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ON THE EQUIVALENCE OF PHRAGMÉN-LINDELÖF PRINCIPLES FOR HOLOMORPHIC AND PLURISUBHARMONIC FUNCTIONS

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Dedicated to Professor Sergio Campanato on his 70th birthday

The existence of solutions of the Cauchy problem for (overdetermined) systems of linear partial differential operators with constant coefficients in some classes of functions or distributions is equivalent to the validity of related Phragmén-Lindelöf-type principles for holomorphic functions on the associated (complex) characteristic variety. Here we prove the equivalence of these Phragmén-Lindelöf principles for holomorphic functions with the analogous ones for plurisubharmonic functions, which are easier to handle in the applications.

Introduction.

Many properties of constant coefficients linear partial differential operators are related to the validity of different kinds of Phragmén-Lindelöf principles (see, for instance, [10], [15], [7], [13], [16], [4], [5], [6]). In [4], [5], [6] we related evolution of (overdetermined) systems of linear p.d.e.'s to Phragmén-Lindelöf principles of the form

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$(Ph-L)$ $\forall \alpha \in \mathbb{N}\ \exists \beta \in \mathbb{N},\ c > 0$ such that
if $f \in \mathcal{O}(V)$ satisfies, for some constants $\alpha_f \in \mathbb{N}$ and $c_f > 0$:

$$\begin{cases} |f(\zeta)| \leq e^{\psi_\alpha(\zeta)} & \forall \zeta \in V \\ |f(\zeta)| \leq c_f e^{\varphi_{\alpha_f}(\zeta)} & \forall \zeta \in V, \end{cases}$$

then it also satisfies:
$|f(\zeta)| \leq c e^{\varphi_\beta(\zeta)} \quad \forall \zeta \in V,$

where $V \subset \mathbb{C}^N$ is an irreducible affine algebraic variety, $\mathcal{O}(V)$ is the space of holomorphic functions on V, and $\{\varphi_\alpha\}_{\alpha\in\mathbb{N}}$, $\{\psi_\alpha\}_{\alpha\in\mathbb{N}}$ are sequences of plurisubharmonic functions related to the spaces of functions or distributions in which we want to establish evolution (i.e. existence of solutions) of the given system.

For a convex locally closed subset K of $\mathbb{R}^N$, and an increasing sequence $\{K_\alpha\}_{\alpha\in\mathbb{N}}$ of compact convex subsets of K with K_α contained in the relative interior of $K_{\alpha+1}$ and $\bigcup_{\alpha=0}^\infty K_\alpha = K$, denoting by $H_{K_\alpha}(y) = \sup_{x\in K_\alpha}\langle x, y\rangle$ the supporting function of K_α, we considered, for example, φ_α (and analogously ψ_α) defined, for $\zeta = (\tau, \eta) \in \mathbb{C}^k_\tau \times \mathbb{C}^n_\eta \simeq \mathbb{C}^N_\zeta$, by:

(0.1) • $\varphi_\alpha(\zeta) = \alpha \log(1 + |\zeta|) + H_{K_\alpha}(\operatorname{Im}\zeta)$
in the case of C^∞ functions (cf. [4]);

(0.2) • $\varphi_\alpha(\zeta) = \alpha|\tau|^{1/r} + \alpha|\zeta^{1/s} + \alpha\log(1 + |\tau|) + H_{K_\alpha}(\operatorname{Im}\zeta)$
for $1 < r \leq +\infty$, $1 < s < +\infty$ (with $1/r = 0$ when $r = +\infty$),
in the case of (small) Gevrey functions, with possibly different scales of regularity in the time-variables t associated to τ and in the spaces-variables x associated to ζ (cf. [5]);

(0.3) • $\varphi_\alpha(\zeta) = \alpha\log(1 + |\zeta|) + \alpha\varphi(\operatorname{Im}\zeta)$
for a convex function φ with suitable properties, in the case of C^∞ functions satisfying certain growth conditions at infinity (cf. [6]).

We prove here the equivalence of the above Phragmén-Lindelöf principle (Ph-L), formulated in terms of holomorphic functions, with the analogous one for weakly plurisubharmonic functions on V, for a general class of sequences $\{\varphi_\alpha\}_{\alpha\in\mathbb{N}}$ (and $\{\psi_\alpha\}_{\alpha\in\mathbb{N}}$) which includes, as special cases, the φ_α's defined in (0.1), (0.2) and (0.3). Our result extends the corresponding one proved in [14] and [8], is suggested by the treatment of analytic convexity in [10] and [1], and is useful in the applications (see, for instance, [4], [3], [5], [6]).

1. Admissible sequences and main result.

Let $V \subset \mathbb{C}^N$ be an irreducible affine algebraic variety of positive dimension. We denote by $P(V)$ the space of weakly plurisubharmonic functions on V (see [9]). The elements u of $P(V)$ are functions $u : V \to [-\infty, +\infty[$, plurisubharmonic on the complement of the singular set $S(V)$ of V, and satisfying

$$u(\zeta) = \limsup_{z \to \zeta} u(z) \quad \forall \zeta \in V.$$

Definition 1.1. *An increasing sequence* $\{\varphi_\alpha\}_{\alpha \in \mathbb{N}}$ *of real valued functions on* V *is called* admissible *if the following conditions are satisfied:*

(i) for every integer $\alpha \geq 0$ *the function* φ_α *is the restriction to* V *of a plurisubharmonic function* $\tilde{\varphi}_\alpha$ *in* $\mathbb{C}^N$;

(ii) for every non-negative integer α *and for every constant* $c > 0$ *there are an integer* $\beta \geq 0$ *and a constant* $c' \geq 0$ *such that*

$$\tilde{\varphi}_\alpha(\zeta) + c \log(e + |\zeta|) \leq \tilde{\varphi}_\beta(\zeta) + c' \quad \forall \zeta \in \mathbb{C}^N ; \tag{1.1}$$

(iii) for every non-negative integer α *there are positive constants* a_α, b_α *and* c_α *such that*

$$|\tilde{\varphi}_\alpha(z + \zeta) - \tilde{\varphi}_\alpha(\zeta)| \leq c_\alpha \quad \text{if } z, \zeta \in \mathbb{C}^N \text{ and } |z| \leq a_\alpha (1 + |\zeta|)^{-b_\alpha} ;$$

(iv) φ_o *is bounded from below on every compact subset of* V.

Note that, in particular, (i) and (ii) imply that

$$\forall \alpha \in \mathbb{N} \ \exists \beta \in \mathbb{N} \text{ s.t. } \lim_{\substack{\zeta \in V \\ |\zeta| \to +\infty}} e^{\tilde{\varphi}_\alpha(\zeta) - \tilde{\varphi}_\beta(\zeta)} = 0,$$

whereas (iii) implies that $\tilde{\varphi}_\alpha$ has polynomial growth:

$$|\tilde{\varphi}_\alpha(\zeta)| \leq c(1 + |\zeta|)^{b_\alpha + 1} \quad \forall \zeta \in \mathbb{C}^N$$

for some constant $c > 0$.

In the following, while considering admissible sequences $\{\varphi_\alpha\}_{\alpha \in \mathbb{N}}$, we will think the functions $\tilde{\varphi}_\alpha$ as given, and write for simplicity φ_α instead of $\tilde{\varphi}_\alpha$.

Note that the sequences $\{\varphi_\alpha\}_{\alpha \in \mathbb{N}}$ defined by (0.1), (0.2), (0.3) are all admissible sequences, under suitable conditions on the convex function φ.

We will prove the following:

Theorem 1.2. *Let $\{\varphi_\alpha\}_{\alpha\in\mathbb{N}}$ and $\{\psi_\alpha\}_{\alpha\in\mathbb{N}}$ be two admissible sequences defined on an irreducible affine algebraic variety $V \subset \mathbb{C}^N$.*

Then the following Phragmén-Lindelöf conditions (Ph-L) and (Ph-L)′ are equivalent:

(Ph-L) $\forall\alpha \in \mathbb{N}\ \exists\beta \in \mathbb{N},\ c > 0$ *such that if $f \in \mathcal{O}(V)$ satisfies, for some constants $\alpha_f \in \mathbb{N}$ and $c_f > 0$:*

$$\begin{cases} |f(\zeta)| \le e^{\psi_\alpha(\zeta)} & \forall\zeta \in V \\ |f(\zeta)| \le c_f e^{\varphi_{\alpha_f}(\zeta)} & \forall\zeta \in V, \end{cases}$$

then it also satisfies:
$|f(\zeta)| \le c e^{\varphi_\beta(\zeta)} \quad \forall\zeta \in V;$

(Ph-L)′ $\forall\alpha \in \mathbb{N}\ \exists\beta \in \mathbb{N},\ c > 0$ *such that if $u \in P(V)$ satisfies, for some constants $\alpha_u \in \mathbb{N}$ and $c_u > 0$:*

$$\begin{cases} u(\zeta) \le \psi_\alpha(\zeta) & \forall\zeta \in V \\ u(\zeta) \le \varphi_{\alpha_u}(\zeta) + c_u & \forall\zeta \in V, \end{cases}$$

then it also satisfies:
$u(\zeta) \le \varphi_\beta(\zeta) + c \quad \forall\zeta \in V.$

Note that the implication (Ph-L)′ $\Rightarrow$ (Ph-L) is trivial since $\log|f|$ is weakly plurisubharmonic on V for every $f \in \mathcal{O}(V)$.

2. Proof of Theorem 1.2.

In this section we prove the implication (Ph-L) $\Rightarrow$ (Ph-L)′ following the general pattern of [10] and [1] for the discussion of analytic convexity. In fact our technique is closely related to that of [14] and [8].

We will borrow some preliminary lemmas from [14] and [8]. For general results on plurisubharmonic functions we refer to [11] and [12].

Let $p_1(\zeta), \dots, p_r(\zeta) \in \mathcal{P}_N = \mathbb{C}[\zeta^1, \dots, \zeta^N]$ be a finite set of generators of the prime ideal $\wp$ of polynomials vanishing on V.

We set

$$|p(\zeta)|^2 = \sum_{j=1}^{r} |p_j(\zeta)|^2 .$$

If n is the dimension of V, by a real linear change of coordinates (cf. [1]) we can assume that:

(1) $V \subset \{\zeta = (z, w) \in \mathbb{C}^k \times \mathbb{C}^n : |z| \le (1 + |w|)\};$

(2) the projection map

$$\pi : V \ni (z, w) \mapsto w \in \mathbb{C}^n$$

is finite and surjective;

(3) the ring $\mathcal{P}_{N/\wp}$ is integral over $\mathbb{C}[w^1, \dots, w^n]$;

(4) for each $j = 1, \dots, k$ there is a polynomial $P_j \in \wp \cap \mathbb{C}[z^j, w^1, \dots, w^n]$ which is monic with respect to z^j;

(5) the quotient field of $\mathcal{P}_{N/\wp}$ contains the field $\mathbb{C}(w^1, \dots, w^n)$ of rational functions of the variables $w^1, \dots, w^n$ and is generated over it by the image of z^1;

(6) if $\Delta(w)$ is the discriminant of P_1 with respect to z^1, there are polynomials $\alpha_j \in \mathbb{C}[z^1, w^1, \dots, w^n]$ for $j = 2, \dots, k$ such that

$$\Delta(w)z^j - \alpha_j(z^1, w) \in \wp \quad \text{for } j = 2, \dots, k .$$

Assume that the minimal polynomial

$$P_1(z^1, w) = (z^1)^m + \beta_1(w)(z^1)^{m-1} + \dots + \beta_{m-1}(w)z^1 + \beta_m(w)$$

of z^1 over $\mathbb{C}(w^1, \dots, w^n)$ has degree m in z^1; then each $\beta_j \in \mathbb{C}[w^1, \dots, w^n]$ has degree $\leq j$ and the projection $\pi : V \to \mathbb{C}^n$ defines an m-sheeted covering over the set $\{w \in \mathbb{C}^n : \Delta(w) \neq 0\}$, i.e. all points of V are of the form $(\vartheta_j(w), w)$ for $j \in 1, \dots, m$, where the different branches $\vartheta_1(w), \dots, \vartheta_m(w)$ of the algebraic function $(z^1, \dots, z^k)$ on V are all distinct when $\Delta(w) \neq 0$.

Therefore the set of singular points of V lies inside the set

$$S_o = S_o(\delta, c) = \{(z, w) \in \mathbb{C}^N : |\Delta(w)| < \delta(1 + |w|)^{-c}\}$$

for every choice of positive constants δ and c.

Given positive constants A and B we define the function

$$\chi(z, w; A, B) = \log|p(\zeta)| + A\{B\log(e + |\zeta|) - \log|\Delta(w)|\} ,$$

which is defined and plurisubharmonic in $\{\zeta = (z, w) \in \mathbb{C}^N : \Delta(w) \neq 0\}$.

Let

$$\Omega(A, B) = \{\zeta = (z, w) \in \mathbb{C}^N : \Delta(w) \neq 0, \ \chi(z, w; A, B) < 0\} .$$

Then the following lemma holds true (see [14], Proposition 4.1):

Lemma 2.1. *We can choose the constants δ, c, A, B in such a way that the following properties are valid:*

(i) the set $\Omega(A, B)$ is a domain of holomorphy in $\mathbb{C}^N$ containing $V \cap \{\Delta(w) \neq 0\}$;

(ii) for each point $\zeta_o = (z_o, w_o) \in \Omega(A, B)$ there is a unique $j_o \in \{1, \dots, m\}$ such that

$$|z_o - \vartheta_{j_o}(w_o)| = \min\{|z_o - \vartheta_j(w_o)| : j = 1, \dots, m\} ;$$

(iii) the map $\rho : \Omega(A, B) \to V$ associating to the point (z_o, w_o) the point $(\vartheta_{j_o}(w_o), w_o) \in V$ closest to (z_o, w_o), as in (ii), is a holomorphic retraction;

(iv) if u is a weakly plurisubharmonic function defined on a neighbourhood of $S_o(\delta, c)$, then we have

$$u(\zeta_o) \leq \max\{u(\zeta) : \zeta \in V \setminus S_o(\delta, c), \ |\zeta - \zeta_o| < 1\} \quad \forall \zeta_o \in V \cap S_o(\delta, c) ;$$

(v) there are positive constants ε_1 and c_1 such that for every $\zeta_o = (\vartheta_j(w_o), w_o) \in V \setminus S_o(\delta, c)$, setting $\varepsilon(\zeta_o) = \varepsilon(w_o) = \varepsilon_1(1+|w_o|)^{-c_1}$, the connected component of ζ_o in $V \cap \{\eta \in \mathbb{C}^n : |\eta - w_o| < 8\varepsilon(\zeta_o)\}$ is the graph of a holomorphic function $\vartheta_j(w_o + \tau)$ defined in the disc $\{|\tau| < 8\varepsilon(\zeta_o)\} \subset \mathbb{C}$, $(j \in \{1, \dots, m\})$, and:

$$\tag{2.1} \begin{cases} \zeta(\tau) = (\vartheta_j(w_o + \tau), w_o + \tau) \in \Omega(A, B) & \text{for } |\tau| < 8\varepsilon(\zeta_o) \\ \{\eta \in \mathbb{C}^N : |\eta - \zeta(\tau)| < 8\varepsilon(\zeta_o)\} \subset \Omega(A, B) & \text{for } |\tau| < 8\varepsilon(\zeta_o) \\ |\zeta(\tau) - \zeta_o| \leq 1 & \text{for } |\tau| < 8\varepsilon(\zeta_o) ; \end{cases}$$

(vi) there are positive constants c_2 and c_3 such that the function

$$\phi(\zeta) = \phi(w) = c_2\{\log|\Delta(w)| + c_3 \log(e + |w|)\}$$

is non-negative if $\zeta = (z, w) \in V \setminus S_o(\delta, c)$ and $|\zeta - \zeta(w)| < \varepsilon(w)$, where $\varepsilon(w)$ and $\zeta(w)$ are defined as in (v);

(vii) if f is a holomorphic function on $\Omega(A, B)$ which satiesfies

$$\int_{\Omega(A,B)} |f(\zeta)|^2 e^{-2(\phi(\zeta)+u(\zeta))} d\zeta < +\infty$$

for a locally bounded function u on $\overline{\Omega}(A, B)$, then there is an entire function F in $\mathbb{C}^N$ such that $F(\zeta) = f(\zeta)$ for $\zeta \in V \cap \Omega(A, B)$ and $F(\zeta) = 0$ on $V \cap \{\Delta(w) = 0\}$.

Moreover, any choice of a smaller δ and larger c, A, B makes the statements (i),...,(vii) still true for different constants $\varepsilon_1, c_1, c_2, c_3$.

Let us now set, for $w \in \mathbb{C}^n$ such that $\zeta = (\vartheta_j(w), w) \in V \setminus S_o(\delta, c)$,

$$B = B(0, \varepsilon(w)) = \{\eta \in \mathbb{C}^n : |\eta| < \varepsilon(w)\}.$$

Then we prove the following lemma (following [8]):

Lemma 2.2. *There exists a positive constant c_4, depending on $V, \delta, c, A, B, \varepsilon_1, c_1, c_2, c_3$, such that, for every weakly plurisubharmonic function u on V satisfying*

$$|u(\zeta)| \leq \rho(1 + |\zeta|)^k \quad \forall \zeta \in V$$

for some positive constants ρ and k, and for every $\zeta_o = (\vartheta_j(w_o), w_o) \in V \setminus S_o(\delta, c)$, there exists a subset E of $B = B(0, \varepsilon(w_o)) \subset \mathbb{C}^n$ with

$$(i) \qquad |E| \leq |B| \max(1, \rho)(1 + |\zeta_o|)^{-k}$$

such that for all $\tau \in B \setminus E$ there exists an entire function f_τ on $\mathbb{C}^N$ satisfying:

$$(ii) \qquad \log |f_\tau(\zeta(\tau))| \geq u(\zeta(\tau)) - c_4 \log(e + |\zeta(\tau)|)$$

for $\zeta(\tau)$ as in (2.1);

$$(iii) \qquad \log |f_\tau(\zeta)| \leq \max\{u(\zeta') : \zeta' \in V,\ |\zeta - \zeta'| \leq 1\} + c_4 \log(e + |\zeta|)$$

for all $\zeta \in V$.

For the proof of Lemma 2.2 we need the following two lemmas (for the proof of which we refer to [8], Propositions 3 and 4, and to [11]):

Lemma 2.3. *Let Ω be a domain of holomorphy in $\mathbb{C}^N$, and let Ψ be a plurisubharmonic function on Ω. Then there exists a constant $C > 0$, depending only on the dimension N, such that, for every $\zeta_o \in \Omega$ and $\varepsilon > 0$ with $B(\zeta_o, \varepsilon) \subset\subset \Omega$ and $\varepsilon \leq \frac{1}{2}(1 + |\zeta_o|)$, there exists a function $f \in \mathcal{O}(\Omega)$ satisfying:*

$$(2.2) \qquad f(\zeta_o) = A(\zeta_o, \varepsilon) := \varepsilon \left\{ \frac{1}{|B(\zeta_o, \varepsilon)|} \int_{B(\zeta_o,\varepsilon)} e^{-2\Psi(\zeta)} d\zeta \right\}^{-1/2},$$

$$(2.3) \qquad \int_\Omega \frac{|f(\zeta)|^2 e^{-2\Psi(\zeta)}}{(1 + |\zeta|^2)^{3N+1}} d\zeta \leq C^2,$$

and

$$(2.4) \qquad |f(\zeta)| \leq C\varepsilon^{-N}(1 + |\zeta|)^{3N+1} \exp\{\tilde{\Psi}(\zeta, \varepsilon)\}, \quad \text{on } \Omega_\varepsilon,$$

where $\Omega_\varepsilon = \{\zeta \in \mathbb{C}^N : \overline{B(\zeta, \varepsilon)} \subset \Omega\}$ and $\tilde{\Psi}(\zeta, \varepsilon) = \max\{\Psi(\zeta + \zeta') : |\zeta'| \leq \varepsilon\}$.

Lemma 2.4. *There exist positive constants* Λ *and* C'*, depending only on the dimension* N*, such that, for every* $\zeta \in \mathbb{C}^N$ *and every non-negative plurisubharmonic function* Ψ *on* $\{|\zeta - \zeta'| \leq \delta\}$ *satisfying*

$$\frac{1}{|B(\zeta,\delta)|}\int_{B(\zeta,\delta)} \Psi(\zeta')d\zeta' \leq \Psi(\zeta) + \Lambda\,,$$

we also have

$$\frac{1}{|B(\zeta, r\delta/2)|}\int_{B(\zeta,r\delta/2)} e^{-2\Psi(\zeta')}d\zeta' \leq C'\exp\{C'r\cdot \sup_{|\zeta'-\zeta|=\delta/2}|\Psi(\zeta')|\}e^{-2\Psi(\zeta)}$$

for all $0 < r \leq 1/2$.

We can now prove Lemma 2.2 (cf. also [8], [14], [2]):

Proof of Lemma 2.2. We define the plurisubharmonic function Ψ on $\Omega(A, B)$ by

$$\Psi(\zeta) = \Psi(z,w) = u\circ\rho(z,w) + c_2\{\log|\Delta(w)| + c_3\log(e + |w|)\}, \tag{2.5}$$

where the constants c_2, c_3 and the map ρ are those of Lemma 2.1.

Let $\zeta_o = (\vartheta_j(w_o), w_o) \in V \setminus S_o(\delta, c)$ be arbitrarily given, with $0 < \delta < \varepsilon(\zeta_o)/4$ for the $\varepsilon(\zeta_0)$ of Lemma 2.1, and set $r = \frac{1}{2}(2 + |\zeta_o|)^{-k}$.

Let $\tau \in B$ and $\zeta(\tau) = (\vartheta_j(w_o + \tau), w_o + \tau)$ satisfy (2.1). We denote by f_τ the holomorphic function on $\Omega(A, B)$, given by Lemma 2.3 for the Ψ defined by (2.5), and ζ_o substituted by $\zeta(\tau)$ and ε by $r\delta/2$ in (2.2):

$$f(\zeta(\tau)) = A\left(\zeta(\tau), r\frac{\delta}{2}\right).$$

Then condition (iii) of Lemma 2.2 is satisfied, because of estimate (2.4) and the definition of Ψ.

Moreover, because of the definition of Ψ and the L^2-estimate for f_τ given by (2.3), point (vii) of Lemma 2.1 implies that f_τ extends from V to an entire function on $\mathbb{C}^N$.

Let $\Lambda > 0$ be the constant of Lemma 2.4 and let us set, for $\zeta \in \Omega(A, B)$,

$$\Psi_\delta(\zeta) := \frac{1}{|B(\zeta,\delta)|}\int_{B(\zeta,\delta)} \Psi(\zeta')d\zeta'\,.$$

We consider the set:

$$E = \{\tau \in B : \Psi_\delta(\zeta(\tau)) > \Psi(\zeta(\tau)) + \Lambda\}\,.$$

If $\tau \in B \setminus E$, with the constant C' of Lemma 2.4, we have:

$$\log|f_\tau(\zeta(\tau))| = \log A(\zeta(\tau), r\delta/2) \geq \tag{2.6}$$

$$\geq \log\frac{r\delta}{2} - \frac{1}{2}\log C' - \frac{C'r}{2}\sup_{|\zeta'-\zeta(\tau)|=\delta/2}|\Psi(\zeta')| + \Psi(\zeta(\tau)).$$

In order to estimate $\sup_{|\zeta'-\zeta(\tau)|=\delta/2}|\Psi(\zeta')|$, let us fix $\zeta' = (z', w') \in \mathbb{C}^N$ with $|\zeta' - \zeta(\tau)| = \delta/2$ and let $\tau' = w' - w_o$. Then

$$|\tau'| \leq |\tau| + |\tau - \tau'| \leq \varepsilon(\zeta_o) + \frac{\delta}{2} \leq \varepsilon(\zeta_o) + \frac{\varepsilon(\zeta_o)}{8} < 2\varepsilon(\zeta_o)\,.$$

Moreover, by (v) and (vi) of Lemma 2.1, we have:

$$\begin{aligned}|\Psi(\zeta')| &\leq |u \circ \rho(\zeta')| + c_2|\log|\Delta(w')| + c_3\log(e + |w'|)| \leq\\ &\leq |u(\zeta(\tau'))| + \rho'\log(e + |w'|) \leq\\ &\leq \rho(1 + |\zeta(\tau')|)^k + \rho'\log(e + |w'|) \leq\\ &\leq \rho(1 + |\zeta_o| + |\zeta(\tau') - \zeta_o|)^k + \rho'(e - 1 + |\zeta(\tau)| + |\zeta' - \zeta(\tau)|) \leq\\ &\leq \rho''(2 + |\zeta_o|)^k.\end{aligned}$$

By the definition of r this implies that

$$r \cdot \sup_{|\zeta'-\zeta(\tau)|=\delta/2}|\Psi(\zeta')| \leq \frac{\rho''}{2}\,,$$

and hence, by (2.6), the definition of Ψ and (vi) of Lemma 2.1:

$$\log|f_\tau(\zeta(\tau))| \geq \Psi(\zeta(\tau)) + \log\frac{\delta}{(1 + |\zeta(\tau)|)^k} - C'' \geq \tag{2.7}$$

$$\geq u(\zeta(\tau)) + \log\frac{\delta}{(1 + |\zeta(\tau)|)^k} - C''$$

for every $\tau \in B \setminus E$.

Next we estimate the measure of E. Note that $\Psi_\delta \geq \Psi$ since Ψ is plurisubharmonic, and

$$\Lambda|E| \leq \int_{|\tau|\leq\varepsilon(\zeta_o)}\{\Psi_\delta(\zeta(\tau)) - \Psi(\zeta(\tau))\}\,d\tau\,. \tag{2.8}$$

Since Ψ, and therefore also Ψ_δ, only depend on w, we can find a positive function $\Phi_\delta(\eta)$ on $\mathbb{C}^n$ with compact support in $|\eta| \leq 2\delta$ and $\int\Phi_\delta(\eta)\,d\eta = 1$, such that

$$\Psi_\delta(\zeta(\tau)) \leq \int\Psi(\zeta(\tau - \eta))\Phi_\delta(\eta)\,d\eta\,.$$

Indeed, denoting by $B_k(z_o, r)$ the ball with center z_o and radius r in $\mathbb{C}^k$, for $\zeta = (\zeta', \tau)$ and $\xi = (z, w)$ in $\Omega(A, B)$ we have:

$$\begin{aligned}
\Psi_\delta(\zeta) = \Psi_\delta(\tau) &= \frac{1}{|B_N(\zeta,\delta)|}\int_{B_N(\zeta,\delta)} \Psi(\xi)\,d\xi = \\
&= \frac{1}{\frac{\pi^N}{N!}\delta^{2N}}\int_{B_N(\zeta,\delta)} \Psi(\vartheta_{j_o}(w), w)\,d\xi = \\
&= \frac{N!}{\pi^N\delta^{2N}}\int_{B_n(\tau,\delta)}\left\{\int_{B_{N-n}(\zeta',\sqrt{\delta^2-|\tau-w|^2})} \Psi(\vartheta_{j_o}(w), w)\,dz\right\}dw = \\
&= \frac{N!}{\pi^N\delta^{2N}}\int_{B_n(\tau,\delta)} \Psi(\vartheta_{j_o}(w), w)\frac{\pi^{N-n}}{(N-n)!}(\delta^2-|\tau-w|^2)^{N-n}\,dw = \\
&= \frac{N!}{\pi^n(N-n)!\delta^{2N}}\int_{B_n(0,\delta)} \Psi(\vartheta_{j_o}(\tau-\eta), \tau-\eta)(\delta^2-|\eta|^2)^{N-n}\,d\eta = \\
&= \frac{N!}{\pi^n(N-n)!\delta^{2N}}\int_0^\delta\left\{\int_{\partial B_n(0,\rho)} \Psi(\vartheta_{j_o}(\tau-\eta), \tau-\eta)\cdot\right.\\
&\qquad\left.\cdot\,(\delta^2-|\eta|^2)^{N-n}dH^{2n-1}(\eta)\right\}d\rho\,,
\end{aligned}$$

where dH^{2n-1} denotes the $(2n-1)$-dimensional Hausdorff measure.

Then, writing ω_n for the measure of the unit sphere in $\mathbb{C}^n$ and $M(\cdot\,,\rho)$ for the mean value of $\Psi(\vartheta_{j_o}(\cdot), \cdot)$ on $\partial B_n(0,\rho)$, we have:

$$\begin{aligned}
\Psi_\delta(\zeta) &= \frac{N!}{\pi^n(N-n)!\delta^{2N}}\int_0^\delta |\partial B_n(0,\rho)|\cdot\\
&\cdot\left\{\frac{1}{|\partial B_n(0,\rho)|}\int_{\partial B_n(0,\rho)} \Psi(\vartheta_{j_o}(\tau-\eta), \tau-\eta)(\delta^2-|\eta|^2)^{N-n}dH^{2n-1}(\eta)\right\}d\rho = \\
&= \frac{N!}{\pi^n(N-n)!\delta^{2N}}\int_0^\delta \omega_n\rho^{2n-1}(\delta^2-|\rho|^2)^{N-n}M(\tau-\cdot\,,\rho)\,d\rho\,.
\end{aligned}$$

Thus we found that, for

$$\chi_\delta(\xi) = \chi_\delta(|\xi|) = \frac{N!(\delta^2-|\xi|^2)^{N-n}}{\pi^n(N-n)!\delta^{2N}}\,,$$

we have

$$\Psi_\delta(\zeta) = \int_0^\delta M(\tau-\cdot\,,\rho)\omega_n\rho^{2n-1}\chi_\delta(\rho)\,d\rho$$

and

$$\int_{B(0,\delta)} \chi_\delta(\xi) d\xi = \int_0^\delta \omega_n \rho^{2n-1} \chi_\delta(\rho)\, d\rho = 1$$

since the mean value 1_δ of 1 is 1.

Then we construct a non-negative radial function $\tilde{\chi}_\delta(\xi)$ on $\mathbb{C}^n$ by:

$$\tilde{\chi}_\delta(\xi) = \begin{cases} 0 & \text{for } |\xi| \leq \delta/2 \\ \chi_\delta(\xi) & \text{for } \delta/2 < |\xi| \leq \delta \\ G_\delta(\xi) & \text{for } \delta < |\xi| \leq \delta + \sigma \\ 0 & \text{for } |\xi| > \delta + \sigma \end{cases}$$

with $0 < \sigma < \delta$ and $G_\delta(\xi) = G_\delta(|\xi|) \geq 0$ such that

$$\int_0^{\delta/2} \omega_n \rho^{2n-1} \chi_\delta(\rho)\, d\rho = \int_\delta^{\delta+\sigma} \omega_n \rho^{2n-1} G_\delta(\rho)\, d\rho,$$

so that we still have

$$\int_{B(0,2\delta)} \tilde{\chi}_\delta(\xi) d\xi = \int_0^{2\delta} \omega_n \rho^{2n-1} \tilde{\chi}_\delta(\rho)\, d\rho = 1\,.$$

Now, since Ψ is plurisubharmonic, the mean value $M(\cdot\,, \rho)$ is an increasing function of ρ, and hence:

$$\begin{aligned}
\Psi_\delta(\zeta) &= \int_0^\delta M(\tau - \cdot\,, \rho) \omega_n \rho^{2n-1} \chi_\delta(\rho)\, d\rho = \\
&= \int_0^{\delta/2} M(\tau - \cdot\,, \rho) \omega_n \rho^{2n-1} \chi_\delta(\rho)\, d\rho + \\
&\qquad + \int_{\delta/2}^{\delta} M(\tau - \cdot\,, \rho) \omega_n \rho^{2n-1} \chi_\delta(\rho)\, d\rho \leq \\
&\leq M\Big(\tau - \cdot\,, \frac{\delta}{2}\Big) \int_0^{\delta/2} \omega_n \rho^{2n-1} \chi_\delta(\rho)\, d\rho + \\
&\qquad + \int_{\delta/2}^{\delta} M(\tau - \cdot\,, \rho) \omega_n \rho^{2n-1} \tilde{\chi}_\delta(\rho)\, d\rho = \\
&= M\Big(\tau - \cdot\,, \frac{\delta}{2}\Big) \int_\delta^{\delta+\sigma} \omega_n \rho^{2n-1} \tilde{\chi}_\delta(\rho)\, d\rho + \\
&\qquad + \int_{\delta/2}^{\delta} M(\tau - \cdot\,, \rho) \omega_n \rho^{2n-1} \tilde{\chi}_\delta(\rho)\, d\rho \leq \\
&\leq \int_\delta^{\delta+\sigma} M(\tau - \cdot\,, \rho) \omega_n \rho^{2n-1} \tilde{\chi}_\delta(\rho)\, d\rho + \\
&\qquad + \int_{\delta/2}^{\delta} M(\tau - \cdot\,, \rho) \omega_n \rho^{2n-1} \tilde{\chi}_\delta(\rho)\, d\rho \leq
\end{aligned}$$

$$\begin{aligned}
&\leq \int_0^{2\delta} M(\tau - \cdot\,, \rho)\omega_n \rho^{2n-1}\tilde{\chi}_\delta(\rho)\, d\rho = \\
&= \int_0^{2\delta} \Big\{ \int_{\partial B_n(0,\rho)} \Psi(\vartheta_{j_o}(\tau-\eta), \tau-\eta) dH^{2n-1}(\eta) \Big\} \tilde{\chi}_\delta(\rho)\, d\rho = \\
&= \int_{B_n(0,2\delta)} \Psi(\vartheta_{j_o}(\tau-\eta), \tau-\eta)\tilde{\chi}_\delta(\eta)\, d\eta\,.
\end{aligned}$$

Finally we choose

$$\Phi_\delta(\eta) = \chi_{B_n(0,2\delta)}(\eta)\tilde{\chi}_\delta(\eta)\,,$$

where $\chi_{B_n(0,2\delta)}(\eta)$ is the characteristic function of $B_n(0, 2\delta)$ in $\mathbb{C}^n$, and we have that

$$\Phi_\delta \geq 0\,, \qquad \int \Phi_\delta(\eta)\, d\eta = 1\,, \qquad \operatorname{supp} \Phi_\delta \subset\subset B_n(0, 2\delta)$$

and

$$\Psi_\delta(\zeta(\tau)) \leq \int \Psi(\zeta(\tau-\eta))\Phi_\delta(\eta)\, d\eta\,.$$

Therefore, denoting by χ_B the characteristic function of $B = B(0, \varepsilon(\zeta_o))$ we have:

$$\begin{aligned}
&\int_{|\tau|\leq\varepsilon(\zeta_o)} \{\Psi_\delta(\zeta(\tau)) - \Psi(\zeta(\tau))\}\, d\tau \leq \\
&\qquad \leq \int_{|\xi|\leq\varepsilon(\zeta_o)} \int \Psi(\zeta(\eta))\Phi_\delta(\xi-\eta)\, d\eta\, d\xi - \int_{|\xi|\leq\varepsilon(\zeta_o)} \Psi(\zeta(\xi))\, d\xi = \\
&\qquad = \iint \chi_B(\xi)\Psi(\zeta(\tau))\Phi_\delta(\xi-\tau)\, d\tau\, d\xi - \int \chi_B(\xi)\Psi(\zeta(\xi))\, d\xi = \\
&\qquad = \iint \chi_B(\tau)\Psi(\zeta(\xi))\Phi_\delta(\tau-\xi)\, d\xi\, d\tau - \\
&\qquad\qquad - \int \chi_B(\xi)\Psi(\zeta(\xi)) \int \frac{\chi_B(\tau)}{|B|}\, d\tau d\xi = \\
&\qquad = \int \Psi(\zeta(\xi)) \Big\{ \int \chi_B(\tau) \Big[\Phi_\delta(\tau-\xi) - \frac{\chi_B(\xi)}{|B|} \Big] d\tau \Big\}\, d\xi = \\
&\qquad = \int \Psi(\zeta(\xi)) h_\delta(\xi)\, d\xi
\end{aligned}$$

with

$$h_\delta(\xi) = \int \chi_B(\tau) \Big\{ \Phi_\delta(\tau-\xi) - \frac{\chi_B(\xi)}{|B|} \Big\}\, d\tau\,.$$

In particular $-1 \leq h_\delta \leq 1$ and h_δ has support in the ring $\varepsilon(\zeta_o) - 2\delta \leq |\xi| \leq \varepsilon(\zeta_o) + 2\delta$. This ring has volume

$$\frac{\pi^n}{n!}[(\varepsilon(\zeta_o) + 2\delta)^{2n} - (\varepsilon(\zeta_o) - 2\delta)^{2n}] \leq \tilde{C}\varepsilon(\zeta_o)^{2n-1}\delta .$$

Therefore, if we set

$$M = M(\zeta_o) = \sup\{\Psi(\zeta(\tau)) : |\tau| \leq 2\varepsilon(\zeta_o)\} ,$$

from (2.8) we obtain that

$$|E| \leq \tilde{C}'|B|\frac{\delta}{\varepsilon(\zeta_o)}M(\zeta_o) .$$

By the definition of Ψ and (2.1) we have that

$$M(\zeta_o) \leq \rho(1 + |\zeta_o|)^k + \bar{C}\log(e + |\zeta_o|) .$$

We finally choose a positive constant $\bar{C}'$ such that, taking

$$\delta = \frac{\varepsilon(\zeta_o)}{\bar{C}'(1 + |\zeta_o|)^{2k}}$$

we have

$$|E| \leq \max(1, \rho)|B|(1 + |\zeta_o|)^{-k} ,$$

which is estimate (i) of Lemma 2.2.

Finally, the value of δ in (2.7), together with (2.1), give estimate (ii) of Lemma 2.2.

This completes the proof. $\square$

We are now ready to prove Theorem 1.2 (cf. also [14], [8], [2]).

Proof of Theorem 1.2. As we already noted in the previous section, the implication (Ph-L)$'$ $\Rightarrow$ (Ph-L) is trivial.

Let us turn to the proof of the opposite inclusion (Ph-L) $\Rightarrow$ (Ph-L)$'$.

Let u be a weakly plurisubharmonic function on V which satisfies the first two inequalities of (Ph-L)$'$, and let us show that it also satisfies the third inequality of (Ph-L)$'$.

We know, by (Ph-L), that this last inequality is satisfied if u is of the form $\log|f|$ with $f \in \mathcal{O}(V)$.

By (iv) of Lemma 2.1, for every $\zeta \in V \cap S_o(\delta, c)$ we have that

$$u(\zeta) \leq \max\{u(\zeta') : \zeta' \in V \setminus S_o(\delta, c),\ |\zeta - \zeta'| < 1\},$$

and hence it is sufficient to prove the third inequality of (Ph-L)$'$ for $\zeta = (\vartheta_j(w), w) \in V \setminus S_o(\delta, c)$.

For these points, by the mean value theorem for plurisubharmonic functions, using the polynomial growth of ψ_α and Lemma 2.2, we have:

$$\begin{aligned}(2.9)\qquad u(\zeta) &\leq \frac{1}{|B|}\int_B u(\zeta(w))\,dw = \\ &= \frac{1}{|B|}\int_E u(\zeta(w))\,dw + \frac{1}{|B|}\int_{B\setminus E} u(\zeta(w))\,dw \leq \\ &\leq \frac{1}{|B|}\int_E \psi_\alpha(\zeta(w))\,dw + \frac{|B\setminus E|}{|B|}\sup\left\{u(\zeta(w)) : w \in B\setminus E\right\} \leq \\ &\leq \frac{|E|}{|B|}\rho_\alpha(1+|\zeta|)^{k_\alpha} + \sup\left\{u(\zeta(w)) : w \in B\setminus E\right\} \leq \\ &\leq \rho_\alpha^2 + \sup\left\{u(\zeta(w)) : w \in B\setminus E\right\}.\end{aligned}$$

By Lemma 2.2 for each $w \in B \setminus E$ there is an entire function f_w such that

$$\log|f_w(\zeta)| \leq \max\{u(\zeta') : \zeta' \in V,\ |\zeta - \zeta'| \leq 1\} + c_4 \log(e + |\zeta|).$$

From the first two inequalities of (Ph-L)$'$ and property (1.1) of $\{\varphi_\alpha\}_{\alpha\in\mathbb{N}}$ and $\{\psi_\alpha\}_{\alpha\in\mathbb{N}}$ we thus deduce that there are constants $\alpha', \alpha'_u \in \mathbb{N}$ and $A', A'_u > 0$ such that

$$\begin{cases}\log|f_w(\zeta)| \leq \psi_{\alpha'}(\zeta) + A' & \forall\zeta \in V\\ \log|f_w(\zeta)| \leq \varphi_{\alpha'_u}(\zeta) + A'_u & \forall\zeta \in V.\end{cases}$$

Therefore, by (Ph-L) there are constants $\beta_\alpha \in \mathbb{N}$ and $B_\alpha > 0$ such that

$$\log|f_w(\zeta)| \leq \varphi_{\beta_\alpha}(\zeta) + B_\alpha \quad \forall\zeta \in V.$$

Using again Lemma 2.2 we thus obtain, for all $w \in B \setminus E$:

$$u(\zeta(w)) - c_4\log(e + |\zeta(w)|) \leq \log|f_w(\zeta(w))| \leq \varphi_{\beta_\alpha}(\zeta(w)) + B_\alpha,$$

and hence, by (1.1),

$$u(\zeta(w)) \leq \varphi_{\beta'_\alpha}(\zeta(w)) + B'_\alpha \quad \forall w \in B \setminus E.$$

Substituting this last inequality in (2.9) we finally have:

$$u(\zeta) \leq \varphi_{\beta'_\alpha}(\zeta) + B''_\alpha \quad \forall\zeta \in V \setminus S_o$$

and hence the thesis. $\square$

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