HÖLDER CONTINUITY OF WEAK SOLUTIONS TO PARABOLIC SYSTEMS WITH CONTROLLED GROWTH NON-LINEARITIES (TWO SPATIAL DIMENSIONS)

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Dedicated to Professor Sergio Campanato on his 70th birthday

1. Introduction. Statement of the main result.

Let $\Omega \subset \mathbb{R}^2$ be open, let $0 < T < +\infty$ and set $Q = \Omega \times (0, T)$. In $Q$ we consider the following system of nonlinear PDE's:

$$
(1.1) \quad \frac{\partial u^i}{\partial t} - D_\alpha a_i^\alpha(x, t, Du) = b_i(x, t, u, Du) \quad (i = 1, \ldots, N),
$$

where

$$
\begin{align*}
    u &= \{u^1, \ldots, u^N\} \quad (N \geq 2), \\
    D_\alpha &= \frac{\partial}{\partial x_\alpha} \quad (\alpha = 1, 2) \\
    Du &= \{D_\alpha u^i\} \quad (= \text{matrix of spatial derivatives}).
\end{align*}
$$

The conditions on the functions $a_i^\alpha : \Omega \times (0, T) \times \mathbb{R}^{2N} \to \mathbb{R}$ are as follows:

$$
(1.2) \quad x \mapsto a_i^\alpha(x, t, \xi) \; \text{is measurable on} \; \Omega \; \forall \{t, \xi\} \in (0, T) \times \mathbb{R}^{2N},
$$

(1) With the exception of Section 2, throughout the paper, a repeated Greek (resp. Latin) index implies summation over 1 and 2 (1, \ldots, N).
\[ |a^a_i(x, s, \eta) - a^a_i(x, t, \xi)| \leq \omega(|s - t|)(1 + |\eta| + |\xi|) + c_0|\eta - \xi| \forall x \in \Omega, \forall \{s, \eta, t, \xi\} \in (0, T) \times \mathbb{R}^{2N}, \]

(1.3) \quad where \ \omega : [0, +\infty) \to (0, +\infty) \ is bounded, nondecreasing

with \ \lim_{h \to 0} \omega(h) = 0, \ and \ c_0 = \text{const};

\[ |a^a_i(x, t, \xi)| \leq c_1(1 + |\xi|) \forall \{x, t, \xi\} \in \Omega \times (0, T) \times \mathbb{R}^{2N} \ (c_1 = \text{const}) \]

(1.4) \quad (\alpha = 1, 2; \ i = 1, \ldots, N), \ and

\[ (a^a_i(x, t, \eta) - a^a_i(x, t, \xi))(\eta^i_a - \xi^i_a) \geq \nu_0|\eta - \xi|^2 \forall \{x, t\} \in \Omega \times (0, T), \forall \eta, \xi \in \mathbb{R}^{2N} \ (\nu_0 = \text{const} > 0). \]

(1.5) \quad The functions \ b_i \ are assumed to satisfy the following conditions:

\[ \{x, t\} \mapsto b_i(x, t, u, \xi) \text{ is measurable on } \Omega \times (0, T) \quad \forall \{u, \xi\} \in \mathbb{R}^N \times \mathbb{R}^{2N}; \]

\[ \{u, \xi\} \mapsto b_i(x, t, u, \xi) \text{ is continuous on } \mathbb{R}^N \times \mathbb{R}^{2N} \quad \forall \{x, t\} \in \Omega \times (0, T); \]

(1.6) \quad controlled growth:

\[ |b_i(x, t, u, \xi)| \leq c_2(1 + |u|^3 + |\xi|^{3/2}) \forall \{x, t, u, \xi\} \in \Omega \times (0, T) \times \mathbb{R}^N \times \mathbb{R}^{2N} \ (i = 1, \ldots, N; c_2 = \text{const}). \]

(1.7) \quad In the present paper, we consider weak solutions \ u \ to (1.1) regardless of whether \ u \ satisfies any boundary and (or) initial conditions. Our goal is to study the interior Hölder continuity of these solutions.

To this end, define

\[ W_{2,0}^1(Q; \mathbb{R}^N) = \left\{ u \in L^2(Q; \mathbb{R}^N) \left| \frac{\partial u}{\partial x_\alpha} \in L^2(Q; \mathbb{R}^N); \alpha = 1, 2 \right\}, \]

\[ W_{2,1}^1(Q; \mathbb{R}^N) = \left\{ u \in W_{2,0}^1(Q; \mathbb{R}^N) \left| \frac{\partial u}{\partial t} \in L^2(Q; \mathbb{R}^N) \right\} = \right. \]

\[ W_{2,1}^1(Q; \mathbb{R}^N) \quad \text{(the usual Sobolev space on } Q), \]

\[ V_{2,0}^1(Q; \mathbb{R}^N) = \left\{ u \in W_{2,0}^1(Q; \mathbb{R}^N) \left| \text{ess sup}_{t \in (0, T)} \int_\Omega |u(x, t)|^2 \, dx < +\infty \right\}. \]

We now introduce the notion of weak solution to (1.1).
**Definition.** Let (1.2), (1.4) and (1.6), (1.7) be satisfied. The vector valued function \( u \in V_{2,0}^1(Q; \mathbb{R}^N) \) is called weak solution to (1.1) if

\[
\begin{aligned}
- \int_Q u^i \frac{\partial \varphi^i}{\partial t} \, dx \, dt + \int_Q a_i^a(x, t, Du) D_a \varphi^i \, dx \, dt = \\
\quad = \int_Q b_i(x, t, u, Du) \varphi^i \, dx \, dt
\end{aligned}
\]

for all \( \varphi \in W_{2,1}^1(Q; \mathbb{R}^N) \) with \( \text{supp}(\varphi) \subset Q \).

The main result of our paper is following

**Theorem.** Let (1.2)–(1.5) and (1.6), (1.7) be satisfied. Then there exists \( \mu \in (0, 1) \) such that: for every weak solution \( u \in V_{2,0}^1(Q; \mathbb{R}^N) \) to (1.1) there holds

\[
u \in C^{\mu, \mu/2}(Q; \mathbb{R}^N) \quad (2).
\]

The interior Hölder continuity of weak solution \( u \in V_{2,0}^1(Q; \mathbb{R}^N) \) (\( n = 2 \)) to (1.1) has been proved in [2], Theorem 7.II, p. 112, under the following more restrictive conditions: uniform continuity of the functions \( x \mapsto a_i^a(x, t, \xi) \), continuous differentiability of the functions \( \xi \mapsto a_i^a(x, t, \xi) \) and strictly controlled growth on \( b_i \), i.e.

\[
|b_i(x, t, u, \xi)| \leq c(1 + |u|^\beta + |\xi|^\gamma) \quad \forall \{x, t, u, \xi\} \in \Omega \times (0, T) \times \mathbb{R}^N \times \mathbb{R}^{2N},
\]

where

\[
1 \leq \beta \leq 3 \quad , \quad 1 \leq \gamma < \frac{3}{2}
\]

\( (i = 1, \ldots, N; \ c = \text{const}) \). Our above result thus sharpens [2], Theorem 7.II, p. 112, and moreover it can be viewed as the “parabolic analogue” of the following well-known result: every weak solution to a nonlinear uniformly elliptic system in two dimensions with measurable coefficients \( x \mapsto a_i^a(x, \xi) \) is Hölder continuous in the interior \( (3) \). This follows merely from the higher integrability of the gradient of the weak solution under consideration and Sobolev’s imbedding theorem (cf. [3], [4] for details).

(2) That is, for every bounded open set \( Q' \) such that \( \overline{Q'} \subset Q \), there holds \( |u(x, t) - u(y, t)| \leq c(|x - y|^\mu + |s - t|^\mu/2) \) for all \( \{x, s\}, \{y, t\} \in Q' \), where the constant \( c \) may depend on \( \text{dist}(Q', \partial Q) \).

(3) Note that this result in fact holds for such systems with coefficients \( a_i^a(x, u, \xi) \).
The interior Hölder continuity of weak solutions to nonlinear parabolic systems with coefficients \( a_i^a(\xi) \) and \( b_i \equiv 0 \) has been proved in [10], and with coefficients \( a_i^a(x, t, u, \xi) \) in [5, 9] \((b_i \equiv 0)\) and in [7] \((b_i = f_i(x, t))\).

The aim of the present paper is to simplify the discussion in [11]. In comparison with [2], the novelty in [11] lies in the use of the interior \( t \)-differentiability of weak solutions to nonlinear parabolic systems and an interpolation inequality. Our paper is organized as follows. In Section 2 we prove the interior \( t \)-differentiability of weak solutions of a class of nonlinear parabolic systems. Then we establish a fundamental inequality for weak solutions to these systems. Section 3 is devoted to the proof of our main result. Here we make use of a generalization of an existence result from [2].

2. Interior estimates on weak solutions to a class of nonlinear parabolic systems.

Let \( x_0 \in \mathbb{R}^n \) \((n \geq 1)\) and \( t_0 \in \mathbb{R} \) be arbitrary, but fixed. Given \( r > 0 \), define

\[
B_r = B_r(x_0) = \{ x \in \mathbb{R}^n \mid |x - x_0| < r \},
\]

\[
Q_r = Q_r(x_0, t_0) = B_r(x_0) \times (t_0 - r^2, t_0).
\]

Let \( R > 0 \) be fixed. In the cylinder \( Q_R = Q_R(x_0, t_0) \) we consider the following system of PDE’s:

\[
\frac{\partial v^i}{\partial t} - D_\alpha a_i^a(x, Dv) = 0 \quad (i = 1, \ldots, N) \tag{4}.
\]

The conditions on the functions \( a_i^a \) are as follows:

\[\begin{align*}
x \mapsto a_i^a(x, \xi) \text{ is measurable on } B_R \quad &\forall \xi \in \mathbb{R}^{nN}, \\
|a_i^a(x, \xi)| \leq c_0(1 + |\xi|) \quad &\forall x \in B_R, \quad \forall \xi \in \mathbb{R}^{nN} \quad (c_0 = \text{const}), \\
|a_i^a(x, \eta) - a_i^a(x, \xi)| \leq c_1|\eta - \xi| \quad &\forall x \in B_R, \quad \forall \eta, \xi \in \mathbb{R}^{nN} \quad (c_1 = \text{const})
\end{align*}\]

\[(a = 1, \ldots, n; \ i = 1, \ldots, N), \text{ and}
\]

\[
\begin{align*}
\begin{cases}
(a_i^a(x, \eta) - a_i^a(x, \xi))(\eta_a^i - \xi_a^i) \geq v_0|\eta - \xi|^2 \\
\forall x \in B_R, \quad \forall \eta, \xi \in \mathbb{R}^{nN} \quad (v_0 = \text{const} > 0).
\end{cases}
\end{align*}
\]

We introduce

\[\tag{4} \text{Unless otherwise stated, in the present section a repeated Greek index implies summation over } 1, \ldots, n.\]
**Definition 2.1.** Let (2.2) and (2.3) be satisfied. The function \( v \in W^{1,0}_2(Q_R; \mathbb{R}^N) \) is called a weak solution to (2.1) if

\[
(2.6) \quad \begin{cases}
- \int_{Q_R} v^i \frac{\partial \varphi^i}{\partial t} \, dx dt + \int_{Q_R} a^i_i(x, Dv) D_{\alpha} \varphi^i \, dx dt = 0 \\
\forall \varphi \in W^{1,1}_2(Q_R; \mathbb{R}^N) \text{ with } \varphi = 0 \text{ a.e. on } \partial Q_R.
\end{cases}
\]

Without any further reference, conditions (2.2)–(2.5) are now assumed to hold throughout the present section.

**Interior differentiability of weak solutions to (2.1).** The following result on the interior \( t \)-differentiability may be of interest in its own right. Our method of proof differs substantially from that of [2], Theorem 3.1, p. 100.

**Theorem 2.2.** Let \( v \in W^{1,0}_2(Q_R; \mathbb{R}^N) \) be a weak solution to (2.1). Then

\[
(2.7) \quad \int_{Q_R} \left| \frac{\partial v}{\partial t} \right|^2 \, dx dt \leq \frac{c}{(R-r)^2} \int_{Q_R} (1 + |Dv|^2) \, dx dt,
\]

\[
(2.8) \quad \int_{Q_R} \left| \frac{\partial v}{\partial t} Dv \right|^2 \, dx dt \leq \frac{c}{(R-r)^4} \int_{Q_R} (1 + |Dv|^2) \, dx dt,
\]

for all \( 0 < r < R \), where the constants \( c \) depend neither on \( r \) nor on \( R \).

Before turning to the proof we present two technical tools.

1) Let \( f \in L^p(Q_r) \) (\( 1 \leq p < +\infty \)). Let \( t_1 \in (t_0 - R^2, t_0) \). For \( \lambda \in (0, t_0 - t_1) \), define the Steklov mean

\[
f_{\lambda}(x, t) = \frac{1}{\lambda} \int_t^{t+\lambda} f(x, s) \, ds \quad \text{for a.e. } \{x, t\} \in B_R \times (t_0 - R^2, t_1).
\]

Then

\[
(2.9) \quad \int_{t_0 - R^2}^{t_1} \int_{B_R} |f_{\lambda}|^p \, dx dt \leq \int_{Q_R} |f|^p \, dx dt \quad \forall \lambda \in (0, t_0 - t_1),
\]

\[
(2.10) \quad f_{\lambda} \to f \text{ in } L^p(B_R \times (t_0 - R^2, t_1)) \text{ as } \lambda \to 0
\]

and

\[
(2.11) \quad \frac{\partial f_{\lambda}}{\partial t} = \frac{1}{\lambda} \left( f(x, t + \lambda) - f(x, t) \right)
\]
for a.e. \((x, t) \in B_R \times (t_0 - R^2, t_1)\) and all \(\lambda \in (0, t_0 - t_1)\).

Let \(v \in W^{1,0}_2(\Omega_R; \mathbb{R}^N)\) be a weak solution to (2.1). Then there holds

\[
\begin{align*}
\int_{B_R} \frac{\partial u^i_\lambda}{\partial t}(x, t)\psi^i(x)\,dx &+ \int_{B_R} (a^a_i(x, Du))_\lambda(t)D_a\psi^i(x)\,dx = 0 \\
\text{for a.e. } t \in (t_0 - R^2, t_1), \text{ for all } \lambda \in (0, t_0 - t_1) \\
\text{and all } \psi \in W^{1}_2(B_R; \mathbb{R}^N) \text{ with } \psi = 0 \text{ a.e. on } \partial B_R
\end{align*}
\]

(cf. [8], [9]).

The integral identity in (2.12) forms the basis for deriving estimates on \(t\)-differences of \(v\) and \(Du\). These estimates will provide (2.7) and (2.8).

2) We need the following technical

**Lemma.** Let \(\sigma\) be a nonnegative bounded function on the interval \([a, b]\) \((-\infty < a < b < +\infty)\). Assume that

\[
\sigma(r) \leq \frac{A}{(R - r)^\theta} + B + \frac{1}{2}\sigma(R)
\]

for all \(r, R\) with \(a < r < R \leq b\), where \(A, B\) and \(\theta\) are (fixed) nonnegative constants.

Then there exists a constant \(C = C(\theta)\) such that

\[
\sigma(r) \leq C\left(\frac{A}{(R - r)^\theta} + B\right) \quad \forall a < r < R \leq b.
\]

A proof of this result may be found in [4].

**Proof of Theorem 2.2.** Define the \(t\)-difference of a function \(f = f(x, t)\) by

\[
\Delta_h f(x, t) = f(x, t + h) - f(x, t), \quad h > 0.
\]

Let \(0 < r < R\). Let \(\xi \in C^\infty_c(B_R)\) be a cut-off function such that \(\xi(x) = 1\) for all \(x \in B_r\), \(0 \leq \xi(x) \leq 1\) and \(|D\xi(x)| \leq \frac{c_0}{R - r}\) for all \(x \in B_R\) \((c_0 = \text{const})\), and let \(\tau \in C^\infty(\mathbb{R})\) be a function satisfying \(\tau(t) = 0\) for all \(t \in (-\infty, t_0 - R^2]\), \(\tau(t) = 1\) for all \(t \in [t_0 - r^2, +\infty)\) and \(0 \leq \tau(t) \leq 1, \ 0 \leq \tau'(t) \leq \frac{c_0}{(R - r)^2}\) for all \(t \in \mathbb{R}\).

Let \(t_1 \in (t_0 - \tau^2, t_0)\) be arbitrary. The function

\[
\psi(x) = (\Delta_h v)(x, t)\xi^2(x)\tau^2(t), \quad x \in B_R, \ t \in (t_0 - R^2, t_1), \ h \in (0, t_0 - t_1)
\]
is admissible in (2.12) (with $\lambda = h$). By (2.11) (with $\lambda = h$), \( \frac{\partial v_h}{\partial t}(x, t) = \frac{1}{h}(\Delta_h v)(x, t) \) for a.e. \( (x, t) \in B_R \times (t_0 - R^2, t_1) \). Integrating the integral identity (2.12) over the interval \((t_0 - R^2, t_1)\) and using (2.3) gives

\[
\int_{t_0 - R^2}^{t_1} \int_{B_R} |\Delta_h v(x, t)|^2 \zeta^2(x) \tau^2(t) \, dx \, dt =
\]

\[
= -h \int_{t_0 - R^2}^{t_1} \int_{B_R} (a_i^a(x, Du))_h(t) \left[ (\Delta_h D_{\alpha} v^i(x, t)) \zeta^2(x) \tau^2(t) + 2(\Delta_h v^i(x, t)) \zeta(x) D_{\alpha} \xi(x) \tau^2(t) \right] \, dx \, dt \leq
\]

\[
\leq c h \int_{t_0 - R^2}^{t_1} \int_{B_R} \left( 1 + |Du(x, \cdot)|_h(t) \left( |\Delta_h Du(x, t)| \zeta^2(x) \tau^2(t) + |\Delta_h v(x, t)| \zeta^2(x) \tau^2(t) \right) \right) \, dx \, dt.
\]

Observing (2.9) and employing Young's inequality we obtain for all $\epsilon > 0$

(2.13) \[
\int_{t_0 - R^2}^{t_1} \int_{B_R} |\Delta_h v|^2 \zeta^2 \tau^2 \, dx \, dt \leq
\]

\[
\leq \epsilon \int_{t_0 - R^2}^{t_1} \int_{B_R} \left( |\Delta_h Du|^2 \zeta^4 \tau^4 + |\Delta_h v|^2 |D\xi|^2 \zeta^2 \tau^4 \right) \, dx \, dt +
\]

\[
+ \frac{ch}{\epsilon} \int_{Q_{R^2}} (1 + |Du|^2) \, dx \, dt.
\]

Here the constant $c$ is independent of $r, R, h$ and $\epsilon$.

Next, as above let $h \in (0, t_0 - t_1)$. We consider the integral identity in (2.12) for $\lambda \in (0, t_0 - t_1 - h)$ and form the $t$-difference $\Delta_h$ therein. Observing that $\Delta_h v_\lambda = (\Delta_h v)_\lambda$ and $\Delta_h \frac{\partial v_\lambda}{\partial t} = \frac{\partial}{\partial t} (\Delta_h v)$ we obtain

\[
\int_{B_R} \frac{\partial}{\partial t} (\Delta_h v^i(x, \cdot))_\lambda(t) \psi^i(x) \, dx + \int_{B_R} [\Delta_h a_i^a(x, Du)]_\lambda(t) D_{\alpha} \psi^i(x) \, dx = 0
\]

for a.e. $t \in (t_0 - R^2, t_1)$. Here we insert $\psi(x) = (\Delta_h v(x, \cdot))_\lambda(t) \zeta^2(x) \tau^2(t)$, where $\zeta$ and $\tau$ are cut-off functions as above. Since

\[
\int_{B_R} \left[ \frac{\partial}{\partial t} (\Delta_h v^i(x, \cdot))_\lambda(t) \right] (\Delta_h v^i(x, \cdot))_\lambda \zeta^2(x) \tau^2(t) \, dx =
\]
\[
\frac{1}{2} \frac{d}{dt} \int_{B_R} |(\Delta_h u(x, \cdot))_\lambda^2 \eta^2(x) \tau^2(t) dx - \int_{B_R} |(\Delta_h u(x, \cdot))_\lambda^2 \eta^2(x) \tau(t) \tau'(t) dx
\]
for a.e. \( t \in (t_0 - R^2, t_1) \), it follows by integration over the interval \( (t_0 - R^2, t_1) \) that
\[
\int_{t_0 - R^2}^{t_1} \int_{B_R} [\Delta_h a^{(x, Dv)}_t]_\lambda(t)(\Delta_h D \alpha v^i(x, \cdot))_\lambda(t) \eta^2(x) \tau^2(t) dx dt \leq
\]
\[
\leq 2 \int_{t_0 - R^2}^{t_1} \int_{B_R} [\Delta_h a^{(x, Dv)}_t]_\lambda(t)(\Delta_h v^i(x, \cdot))_\lambda(t) \eta^2(x) \tau^2(t) dx dt + \int_{t_0 - R^2}^{t_1} \int_{B_R} |(\Delta_h v^i(x, \cdot))_\lambda^2 \eta^2(x) \tau(t) \tau'(t) dx dt.
\]
Letting tend \( \lambda \to 0 \) (cf. (2.10)) and using then (2.4) and (2.5) gives
\[
(2.14) \quad \int_{t_0 - R^2}^{t_1} \int_{B_R} |\Delta_h Dv(x, t)|^2 \eta^2(x) \tau^2(t) dx dt \leq
\]
\[
\leq c \int_{t_0 - R^2}^{t_1} \int_{B_R} |\Delta_h v(x, t)|^2 (|D \eta^2(x) \tau^2(t) + \eta^2(x) \tau(t) \tau'(t) dx dt \leq
\]
\[
\leq \frac{c}{(R^2 - r^2)} \int_{t_0 - R^2}^{t_1} \int_{B_R} |\Delta_h v(x, t)|^2 dx dt,
\]
the constant \( c \) being independent of \( r, R \) and \( h \).
We insert this estimate into the right hand side of (2.13) to obtain
\[
\int_{t_0 - r^2}^{t_1} \int_{B_r} |\Delta_h v|^2 dx dt \leq
\]
\[
\leq \frac{c \varepsilon}{(R^2 - r^2)} \int_{t_0 - R^2}^{t_1} \int_{B_R} |\Delta_h v|^2 dx dt + ch^2 \int_{Q_R} (1 + |Dv|^2) dx dt.
\]
Choosing \( \varepsilon = \frac{(R^2 - r^2)^2}{2c} \) and employing the above technical lemma with
\[
\sigma(r) = \int_{t_0 - r^2}^{t_1} \int_{B_r} |\Delta_h v|^2 dx dt, \quad 0 < r \leq R,
\]
we find
\[(2.15) \quad \int_{t_0 - r^2}^{t_1} \int_{B_r} \left| \Delta_h u \right|^2 dx dt \leq \frac{c h^2}{(R - r)^2} \int_{Q_r} (1 + |Dv|^2) dx dt\]
for all \( h \in (0, t_0 - t_1) \). By a standard argument, (2.15) implies the existence of the weak derivate \( \frac{\partial v}{\partial t} \in L^2(B_r \times (t_0 - r^2, t_1); \mathbb{R}^N) \), which satisfies
\[
\int_{t_0 - r^2}^{t_1} \int_{B_r} \left| \frac{\partial v}{\partial t} \right|^2 dx dt \leq \frac{c}{(R - r)^2} \int_{Q_r} (1 + |Dv|^2) dx dt.
\]
Hence \( \frac{\partial v}{\partial t} \) is defined a.e. on \( B_r \times (t_0 - r^2, t_0) \) and measurable. Taking into account the monotone convergence theorem, we may let tend \( t_1 \to t_0 \) in the latter inequality and obtain (2.7).

Finally, dividing (2.14) by \( h^2 \) (where \( h \in (0, t_0 - t_1) \)) gives
\[
\int_{t_0 - r^2}^{t_1} \int_{B_r} \left| \frac{1}{h} \Delta_h Dv \right|^2 dx dt \leq \frac{c}{(R - r)^2} \int_{t_0 - r^2}^{t_1} \int_{B_r} \left| \frac{1}{h} \Delta_h Dv \right|^2 dx dt \leq \frac{c}{(R - r)^2} \int_{Q_r} \left| \frac{\partial v}{\partial t} \right|^2 dx dt.
\]
It follows that \( Dv \) possesses the weak derivative
\[
\frac{\partial}{\partial t} Dv \in L^2(B_r \times (t_0 - r^2, t_1); \mathbb{R}^{nN}).
\]
As above, (2.8) is readily seen.

\[\square\]

**Remark.** A different method for proving the existence of weak \( t \)-derivative of weak solutions to a class of nonlinear parabolic systems has been developed in [6].

**Local higher integrability of \( Dv \).** We have

**Theorem 2.3.** There exists a \( q > 2 \) such that: for every weak solution \( v \in W_2^{1,0}(Q; \mathbb{R}^N) \) to (2.1) there holds
\[
|Dv| \in L^q(Q_{\rho}(x, s)) \quad \forall Q_{\rho}(y, s) \subset B_R \times (t_0 - R^2, t_0) \quad (5).
\]
In particular, there holds
\[(2.16) \quad \int_{Q_{R/2}} (1 + |Dv|^q) dx dt \leq c R^{(n+2)(1-q/2)} \left( \int_{Q_R} (1 + |Dv|^2) dx dt \right)^{q/2},
\]
where the constant \( c \) does not depend on \( R \).

\(5\) Recall \( Q_{\rho}(y, s) \) := \( B_\rho(y) \times (s - \rho^2, s) \).
The method of proving higher integrability of the gradient of weak
solutions to nonlinear elliptic systems has been developed by M. Giaquinta and
G. Modica. A proof of the analogous result for parabolic systems can be found
in [8]. Theorem 2.3 is a special case of the latter result.

**Fundamental estimate.** The following result is crucial for our proof of the
interior H"older continuity of weak solutions to (1.1).

**Theorem 2.4.** Let $n = 2$. There exists $\lambda \in (0, 1)$ such that: for every weak
solution $v \in L^4(Q_R; \mathbb{R}^N) \cap W^{1,0}_2(Q_R; \mathbb{R}^N)$ to (2.1), there holds

$$
(2.17) \quad \int_{Q_r} (1 + |v|^4 + |Dv|^2) \, dx \, dt \leq c \left( \frac{r}{R} \right)^{2+2\lambda} \int_{Q_r} (1 + |v|^4 + |Dv|^2) \, dx \, dt
$$

for all $r \in (0, R]$, where the constant $c$ is independent of both $r$ and $R$.

**Proof.** It suffices to prove (2.17) for all $r \in \left(0, \frac{R}{2}\right]$.

Let be $q > 2$ the power of integrability of $|Dv|$ obtained in Theorem 2.3.

Define

$$
p := \frac{8q}{3q + 2}, \quad \lambda := 1 - \frac{1}{p}.
$$

It follows

$$
2 < p < q, \quad \frac{1}{p} = \frac{1 - \theta}{2} + \frac{\theta}{q} \quad \text{with} \quad \theta = \frac{1}{4}, \quad 0 < \lambda < 1.
$$

Employing H"older's inequality, Theorem A.4 (with $E = B_{R/2}$, $a = t_0$, $b = t_0 - R^2$, $r = \frac{R}{2}$ there) and (2.8) (with $r = \frac{R}{2}$ there) we obtain for all

$$
(2.18) \quad \int_{Q_r} |Dv|^2 \, dx \, dt \leq |Q_r|^{1-2/p} \left( \int_{Q_r} |Dv|^p \, dx \, dt \right)^{2/p} \leq
$$

$$
\lesssim cr^{2+2\lambda} \, \text{ess sup} \, \left( \int_{B_{R/2}} |Dv|^p \, dx \, dt \right)^{2/p} \leq
$$

$$
\lesssim cr^{2+2\lambda} \left( R \|Dv\|_{L^2(t_0 - R^2/4, t_0; L^q(B_{R/2}))}^{1/2} \|\frac{\partial}{\partial t} Du\|_{L^2(t_0 - R^2/4, t_0; L^2(B_{R/2}))}^{3/2} + \frac{1}{R^2} \|Dv\|_{L^2(t_0 - R^2/4, t_0; L^q(B_{R/2}))}^{1/2} \|Dv\|_{L^2(t_0 - R^2/4, t_0; L^2(B_{R/2}))}^{3/2} \right) \leq
$$
\[
\leq \frac{c r^{2+2\lambda}}{R^2} \left( \int_{Q_r} (1 + |Dv|^2) \, dx \, dt \right)^{3/4} \|Dv\|_{L^2(t_0-R^2/4,t_0; L^6(B_{R/2}))}^{1/2},
\]

To estimate \( \|Dv\|_{L^2(t_0-R^2/4,t_0; L^6(B_{R/2}))} \) we make use of Hölder’s inequality and (2.16) to obtain

\[
\|Dv\|_{L^2(t_0-R^2/4,t_0; L^6(B_{R/2}))}^2 \leq \left( \frac{R^2}{4} \right)^{1-2/q} \left( \int_{Q_{R/2}} |Dv|^q \, dx \, dt \right)^{2/q} \leq c R^{2(2/q-1)} \int_{Q_R} (1 + |Dv|^2) \, dx \, dt.
\]

Inserting this estimate into (2.18) and observing that

\[
-2 + \frac{1}{2} \left( \frac{2}{q} - 1 \right) = -2 - 2\lambda,
\]

it follows

(2.19) \[
\int_{Q_r} |Dv|^2 \, dx \, dt \leq c \left( \frac{r}{R} \right)^{2+2\lambda} \int_{Q_R} (1 + |Dv|^2) \, dx \, dt.
\]

It remains to estimate the integral \( \int_{Q_r} |v|^4 \, dx \, dt \) for \( r \in (0, \frac{R}{2}] \). We do this for \( r \in (0, \frac{R}{4}] \) (the desired estimate for \( r \in (\frac{R}{4}, \frac{R}{2}] \) is readily seen). Let \( \lambda \) be as above. By Hölder’s inequality,

\[
\int_{Q_r} |v|^4 \, dx \, dt \leq c r^{2+2\lambda} \left( \int_{Q_{R/4}} |v|^{8/(1-\lambda)} \, dx \, dt \right)^{(1-\lambda)/2} \leq c r^{2+2\lambda} R^{1-\lambda} \text{ ess sup}_{(t_0-R^2/16,t_0)} \left( \int_{B_{R/4}} |v|^{8/(1-\lambda)} \, dx \right)^{(1-\lambda)/2}.
\]

Next, by Sobolev’s imbedding theorem,

\[
\left( \int_{B_{R/4}} |v(x,t)|^{8/(1-\lambda)} \, dx \right)^{(1-\lambda)/2} \leq c \left\{ R^{-3+\lambda} \left( \int_{B_{R/4}} |v(x,t)|^2 \, dx \right)^{2} + R^{1-\lambda} \left( \int_{B_{R/4}} |Dv(x,t)|^2 \, dx \right)^{2} \right\}
\]
for a.e. $t \in (t_0 - \frac{R^2}{16}, t_0)$. Here the constant $c$ does not depend on $R$. This can be established by a homothetical argument. It follows

$$
\int_{Q_r} |v|^4 \, dx \, dt \leq c \left( \frac{r}{R} \right)^{2+2\lambda} \left\{ \operatorname{ess sup}_{(t_0 - R^2/16, t_0)} \left( \int_{B_{R/4}} |v|^2 \, dx \right)^2 + \right.
$$

$$
+ R^4 \operatorname{ess sup}_{(t_0 - R^2/16, t_0)} \left( \int_{B_{R/4}} |Dv|^2 \, dx \right)^2 \right\}.
$$

We estimate the integrals on the right of (2.20). First, observing that $\frac{\partial v}{\partial t} \in L^2(Q_{R/4}; \mathbb{R}^N)$ (cf. Theorem 2.2), we find

$$
\int_{B_{R/4}} |v(x, t)|^2 \, dx \leq c \left( \frac{1}{R^2} \int_{Q_{R/4}} |v|^2 \, dx \, ds + R^2 \int_{Q_{R/4}} \left| \frac{\partial v}{\partial t} \right|^2 \, dx \, ds \right)
$$

for all $t \in \left[t_0 - \frac{R^2}{16}, t_0\right]$, where the constant $c$ depends neither on $t$ nor on $R$. Thus, by Hölder’s inequality and (2.7) (with $\frac{R}{2}$ in place of $R$ and $r = \frac{R}{4}$ there)

$$
\operatorname{ess sup}_{(t_0 - R^2/16, t_0)} \int_{B_{R/4}} |v|^2 \, dx \leq c \left\{ \int_{Q_R} |v|^4 \, dx \, dt + \left( \int_{Q_{R/2}} (1 + |Dv|^2) \, dx \, dt \right)^2 \right\}.
$$

Secondly, since $\frac{\partial}{\partial t} Dv \in L^2(Q_{R/4}; \mathbb{R}^{2N})$ (cf. Theorem 2.2), we obtain by (2.8) (with $\frac{R}{2}$ in place of $R$ and $r = \frac{R}{4}$ there)

$$
\operatorname{ess sup}_{(t_0 - R^2/16, t_0)} \left( \int_{B_{R/4}} |Dv|^2 \, dx \right)^2 \leq c \left( \frac{1}{R^2} \int_{Q_{R/4}} |Dv|^2 \, dx \, dt + R^2 \int_{Q_{R/4}} \left| \frac{\partial}{\partial t} Dv \right|^2 \, dx \, dt \right)^2 \leq \frac{c}{R^4} \left( \int_{Q_{R/2}} (1 + |Dv|^2) \, dx \, dt \right)^2.
$$
Inserting (2.21) and (2.22) into (2.20) gives

\begin{equation}
\int_{Q_r} |\nu|^4 \, dx \, dt \leq \leq c \left( \frac{r}{R} \right)^{2+2\alpha} \left\{ \int_{Q_R} (1 + |\nu|^4) \, dx \, dt + \left( \int_{Q_{R/2}} |D\nu|^2 \, dx \, dt \right)^2 \right\}
\end{equation}

for all \( r \in \left[ 0, \frac{R}{4} \right] \).

To conclude the proof, we employ the Caccioppoli inequality for weak solutions to (2.1) (cf. [2], [8]) and the Schwarz inequality. Thus

\[ \int_{Q_{R/2}} |D\nu|^2 \, dx \, dt \leq \frac{c}{R^2} \int_{Q_R} |\nu|^2 \, dx \, dt \leq c \left( \int_{Q_R} |\nu|^4 \, dx \, dt \right)^{1/2}. \]

Inserting this estimate into (2.23) and adding the resulting inequality to (2.19) gives (2.17). \( \square \)

3. Proof of the theorem.

We begin with noting the following technical

Lemma. Let \( \sigma : (0, R_0] \to [0, +\infty) \) be a nondecreasing function such that

\[ \sigma(r) \leq A \left[ \left( \frac{r}{R} \right)^\alpha + (2A)^{2\alpha/(\beta-\alpha)} \right] \sigma(R) + BR^\beta \]

for all \( 0 < r \leq R \leq R_0 \), where \( A, B \) and \( \alpha, \beta \) are (fixed) constants satisfying \( A > \frac{1}{2}, B \geq 0 \) and \( \alpha > \beta > 0 \). Then

\[ \sigma(r) \leq C (AR_0^{-\beta} \sigma(R_0) + B)r^\beta \quad \forall r \in (0, R_0], \]

where

\[ C := \max \left\{ \max \left\{ 2, \frac{1}{\tau_0^\beta} \right\}, \frac{1}{\tau_0^\beta} \max \left\{ 1, \frac{1}{\tau_0^\beta - (\alpha + \beta)/2} \right\} \right\}, \]

\[ \tau_0 := (2A)^{2/(\beta-\alpha)}. \]
A proof of this result can be found in [3].

Let \( u \in V^{1,0}_2(Q; \mathbb{R}^N) \) be any weak solution to (1.1). We divide the proof of the theorem into four steps.

1. Let \( Q_R = Q_R(x_0, t_0) = B_R(x_0) \times (t_0 - R^2, t_0) \) be any cylinder such that \( \overline{Q_R} \subset \Omega \times (0, T) \).

The following existence result is a straightforward generalization of [2], Lemma 2.XI, p. 98:

There exists exactly one \( w \in W^{1,0}_2(Q_R; \mathbb{R}^N) \) such that

\[
\begin{aligned}
- \int_{Q_R} w^i \frac{\partial \varphi^i}{\partial t} \, dx dt + \int_{Q_R} a^\alpha_i(x, t_0, Dw + Du) D_\alpha \varphi^i \, dx dt = \\
= \int_{Q_R} \left( -b_i(x, t, u, Du) \varphi^i + a^\alpha_i(x, t, Du) D_\alpha \varphi^i \right) \, dx dt,
\end{aligned}
\]

for all \( \varphi \in W^{1,1}_2(Q_R; \mathbb{R}^N) \) with \( \varphi = 0 \) a.e. on

\( (\partial B_R \times (t_0 - R^2, t_0)) \cup (B_R \times \{t_0\}) \),

(3.2) \[ w = 0 \text{ a.e. on } \partial B_R \times (t_0 - R^2, t_0). \]

Moreover, the function \( w \) possesses the following additional properties:

(3.3) \[ \text{ess sup}_{t \in (t_0 - R^2, t_0)} \int_{B_R} |w(x, t)|^2 \, dx < +\infty, \]

(3.4) \[ \frac{1}{2} \int_{B_R} |w(x, t)|^2 \, dx + \int_{t_0 - R^2}^t \int_{B_R} a^\alpha_i(x, t_0, Dw + Du) D_\alpha w^i \, dx ds \leq \\
\leq \int_{t_0 - R^2}^t \int_{B_R} \left( -b_i(x, s, u, Du) w^i + a^\alpha_i(x, s, Du) D_\alpha w^i \right) \, dx ds 
\]

for a.e. \( t \in (t_0 - R^2, t_0) \).

Observing (3.2) and (3.3), we have the well-known estimate

(3.5) \[ \left( \int_{Q_R} |w|^4 \, dx dt \right)^{1/2} \leq \]
\[ \leq C_0 \left( \text{ess sup}_{t \in (t_0 - R^2, t_0)} \int_{B_R} |w(x, t)|^2 \, dx + \int_{Q_R} |Dw|^2 \, dxdt \right), \]

where the constant \( C_0 \) does not depend on \( R \).

To proceed further, we note that (3.4) is equivalent to

\[ \frac{1}{2} \int_{B_R} |w(x, t)|^2 \, dx + \]

\[ + \int_{t_0 - R^2}^t \int_{B_R} \left[ a_i^\alpha(x, t_0, Du) - a_i^\alpha(x, t_0, Du) \right] \partial_x w^i \, dxds \leq \]

\[ \leq - \int_{t_0 - R^2}^t \int_{B_R} b_i(x, s, Du)w^i \, dxds + \]

\[ + \int_{t_0 - R^2}^t \int_{B_R} \left[ a_i^\alpha(x, s, Du) - a_i^\alpha(x, t_0, Du) \right] \partial_x w^i \, dxds \]

for a.e. \( t \in (t_0 - R^2, t_0) \). From this inequality we obtain by the aid of (1.3), (1.5) and (1.7)

\[ \frac{1}{2} \int_{B_R} |w(x, t)|^2 \, dx + \nu_0 \int_{t_0 - R^2}^t \int_{B_R} |Dw|^2 \, dxds \leq \]

\[ \leq c \int_{t_0 - R^2}^t \int_{B_R} \omega(|s - t_0|)(1 + |Du|)|Dw| \, dxds + \]

\[ + \int_{t_0 - R^2}^t \int_{B_R} (1 + |u|^3 + |Dw|^{3/2})|w| \, dxds \]

for a.e. \( t \in (t_0 - R^2, t_0) \), and therefore with the help of (3.5) by a routine argument

\[ \int_{Q_R} (|w|^4 + |Dw|^2) \, dxdt \leq \chi(R) \int_{Q_R} (1 + |u|^4 + |Du|^2) \, dxdt \]

with

\[ \chi(R) := c \left[ \omega(R) + \left( \int_{Q_R} (1 + |u|^4 + |Du|^2) \, dxdt \right)^{1/2} \right]^2 \cdot \left\{ 1 + \left[ \omega(R) + \left( \int_{Q_R} (1 + |u|^4 + |Du|^2) \, dxdt \right)^{1/2} \right] \int_{Q_R} (1 + |u|^4 + |Du|^2) \, dxdt \right\}^{-1}, \]
where the constant $c$ does not depend on $R$. Clearly, $\chi(R) \to 0$ as $R \to 0$.

2. Define $v := u + w$. Then $v \in V^{1,0}_2(Q_R; \mathbb{R}^N)$. By (3.1),

$$
- \int_{Q_R} v^i \frac{\partial \varphi^i}{\partial t} \, dx \, dt + \int_{Q_R} a^\alpha_i(x, t_0, Dv) D_\alpha \varphi^i \, dx \, dt = 
$$

$$
= - \int_{Q_R} u^i \frac{\partial \varphi^i}{\partial t} \, dx \, dt - \int_{Q_R} w^i \frac{\partial \varphi^i}{\partial t} \, dx \, dt + 
$$

$$
+ \int_{Q_R} a^\alpha_i(x, t_0, Du + Dw) D_\alpha \varphi^i \, dx \, dt = 0
$$

for all $\varphi \in W^{1,1}_2(Q_R; \mathbb{R}^N)$ with $\varphi = 0$ a.e. on $(\partial B_R \times (t_0 - R^2)) \cup (B_R \times \{t_0\})$. Here the functions $a^\alpha_i = a^\alpha_i(x, t_0, \xi)$ satisfy conditions (2.2)–(2.5). Then $v$ is a weak solution to (2.1) (with $a^\alpha_i(x, t_0, \xi)$ in place of $a^\alpha_i(x, \xi)$ there). Thus the fundamental estimate (2.17) holds.

3. Let $0 < r \leq R$. Observing that $u = v - w$, from (2.17) and (3.6) it follows

$$
\int_{Q_r} (1 + |u|^4 + |Du|^2) \, dx \, dt \leq 
$$

$$
\leq c \int_{Q_r} (1 + |v|^4 + |Dv|^2) \, dx \, dt + c \int_{Q_r} (1 + |w|^4 + |Dw|^2) \, dx \, dt \leq 
$$

$$
\leq c \left( \frac{r}{R} \right)^{2+2\lambda} \int_{Q_R} (1 + |v|^4 + |Dv|^2) \, dx \, dt + 
$$

$$
+ \chi(R) \int_{Q_R} (1 + |u|^4 + |Du|^2) \, dx \, dt \leq 
$$

$$
\leq c \left[ \left( \frac{r}{R} \right)^{2+2\lambda} + \chi(R) \right] \int_{Q_R} (1 + |u|^4 + |Du|^2) \, dx \, dt.
$$

Since $\chi(R) \to 0$ as $R \to 0$, for each $0 < \mu < \lambda$ there exists $R_0 > 0$ such that the above technical lemma applies (with $\lambda$ in place of $\alpha$ and $\mu$ in place of $\beta$). We obtain

$$
(3.7) \quad \int_{Q_r} (1 + |u|^4 + |Du|^2) \, dx \, dt \leq c \left( \frac{r}{R_0} \right)^{2+2\mu} \int_{Q_{r_0}} (1 + |u|^4 + |Du|^2) \, dx \, dt
$$

for all $0 < r \leq R_0$. 

4. In [8], the following Poincaré inequality has been established:

\[
\int_{Q_r} |u - u_{Q_r}|^2 \, dx \, dt \leq cr^2 \int_{Q_r} (1 + \|u\|^4 + \|Du\|^2) \, dx \, dt,
\]

where \( u \) is any weak solution to (1.1) and

\[
u_{Q_r} = \frac{1}{|Q_r|} \int_{Q_r} u(y, s) \, dy \, ds
\]

(\( \overline{Q}_r \subset \Omega \times (0, T] \), \( c = \text{const independent of } r \)). Inserting (3.7) into (3.8) we obtain

\[
\int_{Q_r} |u - u_{Q_r}|^2 \, dx \, dt \leq c \left( \frac{r}{R_0} \right)^{4+2\mu} \int_{Q_{R_0}} (1 + \|u\|^4 + \|Du\|^2) \, dx \, dt
\]

for all \( 0 < r \leq R_0 \). Here the constants \( c \) and \( R_0 \) are independent of \( r \). The constant \( c \) depends only on the constants in (1.3), (1.4), (1.5) and (1.7), while \( R_0 \) depends on these constants and on \( t' \in (0, T') \) and \( \text{dist}(\Omega', \partial \Omega) \), too, where \( \Omega' \subset \Omega \) (both \( c \) and \( R_0 \) are independent of \( \{x_0, t_0\} \in \Omega' \times (t', T') \) \( 0 < R_0 < \min\{\text{dist}(\Omega', \partial \Omega), \sqrt{T - t'}\} \)).

From (3.9) it finally follows that \( u \) is Hölder continuous on any subcylinder \( \Omega' \times (t', T) \) (cf. [1]). □

4. Appendix.

For the reader’s convenience, we note some abstract results which have been applied in Sect. 2. Their proofs may be found in [11].

Let \( X \) be a normed space with norm \( \| \cdot \|_X \). Let \( a, b \in \mathbb{R}, a < b \). For \( u \in L^1(a, b; X) \), define

\[
u_{t_0, \rho} = \frac{1}{\rho} \int_{t_0-\rho}^{t_0} u(\sigma) \, d\sigma, \quad t_0 \in (a, b], \quad \rho \in (0, t_0 - a).
\]

**A1.** Let \( u \in L^2(a, b; X) \). Assume there exists \( \delta \in (0, 1) \) such that

\[
\int_{t_0-\rho}^{t_0} \|u(t) - u_{t_0, \rho}\|^2_X \, dt \leq K_0 \rho^{1+2\delta}
\]

for all \( t_0 \in (a, b) \) and all \( \rho \in (0, t_0 - a) \) (\( K_0 = \text{const independent of both } t_0 \) and \( \rho \)).
Then there exists a constant \( c > 0 \) such that

\[
\|u\|_{C^\delta([a,b]; X)} := \max_{t \in [a,b]} \|u(t)\|_X + \sup_{\substack{s,t \in [a,b] \setminus \{a\} \atop s \neq t}} \frac{\|u(s) - u(t)\|_X}{|s - t|^\delta} \leq c \sqrt{K_0}.
\]

A2. Let \( u \in L^2(a,b; X) \) have the distributional derivative \( u' \in L^2(a,b; X) \). Then

\[
\int_{t_0-\rho}^{t_0} \|u(t) - u_{t_0,\rho}\|_X^2 \, dt \leq \rho^2 \int_{t_0-\rho}^{t_0} \|u'(\sigma)\|_X^2 \, d\sigma
\]

for all \( t_0 \in (a,b) \) and all \( \rho \in (0, t_0 - a) \).

Next, let \( Y \) be a normed space with norm \( \| \cdot \|_Y \). Suppose \( X \subset Y \) continuously;

\[
\text{for every } \theta \in (0, 1) \text{ there is a normed space } X_\theta \text{ with normal } \| \cdot \|_{X_\theta} \text{ such that } X \subset X_\theta \subset Y \text{ and } \|z\|_{X_\theta} \leq c \|z\|_X^\theta \|z\|_Y^{1-\theta} \forall z \in Y (c = \text{const}).
\]

A3. Let \( u \in L^2(a,b; X) \) have the distributional derivative \( u' \in L^2(a,b; Y) \). Let \( a_1 \in (a,b) \), \( \theta \in (0, \frac{1}{2}) \).

Then there exists a constant \( c > 0 \) such that

\[
\text{ess sup}_{(t_0-r^2,t_0)} \|u(t)\|_{X_\theta}^2 \leq c r^{2(1-2\theta)} \|u\|_{L^2(t_0-r^2,t_0; X)}^{2\theta} \|u'\|_{L^2(t_0-r^2,t_0; Y)}^{2(1-\theta)} +
\]

\[
+ \frac{c}{r^2} \|u\|_{L^2(t_0-r^2,t_0; X)}^{2\theta} \|u\|_{L^2(t_0-r^2,t_0; Y)}^{2(1-\theta)}
\]

for all \( t_0 \in (a_1, b) \) and all \( r \in (0, \sqrt{t_0 - a_1}) \).

The latter statement will be applied as follows. Let \( E \subset \mathbb{R}^n \) be measurable. Let \( 1 \leq p_0 < p_1 < +\infty \) and \( 0 < \theta < 1 \). Define

\[
\frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1}.
\]

By Hölder’s inequality,

\[
\|f\|_{L^p(E)} \leq \|f\|_{L^{p_0}(E)}^{1-\theta} \|f\|_{L^{p_1}(E)}^\theta \forall f \in L^{p_0}(E) \cap L^{p_1}(E).
\]

To be more specific, let \(|E| (=\text{Lebesgue measure}) < +\infty\), and put \( p_0 = 2, p_1 = q > 2 \) and \( \theta = \frac{1}{4} \). Then from A3 we conclude
A4. Let \( u \in L^2(a, b; L^q(E)) \) have the distributional derivative

\[ u' \in L^2(a, b; L^2(E)). \]

Let \( a_1 \in (a, b) \).

Then there exists a constant \( c > 0 \) such that

\[
\operatorname{ess} \sup_{(t_0-r^2, t_0)} \| u(t) \|_{L^p(E)}^{2/3} \leq c r \| u \|_{L^2(t_0-r^2, t_0; L^q(E))}^{1/2} \| u' \|_{L^2(t_0-r^2, t_0; L^2(E))}^{3/2} + \frac{c}{r^2} \| u \|_{L^2(t_0-r^2, t_0; L^q(E))}^{1/2} \| u \|_{L^2(t_0-r^2, t_0; L^2(E))}^{3/2}
\]

for all \( t_0 \in (a_1, b) \) and all \( r \in (0, \sqrt{t_0 - a_1}) \).

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