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SEMICONCAVITY OF THE VALUE FUNCTION FOR AN EXIT TIME PROBLEM WITH DEGENERATE COST

PIERMARCO CANNARSA - CRISTINA PIGNOTTI

Dedicated to Professor Sergio Campanato on his 70th birthday

A simple exit time problem with degenerate cost is here considered. Using a new technique for constructing admissible trajectories, a semiconcavity result for the value function v is obtained. Such a property of v is then applied to obtain optimality conditions.

1. Introduction.

Optimal exit time problems are usually formulated in the closure $\overline{\Omega}$ of an open domain of $\mathbb{R}^n$, giving a state equation of the form

$$\begin{cases} \dot{y}(t) = f(y(t), \alpha(t)), & t > 0 \\ y(0) = x \in \overline{\Omega}. \end{cases} \tag{1.1}$$

Here f is a vector field and α is a control — i.e. a measurable function taking values in a given closed set B (the control space). Under standard conditions the above Cauchy problem can be uniquely solved; we will denote by $y_x^\alpha(t)$ the solution of (1.1), and say that $y_x^\alpha(t)$ is the trajectory starting at x with control α. Moreover, we will denote by $\tau(x, \alpha)$ the time, even infinite, at which $y_x^\alpha(t)$ reaches the boundary of Ω.

The optimal exit time problem we are interested in, consists in minimizing, over all controls, the cost functional

$$J(x,\alpha) = \int_0^{\tau(x,\alpha)} e^{-\lambda t} L(y_x^\alpha(t), \alpha(t))\, dt + g(y_x^\alpha(\tau(x,\alpha))). \tag{1.2}$$

Here $L : \overline{\Omega} \times B \to \mathbb{R}$ and $g : \overline{\Omega} \to \mathbb{R}$ are given functions, called running cost and terminal cost respectively, $\lambda \geq 0$ is a given number, called discount factor. A well studied example of this class of problems is the minimum time problem, for which one takes $\lambda = 0$, $g \equiv 0$ and $L \equiv 1$.

The Dynamic Programming approach to the above minimization problem is based on the properties of the value function v defined as

$$v(x) = \inf_\alpha J(x,\alpha) \qquad x \in \overline{\Omega}. \tag{1.3}$$

In fact, useful optimality conditions can be formulated in terms of the value function and of its gradients.

Moreover, the value function can be characterized as the unique viscosity solution of the boundary value problem

$$\begin{cases} \lambda v(x) + F(x, Dv(x)) = 0 & \text{in} \quad \Omega \\ v = g & \text{on} \quad \partial\Omega \end{cases} \tag{1.4}$$

where

$$F(x,p) = \sup_a [-f(x,a) \cdot p - L(x,a)].$$

This means, since v is a nonsmooth function, that suitable inequalities hold for the super- and sub-differential of v at any point of Ω (see [14], [5], [4]).

The above considerations motivate our interest in those regularity properties of v that hold for general optimal exit time problems. Among these properties, a special role is reserved for semiconcavity. This is, in fact, the maximal regularity one can expect the solutions of (1.4) to possess when the data are smooth.

We recall that, roughly speaking, a semiconcave function is a function that can be locally represented as the sum of a concave function plus a smooth one. Therefore, semiconcave functions share many differentiability properties of concave functions. For example, they are twice differentiable almost everywhere and possess a nonempty superdifferential at any point. Moreover, sharp Hausdorff estimates from above and below are available for the singular set of a semiconcave function, see [2], [1].

A semiconcavity result for the value function of optimal exit time problems has been obtained in [8], assuming a nondegeneracy condition of the form

$$L(x,a) \geq c > 0 \tag{1.5}$$

for any $x \in \overline{\Omega}$ and any control a. Such an assumption is fundamental for the method of [8]: (1.5) allows to bound the exit time $\tau(x,\alpha)$ with the minimum time function, $T(x)$, in such a way that the regularity properties of $T(x)$, formerly derived in [9], can be used to obtain the semiconcavity of v. Therefore, the approach of [8] cannot be extended to problems with a degenerate running cost.

In this paper we propose a new method to obtain semiconcavity estimates for the value function of optimal exit time problems with degenerate cost. We will consider a simplified model assuming that the control space B is the closed unit ball of $\mathbb{R}^n$, that

$$f(x,a) = a, \qquad \lambda > 0, \qquad g \equiv 0$$

and that $L(x,\alpha) \equiv L(x)$ with

$$\begin{aligned} L(x) &\geq 0 \quad \forall x \in \overline{\Omega}\,, \\ L(x) &= 0 \quad \forall x \in \partial\Omega\,. \end{aligned} \tag{1.6}$$

Then, our optimal exit time problem is equivalent to a constrained optimal control problem in the sense that

$$v(x) = \inf_{\alpha \in \mathcal{A}_x} \int_0^{+\infty} e^{-\lambda t} L(y_x^\alpha(t))\, dt\,, \tag{1.7}$$

where the set of admissible controls is defined as follows

$$\mathcal{A}_x = \{\alpha : y_x^\alpha(t) \in \overline{\Omega}, \forall\, t \geq 0\} \quad \forall x \in \overline{\Omega}.$$

The above observation is crucial for various reasons. First, thanks to the representation formula (1.7) we can construct admissible trajectories to prove the semiconcavity of v. Second, this approach gives an insight into another difficult problem of this theory, namely finding reasonable conditions to ensure the semiconcavity of the value function of optimal control problems with state constraints. For these problems, counterexamples to semiconcavity are known as well as a positive result in the one-dimensional case, see [7].

Once we have proved that the value function is semiconcave, we can apply the properties of this class of functions to derive optimality conditions for our optimal exit time problem. Moreover, using the Hamilton-Jacobi equation (1.4) and the theory of [1], we obtain a propagation result for the singular set of v. In particular, we classify the isolated singularities of the value function and show that nonisolated singularities propagate along Lipschitz arcs.

The outline of the paper is the following. In Section 2 we recall the definition and some properties of semiconcave functions. In Sections 3 and 4 we prove the Hölder continuity and the semiconcavity of the value function. Finally, in Section 5, we give applications to optimality conditions and to the analysis of the singular set of v.

2. Preliminaries.

Let $x_0 \in \mathbb{R}^n$, $r > 0$. As usual, we define

$$B(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| \le r\}.$$

We denote by $\langle p, q \rangle$ or $p \cdot q$ the usual scalar product of two vectors $p, q \in \mathbb{R}^n$.

Let $A \subset \mathbb{R}^n$ and $u : A \to \mathbb{R}$. We recall that u is said to be Hölder continuous of exponent $\theta \in (0, 1]$, if there exists a positive constant C such that

$$|u(x) - u(y)| \le C|x - y|^\theta, \tag{2.1}$$

for every $x, y \in A$. In particular, if $\theta = 1$, we say that u is Lipschitz continuous.

We call *modulus* a nondecreasing upper semicontinuous function $\sigma : [0, +\infty) \to [0, +\infty)$ such that $\lim_{r\to 0^+} \sigma(r) = 0$.

We say that $u : A \to \mathbb{R}$ is *semiconcave* if, for some modulus ω, u satisfies

$$su(x) + (1 - s)u(y) - u(sx + (1 - s)y) \le s(1 - s)|x - y|\omega(|x - y|), \tag{2.2}$$

for any pair $x, y \in A$ such that the segment $[x, y]$ is contained in A and for any $s \in [0, 1]$. In this case, ω is said to be a *semiconcavity modulus* for u in A.

Suppose now that $u : A \to \mathbb{R}$ is a continuous function satisfying, for some modulus $\tilde{\omega}$,

$$u(x) + u(y) - 2u\left(\frac{x + y}{2}\right) \le \frac{|x - y|}{2}\tilde{\omega}(|x - y|), \tag{2.3}$$

for any x, y such that the segment $[x, y]$ is contained in A. Then, one can show that u is semiconcave in A with semiconcavity modulus

$$\omega(r) = \sum_{i=0}^{\infty} \tilde{\omega}\left(\frac{r}{2^i}\right) \qquad (r \ge 0)$$

provided that the right-hand side above is finite. In particular, this occurs if $\tilde{\omega}(r)$ is given by a constant times a positive power of r. In the present paper, we will always prove semiconcavity through property (2.3).

First of all we recall some definitions and some properties of semiconcave functions. We recall that if $u : A \to \mathbb{R}$ and $x \in A$, the *subdifferential* and the *superdifferential* of u at x are, respectively, the sets

$$\nabla^- u(x) = \left\{ p \in \mathbb{R}^n : \liminf_{y \to x} \frac{u(y) - u(x) - p \cdot (y - x)}{|y - x|} \geq 0 \right\},$$

$$\nabla^+ u(x) = \left\{ p \in \mathbb{R}^n : \limsup_{y \to x} \frac{u(y) - u(x) - p \cdot (y - x)}{|y - x|} \leq 0 \right\}.$$

If u is a locally Lipschitz function, then $\nabla^- u(x)$ and $\nabla^+ u(x)$ are compact convex sets. They are both nonempty if and only if u is differentiable at x and, in this case they both contain only the gradient of u. For a locally Lipschitz function $u : A \to \mathbb{R}$, we also define the set of *limiting gradients*

$$\nabla^* u(x) = \big\{ p : \exists \{x_k\} \subset A \quad \text{such that} \quad x_k \to x, \\ u \text{ is differentiable at } x_k, \; p = \lim_{k \to \infty} \nabla u(x_k) \big\},$$

where ∇u denotes the usual gradient of u. This set is nonempty as a corollary of the Rademacher's theorem. In the case of a semiconcave function we have the following result (see [11]).

Theorem 2.1. *Let $u : A \to \mathbb{R}$ be semiconcave. Then u is locally Lipschitz in A, and*

$$\nabla^+ u(x) = co \nabla^* u(x), \qquad \forall x \in A. \tag{2.4}$$

Therefore the superdifferential $\nabla^+ u$ is nonempty at each point. In addition, u is differentiable at x if and only if $\nabla^+ u(x)$ is a singleton.

If $u : A \to \mathbb{R}$ is a semiconcave function, we say that $x \in A$ is a *singular point* for u if u is not differentiable at x. We call *singular arc* an arc consisting of singular points. The following result is proved in [1].

Theorem 2.2. *Let $u : A \to \mathbb{R}$ a semiconcave function with a linear modulus. Suppose that $\partial \nabla^+ u(x) \setminus \nabla^* u(x) \neq \emptyset$. Then there exists a Lipschitz singular arc $\overline{x} : [0, \sigma] \to \mathbb{R}^n$ such that $\overline{x}(0) = x$ and $\overline{x}(s) \neq x$, $\forall s \in (0, \sigma]$.*

The following lemma generalizes a result of [15], where the special case $\beta = 1$ and $\rho = 1$ is obtained.

Lemma 2.3. *Let ω be a modulus and let $\beta > 0$. If there exist $\gamma, A, \rho > 0$ such that*

$$\begin{cases} \omega(R) \le CR^\beta + \gamma\omega(AR) & \forall R \in [0, \rho] \\ \gamma < 1, \quad \gamma A^\beta < 1, \quad \rho \le 1, \end{cases}$$

then there exists a constant $C' = C'(\gamma, \rho, A, C) > 0$ such that $\omega(R) \le C'R^\beta$, for all $R \in [0, \rho]$.

Proof. Suppose first that $A > 1$. Taking $R = \rho/A$, our assumption implies that

$$\omega\Big(\frac{\rho}{A}\Big) \le C\frac{1}{A^\beta} + \gamma\omega(\rho).$$

Similarly, for $R = \rho/A^2$,

$$\omega\Big(\frac{\rho}{A^2}\Big) \le C\frac{1}{A^{2\beta}} + \gamma\omega\Big(\frac{\rho}{A}\Big) \le C\frac{1}{A^{2\beta}} + C\frac{\gamma}{A^\beta} + \gamma^2\omega(\rho) =$$

$$= \frac{C}{A^{2\beta}}(1 + \gamma A^\beta) + \gamma^2\omega(\rho).$$

In general, taking $R = \rho/A^n$ for any $n \ge 1$, we have that

$$\omega\Big(\frac{\rho}{A^n}\Big) \le \frac{C}{A^{n\beta}}[1 + \gamma A^\beta + \cdots + (\gamma A^\beta)^{n-1}] + \gamma^n\omega(\rho) <$$

$$< \frac{C}{A^{n\beta}} \cdot \frac{1}{1 - \gamma A^\beta} + \gamma^n\omega(\rho).$$

Now, let $0 < R \le \rho$ and fix $n \in \mathbb{N}$ so that $\rho/A^{n+1} < R \le \rho/A^n$. Then, by our assumption $\gamma < 1/A^\beta$, we conclude that

$$\omega(R) \le \omega\Big(\frac{\rho}{A^n}\Big) \le$$

$$\le \gamma^n\omega(\rho) + \frac{C}{A^{n\beta}} \cdot \frac{1}{1 - \gamma A^\beta} \le \frac{1}{A^{n\beta}}\Big(\omega(\rho) + \frac{C}{1 - \gamma A^\beta}\Big) =$$

$$= \frac{A^\beta}{A^{n\beta+\beta}}\Big(\omega(\rho) + \frac{C}{1 - \gamma A^\beta}\Big) \le \frac{A^\beta}{\rho^\beta}R^\beta\Big(\omega(\rho) + \frac{C}{1 - \gamma A^\beta}\Big) \le C'R^\beta,$$

where $C' = \frac{A^\beta}{\rho^\beta}[\omega(\rho) + \frac{C}{(1-\gamma A^\beta)}]$. The lemma is thus proved in the case of $A > 1$.

To complete the proof it remains to observe that, if $A \le 1$, then $\omega(AR) \le \omega(R)$ as ω is nondecreasing. So, the conclusion is trivial. □

3. Hölder continuity of the value function.

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with smooth boundary Γ. We denote by $\overline{\Omega}$ its closure.

We consider the distance from Γ

$$\text{dist}(x, \Gamma) = \inf\{|x - y| : y \in \Gamma\}, x \in \mathbb{R}^n, \tag{3.1}$$

and we denote by d the signed distance

$$d(x) = \begin{cases} \text{dist}(x, \Gamma), & x \notin \Omega, \\ -\text{dist}(x, \Gamma), & x \in \Omega. \end{cases} \tag{3.2}$$

Let us set, for $\rho > 0$, $\Gamma_\rho = \{x \in \mathbb{R}^n : |d(x)| \leq \rho\}$.

Through the paper we assume that

$$\exists \rho_0 \in (0, 1) \quad \text{such that} \quad d \in C^{1,1}(\Gamma_{\rho_0}), \tag{3.3}$$

that is d is of class $C^1(\Gamma_{\rho_0})$ and its gradient ∇d is Lipschitz continuous. In particular this occurs if the boundary Γ of Ω is of class C^2, and, more precisely, if there exists $r_0 > 0$ such that

$$\forall x \in \Gamma \; \exists \text{ a diffeomorphism } \varphi : B_+ \subset \mathbb{R}^n \to B(x, r_0) \cap \overline{\Omega}, \varphi \in C^2(B_+), \tag{3.4}$$

where

$$B_+ = B(0, 1) \cap (\mathbb{R}^{n-1} \times [0, +\infty)).$$

Let $L : \overline{\Omega} \to \mathbb{R}$ be a Lipschitz continuous function satisfying (1.6), then the value function v of our problem,

$$v(x) = \inf\left\{ \int_0^{+\infty} e^{-\lambda t} L(y(t))\, dt : y(t) \in \overline{\Omega} \right. \\ \left. \forall t, \; |\dot{y}(t)| \leq 1 \text{ a.e.}, \; y(0) = x \right\}, \tag{3.5}$$

is Hölder continuous in $\overline{\Omega}$.

More precisely we have the following theorem. We recall that ρ_0 is the positive number introduced in (3.3) and λ is a fixed positive number.

Theorem 3.1. *If C_1 is a Lipschitz constant for ∇d in Γ_{ρ_0}, then v is Hölder continuous in $\overline{\Omega}$ of any exponent $0 < \theta \leq 1$ such that*

$$\theta\left(C_1 + \frac{\ln 2}{\rho_0}\right) < \lambda. \tag{3.6}$$

In particular, for any $\lambda > C_1 + \ln 2/\rho_0$ the value function v is Lipschitz continuous.

The Lipschitz continuity of the value function of optimal control problems with state constraints has been proved for more general systems than the one we consider in this paper, see [15], [13], [3]. However, we prefer to give a new, self-contained proof of this result since the technique is similar to the method we will use in the next section to prove semiconcavity. Our proof of Theorem 3.1 is based on a careful use of trajectories of the differential inclusion

$$\tag{3.7} \begin{cases} \dot{y}(t) \in \alpha_0(t) - H(d(y(t)))\langle \alpha_0(t), \nabla d(y(t))\rangle_+ \nabla d(y(t)), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for a. e. } t \geq 0, \\ y(0) = x\,, \end{cases}$$

where α_0 is a piecewise continuous control and H is the set valued map $H : \mathbb{R} \to \mathcal{P}(\mathbb{R})$

$$\tag{3.8} H(s) = \begin{cases} \{1\}, & 0 < s < \rho_0\,, \\ [0, 1], & s = 0, s = \rho_0\,, \\ \{0\}, & s < 0, \ \ s > \rho_0\,. \end{cases}$$

In problem (3.7) and in the sequel, $\langle p, q\rangle_+ = \max\{\langle p, q\rangle, 0\}$ denotes the positive part of $\langle p, q\rangle$, for any $p, q \in \mathbb{R}^n$.

For the proof of Theorem 3.1 we need the following preliminary result.

Lemma 3.2. *Let $x_0 \in \overline{\Omega}$, let $\alpha_0 \in \mathcal{A}_{x_0}$ be a piecewise continuous control, and set $y_0 = y_{x_0}^{\alpha_0}$. Then, for any $x \in \overline{\Omega}$, problem* (3.7) *admits a solution y such that*

(i) $|\dot{y}(t)| \leq 1$ *for a. e.* $t \geq 0$;

(ii) $y(t) \in \overline{\Omega}$ *for every* $t \in [0, \rho_0]$.

Moreover, if $x \in \overline{\Omega} \cap B(x_0, \rho)$ with $\rho \in (0, \rho_0)$, then

$$\tag{3.9} |y(t) - y_0(t)| \leq 2e^{C_1 t}|x - x_0|, \ \forall t \in [0, \rho_0 - \rho]\,,$$

where C_1 is a Lipschitz constant for ∇d in Γ_{ρ_0}.

Proof. Denoting by $\{t^i : i \in \mathbb{N}\}$ the discontinuity points of α_0, let us set, for $t \geq 0$,

$$\tag{3.10} A(t) = \begin{cases} \alpha_0(t) & t \neq t^i, i \in \mathbb{N}\,, \\ B(0, 1) & t = t^i, i \in \mathbb{N}. \end{cases}$$

Let us consider the set valued map $G : [0, +\infty) \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ defined by

$$G(t, w) = \{v \in \mathbb{R}^n : v = a - h\langle b, \nabla d(w)\rangle_+ \nabla d(w), a, b \in A(t), h \in H(d(w))\},$$

for any $(t, w) \in [0, +\infty) \times \mathbb{R}^n$. Notice that, in light of (3.8), $H(d(w))$ vanishes on the region where ∇d may not be defined. Therefore, it is easy to check that G is upper semicontinuous, and that $G(t, w)$ is a nonempty compact convex set for every $(t, w) \in [0, +\infty) \times \mathbb{R}^n$. Hence the existence of a solution, y, to the problem

$$\begin{cases} \dot{y}(t) \in G(t, y(t)), \ t \geq 0 \\ y(0) = x \end{cases}$$

follows from a well known result, see e.g. [12]. Moreover, by the definition of $A(t)$,

$$\dot{y}(t) \in \alpha_0(t) - H(d(y(t)))\langle \alpha_0(t), \nabla d(y(t))\rangle_+ \nabla d(y(t)) \quad \text{for a.e. } t \geq 0,$$

and so $|\dot{y}(t)| \leq 1$ for a.e. $t \geq 0$. Furthermore, a measurable selection $h(t) \in H(d(y(t)))$ exists such that

$$\text{(3.11)} \qquad y'(t) = \alpha_0(t) - h(t)\langle \alpha_0(t), \nabla d(y(t))\rangle_+ \nabla d(y(t)), \text{ for a. e. } t \geq 0.$$

Now, in order to prove (ii), we consider the Lipschitz function $\delta(t) = d(y(t))$, $t \geq 0$. We want to show that $\delta(t) \leq 0$ for any $t \in [0, \rho_0]$. Arguing by contradiction, let us suppose that $t_0 \in (0, \rho_0)$ exists such that $\delta(t_0) > 0$. Let us set

$$t_1 = \inf\{0 \leq t \leq t_0 : \delta(s) > 0, \forall s \in [t, t_0]\}.$$

Then, $\delta(t_1) = 0$ and

$$\delta'(t) = \langle \nabla d(y(t)), \alpha_0(t)\rangle - h(d(y(t)))\langle \alpha_0(t), \nabla d(y(t))\rangle_+ \leq 0, \text{ for a.e. } t \in [t_1, t_0].$$

So,

$$\delta(t_0) = \delta(t_0) - \delta(t_1) = \int_{t_1}^{t_0} \delta'(t)\, dt \leq 0,$$

which contradicts our previous assumption.

Next, let us fix $\rho \in (0, \rho_0)$ and suppose that $x \in \overline{\Omega} \cap B(x_0, \rho)$. From (3.11) it follows that

$$\text{(3.12)} \qquad y_0(t) - y(t) = x_0 - x + \int_0^t h(s)\langle \alpha_0(s), \nabla d(y(s))\rangle_+ \nabla d(y(s))\, ds.$$

Fix $t \leq \rho_0 - \rho$ and define

$$S = \{s \in [0, t] : y(s) \in \Gamma\}.$$

If $S = \emptyset$, then $h(s) = 0$, for any $s \in [0, t]$, and so

$$y_0(t) - y(t) = x_0 - x.$$

If $S \neq \emptyset$, define $s_0 = \inf S$, $s_1 = \sup S$. We claim that

$$y_0(s),\ y(s) \in \Gamma_{\rho_0},\ \forall s \in [s_0, s_1]. \tag{3.13}$$

Indeed, $d(y(s_0)) = 0$ and so $|d(y(s))| \leq s - s_0 \leq t \leq \rho_0 - \rho$ for any $s \in S$. Moreover, $|d(y_0(s_0))| \leq |y_0(s_0) - y(s_0)| \leq |x - x_0| \leq \rho$, and so $|d(y_0(s))| \leq \rho + s - s_0 \leq \rho + t \leq \rho_0$ for any $s \in S$.

Now, in view of (3.13) and (3.11), we can compute the derivative

$$[d(y_0(t)) - d(y(t))]' = \langle \alpha_0(t), \nabla d(y_0(t)) - \nabla d(y(t))\rangle + h(t)\langle \alpha_0(t), \nabla d(y(t))\rangle_+ .$$

Hence, recalling that $d(y_0(s_1)) \leq 0$ and $d(y(s_1)) = 0$, we obtain

$$\int_0^t h(s)\langle \alpha_0(s), \nabla d(y(s))\rangle_+ \, ds = \int_{s_0}^{s_1} h(s)\langle \alpha_0(s), \nabla d(y(s))\rangle_+ \, ds =$$

$$= \int_{s_0}^{s_1} [d(y_0(s)) - d(y(s))]' \, ds - \int_{s_0}^{s_1} \langle \alpha_0(s), \nabla d(y_0(s)) - \nabla d(y(s))\rangle \, ds =$$

$$= d(y_0(s_1)) - d(y(s_1)) - d(y_0(s_0)) + d(y(s_0)) -$$

$$- \int_{s_0}^{s_1} \langle \alpha_0(s), \nabla d(y_0(s)) - \nabla d(y(s))\rangle \, ds \leq$$

$$\leq |x - x_0| + \int_0^t C_1 |y_0(s) - y(s)| \, ds,$$

where C_1 is a Lipschitz constant for ∇d. In any case, we conclude that

$$|y_0(t) - y(t)| \leq 2|x_0 - x| + C_1 \int_0^t |y_0(s) - y(s)| \, ds, \quad 0 \leq t \leq \rho_0 - \rho;$$

and (3.9) easily follows using the Gronwall Lemma. □

Remark 3.3. We observe that (3.11) also yields

$$\frac{1}{2}\frac{d}{dt}|y_0(t) - y(t)|^2 = h(d(y(t)))\langle \nabla d(y(t)), \alpha_0(t)\rangle_+ \langle \nabla d(y(t)), y_0(t) - y(t)\rangle.$$

Now, if Ω is convex, then the right-hand side of the above identity is negative or 0, for a.e. $t \geq 0$. So, $|y_0(t) - y(t)|$ is non-increasing and we have that

$$|y_0(t) - y(t)| \leq |x_0 - x|, \quad \forall t \geq 0.$$

Remark 3.4. We note that the solution y of inclusion (3.7) given by Lemma 3.2 is an admissible trajectory for our control system at x. In fact, in view of (i) in the previous lemma, it suffices to take the trivial control $\alpha(t) = \dot{y}(t)$.

Proof of Theorem 3.1. To begin, let us fix $\theta \in (0, 1]$ so that condition (3.6) is satisfied. We define the continuity modulus of v

$$\sigma_v(R) := \sup\{|v(x) - v(y)| \, : x, y \in \overline{\Omega}, |x - y| \le R\},$$

for any number $R > 0$. Let us fix a number $\rho \in (0, \rho_0)$, and consider two arbitrary points $x_0, x_1 \in \overline{\Omega}$ such that $|x_0 - x_1| \le \rho$. In order to estimate $|v(x_1) - v(x_0)|$ we can assume, without loss of generality, that $v(x_1) > v(x_0)$. Let $\alpha_0 \in \mathcal{A}_{x_0}$ be a piecewise continuous control such that

$$v(x_0) > \int_0^{\rho_0 - \rho} e^{-\lambda t} L(y_0(t))dt + e^{-\lambda(\rho_0 - \rho)} v(y_0(\rho_0 - \rho)) - |x_1 - x_0|^\theta,$$

where $y_0 = y_{x_0}^{\alpha_0}$. Let $y_1(t)$, $t > 0$, be the solution of

$$\begin{cases} \dot{y}(t) \in \alpha_0(t) - H(d(y(t)))\langle \alpha_0(t), \nabla d(y(t))\rangle_+ \nabla d(y(t)) \\ y(0) = x_1 \end{cases}$$

given by Lemma 3.2. Then $y_1(t)$ is an admissible trajectory for our control system and

$$|y_1(t) - y_0(t)| \le 2e^{C_1(\rho_0 - \rho)}|x_1 - x_0|, \ \forall t \in [0, \rho_0 - \rho]. \tag{3.14}$$

By the dynamic programming principle we have that

$$\begin{aligned} v(x_1) - v(x_0) &\le |x_1 - x_0|^\theta + \int_0^{\rho_0 - \rho} e^{-\lambda t}[L(y_1(t)) - L(y_0(t))]\, dt + \\ &\quad + e^{-\lambda(\rho_0 - \rho)}[v(y_1(\rho_0 - \rho)) - v(y_0(\rho_0 - \rho))] \le \\ &\le |x_1 - x_0|^\theta + C_L \int_0^{\rho_0 - \rho} e^{-\lambda t}|y_1(t) - y_0(t)|\, dt + \\ &\quad + e^{-\lambda(\rho_0 - \rho)}\sigma_v(|y_1(\rho_0 - \rho) - y_0(\rho_0 - \rho)|), \end{aligned}$$

where C_L is a Lipschitz constant for L in $\overline{\Omega}$. Now, let $0 \le R \le \rho$. Then, by the above inequality and (3.14) we have that, for some constant $C > 0$,

$$|v(x_1) - v(x_0)| \le CR^\theta + e^{-\lambda(\rho_0 - \rho)}\sigma_v(2e^{C_1(\rho_0 - \rho)}R),$$

for all pairs $x_0, x_1 \in \overline{\Omega}$ satisfying $|x_0 - x_1| \leq R$. Hence, taking the supremum over all such pairs, we obtain

$$\sigma_v(R) \leq CR^\theta + e^{-\lambda(\rho_0-\rho)}\sigma_v(2e^{C_1(\rho_0-\rho)}R), \quad \forall R \in [0, \rho]. \tag{3.15}$$

If θ verifies (3.6) there exists $\rho < \rho_0$ such that $2^\theta e^{-\lambda(\rho_0-\rho)}e^{\theta C_1(\rho_0-\rho)} < 1$. Then, applying Lemma 2.3 to (3.15) we conclude that

$$\sigma_v(R) \leq CR^\theta, \quad \forall R \in [0, \rho],$$

for some positive constant C and the Hölder continuity of v easily follows from such an estimate. If $\lambda > C_1 + \ln 2/\rho_0$ the value function v is Hölder continuous of exponent $\theta = 1$, that is v is Lipschitz continuous. □

Remark 3.5. If Ω is convex, then the value function v is Lipschitz continuous for any $\lambda > 0$. To show this fact it suffices to repeat the above proof recalling Remark 3.3.

Remark 3.6. The regularity results proved in this section make no use of hypothesis (1.6), as the proof clearly shows. Without this assumption, however, our constrained optimal control problem is no longer equivalent to an optimal exit time problem.

4. Semiconcavity of the value function.

In this section we prove the semiconcavity of the value function v under stronger assumptions on the running cost L. More precisely, we shall prove the following result. We recall that ρ_0 was introduced in (3.3) as the radius of a tubular neighborhood of Γ, Γ_{ρ_0}, such that $d \in C^{1,1}(\Gamma_{\rho_0})$.

Also, λ is a fixed positive number.

Theorem 4.1. *Let C_1 be a Lipschitz constant for ∇d in Γ_{ρ_0} and let L be a Lipschitz continuous function satisfying* (1.6). *Moreover, assume that L is semiconcave with linear semiconcavity modulus, that is*

$$L(x) + L(y) - 2L\Big(\frac{x+y}{2}\Big) \leq C_S|x-y|^2 \tag{4.1}$$

for any x, y such that the segment $[x, y]$ is contained in $\overline{\Omega}$, where C_S is a positive constant. Then the value function v is semiconcave on $\overline{\Omega}$ with *semiconcavity modulus* $\omega(r) = cr^\theta$, *for some positive constant* c *depending on* λ *and any exponent* $\theta \in (0,1]$ *satisfying*

$$(1+\theta)\Big(C_1 + \frac{2ln2}{\rho_0}\Big) < \lambda. \tag{4.2}$$

In particular, v *is semiconcave with linear modulus if*

$$\lambda > 2C_1 + 4\ln 2/\rho_0 =: \lambda_0\,. \tag{4.3}$$

As in the previous section we begin with a lemma for a suitable system of differential inclusions.

Lemma 4.2. *Let* $x_0, x_1 \in \overline{\Omega}$ *be such that* $d^2(x_0) + d^2(x_1) > 0$, *and let* $\overline{x} := (x_0 + x_1)/2 \in \overline{\Omega}$. *Let* $\alpha \in \mathcal{A}_{\overline{x}}$ *be a piecewise constant control and set* $\overline{y} = y^\alpha_{\overline{x}}$. *Then, fixed* $\rho \in (0, \rho_0)$, *there exist* $0 < T \le (\rho_0 - \rho)/2$ *and a solution* $(z_0(t), z_1(t))$, $t \in [0,T]$, *of the problem*

$$\begin{cases} \dot z_0(t) \in \alpha(t) - H(d(z_0(t)))\langle \alpha(t), \nabla d(z_0(t))\rangle_+ \nabla d(z_0(t)) + \\ \qquad + H(d(z_1(t)))\langle \alpha(t), \nabla d(z_1(t))\rangle_+ \nabla d(z_1(t)), \\ \dot z_1(t) \in \alpha(t) + H(d(z_0(t)))\langle \alpha(t), \nabla d(z_0(t))\rangle_+ \nabla d(z_0(t)) - \\ \qquad - H(d(z_1(t)))\langle \alpha(t), \nabla d(z_1(t))\rangle_+ \nabla d(z_1(t)), \\ z_0(0) = \quad x_0\,, \\ z_1(0) = \quad x_1\,, \end{cases} \tag{4.4}$$

such that

(*i*) $z_0(t), z_1(t) \in \overline{\Omega}$ *and* $d^2(z_0(t)) + d^2(z_1(t)) > 0 \ \ \forall t \in [0, T)$;

(*ii*) *if* $T < (\rho_0 - \rho)/2$ *then* $z_0(T), z_1(T) \in \Gamma$ *and*

$$T = \inf\{s > 0 \ : \ z_0(s) \in \Gamma, z_1(s) \in \Gamma\};$$

(*iii*) $z_0(t) + z_1(t) - 2\overline{y}(t) = 0, \ \ \forall t \in [0, T]$.

Moreover, if $x_0, x_1 \in \overline{\Omega} \cap B(\overline{x}, \rho/2)$, *then*

$$|z_0(t) - z_1(t)| \le 2e^{C_1 t}|x_0 - x_1|, \quad \forall t \in [0,T], \tag{4.5}$$

where C_1 *is a Lipschitz constant for* ∇d *in* Γ_{ρ_0}.

Proof. Let H be as in (3.8) and $A(t)$ be as in (3.10). Let us consider the set valued map $G : [0, +\infty) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n \times \mathbb{R}^n)$ defined as follows: $(v_1, v_2) \in G(t, w_1, w_2)$ if and only if

$$v_1 = a - h\langle b, \nabla d(w_1)\rangle_+ \nabla d(w_1) + k\langle c, \nabla d(w_2)\rangle_+ \nabla d(w_2),$$
$$v_2 = a + h\langle b, \nabla d(w_1)\rangle_+ \nabla d(w_1) - k\langle c, \nabla d(w_2)\rangle_+ \nabla d(w_2),$$

for some $a, b, c \in A(t)$, $h \in H(d(w_1))$, $k \in H(d(w_2))$.

Then the existence of a solution $(z_0(t), z_1(t))$ to the problem

$$\begin{cases} (\dot{z}_0(t), \dot{z}_1(t)) \in G(t, z_0(t), z_1(t)), \\ (z_0(0), z_1(0)) = (x_0, x_1), \end{cases} \tag{4.6}$$

follows arguing as in the proof of Lemma 3.2. Moreover, by the definition of $A(t)$ the solution $(z_0(t), z_1(t))$ satisfies system (4.4) for a.e. $t \geq 0$. Furthermore, two measurable selections h_0, h_1 of H exist such that:

$$\dot{z}_0(t) = \alpha(t) - h_0(t)\langle \alpha(t), \nabla d(z_0(t))\rangle_+ \nabla d(z_0(t)) + \\ + h_1(t)\langle \alpha(t), \nabla d(z_1(t))\rangle_+ \nabla d(z_1(t)),$$

$$\dot{z}_1(t) = \alpha(t) + h_0(t)\langle \alpha(t), \nabla d(z_0(t))\rangle_+ \nabla d(z_0(t)) - \\ - h_1(t)\langle \alpha(t), \nabla d(z_1(t))\rangle_+ \nabla d(z_1(t)),$$

for a.e. $t \geq 0$. Suppose that a time $t_0 \in (0, (\rho_0 - \rho)/2)$ exists at which $d(z_0(t_0)) > 0$ and define

$$t_1 = \inf\{t \geq 0 : d(z_0(t)) > 0\}.$$

Obviously $d(z_0(t_1)) = 0$. If $z_1(t_1) \in \Omega$, then there exists $\delta > 0$ such that $z_0(t)$ verifies for $t \in [t_1, t_1 + \delta]$

$$\dot{z}_0(t) = \alpha(t) - h_0(t)\langle \alpha(t), \nabla d(z_0(t))\rangle_+ \nabla d(z_0(t)).$$

Therefore, arguing as in Lemma 3.2, we can prove that $z_0(t)$ remains in $\overline{\Omega}$, $\forall t \in [t_1, t_1 + \delta]$, in contradiction with the definition of t_1. If $z_1(t_1) \notin \overline{\Omega}$ let

$$t_2 = \inf\{t \geq 0 : d(z_1(t)) > 0\}.$$

We have that $d(z_1(t_2)) = 0$ and $t_2 < t_1$, then $z_0(t_2) \in \overline{\Omega}$. If $z_0(t_2) \in \Gamma$ we fix $T = t_2$, then (i) and (ii) hold. Otherwise, if $z_0(t_2) \in \Omega$, then there exists $\delta' > 0$ such that $\forall t \in [t_2, t_2 + \delta']$, $z_1(t)$ verifies

$$\dot{z}_1(t) = \alpha(t) - h_1(t)\langle \alpha(t), \nabla d(z_1(t))\rangle_+ \nabla d(z_1(t)).$$

Therefore, Lemma 3.2 ensures that $z_1(t)$ remains in $\overline{\Omega}$, $\forall t \in [t_2, t_2 + \delta']$, in contradiction with the definition of t_2. This implies that $z_1(t_1) \in \Gamma$ and then we obtain (i) and (ii) taking $T = t_1$.

We note that

$$\frac{1}{2}\frac{d}{dt}|z_0(t)+z_1(t)-2\overline{y}(t)|^2 = \langle \dot{z}_0(t)+\dot{z}_1(t)-2\overline{y}(t), z_0(t)+z_1(t)-2\overline{y}(t)\rangle = 0.$$

We have thus proved assertion (iii). Furthermore, the above identity yields

$$|z_0(t) - \overline{y}(t)| = |z_1(t) - \overline{y}(t)|\,, \quad \text{for a.e. } t \geq 0. \tag{4.7}$$

To prove (4.5), fix $t \leq (\rho_0 - \rho)/2$ and define

$$S_{j,t} = \{s \in [0,t) : z_j(s) \in \Gamma\},\ j = 0, 1.$$

If $S_{0,t} \cup S_{1,t} = \emptyset$, then $h_0(s) = h_1(s) = 0$ for any $s \in [0,t)$ and so $z_0(t) - z_1(t) = x_0 - x_1$.

On other hand, let $\sigma_t = \inf S_{0,t} \cup S_{1,t}$ and $\overline{\sigma}_t = \sup S_{0,t} \cup S_{1,t}$. Arguing as in proof of Lemma 3.2 we see that

$$\overline{y}(s),\ z_0(s),\ z_1(s) \in \Gamma_{\rho_0}\,,\ \forall s \in [\sigma_T, \overline{\sigma}_T]. \tag{4.8}$$

We observe that

$$\overline{y}(t) - z_0(t) = \overline{x} - x_0 + \int_0^t h_0(s)\langle\alpha(s), \nabla d(z_0(s))\rangle_+ \nabla d(z_0(s))\,ds -$$
$$- \int_0^t h_1(s)\langle\alpha(s), \nabla d(z_1(s))\rangle_+ \nabla d(z_1(s))\,ds.$$

Then

$$|\overline{y}(t) - z_0(t)| \leq |\overline{x} - x_0| + \int_{\sigma_t}^{\overline{\sigma}_t} \big[h_0(s)\langle\alpha(s), \nabla d(z_0(s))\rangle_+ + h_1(s)\langle\alpha(s), \nabla d(z_1(s))\rangle_+\big]\,ds\,. \tag{4.9}$$

We want to prove that

$$\int_{\sigma_t}^{\overline{\sigma}_t} [h_0(s)\langle\alpha(s), \nabla d(z_0(s))\rangle_+ + h_1(s)\langle\alpha(s), \nabla d(z_1(s))\rangle_+]\,ds \leq \tag{4.10}$$

$$\leq |\overline{x} - x_0| + C_1 \int_{\sigma_t}^{\overline{\sigma}_t} |\overline{y}(s) - z_0(s)|\, ds.$$

We recall that the control α is piecewise constant. Consider an interval $[a, b]$, $b \leq t$, such that α is constant in (a, b). Note that the set

$$\bigcup_{j=0,1} \left\{s \in [a, b] : z_j(s) \in \Gamma\,,\ \langle \alpha(s), \nabla d(z_j(s))\rangle \geq 0\right\}$$

is closed. So, its complementary set is open in $[a, b]$ and it can be viewed as a countable ubion of disjoint intervals. Repeating this argument for any interval in wich α is constant, we obtain that the set

$$W_t = \left\{s \in [0, t) : z_0(s) \in \Omega,\ z_1(s) \in \Omega\right\} \bigcup$$

$$\left(\bigcup_{j=0,1} \left\{s \in [0, t) : z_j(s) \in \Gamma,\ \langle \alpha(s), \nabla d(z_j(s))\rangle < 0\right\}\right)$$

is a countable union of disjoint intervals for any $t \leq T$.

We can estimate

$$\int_{\sigma_t}^{\overline{\sigma}_t} [h_0(s)\langle \alpha(s), \nabla d(z_0(s))\rangle_+ + h_1(s)\langle \alpha(s), \nabla d(z_1(s))\rangle_+]\, ds =$$

$$= \int_{[\sigma_t - \overline{\sigma}_t]\setminus W_t} [h_0(s)\langle \alpha(s), \nabla d(z_0(s))\rangle + h_1(s)\langle \alpha(s), \nabla d(z_1(s))\rangle]\, ds \leq$$

$$\leq \int_{[\sigma_t - \overline{\sigma}_t]\setminus W_t} \langle \alpha(s), w(s)\rangle\, ds$$

where $w(s) = \nabla d(z_0(s))$ if $h_0(s) \neq 0$, $w(s) = \nabla d(z_1(s))$ if $h_1(s) \neq 0$. Then, recalling (4.7),

$$\int_{\sigma_t}^{\overline{\sigma}_t} [h_0(s)\langle \alpha(s), \nabla d(z_0(s))\rangle_+ + h_1(s)\langle \alpha(s), \nabla d(z_1(s))\rangle_+]\, ds \leq$$

$$\leq \int_{[\sigma_t - \overline{\sigma}_t]\setminus W_t} \langle \alpha(s), w(s) - \nabla d(\overline{y}(s))\rangle\, ds + \tag{4.11}$$

$$+ \int_{[\sigma_t - \overline{\sigma}_t]\setminus W_t} \langle \alpha(s), \nabla d(\overline{y}(s))\rangle\, ds \leq$$

$$\leq C_1 \int_{[\sigma_t-\overline{\sigma}_t]\setminus W_t} |\overline{y}(s)-z_0(s)|\,ds + \int_{[\sigma_t-\overline{\sigma}_t]} \left[d(\overline{y}(s))\right]' ds - \\ - \int_{W_t\cap[\sigma_t-\overline{\sigma}_t]} \left[d(\overline{y}(s))\right]' ds\,. \tag{4.12}$$

We know that

$$W_t \cap [\sigma_t - \overline{\sigma}_t] = \bigcup_{j\in\mathbb{N}} I_j$$

where I_j, $j\in\mathbb{N}$, are intervals such that, denoted by a_j and b_j the endpoints, $z_0(a_j)\in\Gamma$ or $z_1(a_j)\in\Gamma$ and $z_0(b_j)\in\Gamma$ or $z_1(b_j)\in\Gamma$. Moreover, in I_j° we have $\dot{z}_0=\alpha$ and $\dot{z}_1=\alpha$. Fix I_j and suppose $z_0(b_j)\in\Gamma$. Then,

$$0 = \int_{I_j} \left[d(\overline{y}(s))\right]' ds - \int_{I_j} \left[d(\overline{y}(s))\right]' ds$$

$$= \int_{I_j} \left[d(\overline{y}(s))\right]' ds - \int_{I_j} \langle \nabla d(\overline{y}(s)) - \nabla d(z_0(s)), \alpha(s)\rangle\, ds - \\ - \int_{I_j} \langle \nabla d(z_0(s)), \alpha(s)\rangle\, ds \leq$$

$$\leq \int_{I_j} \left[d(\overline{y}(s))\right]' ds + C_1 \int_{I_j} |\overline{y}(s)-z_0(s)|\,ds - \int_{I_j} \left[d(z_0(s))\right]' ds \leq$$

$$\leq \int_{I_j} \left[d(\overline{y}(s))\right]' ds + C_1 \int_{I_j} |\overline{y}(s)-z_0(s)|\,ds - d(z_0(b_j)) + d(z_0(a_j)).$$

Since $d(z_0(b_j))=0$ and $d(z_0(a_j))\leq 0$, we obtain

$$-\int_{I_j} \left[d(\overline{y}(s))\right]' ds \leq C_1 \int_{I_j} |\overline{y}(s)-z_0(s)|\,ds\,,$$

and, summing for $j\in\mathbb{N}$,

$$-\int_{W_t\cap[\sigma_t-\overline{\sigma}_t]} \left[d(\overline{y}(s))\right]' ds \leq \int_{W_t\cap[\sigma_t-\overline{\sigma}_t]} |\overline{y}(s)-z_0(s)|\,ds\,. \tag{4.13}$$

Using (4.13) we can obtain from (4.12)

$$\int_{\sigma_t}^{\overline{\sigma}_t} [h_0(s)\langle\alpha(s),\nabla d(z_0(s))\rangle_+ + h_1(s)\langle\alpha(s),\nabla d(z_1(s))\rangle_+]\,ds \leq$$

$$\leq C_1 \int_{\sigma_t}^{\overline{\sigma}_t} |\overline{y}(s) - z_0(s)|\, ds + \int_{\sigma_t}^{\overline{\sigma}_t} [d(\overline{y}(s))]'\, ds \leq$$

$$\leq C_1 \int_{\sigma_t}^{\overline{\sigma}_t} |\overline{y}(s) - z_0(s)|\, ds - d(\overline{y}(\sigma_t)).$$

Therefore, if $z_0(\sigma_t) \in \Gamma$, we have

$$\int_{\sigma_t}^{\overline{\sigma}_t} [h_0(s)\langle \alpha(s), \nabla d(z_0(s))\rangle_+ + h_1(s)\langle \alpha(s), \nabla d(z_1(s))\rangle_+]\, ds \leq$$

$$\leq C_1 \int_{\sigma_t}^{\overline{\sigma}_t} |\overline{y}(s) - z_0(s)|\, ds + d(z_0(\sigma_t)) - d(\overline{y}(\sigma_t))$$

from which follows (4.10). Analogously we can proceed if $z_1(\sigma_t) \in \Gamma$. Now, we can use estimate (4.10) to bound the right-hand side of (4.9). So,

$$|\overline{y}(t) - z_0(t)| \leq 2|\overline{x} - x_0| + C_1 \int_0^t |\overline{y}(s) - z_0(s)|\, ds, \quad 0 \leq t \leq T.$$

Then, by the Gronwall Lemma,

$$|\overline{y}(t) - z_0(t)| \leq 2e^{C_1 t}|\overline{x} - x_0|, \quad 0 \leq t \leq T.$$

Estimate (4.5) easily follows from the above inequality and (4.7). □

Remark 4.3. Note that from (4.10) and (4.5) immediately follows

$$\int_0^t [h_0(s)\langle \alpha(s), \nabla d(z_0(s))\rangle_+ + h_1(s)\langle \alpha(s), \nabla d(z_1(s))\rangle_+]\, ds = O(|x_0 - x_1|).$$

Remark 4.4. We observe that

$$\frac{1}{2}\frac{d}{dt}|z_0(t) - z_1(t)|^2 = -2h_0(t)\langle \alpha(t), \nabla d(z_0(t))\rangle_+ \langle \nabla d(z_0(t)), z_0(t) - z_1(t)\rangle +$$
$$+ 2h_1(t)\langle \alpha(t), \nabla d(z_1(t))\rangle_+ \langle \nabla d(z_1(t)), z_0(t) - z_1(t)\rangle.$$

If Ω is convex, then

$$\langle \nabla d(z_0(t)), z_1(t) - z_0(t)\rangle \leq 0 \quad \text{for a.e.} \quad t \in [0, T],$$

and

$$\langle \nabla d(z_1(t)), z_0(t) - z_1(t)\rangle \leq 0 \quad \text{for a.e.} \quad t \in [0, T].$$

So,

$$|z_0(t) - z_1(t)| \leq |x_0 - x_1|, \ \forall t \in [0, T]. \tag{4.14}$$

Notice that, for convex Ω, (4.14) holds without any Lipschitz assumption for ∇d. Moreover, time T above is subject to no restriction whatsoever and therefore instead of $(\rho_0 - \rho)/2$ we can consider in the lemma above a time large as we want.

Let ρ_0 be as in (3.3) and let $\rho \in (0, \rho_0)$. Then we have the following result.

Lemma 4.5. *Let $\overline{x} \in \overline{\Omega}$ and let $\alpha \in \mathcal{A}_{\overline{x}}$ be a piecewise constant control with $|\alpha(t)| = 1$ or $\alpha(t) = 0$, $\forall t \geq 0$. Then, for any pair $x_0, x_1 \in \overline{\Omega} \cap B(\overline{x}, \rho/2)$, satisfying $\overline{x} = (x_0 + x_1)/2$ and $d^2(x_0) + d^2(x_1) > 0$, there exist numbers $0 < T \leq (\rho_0 - \rho)/2$, $T_0, T_1 \geq 0$, two non-decreasing Lipschitz functions*

$$\varphi^j : [0, T] \to [0, T_j], \quad j = 0, 1,$$

and two trajectories y_0, y_1 admissible at x_0, x_1, respectively, such that:

(i) $\varphi^0(t) + \varphi^1(t) \geq 2t\,,\ \forall t \in [0, T]$;

(ii) $|\varphi^j(t) - t| = O(|h|)$, $j = 0, 1$;

(iii) $y_0(\varphi^0(t)) + y_1(\varphi^1(t)) - 2\overline{y}(t) = 0$, $\forall t \in [0, T]$, *where* $\overline{y} = y^{\alpha}_{\overline{x}}$;

(iv) $|y_0(\varphi^0(t)) - y_1(\varphi^1(t))| \leq 2e^{C_1 t}|x_0 - x_1|$, $\forall t \in [0, T]$;

(v) y_j *is differentiable on* $[0, T_j]$ *with* $|\dot{y}_j| = |\alpha|$ *a. e.,* $j = 0, 1$;

where C_1 is a Lipschitz constant for ∇d in Γ_{ρ_0}.

Proof. Let T be defined as in Lemma 4.2 and $(z_0(t), z_1(t))$, $t \in [0, T]$, be a solution of (4.4). Let us set, for $t \in [0, T]$ and $j = 0, 1$,

$$\psi^j(t) = \begin{cases} 1 & \text{if} \quad \alpha(t) = 0 \\ |\dot{z}_j(t)| & \text{if} \quad \alpha(t) \neq 0 \end{cases} \tag{4.15}$$

and define

$$\varphi^j(t) = \int_0^t \psi^j(s)\, ds, \quad t \in [0, T],\ j = 0, 1. \tag{4.16}$$

Then, $\varphi^j : [0, T] \to [0, T_j]$ are Lipschitz continuous and non-decreasing. Moreover, $\forall t \in [0, T]$,

$$\varphi^0(t) + \varphi^1(t) - 2t = \int_0^t [\psi^0(s) + \psi^1(s) - 2]\, ds =$$

$$= \int_{[0,t]\setminus\alpha^{-1}\{0\}} (|\dot{z}_0(s)| + |\dot{z}_1(s)| - 2|\alpha(s)|)\, ds \geq$$

$$\geq \int_{[0,t]\setminus\alpha^{-1}\{0\}} (|\dot{z}_0(s) + \dot{z}_1(s)| - 2|\alpha(s)|)\, ds = 0.$$

Hence, (i) holds. Now, define

$$S_j = \{t \in [0, T) \ : \ z_j(t) \in \Gamma\}, \ j = 0, 1. \tag{4.17}$$

Obviously, $S_0 \cap S_1 = \emptyset$. If $S_0 \cup S_1 = \emptyset$, then $h_0(t) = h_1(t) = 0, \forall t \in [0, T]$ and so

$$\varphi^0(t) = \varphi^1(t) = t.$$

If, on the contrary, $S_0 \cup S_1 \neq \emptyset$, set $\sigma = \inf S_0 \cup S_1$ and $\overline{\sigma} = \sup S_0 \cup S_1$. From (4.8) we have that

$$\overline{y}(s), \ z_0(s), \ z_1(s) \in \Gamma_{\rho_0}, \ \forall s \in [\sigma, \overline{\sigma}]. \tag{4.18}$$

Therefore, recalling Remark 4.3, $\forall t \in [0, T]$,

$$|\varphi^j(t) - t| = \left| \int_0^t (\psi^j(s) - 1)\, ds \right| \le \left| \int_{[0,t]\cap[\sigma,\overline{\sigma}]} |\dot{z}_j(s) - \alpha(s)|\, ds \right| \le$$

$$\le \int_{[\sigma,\overline{\sigma}]\cap S_0} h_0(s)\langle \alpha(s), \nabla d(z_0(s))\rangle_+ \, ds +$$

$$+ \int_{[\sigma,\overline{\sigma}]\cap S_1} h_1(s)\langle \alpha(s), \nabla d(z_1(s))\rangle_+ \, ds \le c|h|, \qquad c > 0, \ j = 0, 1.$$

We have thus proved (ii). Next, since φ_j are continuous and non-decreasing,

$$(\varphi^j)^{-1}\{s\} = \{t \in [0, T] : \varphi^j(t) = s\} = [a_j(s), b_j(s)], \ s \in [0, T_j], \ j = 0, 1, \tag{4.19}$$

where $a_j(s) = b_j(s)$ whenever $\dot{\varphi}^j(t) \neq 0$ for some $t \in [a_j(s), b_j(s)]$.

Let us define

$$y_j(s) = z_j(a_j(s)), \ s \in [0, T_j], \ j = 0, 1. \tag{4.20}$$

Then, y_0, y_1, are Lipschitz continuous. Indeed for any pair $s', s'' \in [0, T_j]$

$$|y_j(s') - y_j(s'')| = |z_j(a_j(s')) - z_j(a_j(s''))| \le \left| \int_{a_j(s')}^{a_j(s'')} |\dot{z}_j(t)|\, dt \right| \le$$

$$\le \left| \int_{a_j(s')}^{a_j(s'')} \dot{\varphi}^j(s)\, ds \right| = |\varphi^j(a_j(s')) - \varphi^j(a_j(s''))| = |s' - s''|, \ \ j = 0, 1.$$

Now observe that, for any $t \in [0, T]$,

$$z_j(t) = z_j(a_j(\varphi^j(t))), \;\; j = 0, 1,$$

or

$$y_j(\varphi^j(t)) = z_j(t), \;\; t \in [0, T], \;\; j = 0, 1.$$

So, points (iii) and (iv) of the conclusion follow from property (iii) of Lemma 4.2 and from (4.5).

Finally, to prove (v) let us set

$$H_j = \{t \in [0, T] \; : \; \exists \dot{\varphi}^j(t) \neq 0\}, \;\; j = 0, 1. \tag{4.21}$$

Then

$$m\Big(\varphi^j(H_j)\Big) = T_j, \;\; j = 0, 1, \tag{4.22}$$

where m denotes the Lebesgue measure in $\mathbb{R}$. Thus y_j is a. e. differentiable on $[0, T_j]$, $j = 0, 1$, and the identity $|\dot{y}_j| = |\alpha|$ a. e. on $[0, T_j]$, $j = 0, 1$, follows from the definition of y_j. □

Lemma 4.6. *Let* $\Omega \subset \mathbb{R}^n$, $T > 0$ *and* $x \in \overline{\Omega}$. *Suppose that* L *satisfies assumption* (1.6). *Then, there exists an optimal trajectory at* x, $y_x^{\overline{\alpha}}(\cdot)$, *such that for any* $\varepsilon > 0$ *a piecewice constant control* $\alpha(\cdot)$ *exists with* $\alpha = 0$ *or* $|\alpha| = 1$, *and*

$$|y_x^{\overline{\alpha}}(t) - y_x^{\alpha}(t)| < \varepsilon, \;\; \forall t \in [0, T]. \tag{4.23}$$

Proof. By hypothesis (1.6) we can suppose that either $y_x^{\overline{\alpha}}(t) \in \Omega$ for any time $t \in [0, T]$ or there exists a time $\overline{t} \leq T$ such that $y_x^{\overline{\alpha}}(\overline{t}) \in \partial\Omega$. In the latter case, we can take $\overline{\alpha}(t) = 0$, $\forall t \in (\overline{t}, T]$. For fixed $\varepsilon > 0$, choose a time $\widetilde{T} < T$ such that

$$|y_x^{\overline{\alpha}}(\widetilde{T}) - y_x^{\overline{\alpha}}(t)| < \varepsilon, \;\; \forall t \in [\widetilde{T}, T]. \tag{4.24}$$

We can consider a subdivision of the interval $[0, \widetilde{T}]$ in smaller intervals $[t_{i-1}, t_i]$, $i = 1, \ldots, k$, where $t_0 = 0$ and $t_k = \widetilde{T}$. We define on every interval $[t_{i-1}, t_i]$ a trajectory $\overline{y}$ as follows:

$$\overline{y}(t) = y_x^{\overline{\alpha}}(t_{i-1}) + (t - t_{i-1}) \frac{y_x^{\overline{\alpha}}(t_i) - y_x^{\overline{\alpha}}(t_{i-1})}{|y_x^{\overline{\alpha}}(t_i) - y_x^{\overline{\alpha}}(t_{i-1})|},$$

for $t_{i-1} \leq t \leq t_{i-1} + |y_x^{\overline{\alpha}}(t_{i-1}) - y_x^{\overline{\alpha}}(t_i)|$;

$$\overline{y}(t) = y_x^{\overline{\alpha}}(t_i),$$

for $t_{i-1} + |y_x^{\overline{\alpha}}(t_{i-1}) - y_x^{\overline{\alpha}}(t_i)| \leq t \leq t_i$. Obviously we can take the intervals so small that

$$(4.25) \qquad |y_x^{\overline{\alpha}}(t) - \overline{y}(t)| < \varepsilon, \quad \forall t \in [0, \widetilde{T}].$$

Finally, set

$$\overline{y}(t) = y_x^{\overline{\alpha}}(\widetilde{T}), \quad t \in [\widetilde{T}, T].$$

It is easy to see that $\dot{\overline{y}}$ exists a. e. and $\dot{\overline{y}} = \alpha$, where α is a piecewise constant control satisfying $\alpha = 0$ or $|\alpha| = 1$. So, (4.23) immediately follows from (4.24) and (4.25). □

Proof of Theorem 4.1. Let $\overline{x} \in \overline{\Omega}$ and let ρ_0 be as in (3.3). Let $x_0, x_1 \in \overline{\Omega} \cap B(\overline{x}, \rho/2)$ for a positive number $\rho < \rho_0$, with $\overline{x} = (x_0 + x_1)/2$, and take $h = (x_1 - x_0)/2$. Let $\overline{\alpha}(t)$, $t \geq 0$, an optimal control for $\overline{x}$. Fix $\theta \in (0, 1]$ so that condition (4.2) is satisfied. We define a modulus

$$(4.26) \qquad \sigma_v'(R) = \sup\{u(x+h) + u(x-h) - 2u(x) : \\ x \in \overline{\Omega},\ |2h| \leq R, x \pm h \in \overline{\Omega}\},\ R \geq 0.$$

If $x_0, x_1 \in \Gamma$, then from (1.6) it follows that

$$(4.27) \qquad v(x_0) + v(x_1) - 2v(\overline{x}) = -2v(\overline{x}) \leq 0.$$

Otherwise, from Lemma 4.6 it follows that we can consider a piecewise constant control $\alpha \in \mathcal{A}_{\overline{x}}$ such that

$$(4.28) \qquad v(\overline{x}) > \int_0^S e^{-\lambda t} L(\overline{y}(t))\, dt + e^{-\lambda S} v(\overline{y}(S)) - |h|^{1+\theta},$$

where $S = (\rho_0 - \rho)/2$ and $\overline{y} = y_{\overline{x}}^{\alpha}$. Further, we can assume $\alpha = 0$ or $|\alpha| = 1$ for all $t \leq S$.

We know that there exists $T \leq S$ and there exist trajectories y_0, y_1, admissible at x_0 and at x_1 on $[0, T_0]$ and on $[0, T_1]$ respectively, that verify the relations of Lemma 4.2 and Lemma 4.5. Moreover, if $T < S$ then $y_0(T_0)$ and $y_1(T_1)$ belong to Γ.

Let T be as in Lemma 4.2. Then, we claim that

$$v(\overline{x}) > \int_0^T e^{-\lambda t} L(\overline{y}(t))\,dt + e^{-\lambda T} v(\overline{y}(T)) - c|h|^{1+\theta}, \tag{4.29}$$

for some constant $c > 0$. Indeed, by (1.6),

$$\int_0^T e^{-\lambda t} L(\overline{y}(t))\,dt \le \int_0^S e^{-\lambda t} L(\overline{y}(t))\,dt,$$

and, recalling Theorem 3.1, from the definition of T,

$$e^{-\lambda T} v(\overline{y}(T)) \le e^{-\lambda S} v(\overline{y}(S)) + c|h|^{1+\theta}, \quad c > 0.$$

So, (4.29) immediately follows from (4.28).

Now, from the dynamic programming principle

$$v(x_0) + v(x_1) - 2v(\overline{x}) \le$$

$$\le \int_0^{T_0} e^{-\lambda t} L(y_0(t))\,dt + \int_0^{T_1} e^{-\lambda t} L(y_1(t))\,dt - 2\int_0^T e^{-\lambda t} L(\overline{y}(t))\,dt +$$

$$+ e^{-\lambda T_0} v(y_0(T_0)) + e^{-\lambda T_1} v(y_1(T_1)) - 2e^{-\lambda T} v(\overline{y}(T)) + c|h|^{1+\theta}.$$

Recalling (4.22) we can write

$$v(x_0) + v(x_1) - 2v(\overline{x}) \le$$

$$\le \int_{\varphi^0(H_0)} e^{-\lambda t} L(y_0(t))\,dt + \int_{\varphi^1(H_1)} e^{-\lambda t} L(y_1(t))\,dt - 2\int_0^T e^{-\lambda t} L(\overline{y}(t))\,dt +$$

$$+ e^{-\lambda T_0} v(y_0(T_0)) + e^{-\lambda T_1} v(y_1(T_1)) - 2e^{-\lambda T} v(\overline{y}(T)) + c|h|^{1+\theta},$$

where H_j, $j = 0, 1$, are defined in (4.21). Then

$$v(x_0) + v(x_1) - 2v(\overline{x}) \le$$

$$\le \int_{\varphi^0(H_0\cap H_1)} e^{-\lambda t} L(y_0(t))\,dt + \int_{\varphi^0(H_0\setminus H_1)} e^{-\lambda t} L(y_0(t))\,dt +$$

$$+ \int_{\varphi^1(H_0\cap H_1)} e^{-\lambda t} L(y_1(t))\,dt + \int_{\varphi^1(H_1\setminus H_0)} e^{-\lambda t} L(y_1(t))\,dt -$$

$$-2\int_{H_0\cap H_1} e^{-\lambda t}L(\overline{y}(t))\,dt - 2\int_{H_0\setminus H_1} e^{-\lambda t}L(\overline{y}(t))\,dt - 2\int_{H_1\setminus H_0} e^{-\lambda t}L(\overline{y}(t))\,dt +$$
$$+ e^{-\lambda T_0}v(y_0(T_0)) + e^{-\lambda T_1}v(y_1(T_1)) - 2e^{-\lambda T}v(\overline{y}(T)) + c|h|^{1+\theta},$$

that we rewrite as

$$v(x_0) + v(x_1) - 2v(\overline{x}) \le c|h|^{1+\theta} + I + I_0 + I_1 + E, \tag{4.30}$$

where

$$I = \int_{\varphi^0(H_0\cap H_1)} e^{-\lambda t}L(y_0(t))\,dt + \int_{\varphi^1(H_0\cap H_1)} e^{-\lambda t}L(y_1(t))\,dt -$$
$$-2\int_{H_0\cap H_1} e^{-\lambda t}L(\overline{y}(t))\,dt\,,$$
$$I_0 = \int_{\varphi^0(H_0\setminus H_1)} e^{-\lambda t}L(y_0(t))\,dt - 2\int_{H_0\setminus H_1} e^{-\lambda t}L(\overline{y}(t))\,dt,$$
$$I_1 = \int_{\varphi^1(H_1\setminus H_0)} e^{-\lambda t}L(y_1(t))\,dt - 2\int_{H_1\setminus H_0} e^{-\lambda t}L(\overline{y}(t))\,dt,$$
$$E = e^{-\lambda T_0}v(y_0(T_0)) + e^{-\lambda T_1}v(y_1(T_1)) - 2e^{-\lambda T}v(\overline{y}(T)).$$

Using the change of variable $t = \varphi^0(s)$ and $t = \varphi^1(s)$ respectively in the first and in the second integral in I we obtain

$$I = \int_{H_0\cap H_1} \varphi^{0'}(t)e^{-\lambda\varphi^0(t)}L(y_0(\varphi^0(t)))\,dt +$$
$$+\int_{H_0\cap H_1} \varphi^{1'}(t)e^{-\lambda\varphi^1(t)}L(y_1(\varphi^1(t)))\,dt - 2\int_{H_0\cap H_1} e^{-\lambda t}L(\overline{y}(t))\,dt,$$

that we can rewrite as

$$I = \int_{H_0\cap H_1} e^{-\lambda t}\Big[L(y_0(\varphi^0(t))) + L(y_1(\varphi^1(t))) - 2L(\overline{y}(t))\Big]\,dt + \tag{4.31}$$
$$+\int_{H_0\cap H_1} \Big[\varphi^{0'}(t)e^{-\lambda\varphi^0(t)} - e^{-\lambda t}\Big]L(y_0(\varphi^0(t)))\,dt +$$
$$+\int_{H_0\cap H_1} \Big[\varphi^{1'}(t)e^{-\lambda\varphi^1(t)} - e^{-\lambda t}\Big]L(y_1(\varphi^1(t)))\,dt.$$

Since L is semiconcave with a linear semiconcavity modulus we have

$$\int_{H_0\cap H_1} e^{-\lambda t}\Big[L(y_0(\varphi^0(t))) + L(y_1(\varphi^1(t))) - 2L(\overline{y}(t))\Big]\,dt \le c_0|h|^2, \tag{4.32}$$

for some constant $c_0 > 0$. Now, recalling definition (4.17),

$$\begin{aligned}
(4.33)\qquad & \int_{H_0\cap H_1} \Big[\varphi^{0'}(t)e^{-\lambda\varphi^0(t)} - e^{-\lambda t}\Big]L(y_0(\varphi^0(t)))\,dt + \\
&+ \int_{H_0\cap H_1} \Big[\varphi^{1'}(t)e^{-\lambda\varphi^1(t)} - e^{-\lambda t}\Big]L(y_1(\varphi^1(t)))\,dt = \\
&= \int_{H_0\cap H_1\cap S_0} \Big[\varphi^{1'}(t)e^{-\lambda\varphi^1(t)} - e^{-\lambda t}\Big]\Big[L(y_1(\varphi^1(t))) - L(y_0(\varphi^0(t)))\Big]\,dt + \\
&+ \int_{H_0\cap H_1\cap S_0} \Big[\varphi^{0'}(t)e^{-\lambda\varphi^0(t)} + \varphi^{1'}(t)e^{-\lambda\varphi^1(t)} - 2e^{-\lambda t}\Big]L(y_0(\varphi^0(t)))\,dt + \\
&+ \int_{H_0\cap H_1\cap S_1} \Big[\varphi^{0'}(t)e^{-\lambda\varphi^0(t)} - e^{-\lambda t}\Big]\Big[L(y_0(\varphi^0(t))) - L(y_1(\varphi^1(t)))\Big]\,dt + \\
&+ \int_{H_0\cap H_1\cap S_1} \Big[\varphi^{0'}(t)e^{-\lambda\varphi^0(t)} + \varphi^{1'}(t)e^{-\lambda\varphi^1(t)} - 2e^{-\lambda t}\Big]L(y_1(\varphi^1(t)))\,dt + \\
&+ \int_{(H_0\cap H_1)\setminus(S_0\cup S_1)} \Big[e^{-\lambda\varphi^1(t)} - e^{-\lambda t}\Big]\Big[L(y_1(\varphi^1(t))) - L(y_0(\varphi^0(t)))\Big]\,dt + \\
&+ \int_{(H_0\cap H_1)\setminus(S_0\cup S_1)} \Big[e^{-\lambda\varphi^0(t)} + e^{-\lambda\varphi^1(t)} - 2e^{-\lambda t}\Big]L(y_0(\varphi^0(t)))\,dt\,,
\end{aligned}$$

where we used that in $(H_0 \cap H_1) \setminus (S_0 \cup S_1)$ we have

$$\varphi^{0'}(t) = \varphi^{1'}(t) = 1.$$

We observe that

$$\begin{aligned}
(4.34)\qquad & \int_{H_0\cap H_1\cap S_0} \Big[\varphi^{1'}(t)e^{-\lambda\varphi^1(t)} - e^{-\lambda t}\Big]\Big[L(y_1(\varphi^1(t))) - L(y_0(\varphi^0(t)))\Big]\,dt = \\
&= \int_{H_0\cap H_1\cap S_0} e^{-\lambda t}\Big[e^{-\lambda(\varphi^1(t)-t)} - 1\Big]\Big[L(y_1(\varphi^1(t))) - L(y_0(\varphi^0(t)))\Big]\,dt + \\
&+ \int_{H_0\cap H_1\cap S_0} \Big(\varphi^{1'}(t) - 1\Big)e^{-\lambda\varphi^1(t)}\Big[L(y_1(\varphi^1(t))) - L(y_0(\varphi^0(t)))\Big]\,dt \le c_1|h|^2,
\end{aligned}$$

for some constant $c_1 > 0$; where we used that L and $e^{-\lambda t}$ are Lipschitz continuous and that

$$|\varphi^1(t) - t| = O(|h|),$$

and, from Remark 4.3,

$$\int_{H_0 \cap H_1 \cap S_0} \left(\varphi^{1'}(t) - 1\right) dt \leq \int_{H_0 \cap H_1 \cap S_0} (|\dot{z}_1(t)| - |\alpha(t)|)\, dt \leq$$

$$\leq \int_{H_0 \cap H_1 \cap S_0} |\dot{z}_1(t) - \alpha(t)|\, dt \leq \int_{H_0 \cap H_1 \cap S_0} h_0(t)\langle \alpha(t), \nabla d(z_0(t))\rangle_+\, dt = O(|h|).$$

Analogously

$$\int_{H_0 \cap H_1 \cap S_1} \left[\varphi^{0'}(t) e^{-\lambda \varphi^0(t)} - e^{-\lambda t}\right]\left[L(y_0(\varphi^0(t))) - L(y_1(\varphi^1(t)))\right] dt \leq \tag{4.35}$$

$$\leq c_2 |h|^2, \quad c_2 > 0.$$

Recalling that $L|_\Gamma = 0$ we have

$$\int_{H_0 \cap H_1 \cap S_0} \left[\varphi^{0'}(t) e^{-\lambda \varphi^0(t)} + \varphi^{1'}(t) e^{-\lambda \varphi^1(t)} - 2e^{-\lambda t}\right] L(y_0(\varphi^0(t)))\, dt = 0 \tag{4.36}$$

and

$$\int_{H_0 \cap H_1 \cap S_1} \left[\varphi^{0'}(t) e^{-\lambda \varphi^0(t)} + \varphi^{1'}(t) e^{-\lambda \varphi^1(t)} - 2e^{-\lambda t}\right] L(y_1(\varphi^1(t)))\, dt = 0. \tag{4.37}$$

Using the regularity properties of L and $e^{-\lambda t}$ we obtain

$$\int_{(H_0 \cap H_1) \setminus (S_0 \cup S_1)} \left[e^{-\lambda \varphi^1(t)} - e^{-\lambda t}\right]\left[L(y_1(\varphi^1(t))) - L(y_0(\varphi^0(t)))\right] dt \leq \tag{4.38}$$

$$\leq c_3 |h|^2, \quad c_3 > 0,$$

and

$$\int_{(H_0 \cap H_1) \setminus (S_0 \cup S_1)} \left[e^{-\lambda \varphi^0(t)} + e^{-\lambda \varphi^1(t)} - 2e^{-\lambda t}\right] L(y_0(\varphi^0(t)))\, dt \leq \tag{4.39}$$

$$\leq c_4 |h|^2, \quad c_4 > 0.$$

In (4.39) we used also that $\varphi^0(t) + \varphi^1(t) \geq 2t$.

Using (4.34)–(4.39) in (4.33) we have

$$\int_{H_0\cap H_1}\Big[\varphi^{0'}(t)e^{-\lambda\varphi^0(t)}-e^{-\lambda t}\Big]L(y_0(\varphi^0(t)))\,dt+$$

$$+\int_{H_0\cap H_1}\Big[\varphi^{1'}(t)e^{-\lambda\varphi^1(t)}-e^{-\lambda t}\Big]L(y_1(\varphi^1(t)))\,dt\le c_5|h|^2,$$

for some positive constant c_5. This estimate, together with (4.32), implies

$$I\le C_I|h|^2\,,\quad C_I>0\,. \tag{4.40}$$

Now we seek to estimate I_0 and I_1. Using one of the above changes of variable, we obtain

$$I_0=\int_{H_0\setminus H_1}\varphi^{0'}(t)e^{-\lambda\varphi^0(t)}L(y_0(\varphi^0(t)))\,dt-2\int_{H_0\setminus H_1}e^{-\lambda t}L(\overline{y}(t))\,dt=$$

$$=2\int_{H_0\setminus H_1}\Big[e^{-\lambda\varphi^0(t)}-e^{-\lambda t}\Big]L(y_0(\varphi^0(t)))\,dt-$$

$$-2\int_{H_0\setminus H_1}e^{-\lambda t}\Big[L(\overline{y}(t))-L(y_0(\varphi^0(t)))\Big]\,dt.$$

Then, using the Lipschitz continuity of L and $e^{-\lambda t}$,

$$I_0\le c_6 m(H_0\setminus H_1)|h|\,. \tag{4.41}$$

We observe that, in view of Remark 4.3,

$$m(H_0\setminus H_1)=\int_{H_0\setminus H_1}|\alpha(t)|\,dt\le \tag{4.42}$$

$$\le\int_{S_1}h_1(t)\langle\alpha(t),\nabla d(y_1(\varphi^1(t)))\rangle_+\,dt=O(|h|).$$

Thus, by (4.41),

$$I_0\le d_0|h|^2,\ \ d_0>0\,. \tag{4.43}$$

Analogously,

$$I_1\le d_1|h|^2,\ \ d_1>0\,. \tag{4.44}$$

Substituting (4.40), (4.43), (4.44) in (4.30) we have

(4.45) $$v(x_0) + v(x_1) - 2v(\overline{x}) \le C|h|^{1+\theta} + E, \quad C > 0.$$

If $T < S$, then, by (ii) of Lemma 4.2 and hypothesis (1.6),

$$E = -2e^{-\lambda T} v(\overline{y}(T)) \le 0.$$

Therefore

(4.46) $$v(x_0) + v(x_1) - 2v(\overline{x}) \le C|h|^{1+\theta}, \quad C > 0.$$

If $T = S$, then, using Theorem 3.1 and the regularity properties of $e^{-\lambda t}$, we can rewrite E as

(4.47) $$\begin{aligned} E = e^{-\lambda T}[v(y_0(T_0)) + v(y_1(T_1)) - 2v(\overline{y}(T))] +& \\ + \left[e^{-\lambda T_0} - e^{-\lambda T}\right][v(y_0(T_0)) - v(y_1(T_1))] +& \\ + \left[e^{-\lambda T_0} + e^{-\lambda T_1} - 2e^{-\lambda T}\right] v(y_1(T_1)) \le e^{-\lambda S}\sigma_v'(2e^{C_1 S}|x_0 - x_1|) + C|h|^{1+\theta},& \end{aligned}$$

for some positive constant C. Now, in view of (4.27), (4.46) and (4.47) we have

$$v(x_0) + v(x_1) - 2v(\overline{x}) \le CR^{1+\theta} + e^{-\lambda S}\sigma_v'(2e^{C_1 S}R),$$

for any pair x_0, x_1 such that $|x_0 - x_1| \le R$, with $R \in [0, \rho]$.

Taking the supremum over all such pairs, x_0, x_1, we conclude that

$$\sigma_v'(R) \le CR^{1+\theta} + e^{-\lambda S}\sigma_v'(2e^{C_1 S}R), \quad \forall R \in [0, \rho].$$

If θ verifies (4.2), then there exists $\rho < \rho_0$ such that $2^{1+\theta}e^{-\lambda S}e^{(1+\theta)C_1 S} < 1$. So, we can apply Lemma 2.3 to obtain

(4.48) $$\sigma_v'(R) \le \frac{C^*}{2} R^{1+\theta},$$

for some positive constant C^* depending on λ. The above inequality shows that the value function v is semiconcave with a modulus

$$\tilde{\omega}(R) = C^* R^{\theta},$$

where $\tilde{\omega}$ is as in (2.3). In particular if $\lambda > 2C_1 + 4 \ln 2/\rho_0$, then v is semiconcave with a linear semiconcavity modulus. □

Remark 4.7. If Ω is convex, then from Remark 4.4 it follows that for every $\lambda > 0$ the value function v is semiconcave with a linear semiconcavity modulus. Indeed, in the previous argument, we can take an arbitrary large time S.

Remark 4.8. If Ω is unbounded then to obtain the same results we need that the running cost L is bounded in a tubular neighborhood of the boundary Γ.

5. Some applications.

Throughout this section we assume that the running cost L is C^1 in $\overline{\Omega}$ and semiconcave with a linear modulus. Moreover, as in the previous sections, we assume that hypothesis (1.6) is verified and that the boundary Γ of Ω is of class C^2.

As well known the value function v is a viscosity solution of the Hamilton-Jacobi equation

$$\lambda v(x) + F(x, \nabla v(x)) = 0, \tag{5.1}$$

where, in our particular case, the Hamiltonian F is

$$F(x, p) = |p| - L(x), \quad x \in \overline{\Omega},\ p \in \mathbb{R}^n. \tag{5.2}$$

In this section we give some applications of the semiconcavity of the value function v, proving necessary optimality conditions of the same kind as the ones obtained in [6], [10], [8] for different optimal control problems.

First, we can give a simple application of semiconcavity.

Theorem 5.1. *Let $y(\cdot) = y_x^\alpha(\cdot)$ be an optimal trajectory for $x \in \Omega$. Then the value function v is continuously differentiable at $y(t)$ for all times $t \in (0, \tau(x, \alpha))$.*

Proof. Let $t \in (0, \tau(x, \alpha))$. First, we claim that

$$\lambda v(y(t)) + F(y(t), q) = 0, \quad q \in \nabla^+ v(y(t)). \tag{5.3}$$

In fact, by the dynamic programming principle, for any $h \in (0, t)$, we have that

$$e^{-\lambda(t-h)} v(y(t-h)) = \int_{t-h}^{t} e^{-\lambda s} L(y(s))\, ds + e^{-\lambda t} v(y(t)). \tag{5.4}$$

Then,

$$0 = \limsup_{h\to 0} \Big\{ e^{-\lambda(t-h)} \frac{v(y(t-h)) - v(y(t))}{h} + \\ + e^{-\lambda t} \frac{e^{\lambda h} - 1}{h} v(y(t)) - \frac{1}{h} \int_{t-h}^{t} e^{-\lambda s} L(y(s)) ds \Big\}.$$

Hence, recalling the definition of F, the above identity yields

$$0 \le \lambda e^{-\lambda t} v(y(t)) - e^{-\lambda t} \limsup_{h\to 0} \frac{1}{h} \int_{t-h}^{t} \big[- L(y(s)) - q\alpha(s) \big]\, ds \le \\ \le e^{-\lambda t} [F(y(t), q) + \lambda v(y(t))],$$

for any $q \in \nabla^+ v(y(t))$. Therefore

$$\lambda v(y(t)) + F(y(t), q) \geq 0, \quad \forall q \in \nabla^+ v(y(t)).$$

On the other hand, since v is a viscosity solution of (5.1), also the reverse inequality holds. This proves equality (5.3). Moreover such an equality, on account of the special form (5.2) of the Hamiltonian F, also implies that $\nabla^+ v(y(t))$ is a singleton. Owing to the semiconcavity property of v, this ensures that v is continuously differentiable at $y(t)$. □

Before proving the maximum principle we give a preliminary result.

Lemma 5.2. *Let $z \in \partial\Omega$ and let $\nu(z)$ be the outer normal to Ω at z. Let $\alpha^* \in B$ be such that $\alpha^* \cdot \nu(z) > 0$. Then $\tau(x, \alpha^*)$ is finite for any $x \in \Omega$ in a neighborhood of z and*

$$\tau(x, \alpha^*) = O(|x - z|).$$

Proof. The proof is the same of the analogous Lemma in [8], but we write it for the reader's convenience.

Let us consider the signed distance d defined in (3.2). By our hypothesis on the boundary of Ω, d is differentiable near z and $\nabla d(z) = \nu(z)$. Let us define

$$\Phi(x, t) = d(y_x^{\alpha^*}(t)).$$

Then Φ is differentiable for (x, t) near $(z, 0)$ and satisfies

$$\Phi(z, 0) = 0, \quad \Phi_t(z, 0) = \nu(z) \cdot \alpha^* \neq 0.$$

By the implicit function theorem we can find a neighborhood $\mathcal{O}$ of z, a number $\delta > 0$ and a function $s : \mathcal{O} \to [-\delta, \delta]$ such that, for all $x \in \mathcal{O}$ and $t \in [-\delta, \delta]$,

$$d(y_x^{\alpha^*}(t)) = 0 \Longleftrightarrow t = s(x).$$

In addition $s(x) = 0$ if and only if $x \in \partial\Omega$, and therefore $s(x)$ has constant sign in $\mathcal{O} \cap \Omega$. Moreover we have

$$\nabla s(z) = -\frac{\Phi_x(z, 0)}{\Phi_t(z, 0)} = -\frac{\nu(z)}{\nu(z) \cdot \alpha^*}$$

which implies

$$s(x) = \nabla s(z) \cdot (x - z) + o(|x - z|) = -\frac{\nu(z) \cdot (x - z)}{\nu(z) \cdot \alpha^*} + o(|x - z|).$$

In particular, since we assumed $\alpha^* \cdot \nu(z) > 0$, we have that $s(x) > 0$ for $x = z - \varepsilon\nu$ with $\varepsilon > 0$ sufficiently small. By the previous remarks we deduce that $s(x)$ is positive and coincides with $\tau(x, \alpha^*)$, for every $x \in \mathcal{O} \cap \Omega$. This ends the proof. □

Theorem 5.3. *Let $y(\cdot) = y_x^{\overline{\alpha}}(\cdot)$ be an optimal trajectory with exit time $\tau = \tau(x, \overline{\alpha}) \leq \infty$. If $p : [0, \tau] \to \mathbb{R}^n$ solves the adjoint system*

$$(5.5) \qquad \begin{cases} \dot{p}(t) = -e^{-\lambda t}\nabla L(y(t)), \\ p(\tau) = 0, \end{cases}$$

then

$$(5.6) \qquad -p(t) \cdot \overline{\alpha}(t) \geq -p(t) \cdot \alpha, \quad \forall \alpha \in B, \qquad t \in [0, \tau] \text{ a. e..}$$

(In the case of $\tau = +\infty$, $p(\tau)$ has to be understood as $\lim_{t\to+\infty} p(t)$.)

Proof. Let $t \in [0, \tau]$ be a Lebesgue point for $\overline{\alpha}$. We suppose for semplicity $t = 0$, the computations in the general case being entirely analogous. Let us define, for $\varepsilon \in (0, \tau)$, the perturbed control

$$(5.7) \qquad \alpha_\varepsilon(t) = \begin{cases} \alpha & t \in [0, \varepsilon] \\ \overline{\alpha} & t \in [0, \tau] \setminus [0, \varepsilon] \\ \alpha^* & t > \tau \end{cases}$$

where $\alpha^* \in B$ is fixed so that $\alpha^* \cdot \nu(y(\tau)) > 0$. Setting $y_\varepsilon(t) = y_x^{\alpha_\varepsilon}(t)$, $\tau_\varepsilon = \tau(x, \alpha_\varepsilon)$, we have that

$$(5.8) \quad 0 \leq J(x, \alpha_\varepsilon) - J(x, \overline{\alpha}) = \int_0^{\tau_\varepsilon} e^{-\lambda t} L(y_\varepsilon(t))\, dt - \int_0^{\tau} e^{-\lambda t} L(y(t))\, dt.$$

It is easy to see that

$$(5.9) \quad y_\varepsilon(t) - y(t) = \int_0^{\varepsilon} (\alpha_\varepsilon - \overline{\alpha})\, ds = \varepsilon(\alpha - \overline{\alpha}(0)) + o(\varepsilon), \ \varepsilon < t \leq \min\{\tau, \tau_\varepsilon\}.$$

Suppose that $\tau_\varepsilon \geq \tau$ for ε sufficiently small and let $z = y(\tau)$. For $t \in [\tau, \tau_\varepsilon]$, we have

$$|y_\varepsilon(t) - z| \leq \int_\tau^t |\alpha_\varepsilon|\, dt + O(\varepsilon) \leq (\tau_\varepsilon - \tau) + O(\varepsilon),$$

then, by Lemma 5.2, $|y_\varepsilon(s) - z| = O\,(\varepsilon), \forall t \in [\tau, \tau_\varepsilon]$. This implies that

$$(5.10) \qquad \int_\tau^{\tau_\varepsilon} e^{-\lambda s} L(y_\varepsilon(s))\, ds = -\int_{\tau_\varepsilon}^{\tau} e^{-\lambda s} L(z)\, ds + o(\varepsilon) = o(\varepsilon).$$

Therefore, by (5.8) and (5.10),

$$0 \leq \int_0^{\varepsilon} e^{-\lambda t}[L(y_\varepsilon(t)) - L(y(t))]\, dt + \int_\varepsilon^{\tau} e^{-\lambda t}[L(y_\varepsilon(t)) - L(y(t))]\, dt + o(\varepsilon).$$

Dividing by ε and letting $\varepsilon \to 0$ we obtain

$$0 \leq \int_0^\tau e^{-\lambda t} \nabla L(y(t)) \cdot (\alpha - \overline{\alpha}(0))\, dt, \tag{5.11}$$

where we used (5.9). Now observe that, since $p(t)$ solves (5.5),

$$p(t) = \int_t^\tau e^{-\lambda s} \nabla L(y(s))\, ds$$

and then, by (5.11) it follows that $0 \leq p(0) \cdot (\alpha - \overline{\alpha}(0))$, that is (5.6) holds for $t = 0$. This completes the proof in the case that $\tau_\varepsilon \geq \tau$ for small ε.

Otherwise, there exists a sequence $\varepsilon_j \to 0$ as $j \to \infty$, with $\tau_{\varepsilon_j} < \tau$, $\forall j$. In this case, we claim that, possibly taking a subsequence,

$$\tau_{\varepsilon_j} \to \tau \quad \text{as} \quad j \to \infty.$$

If $\tau < \infty$, then sequence $\{\tau_{\varepsilon_j}\}_j$ is bounded. Hence, possibly taking a subsequence, $\{\tau_{\varepsilon_j}\}_j$ converges to some limit $\overline{\tau}$. Since $y_{\varepsilon_j}(\tau_{\varepsilon_j}) \in \Gamma$ and $y_{\varepsilon_j}(\tau_{\varepsilon_j}) \to y(\overline{\tau})$ as $j \to \infty$, $y(\overline{\tau})$ must belong to Γ. This implies $\overline{\tau} \geq \tau$. Therefore, $\tau_{\varepsilon_j} \to \tau$. If $\tau = \infty$, then sequence $\{\tau_{\varepsilon_j}\}_j$ is unbounded. Otherwise, again taking a subsequence, we would have that $\tau_{\varepsilon_j} \to \overline{\tau} < \infty$ with $y(\overline{\tau}) \in \Gamma$, in contrast with the assumption that $\tau = \infty$. So, our claim is proved.

By (5.8), we have

$$0 \leq \int_0^{\varepsilon_j} e^{-\lambda t}[L(y_{\varepsilon_j}(t)) - L(y(t))]\, dt + \int_{\varepsilon_j}^{\tau_{\varepsilon_j}} e^{-\lambda t}[L(y_{\varepsilon_j}(t)) - L(y(t))]\, dt.$$

Dividing by ε_j and letting $j \to \infty$, as above we obtain the result. □

We say that solution $p(t)$ of (5.5) is the *dual arc* associated to the optimal trajectory $y(t)$. We now prove that the dual arc is included in the superdifferential of the value function.

Theorem 5.4. *Let $y(\cdot) = y_x^{\overline{\alpha}}(\cdot)$ be an optimal trajectory and let $p(\cdot)$ the related dual arc. Then*

$$p(t) \in \nabla^+ v(y(t)), \quad t \in [0, \tau(x, \overline{\alpha})).$$

Proof. We restrict ourselves to the case $t = 0$, since the computations in the general case are entirely analogous. We want to show that

$$v(x+h) \leq v(x) + p(0) \cdot h + o(|h|), \quad h \in \mathbb{R}^n. \tag{5.12}$$

Let $\tau = \tau(x, \overline{\alpha})$, $z = y(\tau)$ and let $\nu(z)$ be the outer normal to Ω at z. We define a control

$$\tilde{\alpha}(t) = \begin{cases} \overline{\alpha}(t) & t \in [0, \tau] \\ \alpha^* & t > \tau \end{cases} \tag{5.13}$$

for some $\alpha^* \in B$ with $\alpha^* \cdot \nu(z) > 0$. Let us set $y_h(\cdot) = y^{\tilde{\alpha}}_{x+h}(\cdot)$ and $\tau_h = \tau(x+h, \tilde{\alpha})$. Then

$$v(x+h) - v(x) \leq \int_0^{\tau_h} e^{-\lambda t} L(y_h(t))\, dt - \int_0^{\tau} e^{-\lambda t} L(y(t))\, dt.$$

We observe that

$$y_h(t) - y(t) = h \quad \text{for} \quad t \in [0, \min\{\tau, \tau_h\}]. \tag{5.14}$$

Let us consider a sequence $\{h_j\}_j$ convergent to 0 such that

$$\limsup_{h \to 0} \frac{v(x+h) - v(x) - p(0) \cdot h}{|h|} = \tag{5.15}$$

$$= \lim_{j \to \infty} \frac{v(x+h_j) - v(x) - p(0) \cdot h_j}{|h_j|}.$$

Suppose that $\tau_{h_j} \geq \tau$ for $|h_j|$ sufficiently small. In this case, by Lemma 5.2 it follows that $\tau_{h_j} - \tau = O(|h_j|)$, so

$$y_{h_j}(t) - z = h_j + O(|h_j|) \quad \text{for} \quad t \in [\tau, \tau_{h_j}].$$

Then, recalling (5.14), we have

$$\lim_{j \to \infty} \frac{v(x+h_j) - v(x) - p(0) \cdot h_j}{|h_j|} \leq$$

$$\leq \lim_{j \to \infty} \frac{1}{|h_j|} \left\{ \int_0^{\tau} e^{-\lambda t} [L(y_{h_j}(t)) - L(y(t))]\, dt - p(0) \cdot h_j + o(|h_j|) \right\} \leq$$

$$\leq \lim_{j\to\infty} \frac{1}{|h_j|}\left\{ \int_0^{\tau} e^{-\lambda t}\nabla L(y(t))\cdot h_j\,dt - p(0)\cdot h_j + o(|h_j|)\right\}.$$

Thus, by definition of $p(0)$ and by (5.15) immediately follows

$$\limsup_{h\to 0} \frac{v(x+h)-v(x)-p(0)\cdot h}{|h|} \leq 0.$$

This completes the proof in the case that $\tau_{h_j} \geq \tau$ for $|h_j|$ sufficiently small. Otherwise, possibly taking a subsequence, $\tau_{h_j} < \tau$ for all j. Arguing as in the previous theorem, we can suppose that $\{\tau_{h_j}\}_j$ converges to τ. So, by

$$\lim_{j\to\infty} \frac{v(x+h_j)-v(x)-p(0)\cdot h_j}{|h_j|} \leq$$

$$\leq \lim_{j\to\infty} \frac{1}{|h_j|}\left\{ \int_0^{\tau_{h_j}} e^{-\lambda t}[L(y_{h_j}(t) - L(y(t))]\,dt - p(0)\cdot h_j \right\},$$

the result easily follows using (5.14). □

As an application of the semiconcavity of v we can characterize the dual arc $p(\cdot)$ as follows.

Theorem 5.5. *Let $y(\cdot)$ be an optimal trajectory with exit time τ, and let $p(\cdot)$ be its dual arc. Then $p(t) = \nabla v(y(t))$, $t \in (0,\tau)$ and $p(0) \in \nabla^* v(y(0))$.*

Proof. By Theorem 5.1 v is differentiable at all points $y(t)$ with $t \in [0,\tau)$. Therefore at these points $\nabla^+ v$ contains only the gradient $\nabla v(y(t))$, and so by the previous theorem we have that the dual arc $p(\cdot)$ coincides with $\nabla v(y(\cdot))$. Moreover, by the definition of $\nabla^* v$, it follows that $p(0) \in \nabla^* v(y(0))$. □

Remark 5.6. If $x \in \Omega$ is such that $\lambda v(x) = L(x)$, then $y(t) \equiv x$, $\forall t \geq 0$, is an optimal trajectory for x. Indeed,

$$\int_0^{\infty} e^{-\lambda t} L(x)\,dt = \frac{L(x)}{\lambda} = v(x).$$

This implies, recalling (5.2), that x is a differentiability point for v and $\nabla v(x) = 0$. Obviously, x must be a local minimum point for the running cost L.

Now, given an optimal trajectory $y(\cdot) = y_x^{\overline{\alpha}}(\cdot)$ we define the *critical time* $\bar{t}(y)$ as

$$\bar{t}(y) = \inf\{t : \lambda v(y(t)) = L(y(t))\}. \tag{5.16}$$

Obviously, $\bar{t}(y) = 0$ if $\lambda v(x) = L(x)$ and $\bar{t}(y) = \tau(x,\overline{\alpha})$ if there are no times lower than the exit time at which $\lambda v(y(t)) = L(y(t))$.

We have the following result.

Proposition 5.7. *Let $y(\cdot)$ be an optimal trajectory and let $p(\cdot)$ be its dual arc. Let $\bar{t} = \bar{t}(y)$ be the critical time of y. Then $p(\bar{t}) = 0$ and $p(t) \neq 0$, $\forall t \in [0, \bar{t})$.*

Proof. If $t < \bar{t}$, then $\lambda v(y(t)) - L(y(t)) \neq 0$ and therefore $p(t) = \nabla v(y(t)) \neq 0$. If $\bar{t}$ is lower than the exit time τ, then $\lambda v(y(\bar{t})) = L(y(\bar{t}))$. So, by Remark 5.6, $p(\bar{t}) = \nabla v(y(\bar{t})) = 0$. If $\bar{t} = \tau$, $p(\tau) = 0$ by definition of dual arc. □

Now we can formulate an immediate consequence of the maximum principle (5.6) and of Proposition 5.7.

Proposition 5.8. *Let $y(\cdot)$ be an optimal trajectory with critical time $\bar{t} \neq 0$, and let $p(\cdot)$ be its dual arc. Then the pair (y, p) solves the system*

$$\begin{cases} \dot{y}(t) = -\dfrac{p(t)}{|p(t)|}, \\ \dot{p}(t) = -e^{-\lambda t} \nabla L(y(t)), \end{cases} \tag{5.17}$$

for all $t \in [0, \bar{t})$.

Now we prove that there exists a correspondence between the optimal trajectories starting at a point $x \in \Omega$ and the elements of $\nabla^* v(x)$.

Theorem 5.9. *Let $x \in \Omega$ and let $q \in \nabla^* v(x)$. Consider the solution (y, p) of (5.17) with initial conditions*

$$\begin{cases} y(0) = x, \\ p(0) = q. \end{cases} \tag{5.18}$$

Then, the function

$$\tilde{y}(t) = \begin{cases} y(t) & t \leq \bar{t}, \\ y(\bar{t}) & t > \bar{t}, \end{cases} \tag{5.19}$$

is an optimal trajectory at x and

$$\tilde{p}(t) = \begin{cases} p(t) & t \leq \bar{t}, \\ 0 & t > \bar{t}, \end{cases} \tag{5.20}$$

is the associated dual arc. Conversely, if $\tilde{y}(\cdot)$ is an optimal trajectory at x of the form (5.19), then its dual arc $\tilde{p}(\cdot)$ has the form (5.20) and the pair $(\tilde{y}, \tilde{p})$ solves (5.17) and (5.18) for a suitable choice of $q \in D^* v(x)$.

Proof. Let us suppose first that v is differentiable at x, and let $y(\cdot)$ be an optimal trajectory at x, $p(\cdot)$ being the associated dual arc. By Remark 5.6 we can suppose $y(t) = y(\bar{t})$ and $p(t) = 0$ for $t \geq \bar{t}$. By Theorem 5.5 and Proposition 5.8, $(y(\cdot), p(\cdot))$ is a solution of (5.17) with initial conditions $y(0) = x$, $p(0) = \nabla v(x)$. This proves the assertion at all points x of differentiability for v.

To treat the general case, let $q \in \nabla^* v(x)$. Then there exists a sequence $\{x_k\}_{k \in \mathbb{N}} \subset \Omega$ such that V is differentiable at x_k and

$$x_k \to x,\ \nabla v(x_k) \to q \quad \text{as } k \to \infty.$$

We denote by (y_k, p_k) the solution of (5.17) with initial conditions

$$\begin{cases} y(0) = x_k\,, \\ p(0) = \nabla v(x_k). \end{cases} \tag{5.21}$$

By the first part of the proof y_k is an optimal trajectory for x_k and p_k is the associated dual arc. In addition, (y_k, p_k) converges to (y, p) locally uniformly. Then y is an optimal trajectory and the pair (y, p) solves system (5.17) with initial conditions (5.18). Moreover

$$p(t) = \lim_{k\to\infty} p_k(t) = \lim_{k\to\infty} \nabla v(y_k(t)),\ t \in [0, \bar{t}).$$

Since

$$\lim_{k\to\infty} \nabla v(y_k(t)) \in \nabla^* v(y(t))$$

we have, by Theorem 5.5, that $p(t) = \nabla v(y(t))$, for every $t \in (0, \bar{t})$. Therefore p is the dual arc associated with y. This proves that any solution to system (5.17) with initial conditions (5.18) coincides with an optimal trajectory and its associated dual arc.

Conversely, let $\tilde{y}$ be an optimal trajectory at x of the form (5.19). By Proposition 5.7 and Theorem 5.5, the associated dual arc p has the form (5.20) and the pair (y, p) solves system (5.17) with initial conditions (5.18) for some $q \in \nabla^* v(y(0))$. □

Corollary 5.10. *The value function v is differentiable at a point $x \in \Omega$ if and only if one of the following facts occurs:*

(i) $\lambda v(x) = L(x)$,

(ii) there exists a unique optimal trajectory $y(t)$ at x, on the interval $[0, \bar{t}(y))$.

Finally, we will use of the semiconcavity property to study the propagation of singularities. Let us denote by $\Sigma(v)$ the set of points at which v is not differentiable. In order to apply Theorem 2.2 we need v to be semiconcave with linear modulus. By Theorem 4.1 we know that this further regularity holds if the discount factor λ is sufficiently large.

Theorem 5.11. *Let $\lambda > \lambda_0$ where λ_0 is given by (4.3), and let $x \in \Sigma(v)$ be such that $\dim \nabla^+ v(x) < n$. Then, there exists $\sigma > 0$ and a Lipschitz arc $\overline{x} : [0, \sigma] \to \Sigma(v)$ with $x(0) = x$ and $\overline{x}(s) \neq x$ for all $s \in (0, \sigma]$.*

Proof. Since v satisfies the Hamilton-Jacobi equation (5.2) at all points of differentiability, by definition of $\nabla^* v$,

$$\lambda v(x) + F(x, p) = 0, \quad \forall p \in \nabla^* v(x).$$

Moreover, since v is semiconcave, by Theorem 2.1 it follows that $\nabla^* v(x) \subset \nabla^+ v(x)$. If $\nabla^+ v(x) = \nabla^* v(x)$, then since $\nabla^+ v(x)$ it is a convex set, it is a singleton. But, this contradicts our assumptions. Therefore $\nabla^+ v(x) \setminus \nabla^* v(x) \neq \emptyset$. Since $\dim \nabla^+ v(x) < n$, we also have that $\nabla^+ v(x) = \partial \nabla^+ v(x)$ and the result follows applying Theorem 2.2. □

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Dipartimento di Matematica,
Università di Roma Tor Vergata,
Via della Ricerca Scientifica,
00133 Roma (ITALY),
e-mail: cannarsa@mat.uniroma2.it
pignotti@mat.uniroma2.it