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BOUNDARY REGULARITY RESULTS FOR NON-VARIATIONAL BASIC ELLIPTIC SYSTEMS

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Dedicated to Professor Sergio Campanato on his 70th birthday

Let $u \in H^2(B^+(1), \mathbb{R}^N)$ (N integer ≥ 1) be a solution to the following problem

$$\begin{cases} u = g & \text{on } \Gamma, \\ a(H(u)) = 0 & \text{in } B^+(1), \end{cases}$$

where $a(\xi)$ is a vector of $\mathbb{R}^N$, continuous onto $\mathbb{R}^{n^2 N}$, $\Gamma = \{x \in \mathbb{R}^n : \|x\| < 1, x_n = 0\}$, and $B^+(1) = \{x \in \mathbb{R}^n : \|x\| < 1, x_n > 0\}$. We prove that, if $a(\xi)$ satisfies the conditions $a(0) = 0$ and (C) (see Section 1 below) while $g \in H^3(B^+(1), \mathbb{R}^N)$, then $u \in H^3(B^+(\sigma), \mathbb{R}^N)$, for all $\sigma \in (0, 1)$. Exploiting it we next deduce the Hölder-continuity of the vectors Du and u in $\overline{B^+(\sigma)}$, provided $2 \leq n < 4$ or $2 \leq n < 6$, respectively. These results are basic tools for studying the Hölder-continuity in $\bar{\Omega}$ of the solutions to the Dirichlet problem

$$\begin{cases} u \in H^2(\Omega, \mathbb{R}^N), \\ u = g & \text{on } \partial\Omega, \\ a(H(u)) = 0 & \text{in } \Omega. \end{cases}$$

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1. Introduction.

Let Ω be a bounded open set in $\mathbb{R}^n, n \geq 2$, with generic point $x = (x_1, x_2, \dots, x_n)$. If $u(x)$ is a vector $\Omega \to \mathbb{R}^N$, N integer ≥ 1, we write

$$D_i u = \frac{\partial u}{\partial x_i}, \quad Du = (D_1 u, D_2 u, \dots, D_n u),$$

$$H(u) = \{D_i D_j u\} = \{D_{ij} u\}, \quad i, j = 1, 2, \dots, n;$$

obviously, Du and $H(u)$ are elements of $\mathbb{R}^{nN}$ and $\mathbb{R}^{n^2N}$, respectively.

Let $a(\xi)$ be a vector of $\mathbb{R}^N$, continuous onto $\mathbb{R}^{n^2N}$, satisfying the conditions

$$a(0) = 0; \tag{1.1}$$

(C) *there exist three positive constants* α, γ, *and* δ, *with* $\gamma + \delta < 1$, *such that*

$$\|\sum_{i=1}^n \tau_{ii} - \alpha[a(\tau + \xi) - a(\xi)]\| \leq \gamma \|\tau\| + \delta \|\sum_{i=1}^n \tau_{ii}\|, \quad \forall \tau, \xi \in \mathbb{R}^{n^2N}.$$

These conditions are equivalent to the following "pseudo monotonicity condition" ([1])
(C′) *there exist three positive constants* M, ν, *and* K, *with* $0 < \nu - K < \frac{M^2}{2\nu}$, *such that,* $\forall \tau, \xi \in \mathbb{R}^{n^2N}$, *we have*

$$\|a(\tau + \xi) - a(\xi)\| \leq M \|\tau\|;$$

$$\big(a(\tau + \xi) - a(\xi) | \sum_{i=1}^n \tau_{ii}\big) \geq \nu \|\sum_{i=1}^n \tau_{ii}\|^2 - K \|\tau\|^2.$$

In particular, conditions (1.1) and (C) imply

$$\|a(\tau)\| \leq \frac{c(n)}{\alpha} \|\tau\|, \quad \forall \tau \in \mathbb{R}^{n^2N}.$$

Moreover, if the matrix $\tau \in \mathbb{R}^{n^2N}$ is a solution to the system

$$a(\tau) = 0,$$

([1]) See, for instance, [8], Section 1, and [9], Lemma 3.1.1.

then, using conditions (1.1) and (C) again, and assuming $\xi = 0$ we get

$$\left\| \sum_{i=1}^{n} \tau_{ii} \right\| \leq \frac{\gamma}{1-\delta} \|\tau\|, \tag{1.2}$$

where $\frac{\gamma}{1-\delta} < 1$.

More generally, if $\xi, \tau \in \mathbb{R}^{n^2 N}$ are such that

$$a(\xi) = a(\tau + \xi),$$

then estimate (1.2) holds.

Define

$$H(\Omega, \mathbb{R}^N) = H^2(\Omega, \mathbb{R}^N) \cap H_0^1(\Omega, \mathbb{R}^N),$$

and pick $g \in H^2(\Omega, \mathbb{R}^N)$ (2).
If Ω is of class C^2 and convex, the Dirichlet problem

$$\begin{cases} u - g \in H(\Omega, \mathbb{R}^N), \\ a(H(u)) = 0 \text{ in } \Omega, \end{cases} \tag{1.3}$$

has a unique solution. In fact, setting $w = u - g$, (1.3) is equivalent to

$$\begin{cases} w \in H(\Omega, \mathbb{R}^N), \\ a(H(w) + H(g)) = 0 \text{ in } \Omega, \end{cases} \tag{1.4}$$

and, thanks to Theorem 2.1 in [7], the preceding problem has a unique solution. Moreover, it is known that the solution u to (1.3) is Hölder-continuous in Ω if $n \leq 6$ (3).
Then, in order to obtain the Hölder continuity of u in $\bar{\Omega}$ we clearly need to establish "boundary regularity results". In particular, if Ω is of class C^2 and if $x^0 \in \partial\Omega$, there exists an open neighborhood $\mathcal{B}$ of x^0 such that $\bar{\mathcal{B}}$ is mapped, by a mapping $\mathcal{T}$ of class C^2 together with its inverse, onto the ball $\overline{B(0,1)}$ (4), $\mathcal{T}(\Omega \cap \mathcal{B}) = B^+(1)$, and $\mathcal{T}(\partial\Omega \cap \mathcal{B}) = \Gamma$, where $\Gamma = \{x \in B(0,1) : x_n = 0\}$.
Then if u is the solution to Dirichlet problem (1.3), one has

$$\begin{cases} u \in H^2(\Omega \cap \mathcal{B}, \mathbb{R}^N), \\ u = g \qquad \text{on } \partial\Omega \cap \mathcal{B}, \\ a(H(u)) = 0 \ \text{in} \ \Omega \cap \mathcal{B}. \end{cases} \tag{1.5}$$

(2) $H^2(\Omega, \mathbb{R}^N)$ and $H_0^1(\Omega, \mathbb{R}^N)$ are the usual Sobolev spaces.

(3) See assertions (33) of [5], and [9], Theorem 3.2.26.

(4) If σ is a positive real number, we denote by $B(0,\sigma)$ the open ball $\{x \in \mathbb{R}^n : \|x\| < \sigma\}$, and by $B^+(\sigma)$ the hemisphere $\{x \in B(0,\sigma) : x_n > 0\}$.

Making use of the transformation of co-ordinates $y = \mathcal{T}(x)$, we infer that $W(y) = U(y) - G(y) = u(\mathcal{T}^{-1}(y)) - g(\mathcal{T}^{-1}(y)), y \in B^+(1)$, is a solution to a problem of the type

$$\begin{cases} W \in H^2(B^+(1), \mathbb{R}^N) \\ W = 0 \quad \text{on } \Gamma \\ A(y, DW + DG, H(W) + H(G)) = 0 \ \text{in} \ B^+(1). \end{cases} \tag{1.6}$$

So, it remains to establish $\mathcal{L}^{2,\lambda}$ - regularity results in $B^+(\sigma)$, with $\sigma \in (0,1)$, for the solutions to problem (1.6). The aim of this work is to start such a study by considering at first the case in which the operator A does not depend on y and DU.

2. Differentiability near the boundary.

Let R be a positive real number. In the hemisphere $B^+(R)$, let us consider the problem

$$\begin{cases} u \in H^2(B^+(R), \mathbb{R}^N), \\ u = 0 \quad \text{on } \Gamma_R, \\ a(H(u) + H(g)) = 0 \ \text{in} \ B^+(R), \end{cases} \tag{2.1}$$

where $\Gamma_R = \{x \in B(0,R) : x_n = 0\}$, $g \in H^2(B^+(R), \mathbb{R}^N)$, and $a(\xi)$ is a vector of $\mathbb{R}^N$, continuous onto $\mathbb{R}^{n^2N}$, satisfying conditions (1.1) and (C).

We want to prove the following differentiability theorem (see [3], Section 4, for the case of nonlinear elliptic systems in divergence form)

Theorem 2.1. *If $u \in H^2(B^+(R), \mathbb{R}^N)$ is a solution to problem* (2.1), *under conditions* (1.1) *and* (C), *and if $g \in H^3(B^+(R), \mathbb{R}^N)$, then, for every $r = 1, 2, \dots, n-1$, and $\forall \sigma, \sigma_0 \in (0, R]$, with $\sigma < \sigma_0$, one has*

$$D_r(H(u)) \in L^2(B^+(\sigma), \mathbb{R}^{n^2N}), \tag{2.2}$$

and the following estimate holds

$$\int_{B^+(\sigma)} \|D_r(H(u))\|^2 \, dx \le c\Big\{\frac{1}{(\sigma_0-\sigma)^4}\int_{B^+(\sigma_0)} \|D_r u\|^2 \, dx + \\ + \frac{1}{(\sigma_0-\sigma)^2}\int_{B^+(\sigma_0)} \|H(u)\|^2 dx + \int_{B^+(\sigma_0)} \|D_r(H(g))\|^2 dx\Big\}, \tag{2.3}$$

where the constant c does not depend on σ and σ_0.
In particular, if $0 < \sigma \leq R/2$, it results

$$\int_{B^+(\sigma)} \|D_r(H(u))\|^2\,dx \leq c\Big\{\frac{1}{\sigma^2}\int_{B^+(2\sigma)} \|H(u)\|^2\,dx + \tag{2.4}$$
$$+\int_{B^+(2\sigma)} \|D_r(H(g))\|^2\,dx\Big\}.$$

Proof. Let $\sigma, \sigma_0 \in (0, R]$, with $\sigma < \sigma_0$, and let $\sigma_1 = \frac{\sigma+\sigma_0}{2}$. For every $x \in B^+(\sigma_1)$, $|h| < \frac{\sigma_0-\sigma}{2}$, and $r = 1, 2, \ldots, n-1$, we put

$$\tau_{r,h}u(x) = u(x + he^r) - u(x),$$

where $\{e^r\}_{r=1,2,\ldots,n}$ is the canonic basis of $\mathbb{R}^n$.
We proceed exactly as in the interior differentiability case (see [5]).
Let $\vartheta(x) \in C_0^\infty(\mathbb{R}^n)$ be a function fulfilling the conditions

$$0 \leq \vartheta \leq 1,\ \vartheta = 1 \text{ in } B(\sigma),\ \vartheta = 0 \text{ in } \mathbb{R}^n \setminus B(\sigma_1),\ |D^\alpha\vartheta| \leq c(\sigma_0 - \sigma)^{-|\alpha|},$$

$\forall \alpha : |\alpha| \leq 2$ (5).
Set $w = u + g$. From system (2.1) we deduce that

$$\tau_{r,h}a(H(w)) = a(H(w(x + he^r))) - a(H(w(x))) = 0 \quad \text{in } B^+(\sigma_1)$$

and hence

$$a(H(\tau_{r,h}w) + H(w)) - a(H(w)) = 0 \quad \text{in } B^+(\sigma_1). \tag{2.5}$$

By (2.5) estimate (1.2) holds for the matrix $\tau = H(\tau_{r,h}w)$. Consequently,

$$\|\vartheta\Delta(\tau_{r,h}w)\| \leq \frac{\gamma}{1-\delta}\|\vartheta H(\tau_{r,h}w)\| \quad \text{in } B^+(\sigma_1). \tag{2.6}$$

Since $\frac{\gamma}{1-\delta} < 1$, we get

$$\|\vartheta\Delta(\tau_{r,h}u)\| - \|\vartheta\Delta(\tau_{r,h}g)\| \leq \|\vartheta\Delta(\tau_{r,h}u) + \vartheta\Delta(\tau_{r,h}g)\| \leq$$

(5) $D^\alpha = D_1^{\alpha_1}D_2^{\alpha_2}\ldots D_n^{\alpha_n}$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n$, α_i integer ≥ 0.

$$\le \frac{\gamma}{1-\delta}\|\vartheta H(\tau_{r,h}u)\| + \|\vartheta H(\tau_{r,h}g)\|$$

that is

$$\|\vartheta \Delta(\tau_{r,h}u)\| \le \frac{\gamma}{1-\delta}\|\vartheta H(\tau_{r,h}u)\| + 2\|\tau_{r,h}H(g)\| \quad \text{in } B^+(\sigma_1). \tag{2.7}$$

Now, setting

$$\mathcal{U} = \vartheta\tau_{r,h}u, \quad \text{in } B^+(\sigma_1),$$

we obtain $\mathcal{U}(x) = \vartheta(x)[u(x+he^r) - u(x)] = 0$, $\forall x \in \Gamma_{\sigma_1}$ ([6]), because $u = 0$ on Γ_R, and

$$\mathcal{U} \in H^2(B^+(\sigma_1), \mathbb{R}^N) \cap H_0^1(B^+(\sigma_1), \mathbb{R}^N).$$

Moreover, we have

$$\Delta\mathcal{U} = \vartheta\Delta(\tau_{r,h}u) + A(u), \tag{2.8}$$

$$H(\mathcal{U}) = \vartheta H(\tau_{r,h}u) + B(u), \tag{2.9}$$

where

$$A(u) = \Delta\vartheta \cdot \tau_{r,h}u + 2\sum_{i=1}^{n} D_i\vartheta D_i(\tau_{r,h}u), \tag{2.10}$$

$$B(u) = \{D_{ij}\vartheta \cdot \tau_{r,h}u + D_i\vartheta D_j(\tau_{r,h}u) + D_j\vartheta D_i(\tau_{r,h}u)\}_{i,j=1,2,\dots,n}. \tag{2.11}$$

From (2.7), (2.8), and (2.9) we obtain

$$\|\Delta\mathcal{U}\| \le \frac{\gamma}{1-\delta}\|H(\mathcal{U})\| + \|A(u)\| + \|B(u)\| + 2\|\tau_{r,h}H(g)\| \text{ in } B^+(\sigma_1).$$

Hence, given any $\varepsilon > 0$, we get

$$\|\Delta\mathcal{U}\|^2 \le (1+\varepsilon)\Big(\frac{\gamma}{1-\delta}\Big)^2\|H(\mathcal{U})\|^2 + c(\varepsilon)(\|A(u)\|^2 + \tag{2.12}$$

$$+\|B(u)\|^2 + \|\tau_{r,h}H(g)\|^2) \quad \text{in } B^+(\sigma_1).$$

([6]) $\Gamma_{\sigma_1} = \{x \in B(0,\sigma_1) : x_n = 0\}$.

Integrating on $B^+(\sigma_1)$ and using the Miranda-Talenti estimate (see, for instance, [5], Lemma 1), besides (2.12) yields

$$\int_{B^+(\sigma_1)} \|H(\mathcal{U})\|^2\,dx \le (1+\varepsilon)\Big(\frac{\gamma}{1-\delta}\Big)^2 \int_{B^+(\sigma_1)} \|H(\mathcal{U})\|^2\,dx + $$

$$+\,c(\varepsilon)\int_{B^+(\sigma_1)} (\|A(u)\|^2 + \|B(u)\|^2 + \|\tau_{r,h} H(g)\|^2)\,dx.$$

Since $\frac{\gamma}{1-\delta} < 1$, for $\varepsilon \in \big(0, \big(\frac{1-\delta}{\gamma}\big)^2 - 1\big)$ and by virtue of (2.9), we deduce

$$\int_{B^+(\sigma)} \|\tau_{r,h} H(u)\|^2\,dx \le c(\gamma,\delta)\int_{B^+(\sigma_1)} (\|A(u)\|^2+\|B(u)\|^2+\|\tau_{r,h} H(g)\|^2)\,dx.$$

From this, using a well-known lemma (see [2], Chap. I, Lemma 3.VI), it follows

$$(2.13)\quad \int_{B^+(\sigma)} \|\tau_{r,h} H(u)\|^2\,dx \le c(\gamma,\delta)\Big\{\int_{B^+(\sigma_1)} (\|A(u)\|^2 + \|B(u)\|^2)\,dx + $$

$$+\,|h|^2 \int_{B^+(\sigma_0)} \|D_r H(g))\|^2\,dx\Big\}.$$

We shall now evaluate the first integral in the right-hand side of (2.13). Using Lemma 3.VI of Chap. I in [2], we get

$$(2.14)\quad \int_{B^+(\sigma_1)} \|A(u)\|^2\,dx \le c(\sigma_0-\sigma)^{-4}\int_{B^+(\sigma_1)} \|\tau_{r,h} u\|^2\,dx + $$

$$+\,c(\sigma_0-\sigma)^{-2}\int_{B^+(\sigma_1)} \|\tau_{r,h} Du\|^2\,dx \le\; c(\sigma_0-\sigma)^{-4}|h|^2\cdot$$

$$\cdot\int_{B^+(\sigma_0)} \|D_r u\|^2\,dx + c(\sigma_0-\sigma)^{-2}|h|^2\int_{B^+(\sigma_0)} \|D_r(Du)\|^2\,dx.$$

The integral of $\|B(u)\|^2$ is estimated in an analogous way, and we have

$$(2.15)\quad \int_{B^+(\sigma_1)} \|B(u)\|^2\,dx \le c(\sigma_0-\sigma)^{-4}|h|^2\int_{B^+(\sigma_0)} \|D_r u\|^2\,dx + $$

$$+\;c(\sigma_0-\sigma)^{-2}|h|^2\int_{B^+(\sigma_0)} \|D_r(Du)\|^2\,dx.$$

Finally, (2.13), (2.14), and (2.15) lead to

$$\int_{B^+(\sigma)} \|\tau_{r,h} H(u)\|^2 \, dx \le c|h|^2 \Big\{ (\sigma_0 - \sigma)^{-4} \int_{B^+(\sigma_0)} \|D_r u\|^2 \, dx +$$

$$+ (\sigma_0 - \sigma)^{-2} \int_{B^+(\sigma_0)} \|D_r(Du)\|^2 \, dx + \int_{B^+(\sigma_0)} \|D_r(H(g))\|^2 \, dx \Big\}.$$

So, by virtue of Nirenberg's Lemma (see, for instance, [6], Lemma 2.I), we can conclude that there exists $D_r(H(u)) \in L^2(B^+(\sigma), \mathbb{R}^{n^2 N})$, $r = 1, 2, \dots, n-1$, and

$$\int_{B^+(\sigma)} \|D_r(H(u))\|^2 \, dx \le \frac{c}{(\sigma_0 - \sigma)^2} \Big\{ \frac{1}{(\sigma_0 - \sigma)^2} \int_{B^+(\sigma_0)} \|D_r u\|^2 \, dx + \int_{B^+(\sigma_0)} \|D_r(Du)\|^2 \, dx \Big\} + c \int_{B^+(\sigma_0)} \|D_r(H(g))\|^2 \, dx. \tag{2.16}$$

Thus, (2.2) and (2.3) are proved.

Now, if $0 < \sigma \le \frac{R}{2}$, estimate (2.16), written for $\sigma_0 = 2\sigma$, gives

$$\int_{B^+(\sigma)} \|D_r(H(u))\|^2 \, dx \le c\sigma^{-2} \Big\{ \sigma^{-2} \int_{B^+(2\sigma)} \|D_r u\|^2 \, dx + \int_{B^+(2\sigma)} \|D_r(Du)\|^2 dx \Big\} + c \int_{B^+(2\sigma)} \|D_r(H(g))\|^2 \, dx. \tag{2.17}$$

On the other hand, from the condition $D_r u = 0$ on Γ_R, and taking into account Poincarè's inequality, one has

$$\int_{B^+(2\sigma)} \|D_r u\|^2 \, dx \le c\sigma^2 \int_{B^+(2\sigma)} \|D_n(D_r u)\|^2 \, dx \, . \tag{2.18}$$

Finally, estimate (2.4) is consequence of (2.17) and (2.18). □

In the case $r = n$ the following result holds

Theorem 2.2. *If $u \in H^2(B^+(R), \mathbb{R}^N)$ is a solution to problem* (2.1), *under conditions* (1.1) *and* (C), *and if $g \in H^3(B^+(R), \mathbb{R}^N)$, then, $\forall \sigma, \sigma_0 \in (0, R]$, with $0 < \sigma_0 - \sigma \le 1$, one has*

$$D_n(D_{nn} u) \in L^2(B^+(\sigma), \mathbb{R}^N), \tag{2.19}$$

and the following estimate holds

$$(2.20)\qquad \int_{B^+(\sigma)} \|D_n(D_{nn}u)\|^2\,dx \le \tilde{\gamma}^2 c\Big\{\frac{1}{(\sigma_0-\sigma)^4}\sum_{r=1}^{n-1}\int_{B^+(\sigma_0)}\|D_r u\|^2\,dx + \frac{1}{(\sigma_0-\sigma)^2}\int_{B^+(\sigma_0)}\|H(u)\|^2\,dx + \int_{B^+(\sigma_0)}\|D(H(g))\|^2\,dx\Big\},$$

where the constant c does not depend on σ and σ_0, while $\tilde{\gamma}$ is the constant that arises from the application of Lemma 9.3 in [1].
In particular, if $0 < \sigma \le \min(1, \frac{R}{2})$, it results

$$(2.21)\qquad \int_{B^+(\sigma)} \|D_n(D_{nn}u)\|^2 dx \le \tilde{\gamma}^2 c\Big\{\frac{1}{\sigma^2}\int_{B^+(2\sigma)}\|H(u)\|^2\,dx + \int_{B^+(2\sigma)}\|D(H(g))\|^2\,dx\Big\}.$$

Proof. Let $\sigma, \sigma_0 \in (0, R]$, with $0 < \sigma_0 - \sigma \le 1$, let $\sigma_1 = \frac{\sigma+\sigma_0}{2}$, and let $|h| < \frac{\sigma_0-\sigma}{2}$. Write $w = u + g$, $\tau_{n,-h}B^+(\sigma_1) = \{x \in \mathbb{R}^n : x - he^n \in B^+(\sigma_1)\}$, $\mathcal{B}^+(\sigma_1, -h) = B^+(\sigma_1) \cap \tau_{n,-h}B^+(\sigma_1)$. From system (2.1), we have

$$\tau_{n,-h}a(H(w)) = a(H(w(x-he^n))) - a(H(w(x))) = 0 \text{ in } \mathcal{B}^+(\sigma_1, -h).$$

Hence, for every $\nu = 1, 2, \dots, N$ (7)

$$\tau_{n,-h}a^\nu(H(w)) = a^\nu(H(w(x-he^n))) - a^\nu(H(w(x))) = \sum_{\mu=1}^{N}\sum_{i,j=1}^{n}\Big(\int_0^1 \frac{\partial a^\nu(H(w)+t\tau_{n,-h}H(w))}{\partial\xi_{ij}^\mu}dt\Big)\tau_{n,-h}D_{ij}w^\mu = 0.$$

From this it follows

$$(2.22)\qquad \sum_{i,j=1}^{n} A_{ij}(x)\tau_{n,-h}D_{ij}w = 0 \quad \text{in } \mathcal{B}^+(\sigma_1, -h),$$

(7) Conditions (1.1) and (C) ensure that the function $\xi \to a(\xi)$ is differentiable almost everywhere in $\mathbb{R}^{n^2N}$ (see, in the case $N = 1$ and $n = 2$, [10], and, for the general case, [9], Lemma 3.1.2).

where $A_{ij}(x)$ is the $N \times N$ matrix defined by

$$(2.23) \qquad A_{ij}(x) = \left\{ \int_0^1 \frac{\partial a^\nu(H(w) + t\tau_{n,-h}H(w))}{\partial \xi_{ij}^\mu} \, dt \right\}_{\nu,\mu=1,2,\dots,N},$$

for $i, j = 1, 2, \dots, n$, and $x \in \mathcal{B}^+(\sigma_1, -h)$.
Under the hypotheses made above, there exist two constants $\tilde{M}$ and $\tilde{\nu}$, $\tilde{M} \geq \tilde{\nu} > 0$, such that, for almost all $x \in \mathcal{B}^+(\sigma_1, -h)$, $\forall \eta \in \mathbb{R}^N$, and $\forall \lambda \in \mathbb{R}^n$, it results (see [4], [9], and [10])

$$(2.24) \qquad \sum_{i,j=1}^n \|A_{ij}(x)\|^2 \leq \tilde{M}^2,$$

$$(2.25) \qquad \sum_{i,j=1}^n \lambda_i \lambda_j (A_{ij}(x)\eta|\eta) \geq \tilde{\nu}\|\lambda\|^2\|\eta\|^2.$$

In particular, by (2.25),

$$(A_{nn}(x)\eta|\eta) \geq \tilde{\nu}\|\eta\|^2,$$

for almost all $x \in \mathcal{B}^+(\sigma_1, -h)$, and $\forall \eta \in \mathbb{R}^N$, so that

$$\det A_{nn} \neq 0 \quad \text{and} \quad \|A_{nn}^{-1}(x)\| \leq \frac{\sqrt{N}}{\tilde{\nu}}, \forall x \in \mathcal{B}^+(\sigma_1, -h).$$

Now, using (2.22), yields

$$(2.26) \qquad \tau_{n,-h}D_{nn}u = A_{nn}^{-1}\Big[- \sum_{\substack{i,j=1 \\ i+j<2n}}^n A_{ij}(x)\tau_{n,-h}D_{ij}u - \\ - \sum_{i,j=1}^n A_{ij}(x)\tau_{n,-h}D_{ij}g \Big] \text{ in } \mathcal{B}^+(\sigma_1, -h).$$

On the other hand, for every $\varphi \in C_0^\infty(B^+(\sigma_1), \mathbb{R}^N)$, we have

$$\int_{B^+(\sigma_1)} (D_{nn}u|\tau_{n,h}\varphi)\, dx = \int_{\tau_{n,-h}B^+(\sigma_1)} (D_{nn}u(x - he^n)|\varphi(x))\, dx - \\ - \int_{B^+(\sigma_1)} (D_{nn}u(x)|\varphi(x))\, dx =$$

$$= \int_{\mathcal{B}^+(\sigma_1,-h)} (D_{nn}u(x-he^n)|\varphi(x))\,dx - \int_{B^+(\sigma_1)} (D_{nn}u(x)|\varphi(x))\,dx =$$

$$= \int_{\mathcal{B}^+(\sigma_1,-h)} (\tau_{n,-h}D_{nn}u|\varphi)\,dx - \int_{B^+(\sigma_1)\setminus\mathcal{B}^+(\sigma_1,-h)} (D_{nn}u|\varphi)\,dx.$$

If $|h|$ is small enough, the last integral vanishes because φ has a compact support in $B^+(\sigma_1)$. Then, taking into account (2.26) and (2.24), if $|h|$ is small enough we get ([8])

$$(2.27)\quad \Big|\int_{B^+(\sigma_1)} (D_{nn}u|\tau_{n,h}\varphi)\,dx\Big| = \Big|\int_{\mathcal{B}^+(\sigma_1,-h)} \Big(-\sum_{\substack{i,j=1\\ i+j<2n}}^{n} A_{ij}(x)\tau_{n,-h}D_{ij}u -$$

$$-\sum_{i,j=1}^{n} A_{ij}(x)\tau_{n,-h}D_{ij}g\,|(A_{nn}^{-1})^*\varphi\Big)\,dx\Big| \le c(\tilde{\nu},\tilde{M})\Big(\int_{B^+(\sigma_1)} \|\varphi(x)\|^2\,dx\Big)^{\frac{1}{2}}\cdot$$

$$\cdot\Big\{\sum_{\substack{i,j=1\\ i+j<2n}}^{n} \int_{\mathcal{B}^+(\sigma_1,-h)} \|\tau_{n,-h}D_{ij}u\|^2\,dx + \sum_{i,j=1}^{n} \int_{\mathcal{B}^+(\sigma_1,-h)} \|\tau_{n,-h}D_{ij}g\|^2\,dx\Big\}^{\frac{1}{2}} \le$$

$$\le c(\tilde{\nu},\tilde{M})|h|\Big(\int_{B^+(\sigma_1)} \|\varphi(x)\|^2\,dx\Big)^{\frac{1}{2}}\cdot$$

$$\cdot\Big\{\sum_{\substack{i,j=1\\ i+j<2n}}^{n} \int_{B^+(\sigma_1)} \|D_n(D_{ij}u)\|^2\,dx + \sum_{i,j=1}^{n} \int_{B^+(\sigma_1)} \|D_n(D_{ij}g)\|^2\,dx\Big\}^{\frac{1}{2}}.$$

On the other hand, by (2.3), it follows, for every $i, j = 1, 2, \ldots, n$, with $i + j < 2n$

$$(2.28)\quad \int_{B^+(\sigma_1)} \|D_n(D_{ij}u)\|^2\,dx \le c\Big\{\frac{1}{(\sigma_0-\sigma)^4}\sum_{r=1}^{n-1}\int_{B^+(\sigma_0)} \|D_r u\|^2\,dx +$$

$$+\frac{1}{(\sigma_0-\sigma)^2}\int_{B^+(\sigma_0)} \|H(u)\|^2\,dx + \int_{B^+(\sigma_0)} \|D(H(g))\|^2\,dx\Big\} = c\mathcal{M}^2,$$

where the constant c does not depend on σ and σ_0.

From (2.27) and (2.28), we deduce

$$\Big|\int_{B^+(\sigma_1)} (D_{nn}u|\tau_{n,h}\varphi)\,dx\Big| \le c\,\mathcal{M}\,|\,h\,|\Big(\int_{B^+(\sigma_1)} \|\varphi(x)\|^2\,dx\Big)^{\frac{1}{2}},$$

([8]) $(A_{nn}^{-1})^*$ is the adjoint of the matrix A_{nn}^{-1}.

while, dividing both sides by $|h|$ and letting $h \to 0$, we obtain, for every $\varphi \in C_0^\infty(B^+(\sigma_1), \mathbb{R}^N)$

$$\left|\int_{B^+(\sigma_1)} (D_{nn}u|D_n\varphi)\,dx\right| \le c\mathcal{M}\Big(\int_{B^+(\sigma_1)} \|\varphi(x)\|^2\,dx\Big)^{\frac{1}{2}}. \tag{2.29}$$

Now, through (2.29) and Lemma 9.3 in [1] (see also [3], Lemma 2.III), we achieve condition (2.19) and the following inequality

$$\int_{B^+(\sigma)} \|D_n(D_{nn}u)\|^2\,dx \le \tilde{\gamma}^2 c\Big\{\mathcal{M}^2 + \sum_{i=1}^{n-1}\int_{B^+(\sigma_1)} \|D_i(D_{nn}u)\|^2\,dx + \\ + \int_{B^+(\sigma_1)} \|D_{nn}u\|^2\,dx\Big\}, \tag{2.30}$$

where the constant c does not depend on σ and σ_0, while $\tilde{\gamma}$ is the constant that arises from the application of Lemma 9.3 in [1].
From this, thanks to estimate (2.28) and the hypothesis $\sigma_0 - \sigma \le 1$, inequality (2.20) follows.

Now, if $0 < \sigma \le \min(1, \frac{R}{2})$, estimate (2.20), written for $\sigma_0 = 2\sigma$, gives

$$\int_{B^+(\sigma)} \|D_n(D_{nn}u)\|^2\,dx \le \tilde{\gamma}^2 c\Big\{\sigma^{-4}\sum_{r=1}^{n-1}\int_{B^+(2\sigma)} \|D_r u\|^2\,dx + \\ + \sigma^{-2}\int_{B^+(2\sigma)} \|H(u)\|^2\,dx + \int_{B^+(2\sigma)} \|D(H(g))\|^2\,dx\Big\}. \tag{2.31}$$

On the other hand, from the condition $D_r u = 0$ on Γ_R, $r = 1, 2, \dots, n-1$, and taking into account Poincarè's inequality one has

$$\int_{B^+(2\sigma)} \|D_r u\|^2\,dx \le c\sigma^2\int_{B^+(2\sigma)} \|D_n(D_r u)\|^2\,dx, \qquad r = 1, 2, \dots, n-1. \tag{2.32}$$

By (2.31) and (2.32), we obtain

$$\int_{B^+(\sigma)} \|D_n(D_{nn}u)\|^2\,dx \le \tilde{\gamma}^2 c\Big\{\sigma^{-2}\sum_{r=1}^{n-1}\int_{B^+(2\sigma)} \|D_n(D_r u)\|^2\,dx + \\ + \sigma^{-2}\int_{B^+(2\sigma)} \|H(u)\|^2\,dx + \int_{B^+(2\sigma)} \|D(H(g))\|^2\,dx\Big\},$$

which leads to (2.21). □

Obviously, using Theorems 2.1 and 2.2 we can state the following differentiability result

Theorem 2.3. *If $u \in H^2(B^+(R), \mathbb{R}^N)$ is a solution to problem* (2.1), *under conditions* (1.1) *and* (C), *and if $g \in H^3(B^+(R), \mathbb{R}^N)$, then, for every $\sigma \in (0, R)$, one has*

$$H(u) \in H^1(B^+(\sigma), \mathbb{R}^{n^2N}).$$

3. Results on Hölder continuity.

$\mathcal{L}^{2,\lambda}$ - regularity and hence Hölder continuity results concerning the vectors Du and u can be derived from Theorem 2.3. In fact, we have the following

Theorem 3.1. *If $u \in H^2(B^+(R), \mathbb{R}^N)$ is a solution to problem* (2.1), *under conditions* (1.1) *and* (C), *and if $g \in H^3(B^+(R), \mathbb{R}^N)$, then, for every $\sigma \in (0, R), \forall q > 2$, and $\forall \lambda \in (n, n+2)$, one has*

$$H(u) \in L^{2,2-\frac{4}{q}}(B^+(\sigma), \mathbb{R}^{n^2N}), \quad Du \in \mathcal{L}^{2,4-\frac{4}{q}}(B^+(\sigma), \mathbb{R}^{nN}), \tag{3.1}$$

$$u \in \mathcal{L}^{2,6-\frac{4}{q}}(B^+(\sigma), \mathbb{R}^N), \quad \text{if } n \geq 4, \tag{3.2}$$

$$u \in \mathcal{L}^{2,\lambda}(B^+(\sigma), \mathbb{R}^N), \quad \text{if } n = 2 \text{ or } n = 3. \tag{3.3}$$

Proof. Fixing $\sigma \in (0, R)$, by virtue of Theorem 2.3, we get

$$H(u) \in H^1(B^+(\sigma), \mathbb{R}^{n^2N}).$$

Thus, if $n > 2$, the Sobolev imbedding Theorem, provides

$$H(u) \in L^{2^*}(B^+(\sigma), \mathbb{R}^{n^2N}), \tag{3.4}$$

where $\frac{1}{2^*} = \frac{1}{2} - \frac{1}{n}$.
Now, $\forall x^0 \in \overline{B^+(\sigma)}$ and $\forall \rho \in (0, R-\sigma)$, set $B^+_\sigma(x^0, \rho) = B^+(\sigma) \cap B(x^0, \rho)$ (9). Thanks to condition (3.4) and Hölder's inequality, one has

$$\int_{B^+_\sigma(x^0,\rho)} \|H(u)\|^2\, dx \leq c\rho^2 \Big(\int_{B^+(\sigma)} \|H(u)\|^{2^*}\, dx \Big)^{\frac{2}{2^*}}. \tag{3.5}$$

(9) $B(x^0, \rho) = \{x \in \mathbb{R}^n : \|x - x^0\| < \rho\}$.

If $n = 2$, by the Sobolev imbedding Theorem again, it follows

$$H(u) \in L^q(B^+(\sigma), \mathbb{R}^{n^2N}), \ \forall q > 2;$$

therefore, $\forall x^0 \in \overline{B^+(\sigma)}$, $\forall \rho \in (0, R - \sigma)$, and $\forall q > 2$, we get

$$\int_{B_\sigma^+(x^0,\rho)} \|H(u)\|^2 \, dx \leq c\rho^{2-\frac{4}{q}} \Big(\int_{B^+(\sigma)} \|H(u)\|^q \, dx \Big)^{\frac{2}{q}}. \tag{3.6}$$

Clearly, if $n \geq 2$, by (3.5) and (3.6), one has

$$\int_{B_\sigma^+(x^0,\rho)} \|H(u)\|^2 \, dx \leq c\rho^{2-\frac{4}{q}} M_{\sigma,q}, \tag{3.7}$$

$\forall x^0 \in \overline{B^+(\sigma)}$, $\forall \rho \in (0, \min\,(1, R - \sigma))$, and $\forall q > 2$, where

$$M_{\sigma,q} = \begin{cases} \Big(\int_{B^+(\sigma)} \|H(u)\|^{2^*} \, dx \Big)^{\frac{2}{2^*}}, & \text{if } n > 2, \\ \Big(\int_{B^+(\sigma)} \|H(u)\|^q \, dx \Big)^{\frac{2}{q}}, & \text{if } n = 2. \end{cases}$$

Hence, the first assertion in (3.1) is true.
Finally, conditions

$$Du \in \mathcal{L}^{2,4-\frac{4}{q}}(B^+(\sigma), \mathbb{R}^{nN}), \ \forall q > 2, \tag{3.8}$$

(3.2), and (3.3) follow from the first assertion in (3.1) and Poincarè's inequality (see [2], Chap. I, Theorem 3.IV). □

Results on the Hölder continuity of the solutions to problem (2.1) and their gradient in $\overline{B^+(\sigma)}$, $0 < \sigma < R$, can be immediately obtained from Theorem 3.1.
In fact, if $2 \leq n < 4$, then there exists $q > 2$ such that $4 - \frac{4}{q} > n$, and hence, by (3.8) and well-known properties of isomorphism between the spaces $\mathcal{L}^{2,\lambda}$ and $C^{0,\alpha}$, one has

$$Du \textit{ is Hölder-continuous in } \overline{B^+(\sigma)}.$$

If $4 \leq n < 6$, then there exists $q > 2$ such that $6 - \frac{4}{q} > n$, and hence, by (3.2), it follows

$$u \textit{ is Hölder-continuous in } \overline{B^+(\sigma)}.$$

Finally, if $2 \leq n \leq 3$, (3.3) holds, and this ensures that $u \in C^{0,\alpha}(\overline{B^+(\sigma)}, \mathbb{R}^N)$, $\forall \alpha \in (0, 1)$.

REFERENCES

[1] S. Agmon, *Lectures on Elliptic Boundary Value Problems,* Van Nostrand Mathematical Studies, Van Nostrand, Princeton, N. J., 1965.

[2] S. Campanato, *Sistemi ellittici in forma divergenza. Regolarità all'interno,* Quaderni Scuola Norm. Sup. Pisa, 1980.

[3] S. Campanato, *A Maximum Principle for Non-linear Elliptic Systems: Boundary Fundamental Estimates,* Adv. in Math., 66 (1987), pp. 291–317.

[4] S. Campanato, $\mathcal{L}^{2,\lambda}$ *Theory for non Linear non Variational Differential Systems,* Rend. Mat., (7) 10 (1990), pp. 531–549.

[5] S. Campanato, *Non variational basic elliptic systems of second order,* Rend. Sem. Mat. Fis. Milano, 60 (1990), pp. 113–131.

[6] S. Campanato-P. Cannarsa, *Differentiability and Partial Hölder Continuity of the Solutions of Non-Linear Elliptic Systems of Order 2m with Quadratic Growth,* Ann. Sc. Norm. Sup. Pisa, (4) 8 (1981), pp. 285–309.

[7] M.S. Fanciullo, $\mathcal{L}^{2,\lambda}$ *regularity for second order non linear non variational elliptic systems,* Le Matematiche, 50 (1995), pp. 163–172.

[8] M. Marino-A. Maugeri, $\mathcal{L}^{2,\lambda}$ *- Theory for Nonvariational Basic Parabolic Systems,* Proc. 8th Int. Coll. Diff. Equations, D. Bainov (Ed.), VSP, Utrecht, 1998, pp. 293–300.

[9] A. Maugeri-D.K. Palagachev-L.G. Softova, *Elliptic and Parabolic Equations with Discontinuous Coefficients,* Wiley-VCH, Berlin, 2000.

[10] D. K. Palagachev, *Global strong solvability of Dirichlet problem for a class of nonlinear elliptic equations in the plane,* Le Matematiche, 48 (1993), pp. 311–321.

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