{1\delta}$, $\theta'''_{2\delta}$. Necessarily, since $F_{1\delta}(\theta''') > 0$, we have

$$\theta''' < \theta'''_{1\delta} < 1 < \theta'''_{2\delta}. \tag{3.66}$$

The value at the local minimum $\theta_{1\delta}$ is not necessarily negative. It is so if δ is sufficiently small, since $\theta_{1\delta}$ is close to θ_1, and $F_{1\delta}(\theta_{1\delta}) \sim F_1(\theta_1) < 0$. In that case there will be two negative roots of $F_{1\delta}(\theta)$, denoted by θ'_δ, θ''_δ, and from (3.65) necessarily one has

$$\theta' < \theta'_\delta < \theta_{1\delta} < \theta''_\delta < \theta''. \tag{3.67}$$

So

$$F_{1\delta} > 0 \ \text{ for } \ \theta \in (-\infty, \theta'_\delta)\,, \qquad \theta \in (\theta''_\delta, \theta'''_{1\delta})\,, \qquad \theta > \theta''_{2\delta}\,, \tag{3.68}$$

and the interval $\theta'_\delta, \theta''_\delta$ may be void.

For $\delta = 0$, we have $\theta'_\delta = \theta'$, $\theta''_\delta = \theta''$, $\theta'''_{1\delta} = \theta'''$, $\theta'''_{2\delta} = +\infty$ and $\bar{\theta}_\delta = \bar{\theta}$, $\theta'_\delta = +\infty$. We recover the situation of paragraph 3.2. For $\delta = +\infty$, the numbers $\theta'_\delta, \theta''_\delta$ do not exist and $\theta'''_{1\delta} = \theta'''_{2\delta} = 1$, hence $F_{1\delta}(\theta) > 0, \forall \theta, \theta \neq 1$. Also $\bar{\theta}_\delta = \bar{\bar{\theta}}_\delta = -\frac{1}{N-1}$, and $F_{2\delta}(\theta) > 0$, $\forall \theta \neq 1\,, \theta \neq -\frac{1}{N-1}$.
We shall also use the property

$$
\begin{aligned}
&\text{If } \ \bar{\theta}'_\delta \leq \bar{\theta}'_0 \ \text{ then } \ \theta'_0 \leq \theta'''_{1\delta} \ \text{(equality only when } \bar{\theta}'_\delta = \theta'_0) \\
&\text{If } \ \bar{\theta}'_\delta > \theta'_0 \text{ then } \ \theta'_0 > \theta'''_{1\delta}.
\end{aligned} \tag{3.69}
$$

Indeed, in the first case we have by definition of $\bar{\theta}'_\delta$, $F_{2\delta}(\theta'_0) \geq 0$, hence $F_{1\delta}(\theta'_0) = F_{2\delta}(\theta'_0) \geq 0$. Now, if $\theta'''_{1\delta} < \theta'_0$, we have $F_{1\delta}(\theta'_0) < 0$, which leads to a contradiction.

The second case is proven in a similar way.

Collecting results, thanks to (3.62), (3.68) and the definition (3.59) as well as (3.69) we get

(3.70)

$$F_\delta(\theta) > 0 \text{ for } \theta \in (-\infty, \theta_\delta'),\ (\theta_\delta'', \bar\theta_\delta),\ (\bar\theta_\delta', \theta_{1\delta}'''),\ (\theta_{2\delta}''', \infty),\ \text{if } \bar\theta_\delta' < \theta_0',$$

$$F_\delta(\theta) > 0 \text{ for } \theta \in (-\infty, \theta_\delta'),\ (\theta_\delta'', \bar\theta_\delta),\ (\theta_{2\delta}''', \infty),\ \text{if } \bar\theta_\delta \geq \theta_0'.$$

For $\delta = 0$ this yields $F_\delta(\theta) > 0$, for $\theta < \bar\theta$, and if $\delta = +\infty$, we get $F_\delta(\theta) > 0\ \forall\theta, \theta \neq 1, \theta \neq -\frac{1}{N-1}$. Finally in the case $N = 2$, we have

$$F_{2\delta}(\theta) = \frac{(\theta+1)}{2}[\delta(1-\theta)^2 + 1 - 2\theta],$$

$$F_{1\delta}(\theta) = \frac{(\theta+1)}{2}[\delta(1+\theta)^2 - 2\theta - 3]$$

hence

$$\theta' = \theta_\delta'' = -1,$$

$$\theta_{1\delta}''' = \frac{1+\delta-\sqrt{1+\delta}}{\delta}, \quad \theta_{2\delta}''' = \frac{1+\delta+\sqrt{1+\delta}}{\delta}$$

$$\bar\theta_\delta = \frac{1-\delta-\sqrt{1-\delta}}{\delta}, \quad \bar\theta_\delta' = \frac{1-\delta+\sqrt{1+\delta}}{\delta}.$$

We now examine how to assert

(3.71) $$\sum_\nu M_\nu(p) \leq -c_1|p|^2$$

and

(3.72) $$\tilde M_\nu(p) \geq -k|\bar v_\nu|^2,$$

where $\tilde M_\nu(p)$ is defined analogously to $\tilde L_\nu(p)$, see (3.45). Here the risk factor is not helping, so δ cannot be too large.

Let us investigate the conditions on θ. We must obtain $\hat F_\delta(\theta) < 0$, with

(3.73) $$\hat F_\delta(\theta) = \begin{cases} F_{1\delta}(\theta) & \text{if } \theta_0 \leq \theta \leq \theta_0' \\ F_{2\delta}(\theta) & \text{if } \theta \leq \theta_0 \text{ or } \theta \geq \theta_0'. \end{cases}$$

We may assert, using (3.62), (3.68), (3.69), and the definition (3.73), that

(3.74) $\hat F_\delta(\theta) \geq 0,\ \forall\theta$, when $\bar\theta_\delta' \leq \theta_0'$ so there is no validity interval

$$\hat{F}_\delta(\theta) < 0 \text{ if } \theta \in (\theta'''_{1\delta}, \bar{\theta}'_\delta)\,, \text{ when } \bar{\theta}'_\delta > \theta'_0\,. \tag{3.75}$$

For $\delta = 0$, we get (θ''', ∞). For $\delta = \infty$, there is no validity interval.

Let us now check (3.72). A tedious but easy calculation leads to the formula

$$(N-1)\theta|p_\nu|^2 - \sum_{\mu\neq\nu}|p_\mu|^2 = |\bar{v}_\nu|^2\left\{\frac{1}{N-1} + \theta(\theta-1)(\theta(N-1)-N+3)\right\} - \tag{3.76}$$
$$-\frac{(\theta-1)}{N-1}|\sum_{\mu\neq\nu}\bar{v}_\mu|^2 - (\theta-1)^2\sum_{\mu\neq\nu}|\bar{v}_\mu|^2 - 2\theta(1-\theta)\bar{v}_\nu\sum_{\mu\neq\nu}\bar{v}_\mu$$

therefore

$$\tilde{M}_\nu(p) = |\bar{v}_\nu|^2\Big\{(N-1)\theta\Big(\frac{1}{2}-\theta\Big) + \frac{1}{2}\frac{1-\theta}{N-1} + \tag{3.77}$$
$$+\frac{\delta}{2}\Big[\theta(\theta-1)(\theta(N-1)-N+3) + \frac{1}{N-1}\Big]\Big\} +$$
$$+\frac{|\sum_{\mu\neq\nu}\bar{v}_\mu|^2}{2(N-1)}(1-\theta)(1+\delta) + (1-\theta)\Big(\frac{1}{N-1} - \delta\theta\Big)\bar{v}_\nu\sum_{\mu\neq\nu}\bar{v}_\mu +$$
$$+\Big(\Big(\theta-\frac{1}{2}\Big) - \frac{\delta}{2}(\theta-1)^2\Big)\sum_{\mu\neq\nu}|\bar{v}_\mu|^2\,.$$

Assume that

$$\theta - \frac{1}{2} - \frac{\delta}{2}(\theta-1)^2 > 0 \tag{3.78}$$

and using (3.48) we get finally the estimate

$$\tilde{M}_\nu(p) \geq |\bar{v}_\nu|^2\Big\{(N-1)\theta\Big(\frac{1}{2}-\theta\Big) + \frac{1}{2}\frac{1-\theta}{N-1} + \tag{3.79}$$
$$+\frac{\delta}{2}\Big[\theta(\theta-1)(\theta(N-1)-N+3) + \frac{1}{N-1}\Big]\Big\} +$$
$$+(1-\theta)\Big(\frac{1}{N-1} - \delta\theta\Big)\bar{v}_\nu\sum_{\mu\neq\nu}\bar{v}_\mu + \frac{\theta(1-\delta(\theta-1))}{2(N-1)}|\sum_{\mu\neq\nu}\bar{v}_\mu|^2$$

and if we also assume

$$1 - \delta(\theta-1) > 0 \tag{3.80}$$

we deduce from (3.79) the estimate (3.72).

Collecting results we can state the

Lemma 3.3. *If θ belongs to the validity intervals defined in* (3.70), *then we have* (3.53), (3.55). *On the other hand, if $\theta_0' < \bar{\theta}_\delta'$ and $\theta \in (\theta_{1\delta}''', \bar{\theta}_\delta')$ and satisfies the properties* (3.78), (3.80), *then* (3.71), (3.72) *are satisfied.*

4. Nonlinear system of partial differential equations.

4.1. *Setting of the problem.*

Here we consider the following system of equations:

$$-\frac{1}{2}\Delta u_\nu = H_\nu(x, Du) \qquad (4.1)$$
$$u_\nu|_{\partial\mathcal{O}} = 0\,.$$

Firstly, the functions $H_\nu(x, p)$ satisfy

$$H_\nu(x, p) \ \ \textit{are Caratheodory functions}. \qquad (4.2)$$

We shall make an important use of linear manipulations on the equations (4.1). Consider an $N \times N$ matrix Γ, which is invertible, and define

$$H^\Gamma(x, p) = \Gamma H(x, \Gamma^{-1} p), \qquad (4.3)$$

where $H(x, p)$ represents the vector $H_\nu(x, p)$. Setting

$$z = \Gamma u\,, \qquad (4.4)$$

then we see that z is the solution of

$$-\frac{1}{2}\Delta z_\nu = H_\nu^\Gamma(x, Dz) \qquad (4.5)$$
$$z_\nu|_{\partial\mathcal{O}} = 0\,.$$

So our original problem (4.1) is imbedded in a family of equivalent problems, indexed by the transformation Γ. All matrices Γ to be considered will be *invertible*, so we shall not mention it explicitly.

A matrix Γ satisfies the maximum principle if

$$\Gamma u \geq 0 \Rightarrow u \geq 0\,. \qquad (4.6)$$

We begin by starting two alternative sets of assumptions:

(4.7) $$\sum_{\nu} H_\nu(x, p) \geq -\lambda .$$

(4.8) There exists a matrix Γ which satisfies the maximum principle and
$H_\nu^\Gamma(x, p) \leq \lambda_\nu + \lambda_\nu^0 |p_\nu|^2 , \quad \lambda_\nu, \lambda_\nu^0$ not too large

or

(4.9) $$\sum_{\nu} H_\nu(x, p) \leq \lambda .$$

(4.10) There exists a matrix Γ which satisfies the maximum principle and
$H_\nu^\Gamma(x, p) \geq -\lambda_\nu - \lambda_\nu^0 |p_\nu|^2 , \quad \lambda_\nu , \lambda_\nu^0$ not too large.

We furthermore assume

(4.11) There exists a matrix Γ such that
$$H_\nu^\Gamma(x, p) = Q(x, p) \cdot p_\nu + H_\nu^0(x, p)$$

with

(4.12) $$|Q(x, p)| \leq k + K|p|,$$

(4.13) $$|H_\nu^0(x, p)| \leq k_\nu + K_\nu \sum_{\mu \leq \nu} |p_\mu|^2.$$

Remark 4.1. If we pick $\nu = N$ in (4.13) we obtain

$$|H_N^0(x, p)| \leq k_N + K_N |p|^2$$

which is a general quadratic growth assumption. So if $H(x, p)$ has a general quadratic growth, it is sufficient to check (4.11), (4.12), (4.13) for $\nu = 1, \ldots , N - 1$. We may define

(4.14) $$H_N^0(x, p) = H_N^\Gamma(x, p) - Q(x, p) \cdot p_N$$

and (4.11), (4.13) will be satisfied automatically.

Our objective is to prove the

Theorem 4.1. *We assume that the functions* $H_\nu(x, p)$ *satisfy* (4.2), (4.11), (4.12), (4.13) *and one or the other of the two sets of assumptions* (4.7), (4.8) *or* (4.9), (4.10). *Then there exists a solution of* (4.1) *which is in* $W^{2,s}(\mathcal{O})$, $\forall s$ *such that* $2 \leq s < \infty$.

The proof will be done by explaining how to obtain an a priori estimates first, then an approximation argument will be used. We shall consider the Green function associated with any point $\xi \in \mathcal{O}$, corresponding to the operator $-\frac{1}{2}\Delta$, called G^ξ. It is the solution of the equation (written formally)

$$-\frac{1}{2}\Delta G^\xi = \delta(x - z) \qquad (4.15)$$
$$G^\xi|_{\partial\mathcal{O}} = 0\,.$$

The Green function is positive and satisfies the following estimates

$$\|G^\xi\|_{L^q(\mathcal{O})} \leq C, \quad \forall\xi, \quad 1 \leq q < \frac{n}{n-2} \qquad (4.16)$$

$$\|G^\xi\|_{W_0^{1,r}(\mathcal{O})} \leq C, \quad \forall\xi, \quad 1 \leq r < \frac{n}{n-1}. \qquad (4.17)$$

We shall denote by $\|G\|_{L^q}$, $\|G\|_{W^{1,r}}$ the bounds on the right-hand side of (4.16), (4.17).

4.2. L^∞ *a priori estimate.*

We assume (4.3), (4.4). We shall indicate briefly the changes which are necessary when (4.5), (4.6) apply. Summing up the equations (4.1) yields

$$-\frac{1}{2}\Delta\sum_\nu u_\nu = \sum_\nu H_\nu(x, Du) \geq -\lambda.$$

We test with $\left(\sum_\nu u_\nu\right)^- G^\xi$, hence we get

$$\frac{1}{2}\int_\mathcal{O} D\sum_\nu u_\nu D\Big(\Big(\sum_\nu u_\nu\Big)^- G^\xi\Big)\,dx \geq -\lambda\int_\mathcal{O}\Big(\sum_\nu u_\nu\Big)\ G^\xi\,dx$$

hence as easily seen

$$\frac{1}{4}\int_\mathcal{O} D\Big(\Big(\sum_\nu u_\nu\Big)^-\Big)^2 DG^\xi\,dx + \frac{1}{2}\int_\mathcal{O}\Big|D\Big(\sum_\nu u_\nu\Big)^-\Big|^2 G^\xi\,dx \leq$$

$$\leq \lambda \int_{\mathcal{O}} \Big(\sum_{\nu} u_{\nu}\Big)^{-} G^{\xi}\, dx\,.$$

From the definition of the Green function we obtain

$$\frac{1}{2}\bigg(\Big(\sum_{\nu} u_{\nu}\Big)^{-}(\xi)\bigg)^{2} \leq \lambda \int_{\mathcal{O}} \Big(\sum_{\nu} u_{\nu}\Big)^{-} G^{\xi}\, dx\,.$$

Suppose ξ is a point where $\left(\sum_{\nu} u_{\nu}\right)^{-}$ reaches a positive maximum (necessarily in $\mathcal{O}$), then we get

$$\Big\|\Big(\sum_{\nu} u_{\nu}\Big)^{-}\Big\|_{\infty} \leq 2\lambda \int_{\mathcal{O}} G^{\xi}\, dx \leq C.$$

Therefore we have proven the first L^{∞} estimate

$$\sum_{\nu} u_{\nu} \geq -c\,. \tag{4.18}$$

We consider now the matrix Γ intervening in the assumption (4.8) and set

$$\tilde{u} = \Gamma u\,.$$

From (4.5) we know that

$$\begin{aligned} -\frac{1}{2}\Delta\tilde{u}_{\nu} &= H_{\nu}^{\Gamma}(x, D\tilde{u}) \\ \tilde{u}_{\nu}|_{\partial\mathcal{O}} &= 0\,. \end{aligned} \tag{4.19}$$

Let us set

$$E_{\nu} = \exp 2\lambda_{\nu}^{0}\tilde{u}_{\nu}\,, \tag{4.20}$$

then

$$-\frac{1}{2}\Delta E_{\nu} = -E_{\nu}[2(\lambda_{\nu}^{0})^{2}|D\tilde{u}_{\nu}|^{2} + \lambda_{\nu}^{0}\Delta\tilde{u}_{\nu}]$$

and from (4.19) and the property (4.8) this yields

$$-\frac{1}{2}\Delta E_{\nu} \leq 2\lambda_{\nu}\lambda_{\nu}^{0}E_{\nu}.$$

Using $E_\nu G^\xi$ as a test function we get

$$\frac{1}{2}\int_{\mathcal{O}} |DE_\nu|^2 G^\xi \, dx + \frac{1}{4}\int_{\mathcal{O}} D(E_\nu^2 - 1) DG^\xi \, dx \leq$$

$$\leq 2\lambda_\nu \lambda_\nu^0 \int_{\mathcal{O}} E_\nu^2 G^\xi \, dx$$

hence (noting that $E_\nu^2 - 1$ vanishes on $\partial\mathcal{O}$)

$$\frac{1}{2}(E_\nu^2(\xi) - 1) \leq 2\lambda_\nu \lambda_\nu^0 \int_{\mathcal{O}} E_\nu^2 G^\xi \, dx$$

(4.21) $$E_\nu^2(\xi) \leq 1 + 4\lambda_\nu \lambda_\nu^0 \int_{\mathcal{O}} E_\nu^2 G^\xi \, dx \, .$$

If ξ is chosen to be the maximum of $\tilde{u}_\nu$, assumed to be positive, hence $\xi \in \partial\mathcal{O}$, we deduce from (4.21)

$$\|E_\nu^2\|_\infty \leq 1 + 4\lambda_\nu \lambda_\nu^0 \|E_\nu^2\|_\infty \|G\|_{L^1}$$

and using the assumption (4.8), provided that $\lambda_\nu \lambda_\nu^0$ is sufficiently small so that

(4.22) $$4\lambda_\nu \lambda_\nu^0 \|G\|_{L^1} < 1$$

we obtain

(4.23) $$\|E_\nu^2\|_\infty \leq \frac{1}{1 - 4\lambda_\nu \lambda_\nu^0 \|G\|_{L^1}}$$

hence also

(4.24) $$\tilde{u}_\nu \leq C \, .$$

Since Γ satisfies the maximum principle, we get also for the original functions u_ν

(4.25) $$u_\nu \leq c$$

which together with (4.18) implies

(4.26) $$\|u_\nu\|_{L^\infty} \leq c \, .$$

Let us indicate the changes to be performed when (4.9), (4.10) apply. We test with $\left(\sum_\nu u_\nu\right)^+ G^\xi$ to obtain first, thanks to (4.9)

$$\sum_\nu u_\nu \leq c\,. \tag{4.27}$$

Introduce again the functions

$$\tilde{u} = \Gamma u$$

where Γ refers to the matrix intervening (4.10).
We set this time

$$E_\nu = \exp(-2\lambda_\nu^0 \tilde{u}_\nu) \tag{4.28}$$

and perform computation similar to those for obtaining (4.21). Making use of (4.22), thanks to assumption (4.10) we deduce

$$\|E_\nu^2\|_\infty \leq c$$

hence

$$\tilde{u}_\nu \geq -c \tag{4.29}$$

and since Γ satisfies the maximum principle, this implies

$$u_\nu \geq c \tag{4.30}$$

which together with (4.27) means again

$$\|u_\nu\|_\infty \leq c\,. \tag{4.31}$$

4.3. *H_0^1 estimate.*

To obtain the H_0^1 estimate, we shall make use of the special structure (4.11), (4.12), (4.13). We omit this notation Γ, to simplify the writing and by virtue of the L^∞ estimate, we have

$$|u_\nu(x)| \leq \rho\,. \tag{4.32}$$

To obtain an a priori estimate for H_0^1, one uses a specific test function. Set

$$\beta(s) = e^s - s - 1$$

and

$$F = \prod_{\nu=1}^{N} \exp(\beta(\gamma_\nu u_\nu))$$

where γ is a positive constant to be defined later. We have

$$DF = F \sum_{\nu=1}^{N} \gamma_\nu \beta'(\gamma_\nu u_\nu) Du_\nu \, .$$

We test (4.1) with $F\gamma_\nu \beta'(\gamma_\nu u_\nu)$, which vanishes on the boundary, integrate by parts and add up. We get

$$\sum_\nu \frac{1}{2} \int_{\mathcal{O}} \gamma_\nu^2 |Du_\nu|^2 e^{\gamma_\nu u_\nu} F \, dx + \frac{1}{2} \int_{\mathcal{O}} \frac{|DF|^2}{F} dx =$$

$$= \int_{\mathcal{O}} Q \cdot DF \, dx + \int_{\mathcal{O}} \sum_\nu \gamma_\nu H_\nu^0(Du) F (e^{\gamma_\nu u_\nu} - 1) \, dx$$

hence also

$$\sum_\nu \frac{1}{2} \int_{\mathcal{O}} \gamma_\nu^2 |Du_\nu|^2 e^{\gamma_\nu u_\nu} F \, dx \leq \frac{1}{2} \int_{\mathcal{O}} FQ \cdot Q \, dx + \tag{4.33}$$

$$+ \int_{\mathcal{O}} \sum_\nu \gamma_\nu H_\nu^0(Du) F (e^{\gamma_\nu \mu_\nu} - 1) \, dx \, .$$

In order to get comparable terms on both sides, we introduce the function

$$X = \prod_{\nu=1}^{N} (\exp \beta(\gamma_\nu u_\nu) + \exp \beta(-\gamma_\nu u_\nu)) \tag{4.34}$$

and the related quantities

$$X_\nu = X \frac{e^{\gamma_\nu u_\nu} \exp \beta(\gamma_\nu u_\nu) + e^{-\gamma_\nu u_\nu} \exp \beta(-\gamma_\nu u_\nu)}{\exp \beta(\gamma_\nu u_\nu) + \exp \beta(-\gamma_\nu u_\nu)} \, ,$$

$$\tilde{X}_\nu = X \frac{(e^{\gamma_\nu u_\nu} - 1) \exp \beta(\gamma_\nu u_\nu) - (e^{-\gamma_\nu u_\nu} - 1) \exp \beta(-\gamma_\nu u_\nu)}{\exp \beta(\gamma_\nu u_\nu) + \exp \beta(-\gamma_\nu u_\nu)} \, .$$

We have the inequalities

$$2^N \leq X \leq X_\nu \leq X e^{\gamma_\nu |u_\nu|},$$

$$|\tilde{X}_\nu| \leq X_\nu \,.$$

Applying the relations (4.33) with γ_ν changed one by one into $-\gamma_\nu$, and summing up the 2^N relations obtained in this way, we get the inequality

$$\frac{1}{2}\sum_\nu \int_{\mathcal{O}} \gamma_\nu^2 |Du_\nu|^2 X_\nu \, dx \leq \frac{1}{2}\int_{\mathcal{O}} XQ \cdot Q \, dx +$$

$$+ \int_{\mathcal{O}} \sum_\nu \gamma_\nu H_\nu^0(Du)\tilde{X}_\nu \, dx$$

hence

$$\frac{1}{2}\sum_\nu \int_{\mathcal{O}} \gamma_\nu^2 |Du_\nu|^2 X_\nu \, dx \leq \frac{1}{2}\int_{\mathcal{O}} XQ \cdot Q \, dx + \int_{\mathcal{O}} \sum_\nu \gamma_\nu |H_\nu^0(Du)| X_\nu \, dx \leq$$

$$\leq \frac{1}{2}\int_{\mathcal{O}} XQ \cdot Q \, dx + \int_{\mathcal{O}} \sum_\nu \gamma_\nu (k_\nu + K_\nu |Du_\nu|^2) X_\nu \, dx +$$

$$+ \int_{\mathcal{O}} \sum_\nu |Du_\nu|^2 \sum_{\mu>\nu} \gamma_\mu K_\mu X_\mu \, dx \leq$$

$$\leq \int_{\mathcal{O}} k^2 X \, dx + K^2 \sum_\nu \int_{\mathcal{O}} X_\nu |Du_\nu|^2 \, dx + \sum_\nu \gamma_\nu \int_{\mathcal{O}} (k_\nu + K_\nu |Du_\nu|^2) X_\nu \, dx +$$

$$+ \int_{\mathcal{O}} \sum_\nu |Du_\nu|^2 \sum_{\mu>\nu} \gamma_\mu K_\mu X_\mu \, dx \,.$$

We obtain

$$\sum_\nu \int_{\mathcal{O}} |Du_\nu|^2 X_\nu \Big[\frac{1}{2}\gamma_\nu^2 - K^2 - \gamma_\nu K_\nu\Big] \, dx \leq \int_{\mathcal{O}} (k^2 X + \sum_\nu \gamma_\nu k_\nu X_\nu) \, dx +$$

$$+ \sum_\nu \int_{\mathcal{O}} |Du_\nu|^2 \sum_{\mu>\nu} \gamma_\mu K_\mu X_\mu \, dx \,.$$

Finally

$$\sum_\nu \int_{\mathcal{O}} X |Du_\nu|^2 \Big[\frac{1}{2}\gamma_\nu^2 - K^2 - \gamma_\nu K_\nu - \sum_{\mu>\nu} \gamma_\mu K_\mu e^{\rho\gamma_\mu}\Big] \, dx \leq$$

$$\leq \int_{\mathcal{O}} X(k^2 + \sum_{\nu} \gamma_\nu k_\nu e^{\rho\gamma_\nu})\, dx$$

with $\rho = \|u\|_\infty$. Therefore if we choose the constants γ_ν so that

$$\frac{1}{2}\gamma_\nu^2 - K^2 - \gamma_\nu K_\nu - \sum_{\mu>\nu} \gamma_\mu e^{\rho\gamma_\mu} > 0 \tag{4.35}$$

we get

$$\int_{\mathcal{O}} |Du|^2\, dx \leq K_0(\rho)\,. \tag{4.36}$$

It is possible to choose the constants γ_ν in order to fulfill (4.35). This can be done backwards, starting with γ_N.

4.4. *C^δ and $W^{1,p}$ estimates.*

The special structure permits to obtain additional estimates in C^δ and $W^{1,p}$, $p > 2$. We perform first a calculation similar to that leading to the H_0^1 estimate. To u_ν we associate a constant c_ν which is arbitrary except

$$|c_\nu| \leq \rho \tag{4.37}$$

and set now

$$F = \prod_{\nu=1}^{N} \exp \beta(\gamma_\nu(u_\nu - c_\nu))\,.$$

Let also ψ be a function such that

$$\psi \geq 0,\ \psi \in C^1(\bar{\mathcal{O}}),\ \psi|_{\partial\mathcal{O}} = 0 \quad \text{if and only if one of the constants} \quad c_\nu \neq 0. \tag{4.38}$$

We test (4.1) with $F\gamma_\nu\beta'(\gamma_\nu(u_\nu - c_\nu))\psi$, which vanishes on the boundary of $\mathcal{O}$. We obtain instead of (4.33)

$$\sum_\nu \frac{1}{2}\int_{\mathcal{O}} \gamma_\nu^2 |Du_\nu|^2 e^{\gamma_\nu(u_\nu - c_\nu)} F\psi\, dx + \frac{1}{2}\int_{\mathcal{O}} DF\cdot D\psi\, dx \leq \tag{4.39}$$

$$\leq \frac{1}{2}\int_{\mathcal{O}} FQ\cdot Q\psi\, dx + \int_{\mathcal{O}} \sum_\nu \gamma_\nu H_\nu^0(Du) F(e^{\gamma_\nu(u_\nu - c_\nu)} - 1)\psi\, dx\,.$$

Introduce now

$$X = \prod_{\nu=1}^{N} (\exp \beta(\gamma_\nu (u_\nu - c_\nu)) + \exp \beta(-\gamma_\nu (u_\nu - c_\nu)), \tag{4.40}$$

$$X_\nu = X \frac{e^{\gamma_\nu (u_\nu - c_\nu)} \exp \beta(\gamma_\nu (u_\nu - c_\nu)) + e^{-\gamma_\nu (u_\nu - c_\nu)} \exp \beta(-\gamma_\nu (u_\nu - c_\nu))}{\exp \beta(\gamma_\nu (u_\nu - c_\nu)) + \exp \beta(-\gamma_\nu (u_\nu - c_\nu))},$$

$$\tilde{X}_\nu = X \frac{(e^{\gamma_\nu (u_\nu - c_\nu)} - 1) \exp \beta(\gamma_\nu (u_\nu - c_\nu)) - (e^{-\gamma_\nu (u_\nu - c_\nu)} - 1) \exp \beta(-\gamma_\nu (u_\nu - c_\nu))}{\exp \beta(\gamma_\nu (u_\nu - c_\nu)) + \exp \beta(-\gamma_\nu (u_\nu - c_\nu))}$$

and note that

$$DX = \sum_\nu \gamma_\nu \tilde{X}_\nu D u_\nu \,. \tag{4.41}$$

So writing (4.39) for all combinations of γ_ν and $-\gamma_\nu$, and adding up yields

$$\frac{1}{2} \sum_\nu \int_{\mathcal{O}} \gamma_\nu^2 |Du_\nu|^2 X_\nu \psi \, dx + \frac{1}{2} \int_{\mathcal{O}} DX \cdot D\psi \, dx \le \frac{1}{2} \int_{\mathcal{O}} XQQ\psi \, dx + \tag{4.42}$$

$$+ \sum_\nu \int_{\mathcal{O}} \gamma_\nu H_\nu^0 (Du) \tilde{X}_\nu \psi \, dx \,.$$

Picking the constants γ_ν so that (4.35) holds we deduce that there exists two positive constants k_0, K_0, so that

$$k_0 \int_{\mathcal{O}} |Du|^2 \psi \, dx + \int_{\mathcal{O}} DX \cdot D\psi \, dx \le K_0 \int_{\mathcal{O}} \psi \, dx \,. \tag{4.43}$$

We first apply (4.43) as follows: Let $B_R(x_0)$, $x_0 \in \mathbb{R}^n$, be the ball of center x_0 and radius R. Let τ be a cut off function such that

$$\tau = \begin{cases} 1 & \text{on } B_1(0) \\ 0 & \text{outside } B_2(0) \end{cases}$$

and $0 \le \tau \le 1$, $\tau \in C^\infty$. We denote

$$\tau_R(x) = \tau\left(\frac{x - x_0}{R}\right).$$

We define the constants c_ν as follows:

$$c_\nu = c_\nu^R = \begin{cases} \dfrac{1}{|B_{2R}|}\displaystyle\int_{B_{2R}} u_\nu \, dx & \text{if } B_{2R} \subset \mathcal{O} \\ 0 & \text{if } B_{2R} \cap (\mathbb{R}^n - \mathcal{O}) \neq \emptyset, \end{cases} \tag{4.44}$$

and we take

$$\psi = \tau_R^2 \tag{4.45}$$

so (4.38) is verified. We extend u by 0 outside $\mathcal{O}$, and we note that

$$|\tilde{X}_\nu| \leq C|u_\nu - c_\nu^R|,$$

$$|DX| \leq C|u - c^R|\,|Du|,$$

where c^R represents the vector c_ν^R. Therefore we deduce from (4.43)

$$\int_{B_R} |Du|^2 \, dx \leq c \int_{B_{2R}} |Du| \frac{|u - c^R|}{R} \, dx + cR^n. \tag{4.46}$$

Such a property implies that

$$u_\nu \in W_0^{1,p}(\mathcal{O}), \qquad 2 \leq p < 2 + \varepsilon. \tag{4.47}$$

It is a consequence of results of Gehring and Giaquinta-Modica. Indeed, from Hölder's inequality and Poincaré's inequality, we have

$$\int_{B_{2R}} |Du| \frac{|u - c^R|}{R} \, dx \leq \frac{c}{R} \left(\int_{B_{2R}} |Du|^{\frac{2n}{n+1}} \, dx \right)^{\frac{n+1}{2n}} \left(\int_{B_{2R}} |u - c^R|^{\frac{2n}{n-1}} \, dx \right)^{\frac{n-1}{2n}}$$

$$\leq \frac{c}{R} \left(\int_{B_{6R}} |Du|^{\frac{2n}{n+1}} \, dx \right)^{\frac{n+1}{n}}.$$

Setting $z = |Du|^{\frac{2n}{n+1}}$, we have the inequality

$$\oint_{B_R} z^{\frac{n+1}{n}} \, dx \leq \left(\oint_{B_{6R}} z \, dx \right)^{\frac{n+1}{n}} + c, \tag{4.48}$$

where $\oint_{B_R} = \frac{1}{R^n} \int_{B_R}$.

This is the reverse Hölder's inequality, which implies Gehring's result, namely $z^{\frac{n+1}{n}+\varepsilon}$ is integrable for some positive ε, hence (4.47), see [7], [8].

The obtaining of C^δ is more delicate: The idea is to estimate the Morrey norm, that is to say to check that

$$\int_{B_R(x_0)} |Du|^2\,dx \le K R^{n-2+2\delta}\,, \quad \forall R\,,\ \forall x_0\,. \tag{4.49}$$

This implies C^δ. Indeed one relies on Morrey's result

$$\sup_{\substack{x,y\\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^\delta} \le C\left(\frac{\sup_{x_0,R} \int_{B_R(x_0)} |Du|^2\,dx}{R^{n-2+2\delta}}\right)^{1/2}. \tag{4.50}$$

To check (4.49) we shall prove the inequality

$$\int_{B_R} |Du|^2|x-x_0|^{2-n}\,dx \le C\int_{B_{\sigma R}-B_R} |Du|^2|x-x_0|^{2-n}\,dx + CR^\beta, \tag{4.51}$$

where $\sigma > 2$, $\beta > 0$. One then relies on the hole filling technique of Widman [13], to assert that (4.49) holds for some δ such that $2\delta < \beta$. Indead (4.51) implies

$$\int_{B_R} |Du|^2|x-x_0|^{2-n}\,dx \le \theta\int_{B_{\sigma R}} |Du|^2|x-x_0|^{2-n}\,dx + CR^\beta \tag{4.52}$$

with $\theta < 1$, and setting

$$\varphi(R) = R^{-2\delta}\int_{B_R} |Du|^2|x-x_0|^{2-n}\,dx$$

with $2\delta < \beta$, $\mu = \theta\sigma^{2\delta} < 1$, we deduce

$$\varphi(R) \le \mu\varphi(\sigma R) + C \qquad \text{for } R < R_0$$

which implies $\varphi(R) \le C$, provided $\varphi(R_0) < \infty$.

Proof of (4.51). We consider the Green function $G = G^{x_0}$, solution of

$$-\frac{1}{2}\Delta G^{x_0} = \delta(x-x_0), \qquad x_0 \in Q,$$
$$G|_{\partial Q} = 0,$$

where $Q \supset \bar{\mathcal{O}}$. We know that (see (4.16), (4.17))

$$G^{x_0} \in L^q(Q) \cap W_0^{1,n}(Q), \quad 1 \le q < \frac{n}{n-2}, 1 \le n < \frac{n}{n-1} \tag{4.53}$$

and also

$$c_0 |x - x_0|^{2-n} \le G^{x_0} \le c_1 |x - x_0|^{2-n} \tag{4.54}$$

in a neighborhood of x_0 strictly included in Q. So this estimate will be valid on $\bar{Q}$. We apply (4.43) with

$$\psi = G^{x_0} \tau_R^2 \tag{4.55}$$

and

$$c_\nu = c_\nu^R = \begin{cases} \dfrac{1}{|B_{2R} - B_{R/2}|} \displaystyle\int_{B_{2R} - B_{R/2}} u_\nu \, dx & \text{if } B_{2R} \subset \mathcal{O} \\ 0 & \text{if } B_{2R} \cap (\mathbb{R}^n - \mathcal{O}) \neq \emptyset. \end{cases} \tag{4.56}$$

Consider the various terms in (4.43). We first have

$$k_0 \int_{\mathcal{O}} |Du|^2 G^{x_0} \tau_R^2 \, dx \ge c \int_{B_R} |Du|^2 |x - x_0|^{2-n} \, dx. \tag{4.57}$$

We next have

$$\int_{\mathcal{O}} G^{x_0} \tau_R^2 \, dx \le C R^{\frac{n}{q'}} = C R^\beta, \qquad \beta < 2. \tag{4.58}$$

Next

$$D\psi = DG\tau_R^2 + 2G\tau_R D\tau_R.$$

Consider

$$I = \int_{\mathcal{O}} DX D\tau_R G \tau_R \, dx$$

and using

$$|dX| \le C|u - c^R| \, |Du|, \tag{4.59}$$

we have

$$|I| \le \int_{B_{2R} - B_R} |Du| \frac{|u - c^R|}{R} G \, dx \le$$

$$\leq c \int_{B_{2R}-B_R} |Du|^2 \, |x - x_0|^{2-n} \, dx + c \int_{(B_{2R}-B_R)\cap\mathcal{O}} \frac{|u - c^R|^2}{R^2} |x - x_0|^{2-n} \, dx \, .$$

But

$$\int_{(B_{2R}-B_R)\cap\mathcal{O}} \frac{|u - c^R|^2}{R^2} |x - x_0|^{2-n} \, dx \leq \frac{c}{R^n} \int_{(B_{2R}-B_R)\cap\mathcal{O}} |u - c^R|^2 \, dx \leq$$

$$\leq \frac{c}{R^n} \int_{(B_{2R}-B_{R/2})\cap\mathcal{O}} |u - c^R|^2 \, dx \, ,$$

and by Poincaré's inequality

$$\int_{(B_{2R}-B_{R/2})\cap\mathcal{O}} |u - c^R|^2 \leq CR^2 \int_{B_{\sigma R}-B_{R/2}} |Du|^2 \, dx \, , \quad \sigma > 2,$$

hence collecting results, one has

$$|I| \leq C \int_{B_{\sigma R}-B_{R/2}} |Du|^2 \, |x - x_0|^{2-n} \, dx \, . \tag{4.60}$$

The other term

$$II = \int_{\mathcal{O}} DXDG\tau_R^2 \, dx \, , \tag{4.61}$$

which involves DG, is more complicated to estimate. We have to change X into $X - 2^N$:

$$II = \int_{\mathcal{O}} DG \cdot D((X - 2^N)\tau_R^2) \, dx - 2 \int_{\mathcal{O}} DGD\tau_R(X - 2^N)\tau_R \, dx \, .$$

Without loss of generality we can assume that $B_{2R} \subset Q$, hence using $X \geq 2^N$, we have from the definition of the Green function

$$II > -c \int_{(B_2R-B_R)\cap\mathcal{O}} |DG| \frac{|u - c^R|^2}{R} \tau_R \, dx \, ,$$

where we have used the property

$$|X^N - 2^N| \leq C|u - c^R|^2 \, . \tag{4.62}$$

We estimate II from below as follows:

$$II > -c\int_{(B_{2R}-B_R)\cap\mathcal{O}} G\frac{|u-c^R|^2}{R^2}\,dx - c\int_{(B_2R-B_R)\cap\mathcal{O}} G^{-1}|DG|^2\,|u-c^R|^2\tau_R^2\,dx\,.$$

The first term in the right-hand side is estimated by the right-hand side of (4.60).

We need thus to estimate the term

$$III = \int_{(B_2R-B_R)\cap\mathcal{O}} G^{-1}|DG|^2\,|u-c^R|^2\tau_R^2\,dx\,.$$

To estimate this quantity, we introduce a new cut off function satisfying

$$\xi = \begin{cases} 0 & \text{for } |x| \le \frac{1}{2} \\ \tau & \text{for } |x| > 1\,, \end{cases}$$

and we set

$$\xi_R(x) = \xi\left(\frac{x-x_0}{R}\right).$$

Thus

$$\xi_R = \tau_R \text{ on } B_{2R} - B_R\,.$$

From the Green function equation we deduce, by testing with $G^{-1}|u-c^R|^2\xi_R^2$, which vanishes in x_0, and on the boundary of $\mathcal{O}$

$$\text{(4.63)}\quad \frac{1}{4}\int_{\mathcal{O}} |DG|^2 G^{-\frac{3}{2}}|u-c^R|^2\xi_R^2\,dx = \int_{\mathcal{O}} D(|u-c^R|^2\xi_R^2)\cdot DGG^{-\frac{1}{2}}\,dx.$$

On the other hand

$$\int_{\mathcal{O}}(-\frac{1}{2}\Delta u_\nu)(u_\nu - c_\nu^R)G^{\frac{1}{2}}\xi_R^2\,dx = \frac{1}{2}\int_{\mathcal{O}} |Du_\nu|^2 G^{\frac{1}{2}}\xi_R^2\,dx +$$
$$+\frac{1}{8}\int_{\mathcal{O}} D(\xi_R^2|u-c^R|^2)DGG^{-\frac{1}{2}}\,dx +$$
$$+\int_{\mathcal{O}} Du_\nu D\xi_R\xi_R(u_\nu - c_\nu^R)G^{\frac{1}{2}}\,dx -$$
$$-\frac{1}{4}\int_{\mathcal{O}} D\xi_R DG|u-c^R|^2G^{-\frac{1}{2}}\xi_R\,dx\,,$$

so

$$\frac{1}{8}\int_{\mathcal{O}} D(\xi_R^2|u-c^R|^2)DGG^{-\frac{1}{2}}\,dx \le \int_{\mathcal{O}} H_\nu(u_\nu - c_\nu^R)G^{\frac{1}{2}}\xi_R^2\,dx -$$

$$-\int_{\mathcal{O}} Du_\nu D\xi_R \xi_R (u_\nu - c_\nu^R) G^{\frac{1}{2}}\, dx + \frac{1}{4}\int_{\mathcal{O}} D\xi_R DG |u - c^R|^2 G^{-\frac{1}{2}} \xi_R\, dx\,,$$

and from the quadratic growth of H_ν

$$\frac{1}{8}\int_{\mathcal{O}} D(\xi_R^2 |u - c^R|^2) DGG^{-\frac{1}{2}}\, dx \le CR^{n(1-\frac{1}{2q})} + CR^{\frac{2-n}{2}} \int_{B_{2R}-B_{R/2}} |Du|^2\, dx +$$

$$+ CR^{\frac{2-n}{2}} \int_{(B_{2R}-B_{R/2})\cap\mathcal{O}} \frac{|u - c^R|^2}{R^2}\, dx + \frac{1}{4}\int_{\mathcal{O}} D\xi_R DG |u - c^R|^2 G^{-1/2} \xi_R\, dx\,.$$

Furthermore

$$\int_{\mathcal{O}} D\xi_R DG |u - c^R|^2 G^{-1/2} \xi_R\, dx \le C\delta \int_{\mathcal{O}} |DG|^2 G^{-3/2} |u - c^R|^2 \xi_R^2\, dx +$$

$$+ \frac{c}{\delta} R^{\frac{2-n}{2}} \int_{(B_{2R}-B_{R/2})\cap\mathcal{O}} \frac{|u - C^R|^2}{R^2}\, dx\,.$$

Collecting results and choosing δ sufficiently small, we have

$$\int_{\mathcal{O}} |DG|^2 G^{-3/2} |u - c^R|^2 \xi_R^2\, dx \le CR^{n(1-\frac{1}{2q})} +$$

$$+ CR^{\frac{2-n}{2}} \int_{(B_{2R}-B_{R/2})\cap\mathcal{O}} \frac{|u - c^R|^2}{R^2}\, dx + CR^{\frac{2-n}{2}} \int_{B_{2R}-B_{R/2}} |Du|^2\, dx\,,$$

and from Poincaré's inequality it follows

$$\int_{\mathcal{O}} |DG|^2 G^{-3/2} |u - c^R|^2 \xi_R^2\, dx \le CR^{n(1-\frac{1}{2q})} + CR^{\frac{2-n}{2}} \int_{B_{\sigma R}-B_{R/2}} |Du|^2\, dx\,.$$

Going back to the definition of III and recalling that $\xi_R = \tau_R$ on $B_{2R} - B_R$ we get

$$III \le CR^{\frac{2-n}{2}} \int_{\mathcal{O}} G^{-\frac{3}{2}} |DG|^2\, |u - c^R|^2 \xi_R^2\, dx\,,$$

and from the previous estimate

$$III \le CR^{2-n} \int_{B_{\sigma R}-B_{R/2}} |Du|^2\, dx + CR^{1+\frac{n}{2q'}} \le$$

$$\le C \int_{B_{\sigma R}-B_{R/2}} |Du|^2\, |x - x_0|^{2-n}\, dx + CR^{1+\frac{n}{2q'}}\,.$$

Therefore we can state the following inequality:

$$III \geq -\int_{B_{\sigma R}-B_{R/2}} |Du|^2 \, |x-x_0|^{2-n} \, dx - CR^{1+\frac{n}{2q'}} ,$$

and from (4.43) we immediately get (4.51) with $\beta = \frac{n}{q'}$, since $\frac{n}{q'} < 1 + \frac{n}{2q'}$.

To apply the Hole filling technique it remains to verify

$$\int_{\mathcal{O}} |Du|^2 \, |x-x_0|^{2-n} \, dx < C . \tag{4.64}$$

For that we use again (4.43) with $c_\nu = 0$ and $\psi = G$. Since

$$\int_{\mathcal{O}} DX \cdot DG \, dx > 0 ,$$

the result (4.65) is obvious.

4.5. $W^{2,s}$*-estimates.*

From the linear theory and $W^{1,p}$, C^δ estimates, for some $p > 2$, $\delta > 0$ we can derive $W^{2,s}$-estimates for any s. This is thanks to an interpolation result and a boot strap argument. Indeed, if $u^\nu \in W^{1,p_0} \cap C^\delta$, $p_0 > 2$, $\delta > 0$, then $\Delta u^\nu \in L^{\frac{p_0}{2}}$. Therefore $u^\nu \in W^{2,\frac{p_0}{2}}$ from linear regularity theory. It follows from Miranda-Nirenberg's interpolation theorem, see [12], that

$$u^\nu \in W^{1,p_1} , \qquad \text{with } \frac{1}{p_1} = \frac{1}{p_0} - \frac{\delta}{2n} ,$$

provided $p_0 < \frac{2n}{\delta}$, and thus $p_1 > p_0$. After a finite number of steps we get $p_1 \geq \frac{2n}{\delta}$, and it follows that

$$u^\nu \in W_0^{1,p}(\mathcal{O}) , \qquad p > 2n ,$$

and from the linear theory again

$$u^\nu \in W^{2,s}(\mathcal{O}) , \qquad s > n .$$

From Sobolev's imbedding theorem, $u^\nu \in W^{1,n}$, $\forall n$, and from the linear theory again $u^\nu \in W^{2,s}$, $\forall s$.

4.6. *The case of a positive* 0*-order term.*

As a preliminary to problem (4.1) we consider the problem

$$-\frac{1}{2}\Delta u_\nu + \alpha u_\nu = H_\nu(x, Du) \tag{4.65}$$
$$u_\nu|_{\partial\mathcal{O}} = 0$$

with $\alpha > 0$. For such a problem we can weaken the assumptions (4.7), (4.8) and (4.9), (4.10) as follows:

$$\sum_\nu H_\nu(x, p) \leq -\lambda - \bar{\lambda}|\sum_\nu p_\nu|^2, \quad \bar{\lambda} \geq 0, \quad \lambda \geq 0. \tag{4.66}$$

There exists a matrix Γ which satisfies the maximum principle and

$$H_\nu^\Gamma(x, p) \leq \lambda_\nu + \lambda_\nu^0|p_\nu|^2, \quad \lambda_\nu, \lambda_\nu^0 \geq 0, \tag{4.67}$$

$$\sum_\nu H_\nu(x, p) \leq \lambda + \bar{\lambda}|\sum_\nu p_\nu|^2, \quad \bar{\lambda} \geq 0, \ \lambda \geq 0. \tag{4.68}$$

There exists a matrix Γ which satisfies the maximum principle and

$$H_\nu^\Gamma(x, p) \geq -\lambda_\nu - \lambda_\nu^0|p_\nu|^2, \quad \lambda_\nu, \ \lambda_\nu^0 \geq 0. \tag{4.69}$$

Theorem 4.2. *We assume that the functions* $H_\nu(x, p)$ *satisfy* (4.2), (4.11), (4.12), (4.13) *and one or the other of the two sets of assumptions* (4.66), (4.67) *or* (4.68), (4.69). *Then for* $\alpha > 0$ *there exists a solution of* (4.65) *which is in* $W^{2,s}(\mathcal{O})$, $\forall s$, *such that* $2 \leq s < \infty$.

If one considers the developments of Sections 4.2, 4.3, 4.4. 4.5, the only thing which fails is the treatment of the L^∞-estimate. But to recover the L^∞-estimate in the present framework is very easy since we can rely on maximum principle arguments. Indeed, assuming first (4.66), (4.67), then we have

$$-\frac{1}{2}\Delta\sum_\nu u_\nu + \alpha(\sum_\nu u_\nu) \geq -\lambda - \bar{\lambda}|D\sum_\nu u_\nu|^2,$$

and if ξ is a point of negative minimum of $\sum_\nu u_\nu$, necessarily $D\sum_\nu u_\nu(\xi) = 0$, $\Delta\sum_\nu u_\nu(\xi) \geq 0$, hence

$$\sum_\nu u_\nu(\xi) \geq -\frac{\lambda}{\alpha}.$$

Therefore

$$\sum_{\nu} u_\nu(x) \geq -\frac{\lambda}{\alpha}. \tag{4.70}$$

Next considering

$$\tilde{u} = \Gamma u, \tag{4.71}$$

we get the system

$$-\frac{1}{2}\Delta\tilde{u}_\nu + \alpha\tilde{u}_\nu = H^\Gamma(x, D\tilde{u}),$$

hence from the assumption (4.67), it follows

$$-\frac{1}{2}\Delta\tilde{u}_\nu + \alpha\tilde{u}_\nu \leq \lambda_\nu + \lambda_\nu^0|D\tilde{u}_\nu|^2,$$

and if ξ_ν is a point of positive maximum, necessarily

$$\tilde{u}_\nu(\xi_\nu) \leq \frac{\lambda_\nu}{\alpha}.$$

Therefore

$$\tilde{u}_\nu(x) \leq \frac{\lambda_\nu}{\alpha}, \tag{4.72}$$

and since Γ satisfies the maximum principle, we get

$$u_\nu(x) \leq \frac{1}{\alpha}\sum_{\mu}(\Gamma^{-1})_{\nu\mu}\lambda_\nu, \tag{4.73}$$

which combined with (4.70) yields the result.
The case of the assumptions (4.68), (4.69) is treated in a similar way.

Once we know L^∞-estimates on αu_ν, then this term can be incorporated in the Lagrangian and the developments of Sections 4.3, 4.4, 4.5 can be made. Of course all the estimates on the functions u_ν depend on α.

Proof of Theorem 4.2. One considers an approximation as follows:

$$\begin{aligned} -\frac{1}{2}\Delta u_\nu + \alpha u_\nu &= \frac{H_\nu(x, Du)}{1+\varepsilon|H(x, Du)|} = H_\nu^\varepsilon(x, Du) \\ u_\nu|_{\partial\mathcal{O}} &= 0, \end{aligned} \tag{4.74}$$

where $H(x, p)$ represents the vector $H_\nu(x, p)$. Clearly the right-hand side is bounded in L^∞ by a constant depending on ε.

To show that (4.74) has a solution in $W^{2,s}(\mathcal{O}) \cap W_0^{1,s}(\mathcal{O})$, $\forall 2 \leq s < \infty$, one relies on Schauder's fixed point Theorem. Indeed consider the set, for s arbitrary

$$K_\varepsilon = \left\{ z \in (W_0^{1,s}(\mathcal{O}))^N \mid \|z\|_{(W^{2,s})^N} \leq C \right\}, \tag{4.75}$$

which is compact in $(W_0^{1,s}(\mathcal{O}))^N$. Consider next the map from $(W_0^{1,s})^N$ into itself, $T^\varepsilon \zeta = z$, by solving

$$-\frac{1}{2}\Delta z_\nu + \alpha z_\nu = H_\nu^\varepsilon(x, D\xi)$$
$$z_\nu|_{\partial\mathcal{O}} = 0 .$$

By choosing conveniently the constant C in the definition of K_ε, one can check that T^ε maps K_ε into itself. Moreover, from (4.2), T^ε is continuous. Hence the fixed point property applies. Now it is easy to check that H^ε satisfies all the assumptions of H, with the same constants, hence independent of ε. Hence the solution u^ε of (4.74) remains bounded in $(W^{2,s}(\mathcal{O}))^N$ norm, independently of ε. Hence letting ε tend to 0, one obtains a solution of (4.65). □

Proof of Theorem 4.1. We can proceed on the solution u^α of (4.65) in $W^{2,s}$ which has been obtained with the development of Sections 4.2, 4.3, 4.4, 4.5. Thanks to the assumptions of Theorem 4.1 we can obtain estimates in $W^{2,s}$, which are independent of α. We can then let α tend to 0 to complete the proof. □

5. Hamiltonians arising from games.

5.1. *Set up: case* (4.7), (4.8).

We define here the Hamiltonians $H_\nu(x, p)$ as follows:

$$H_\nu(x, p) = f_\nu(x) + g \cdot p_\nu + L_\nu(p) + \frac{\delta}{2}|p_\nu|^2, \tag{5.1}$$

where $L_\nu(p)$ has been defined by (3.8) and f_ν and g are the functions which arise in (2.10), (2.16). We first check (4.7), (4.8) with $\Gamma = I$. Using (3.53), (3.70) and formula (5.1), we obtain (4.8) with

$$\lambda_\nu^0 = \frac{1 - 2\theta}{2\theta^2} + \frac{\delta}{2} + \frac{\varepsilon}{2}, \tag{5.2}$$
$$\lambda_\nu = \|f_\nu\| + \frac{1}{2\varepsilon}\|g\|^2 .$$

Using (3.55) we have

$$\sum_\nu H_\nu(x,p) \geq -\|\sum_\nu f_\nu\| - \frac{N}{2\varepsilon}\|g\|^2 - \frac{\varepsilon}{2}|p|^2 + \frac{F_\delta(\theta)|p|^2}{(1-\theta)^2(1+(N-1)\theta)^2},$$

we obtain (4.7) with

$$\lambda = \|\sum_\nu f_\nu\| + \frac{N}{2\varepsilon}\|g\|^2 \tag{5.3}$$

provided θ belongs to the validity intervals defined in (3.70). If the product $\lambda_\nu\lambda_\nu^0$ satisfies the condition (4.22), then the properties (4.7), (4.8) are satisfied.

Recalling that $F_\delta(\theta) \geq F(\theta)$, we may pick $\theta < \theta'$ or $\theta'' < \theta < \bar{\theta}$, and we get a condition of smallness on δ (note that $\theta < 0$), from (4.22).

Let us verify (4.11), (4.12), (4.13). We define the matrix Γ as follows:

$$\begin{aligned}
&\Gamma_{\nu\mu} = \delta_{\nu\mu} \text{ if } \nu = 1,\dots,N,\ \mu = 1,\dots,N-1,\\
&\Gamma_{\nu N} = -1 \text{ if } \nu = 1,\dots,N-1,\\
&\Gamma_{NN} = 1,
\end{aligned}$$

hence setting $\tilde{p} = \Gamma p$, we have

$$H_\nu^\Gamma(x,\tilde{p}) = H_\nu(x,p) - H_N(x,p), \qquad \forall \nu < N,$$

$$H_N^\Gamma(x,\tilde{p}) = H_N(x,p).$$

Using (3.12) it follows, after easy computations,

$$H_\nu^\Gamma(x,\tilde{p}) = Q(\tilde{p})\tilde{p}_\nu + H_\nu^0(x,\tilde{p}),$$

with

$$Q(\tilde{p}) = \frac{2\theta-1}{(1-\theta)^2(1+(N-1)\theta)}\sum_\mu \tilde{p}_\mu + \left(\frac{1-2\theta}{(1-\theta)^2} + \delta + N - 1\right)p_N,$$

$$H_\nu^0(x,\tilde{p}) = f_\nu - f_N + g\cdot\tilde{p}_\nu + \frac{1}{2}\left(\delta + \frac{1-2\theta}{(1-\theta)^2}\right)|\tilde{p}_\nu|^2,\ \nu < N,$$

$$H_N^\Gamma(x,\tilde{p}) = H_N(x,p) - Q(\tilde{p})\tilde{p}_N$$

and the assumptions (4.11), (4.12), (4.13) are easily verified.

5.2. *Set up: Case* (4.9), (4.10).

We are going to check satisfying (4.9), (4.10) with the following Γ:

$$\Gamma_{\nu\nu} = (N-1)\theta\,, \quad \Gamma_{\nu\mu} = -1 \qquad \text{if } \mu \neq \nu\,. \tag{5.4}$$

This matrix satisfies the maximum principle, provided $\theta > 1$. Indeed its inverse is

$$(\Gamma^{-1})_{\nu\nu} = \frac{(N-1)\theta - (N-2)}{(N-1)(\theta-1)((N-1)\theta+1)}\,,$$

$$(\Gamma^{-1})_{\nu\mu} = \frac{1}{(N-1)(\theta-1)((N-1)\theta+1)}\,, \qquad \mu \neq \nu\,,$$

which are positive, under the assumption on θ. This guarantees the maximum principle. We set

$$\tilde{p} = \Gamma p\,,$$

and we have (see (3.45), (3.70))

$$M_\nu^\Gamma(\tilde{p}) = \tilde{M}_\nu(p)\,.$$

According to formula (3.79) and recalling (3.44), we get

$$M_\nu^\Gamma(\tilde{p}) \geq -k_0|\tilde{p}_\nu|^2 \tag{5.5}$$

with

$$k_0 = \frac{1}{2(1-\theta^2)^2(1+(N-1)\theta)^2}\left\{\frac{(1-\theta)^2(1-\delta\theta(N-1))^2}{(N-1)\theta(1-\delta(\theta-1))} + \right. \tag{5.6}$$

$$\left. + \ (N-1)\theta(2\theta-1)\frac{\theta-1}{N-1} - \delta\left(\theta(\theta-1)(\theta(N-1)-N+3) + \frac{1}{N-1}\right)\right\}^+$$

provided we assume

$$\theta - \frac{1}{2} - \frac{\delta}{2}(\theta-1)^2 > 0 \tag{5.7}$$

$$1 - \delta(\theta-1) > 0\,. \tag{5.8}$$

Next,

$$H_\nu^\Gamma(x,\tilde{p}) = (N-1)\theta f_\nu(x) - \sum_{\mu\neq\nu} f_\mu(x) + g\cdot\tilde{p}_\nu + M_\nu^\Gamma(\tilde{p}) \geq$$

$$\geq -\lambda_\nu - \lambda_\nu^0 |\tilde{p}_\nu|^2$$

with

$$(5.9)\qquad \begin{aligned} \lambda_\nu &= \|(N-1)\theta f_\nu - \sum_{\mu\neq\nu} f_\mu\| + \frac{1}{2\varepsilon}\|g\|^2 \\ \lambda_\nu^0 &= k_0 + \frac{\varepsilon}{2}, \end{aligned}$$

and (4.10) is satisfied, provided $\lambda_\nu \lambda_\nu^0$ verifies (4.22).
Next we have

$$\sum_\nu H_\nu(x,p) \leq \|\sum_\nu f_\nu\| + \frac{N}{2\varepsilon}\|g\|^2 + \frac{\varepsilon}{2}|p|^2 + \frac{\hat{F}_\delta(\theta)|p|^2}{(1-\theta)^2(1+(N-1)\theta)^2},$$

and we obtain (4.9) provided the conditions of Lemma 3.3 are fulfilled, namely $\theta_0' < \bar{\theta}_\delta'$ and $\theta \in (\theta_{1\delta}''', \bar{\theta}_\delta')$. Since $\theta > 1$, and thanks to (3.66), this reduces to $\bar{\theta}_\delta' > 1$, and $\theta \in (1, \bar{\theta}_\delta')$. Considering (3.61), this means

$$(5.10)\qquad \delta < \frac{1+2N(N-1)}{N^2},$$

$$(5.11)\qquad 1 < \theta < \frac{1}{\delta}\left(1 - \frac{\delta}{N-1} + \sqrt{\frac{1+\delta}{(N-1)^2}}\right).$$

Summing up we can assert the

Proposition 5.1. *For the Hamiltonians* (5.1), *the assumptions* (4.11), (4.12), (4.13) *are satisfied. The conditions* (4.7), (4.8) *are verified with* λ_ν^0, λ_ν *given by* (5.2). λ *given by* (5.3) *with* $\Gamma = I$, *provided* θ *belongs to the validity intervals defined in* (3.70) *and* $\lambda_\nu \lambda_\nu^0$ *verifies the condition* (4.22). *On the other hand, the conditions* (4.9), (4.10) *are satisfied with* λ_ν^0, λ_ν *given by* (5.9), λ *given by* (5.3), *with* Γ *defined by* (5.4), *provided* $\theta > 1$ *and* δ *verify* (5.7), (5.8), (5.10), (5.11), $\lambda_\nu^0 \lambda_\nu$ *verifies the condition* (4.22).

6. Solution of the stochastic game problem.

6.1. *Bellman system.*

We relate to problem (2.17) the system of Bellman equations

$$(6.1)\qquad \begin{aligned} &-\frac{1}{2}\Delta u_\nu = f_\nu + g\cdot Du_\nu + L_\nu(Du) + \frac{\delta}{2}|Du_\nu|^2 \\ &u_\nu|_{\partial\mathcal{O}} = 0\,, \end{aligned}$$

and we assume that all conditions of Proposition 5.1 are satisfied. Therefore, according to Theorem 4.1, there exists a solution in $(W^{2,s}(\mathcal{O})\cap W_0^{1,s}(\mathcal{O}))^N$, $\forall 1 \leq s < \infty$. We define the feedbacks

$$(6.2)\qquad \hat{v}_\nu(x) = v_\nu(Du(x)),$$

where $v_\nu(p)$ are given by (3.6). These functions are bounded and continuous. To such a feedback we associate the stochastic processes

$$(6.3)\qquad \hat{v}_\nu(t) = \hat{v}_\nu(x(t)),$$

where $x(t)$ is the process defined by (2.4). Our objective is to check that such a vector process $\hat{v}(t)$ is the Nash point of the functionals $J_\nu(x, v(.))$ defined by (2.16), in the sense of (2.17). We state then

Theorem 6.1. *We assume all the conditions of Proposition 5.1, so that there exists a solution of the system* (6.1) *in the space* $(W^{2,s}(\mathcal{O})\cap W_0^{1,s}(\mathcal{O}))^N$. *Then the control defined by* (6.3), *corresponding to the feedback defined by* (6.2) *is a Nash point of the functionals* $J_\nu(x, v(.))$ *defined by* (2.16) *in the sense of* (2.17). *Moreover one has*

$$(6.4)\qquad u_\nu(x) = J_\nu(x, \hat{v}(.))\,.$$

6.2. *Proof of Theorem 6.1.*

Note that from the Defintion (6.2) we have

$$(6.5)\qquad \begin{aligned} L_\nu(Du) &= \frac{1}{2}|\hat{v}_\nu(Du)|^2 + \theta\hat{v}_\nu(Du)\cdot\bar{\hat{v}}_\nu(Du) \leq \\ &\leq \frac{1}{2}|v_\nu|^2 + \theta v_\nu\cdot\bar{\hat{v}}_\nu(Du), \qquad \forall x\,,\ \forall v_\nu\,. \end{aligned}$$

Applying this inequality to $x = x(t)$, yields, from (6.3)

$$L_\nu(Du(x(t))) = \frac{1}{2}|\hat{v}_\nu(t)|^2 + \theta\hat{v}_\nu(t)\cdot\bar{\hat{v}}_\nu(t) \leq \tag{6.6}$$

$$\leq \frac{1}{2}|v_\nu(t)|^2 + \theta v_\nu(t)\cdot\bar{\hat{v}}_\nu(t)\,,$$

for any control v_ν.

Consider the probability $P_{x,\hat{v}}$ defined by (2.8), where $\hat{v}(.)$ is the process defined by (6.3). Note that for the system $(\Omega, \mathcal{A}, \mathcal{F}^t, P_{x,\hat{v}}, w_{x,\hat{v}})$ we have (see (2.10))

$$dx = \Big(g(x(t)) + \sum_\mu \hat{v}_\mu(t)\Big)\,dt + dw_{x,\hat{v}}(t)\,, \quad x(0) = x\,. \tag{6.7}$$

From Ito's formula we have

$$du_\nu(x(t)) = \Big(Du_\nu(x(t))\cdot(g(x(t)) + \sum_\mu \hat{v}_\mu(t)) +$$

$$+ \frac{1}{2}\Delta u_\nu(x(t))\Big)\,dt + Du_\nu(x(t))\,dw_{x,\hat{v}}(t)$$

and from the Bellman equation (6.1) we deduce

$$du_\nu(x(t)) = -f_\nu(x(t)) - \frac{1}{2}|\hat{v}_\nu(t)|^2 - \theta\hat{v}_\nu(t)\cdot\bar{\hat{v}}_\nu(t) - \tag{6.8}$$

$$-\frac{\delta}{2}|Du_\nu(x(t))|^2 + Du_\nu(x(t))\cdot dw_{x,\hat{v}}(t)\,.$$

Integrating between 0 and τ_x yields

$$u_\nu(x) = \int_0^{\tau_x}\Big[f_\nu(x(t)) + \frac{1}{2}|\hat{v}_\nu(t)|^2 + \theta\hat{v}_\nu(t)\cdot\bar{\hat{v}}_\nu(t)\Big]\,dt - \tag{6.9}$$

$$-\int_0^{\tau_x} Du_\nu(x(t))dw_{x,\hat{v}}(t) + \frac{\delta}{2}\int_0^{\tau_x}|Du_\nu(x(t))|^2\,dt$$

hence

$$\exp\delta\int_0^{\tau_x}\Big[f_\nu(x(t)) + \frac{1}{2}|\hat{v}_\nu(t)|^2 + \theta\hat{v}_\nu(t)\cdot\bar{\hat{v}}_\nu(t)\Big]\,dt = \tag{6.10}$$

$$= \exp \delta u_\nu(x) \exp \left(\int_0^{\tau_x} \delta D u_\nu(x(t)) dw_{x,\hat{v}}(t) - \frac{1}{2} \int_0^{\tau_x} |\delta D u_\nu(x(t))|^2 dt \right).$$

Since

$$\exp \left(\int_0^t \delta D u_\nu(x(s)) dw_{x,\hat{v}}(s) - \frac{1}{2} \int_0^t |\delta D u_\nu(x(s))|^2 ds \right)$$

is an $\mathcal{F}^t$, $P_{x,\hat{v}}$ martingale, and τ_x is a stopping time, we deduce the formula

$$\text{(6.11)} \quad \exp \delta u_\nu(x) = E_{x,\hat{v}} \exp \delta \int_0^{\tau_x} \left[f_\nu(x(t)) + \frac{1}{2} |\hat{v}_\nu(t)|^2 + \theta \hat{v}_\nu(t) \cdot \bar{\hat{v}}(t) \right] dt$$

hence (6.4) is demonstrated.

Next considering an arbitrary control $v(.)$, we manufacture the control $(v_\nu(.), \hat{v}^\nu(.))$, in which we take for all components $\mu \neq \nu$ the control $\hat{v}_\mu(.)$ defined by (6.3), and for the component ν the control $v_\nu(.)$. Performing similar calculations as above and taking account this time of the inequality (6.6) we can check that

$$u_\nu(x) \leq J_\nu(x, v_\nu, \hat{v}_\nu),$$

which establishes the inequality (2.17).
The proof has been completed. □

Appendix 1.

Discussion on the smallness of $\lambda_\nu \lambda_\nu^0$.

First we argue that a smallness condition on $\lambda_\nu \lambda_\nu^0$ (see (4.8) and (4.10)) cannot avoided. Consider indeed the case of $N = 1$, and in dimension one the equation

$$-\frac{1}{2} u'' = \lambda + \lambda^0 u'^2$$
$$u(0) = u(1) = 1.$$

For $\lambda^0 \leq 0$ it is easy to check that there exists a unique bounded solution. For $\lambda^0 > 0$ the unique possible solution is

$$u(x) = \frac{1}{2\lambda^0} \log \left| \frac{\cos \sqrt{\lambda^0 \lambda}(2x-1)}{\cos \sqrt{\lambda^0 \lambda}} \right|,$$

which is bounded if and only if $\lambda^0 \lambda < \frac{\pi^2}{4}$.

Let us compare with the limitation obtained in (4.22). In the present example, the Green function $G^{\xi}(x)$ is given by

$$G^{\xi}(x) = \begin{cases} 2(1-\xi)x & \text{if } 0 \leq x \leq \xi \\ 2\xi(1-x) & \text{if } \xi \leq x \leq 1 \end{cases}$$

and

$$\int_0^1 G^{\xi}(x)\,dx = \xi(1-\xi) \leq \frac{1}{4}\,.$$

Therefore, condition (4.22) means here $\lambda^0\lambda < 1$. So we do not get the optimum limitation by this method, as can be expected.

So the issue of the improvement of the limitation on $\lambda^0\lambda$ is an interesting question. We shall check in the following result that an improvement can arise from the presence of an adequate drift g. This is reminiscent of what occurs in the finite horizon problem (see [6]).

A variant of Theorem 4.1.

We consider here the problem

$$(1) \qquad \begin{aligned} &-\frac{1}{2}\Delta u_\nu - g \cdot Du_\nu = H_\nu(x, Du) \\ &u_\nu|_{\partial\mathcal{O}} = 0\,, \end{aligned}$$

which, compared to (4.1), has a linear first order term $-g \cdot Du_\nu$. Of course, the usual treatment would be to incorporate it in the Hamiltonian H_ν, but as far as condition (4.8) is concerned, this implies a deterioration of the smallness condition on $\lambda_\nu\lambda_\nu^0$. In fact, we shall see here that this term can relax to a large extent the limitation on $\lambda_\nu\lambda_\nu^0$, provided div $g > 0$.

We proceed as follows: Considering the function E_ν in (4.20), we can check that

$$(2) \qquad -\frac{1}{2}\Delta E - gDE_\nu \leq 2\lambda_\nu\lambda_\nu^0 E_\nu\,.$$

Since everything relates to the ν equation, we shall drop the symbol ν in the sequel. So, also

$$(3) \qquad \begin{aligned} &-\frac{1}{2}\Delta(E-1) - g \cdot D(E-1) \leq 2\lambda\lambda^0(E-1) + 2\lambda\lambda^0 \\ &(E-1)|_{\partial\mathcal{O}} = 0\,. \end{aligned}$$

We first check that under some smallness conditions we can obtain an estimate on $E-1$ in $L^2(\mathcal{O})$. Indeed, testing (3) with $(E-1)^+$ yields

$$\frac{1}{2}\int_{\mathcal{O}} |D(E-1)^+|^2\,dx + \int_{\mathcal{O}} \frac{\operatorname{div} g}{2}\left[(E-1)^+\right]^2 dx \leq$$

$$\leq 2\lambda\lambda^0 \int_{\mathcal{O}} \left[(E-1)^+\right]^2 dx + 2\lambda\lambda^0 \int_{\mathcal{O}} (E-1)^+\,dx\,.$$

From Poincaré's inequality we get

$$\int_{\mathcal{O}} (k_0+\operatorname{div} g)\left[(E-1)^+\right]^2 dx \leq 4\lambda\lambda^0 \int_{\mathcal{O}} \left[(E-1)^+\right]^2 dx + 4\lambda\lambda^0 \int_{\mathcal{O}} (E-1)^+\,dx\,,$$

and if the smallness condition

(4) $$4\lambda\lambda^0 < k_0 + \inf \operatorname{div} g$$

is satisfied, then clearly

(5) $$\int_{\mathcal{O}} (E-1)^2\,dx \leq |\mathcal{O}| + \int_{\mathcal{O}} \left[(E-1)^+\right]^2 dx \leq c\,,$$

hence also

(6) $$\int_{\mathcal{O}} E^2\,dx \leq c\,.$$

From that knowledge, we are going to check that E is bounded in L^∞, without using any more any smallness condition. We test (3) with EG^ξ, using again the Green function (4.15) (although the Green function related to the operator $-\frac{1}{2}\Delta - g\cdot D$ is also possible). We obtain

(7) $$\frac{1}{2}\int_{\mathcal{O}} |DE|^2 G^\xi\,dx + \frac{1}{4}\int_{\mathcal{O}} D(E^2-1)DG^\xi\,dx \leq$$

$$\leq \int_{\mathcal{O}} E^2 G^\xi \left(2\lambda\lambda^0 - \frac{1}{2}\operatorname{div} g\right) dx -$$

$$-\int_{\mathcal{O}} \frac{1}{2} E^2 g\cdot \operatorname{grad} G^\xi\,dx + \frac{1}{2}\int_{\mathcal{O}} (G^\xi \operatorname{div} g + g\cdot \operatorname{grad} G^\xi)\,dx\,,$$

and from the definition of the Green function we deduce

$$E^2(\xi) \le 1 + \int_{\mathcal{O}} E^2 G^\xi (4\lambda\lambda^0 - \mathrm{div} g)\, dx - \int_{\mathcal{O}} E^2 \mathrm{grad}\, G^\xi\, dx + \\ + \int_{\mathcal{O}} (G^\xi \mathrm{div} g + g \cdot \mathrm{grad} G^\xi)\, dx\,. \tag{8}$$

We now make use of (6). Consider the set $\{x : E(x) > L\}$, where L is large > 1. We split the integrals on the right-hand side of (8) into the integral on this set and on its complement. Assuming ξ is chosen to be the maximum of E, supposed to be larger that 1 (otherwise the L^∞ bound is obvious), we can then deduce from (8) the inequality

$$\|E^2\|_{L^\infty} \le 1 + L^2(4\lambda\lambda^0 + \|\mathrm{div} g\|) \int_{\mathcal{O}} G^\xi\, dx + \\ + L^2\|g\| \int_{\mathcal{O}} |DG^\xi|\, dx + \|E^2\|_\infty (4\lambda\lambda^0 + \|\mathrm{div} g\|) \int_{\{E>L\}} G^\xi\, dx + \\ + \|E^2\|_\infty \|g\| \int_{\{E>L\}} |DG^\xi|\, dx + C\,. \tag{9}$$

Now from (6) we have

$$\mathrm{Meas}\, \{E > L\} \le \frac{C}{L^2}\,, \tag{10}$$

and from the estimates (4.16), (4.17) we can assert, thanks to Hölder's inequality,

$$\int_{\{E>L\}} G^\xi\, dx \le C\|G^\xi\|_{L^q} \frac{1}{L^{2q'}}\,, \tag{11}$$

$$\int_{\{E>L\}} |DG^\xi|\, dx \le C\|G^\xi\|_{L^{r'}} \frac{1}{L^{2r'}}\,. \tag{12}$$

By picking L sufficiently large, the coefficient in front of $\|E^2\|_\infty$ in the right-hand side of (9) can be made as small as we wish, hence strictly smaller than 1. Then (9) yields an estimate on the L^∞ norm of E.
The rest of the proof of Theorem 4.1 is unchanged.

Considering the limitation (4), it can be very good if $\inf \mathrm{div}\, g$ is very large. However this estimate, which can be also used when $g = 0$, is not very good in that case. For example in our one dimensional example, the Poincaré constant $k_0 = 1$, hence $\lambda\lambda^0 < \frac{1}{4}$, which is much worse than 1 obtained from (4.22).

Appendix 2.

Global smallness condition.

If we consider (4.7) and (4.8) with $\Gamma = I$, we get the property

$$-\lambda - \sum_{\mu \neq \nu} \lambda_\mu - \sum_{\mu \neq \nu} \lambda_\mu^0 |p_\mu|^2 \leq H_\nu(x, p) \leq \lambda_\nu + \lambda_\nu^0 |p_\nu|^2 , \tag{1}$$

and $\lambda_\nu \lambda_\nu^0$ not too large. This is reminiscent, although *not equivalent* to a global smallness assumption on k, K, where

$$|H(x, p)| \leq k|p|^2 + K . \tag{2}$$

In that case things simplify greatly (see [9]).

For the sake of completeness we sketch the main arguments in the case of a global small assumption. First to obtain L^∞ estimates, we test the equation

$$-\frac{1}{2} \Delta u_\nu = H_\nu(x, Du) \tag{3}$$

with $G^\xi u_\nu$, which yields

$$\int_{\mathcal{O}} D|u|^2 DG^\xi \, dx + \frac{1}{2} \int_{\mathcal{O}} G^\xi |Du|^2 \, dx = \int_{\mathcal{O}} G^\xi |u| \, |H(x, Du)| \, dx$$

hence if ξ is a point where $|u|$ reaches its maximum, we have

$$\|u\|_\infty^2 + \frac{1}{2} \int_{\mathcal{O}} G^\xi |Du|^2 \, dx \leq \|u\|_\infty \int_{\mathcal{O}} G^\xi (K + k|Du|^2) \, dx$$

thus if we have

$$2k\|u\|_\infty < 1 , \tag{4}$$

then it follows

$$\|u_\infty\| < K \int_{\mathcal{O}} G^\xi \, dx ,$$

so it is sufficient to assume

$$2kK\|G\|_{L^1} < 1 . \tag{5}$$

The H_0^1 estimate follows, since testing (3) with u_ν yields

$$\frac{1}{2}\int_{\mathcal{O}} |Du|^2\,dx \le \|u\|_\infty \int_{\mathcal{O}} |H|\,dx \le$$

$$\le \|u\|_\infty \left(\int_{\mathcal{O}} k|Du|^2\,dx + K \text{ Meas } \mathcal{O} \right),$$

and we make use of (4) to derive the H_0^1 estimate.

To obtain the C^δ and $W^{1,p}$ estimate, $2 \le p < 2+\varepsilon$, we test with $(u_\nu - c_\nu^R)\tau_R^2$, where τ_R, c_ν^R have been defined in (4.44), (4.45). We obtain

$$\frac{1}{2}\int_{\mathcal{O}} |Du|^2\tau_R^2\,dx + \int_{\mathcal{O}} (u_\nu - c_\nu^R) Du_\nu D\tau_R \tau_R\,dx \le$$

$$\le \|u - c^R\|_\infty \int_{\mathcal{O}} (k|Du|^2 + K)\tau_R^2\,dx \le 2\|u\|_\infty \int_{\mathcal{O}} (k|Du|^2 + K)\tau_R^2\,dx\,,$$

so if we assume a more stringent assumption that (5), namely

$$4kK\|G\|_{L^1} < 1\,, \tag{6}$$

which implies

$$4k\|u\|_\infty < 1\,,$$

then we get

$$c_0 \int_{B_R} |Du|^2\,dx \le C \int_{B_{2R}} |Du| \frac{|u - c^R|}{R}\,dx + CR^n\,, \tag{7}$$

which is the condition (4.46).

Similarly we can obtain (4.51) by testing with $(u_\nu - c_\nu^R)G^{x_0}\tau_R^2$, where G^{x_0} has been defined in (4.53). We get this time

$$\frac{1}{2}\int_{\mathcal{O}} |Du|^2 G^{x_0}\tau_R^2\,dx + \int_{\mathcal{O}} (u_\nu - c_\nu^R) Du_\nu D\tau_R \tau_R G^{x_0}\,dx \le$$

$$\le 2\|u\|_\infty \int_{\mathcal{O}} (k|Du|^2 + K) G^{x_0}\tau_R^2\,dx\,,$$

and by virtue of (6),

$$c_0 \int_{\mathcal{O}} |Du|^2 G^{x_0}\tau_R^2\,dx \le \int_{\mathcal{O}} |u - c^R|\,|Du|\,|D\tau_R|\tau_R G^{x_0}\,dx + C\int_{\mathcal{O}} G^{x_0}\tau_R^2\,dx\,.$$

Then making use of the properties of the Green function (see (4.54)), using Hölder's inequality, then Poincaré's inequality, we derive (4.51) easily.

It is also possible to get rid of the more stringent assumption (6) and to assume only (5), see [9].

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