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HÖLDER CONTINUITY OF WEAK SOLUTIONS TO PARABOLIC SYSTEMS WITH CONTROLLED GROWTH NON-LINEARITIES (TWO SPATIAL DIMENSIONS)

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Dedicated to Professor Sergio Campanato on his 70th birthday

1. Introduction. Statement of the main result.

Let $\Omega \subset \mathbb{R}^2$ be open, let $0 < T < +\infty$ and set $Q = \Omega \times (0, T)$. In Q we consider the following system of nonlinear PDE's:

$$\frac{\partial u^i}{\partial t} - D_\alpha a_i^\alpha(x, t, Du) = b_i(x, t, u, Du) \quad (i = 1, \dots, N), \ (^1) \tag{1.1}$$

where

$$u = \{u^1, \dots, u^N\} \quad (N \geq 2),$$
$$D_\alpha = \frac{\partial}{\partial x_\alpha} \quad (\alpha = 1, 2)$$
$$Du = \{D_\alpha u^i\} \quad (= \text{matrix of spatial derivatives}).$$

The conditions on the functions $a_i^\alpha : \Omega \times (0, T) \times \mathbb{R}^{2N} \to \mathbb{R}$ are as follows:

$$x \mapsto a_i^\alpha(x, t, \xi) \ \textit{is measurable on} \ \Omega \ \forall \{t, \xi\} \in (0, T) \times \mathbb{R}^{2N}, \tag{1.2}$$

(1) With the exception of Section 2, throughout the paper, a repeated Greek (resp. Latin) index implies summation over 1 and 2 $(1, \dots, N)$.

$$(1.3)\quad \begin{cases} |a_i^\alpha(x,s,\eta) - a_i^\alpha(x,t,\xi)| \le \omega(|s-t|)(1+|\eta|+|\xi|) + c_0|\eta-\xi| \\ \forall x \in \Omega, \forall \{s,\eta\}, \{t,\xi\} \in (0,T)\times\mathbb{R}^{2N}, \\ \textit{where } \omega : [0,+\infty) \to (0,+\infty) \textit{ is bounded, nondecreasing} \\ \textit{with } \lim_{h\to 0}\omega(h) = 0, \textit{ and } c_0 = \text{const}; \end{cases}$$

$$(1.4)\quad |a_i^\alpha(x,t,\xi)| \le c_1(1+|\xi|)\ \forall \{x,t,\xi\} \in \Omega\times(0,T)\times\mathbb{R}^{2N}\ (c_1 = \text{const})$$

($\alpha = 1,2;\ i = 1,\dots,N$), and

$$(1.5)\quad \begin{cases} (a_i^\alpha(x,t,\eta) - a_i^\alpha(x,t,\xi))(\eta_\alpha^i - \xi_\alpha^i) \ge \nu_0|\eta-\xi|^2 \\ \forall \{x,t\} \in \Omega\times(0,T), \forall \eta,\xi \in \mathbb{R}^{2N} \quad (\nu_0 = \text{const} > 0). \end{cases}$$

The functions b_i are assumed to satisfy the following conditions:

$$(1.6)\quad \begin{cases} \{x,t\} \mapsto b_i(x,t,u,\xi) \textit{ is measurable on } \Omega\times(0,T) \\ \forall \{u,\xi\} \in \mathbb{R}^N\times\mathbb{R}^{2N}; \\ \{u,\xi\} \mapsto b_i(x,t,u,\xi) \textit{ is continuous on } \mathbb{R}^N\times\mathbb{R}^{2N} \\ \forall \{x,t\} \in \Omega\times(0,T); \end{cases}$$

$$(1.7)\quad \begin{cases} \textit{controlled growth:} \\ |b_i(x,t,u,\xi)| \le c_2(1+|u|^3+|\xi|^{3/2})\ \forall \{x,t,u,\xi\} \in \\ \in \Omega\times(0,T)\times\mathbb{R}^N\times\mathbb{R}^{2N}\quad (i = 1,\dots,N;\ c_2 = \text{const}). \end{cases}$$

In the present paper, we consider weak solutions u to (1.1) regardless of whether u satisfies any boundary and (or) initial conditions. Our goal is to study the interior Hölder continuity of these solutions.

To this end, define

$$W_2^{1,0}(Q;\mathbb{R}^N) = \left\{u \in L^2(Q;\mathbb{R}^N) \,\middle|\, \frac{\partial u}{\partial x_\alpha} \in L^2(Q;\mathbb{R}^N);\ \alpha = 1,2\right\},$$

$$W_2^{1,1}(Q;\mathbb{R}^N) = \left\{u \in W_2^{1,0}(Q;\mathbb{R}^N) \,\middle|\, \frac{\partial u}{\partial t} \in L^2(Q;\mathbb{R}^N)\right\} =$$

$$= W_2^1(Q;\mathbb{R}^N)\ \text{(the usual Sobolev space on } Q\text{)},$$

$$V_2^{1,0}(Q;\mathbb{R}^N) = \left\{u \in W_2^{1,0}(Q;\mathbb{R}^N) \,\middle|\, \operatorname*{ess\,sup}_{t\in(0,T)} \int_\Omega |u(x,t)|^2\,dx < +\infty\right\}.$$

We now introduce the notion of weak solution to (1.1).

Definition. *Let* (1.2), (1.4) *and* (1.6), (1.7) *be satisfied. The vector valued function* $u \in V_2^{1,0}(Q;\mathbb{R}^N)$ *is called weak solution to* (1.1) *if*

$$(1.8)\qquad \begin{cases} -\displaystyle\int_Q u^i \frac{\partial \varphi^i}{\partial t}\,dxdt + \int_Q a_i^\alpha(x,t,Du) D_\alpha \varphi^i\,dxdt = \\ \quad = \displaystyle\int_Q b_i(x,t,u,Du)\varphi^i\,dxdt \\ \textit{for all } \varphi \in W_2^{1,1}(Q;\mathbb{R}^N) \textit{ with } \operatorname{supp}(\varphi) \subset Q. \end{cases}$$

The main result of our paper is following

Theorem. *Let* (1.2)–(1.5) *and* (1.6), (1.7) *be satisfied. Then there exists* $\mu \in (0,1)$ *such that: for every weak solution* $u \in V_2^{1,0}(Q;\mathbb{R}^N)$ *to* (1.1) *there holds*

$$u \in C^{\mu,\mu/2}(Q;\mathbb{R}^N)\ (^2).$$

The interior Hölder continuity of weak solution $u \in V_2^{1,0}(Q;\mathbb{R}^N)$ $(n=2)$ to (1.1) has been proved in [2], Theorem 7.II, p. 112, under the following more restrictive conditions: uniform continuity of the functions $x \mapsto a_i^\alpha(x,t,\xi)$, continuous differentability of the functions $\xi \mapsto a_i^\alpha(x,t,\xi)$ and strictly controlled growth on b_i, i.e.

$$|b_i(x,t,u,\xi)| \le c(1+|u|^\beta + |\xi|^\gamma)\ \forall \{x,t,u,\xi\} \in \Omega \times (0,T) \times \mathbb{R}^N \times \mathbb{R}^{2N},$$

where

$$1 \le \beta \le 3\,,\ 1 \le \gamma < \frac{3}{2}$$

$(i = 1,\dots,N;\ c = \text{const})$. Our above result thus sharpens [2], Theorem 7.II, p. 112, and moreover it can be viewed as the "parabolic analogue" of the following well-known result: every weak solution to a nonlinear uniformly elliptic system in two dimensions with measurable coefficients $x \mapsto a_i^\alpha(x,\xi)$ is Hölder continuous in the interior (³). This follows merely from the higher integrability of the gradient of the weak solution under consideration and Sobolev's imbedding theorem (cf. [3], [4] for details).

(²) That is, for every bounded open set Q' such that $\overline{Q'} \subset Q$, there holds $|u(x,t) - u(y,t)| \le c(|x-y|^\mu + |s-t|^{\mu/2})$ for all $\{x,s\}, \{y,t\} \in Q'$, where the constant c may depend on $\operatorname{dist}(Q', \partial Q)$.

(³) Note that this result in fact holds for such systems with coefficients $a_i^\alpha(x,u,\xi)$.

The interior Hölder continuity of weak solutions to nonlinear parabolic systems with coefficients $a_i^\alpha(\xi)$ and $b_i \equiv 0$ has been proved in [10], and with coefficients $a_i^\alpha(x, t, u, \xi)$ in [5], [9] ($b_i \equiv 0$) and in [7] ($b_i = f_i(x,t)$).

The aim of the present paper is to simplify the discussion in [11]. In comparison with [2], the novelty in [11] lies in the use of the interior t-differentiability of weak solutions to nonlinear parabolic systems and an interpolation inequality. Our paper is organized as follows. In Section 2 we prove the interior t-differentiability of weak solutions of a class of nonlinear parabolic systems. Then we establish a fundamental inequality for weak solutions to these systems. Section 3 is devoted to the proof of our main result. Here we make use of a generalization of an existence result from [2].

2. Interior estimates on weak solutions to a class of nonlinear parabolic systems.

Let $x_0 \in \mathbb{R}^n$ ($n \geq 1$) and $t_0 \in \mathbb{R}$ be arbitrary, but fixed. Given $r > 0$, define

$$B_r = B_r(x_0) = \{x \in \mathbb{R}^n \mid |x - x_0| < r\},$$

$$Q_r = Q_r(x_0, t_0) = B_r(x_0) \times (t_0 - r^2, t_0).$$

Let $R > 0$ be fixed. In the cylinder $Q_R = Q_R(x_0, t_0)$ we consider the following system of PDE's:

$$\frac{\partial v^i}{\partial t} - D_\alpha a_i^\alpha(x, Dv) = 0 \quad (i = 1, \dots, N)\,(^4). \tag{2.1}$$

The conditions on the functions a_i^α are as follows:

$$x \mapsto a_i^\alpha(x, \xi) \text{ is measurable on } B_R \quad \forall \xi \in \mathbb{R}^{nN}, \tag{2.2}$$

$$|a_i^\alpha(x, \xi)| \leq c_0(1 + |\xi|) \quad \forall x \in B_R, \quad \forall \xi \in \mathbb{R}^{nN} \ (c_0 = \text{const}), \tag{2.3}$$

$$\begin{cases} |a_i^\alpha(x, \eta) - a_i^\alpha(x, \xi)| \leq c_1 |\eta - \xi| \\ \forall x \in B_R, \quad \forall \eta, \xi \in \mathbb{R}^{nN} \quad (c_1 = \text{const}) \end{cases} \tag{2.4}$$

($\alpha = 1, \dots, n$; $i = 1, \dots, N$), and

$$\begin{cases} (a_i^\alpha(x, \eta) - a_i^\alpha(x, \xi))(\eta_\alpha^i - \xi_\alpha^i) \geq \nu_0 |\eta - \xi|^2 \\ \forall x \in B_R, \ \forall \eta, \xi \in \mathbb{R}^{nN} \quad (\nu_0 = \text{const} > 0). \end{cases} \tag{2.5}$$

We introduce

(^{4}) Unless otherwise stated, in the present section a repeated Greek index implies summation over $1, \dots, n$.

Definition 2.1. *Let* (2.2) *and* (2.3) *be satisfied. The function* $v \in W_2^{1,0}(Q_R; \mathbb{R}^N)$ *is called a weak solution to* (2.1) *if*

$$\begin{cases} -\int_{Q_R} v^i \frac{\partial \varphi^i}{\partial t}\, dxdt + \int_{Q_R} a_i^\alpha(x, Dv) D_\alpha \varphi^i\, dxdt = 0 \\ \forall \varphi \in W_2^{1,1}(Q_R; \mathbb{R}^N) \text{ with } \varphi = 0 \text{ a.e. on } \partial Q_R. \end{cases} \tag{2.6}$$

Without any further reference, conditions (2.2)–(2.5) are now assumed to hold throughout the present section.

Interior differentiability of weak solutions to (2.1). The following result on the interior t-differentiability may be of interest in its own right. Our method of proof differs substantially from that of [2], Theorem 3.I, p. 100.

Theorem 2.2. *Let* $v \in W_2^{1,0}(Q_R; \mathbb{R}^N)$ *be a weak solution to* (2.1). *Then*

$$\int_{Q_r} \left|\frac{\partial v}{\partial t}\right|^2 dxdt \le \frac{c}{(R-r)^2} \int_{Q_R} (1 + |Dv|^2)\, dxdt, \tag{2.7}$$

$$\int_{Q_r} \left|\frac{\partial}{\partial t} Dv\right|^2 dxdt \le \frac{c}{(R-r)^4} \int_{Q_R} (1 + |Dv|^2)\, dxdt, \tag{2.8}$$

for all $0 < r < R$, *where the constants* c *depend neither on* r *nor on* R.

Before turning to the proof we present two technical tools.

1) Let $f \in L^p(Q_r)$ $(1 \le p < +\infty)$. Let $t_1 \in (t_0 - R^2, t_0)$. For $\lambda \in (0, t_0 - t_1)$, define the Steklov mean

$$f_\lambda(x, t) = \frac{1}{\lambda} \int_t^{t+\lambda} f(x, s) \mathrm{d}s \quad \text{for a.e. } \{x, t\} \in B_R \times (t_0 - R^2, t_1).$$

Then

$$\int_{t_0 - R^2}^{t_1} \int_{B_R} |f_\lambda|^p\, dxdt \le \int_{Q_R} |f|^p\, dxdt \quad \forall \lambda \in (0, t_0 - t_1), \tag{2.9}$$

$$f_\lambda \to f \text{ in } L^p(B_R \times (t_0 - R^2, t_1)) \text{ as } \lambda \to 0 \tag{2.10}$$

and

$$\frac{\partial f_\lambda}{\partial t} = \frac{1}{\lambda} \left(f(x, t + \lambda) - f(x, t)\right) \tag{2.11}$$

for a.e. $\{x,t\} \in B_R \times (t_0 - R^2, t_1)$ and all $\lambda \in (0, t_0 - t_1)$.

Let $v \in W_2^{1,0}(Q_R; \mathbb{R}^N)$ be a weak solution to (2.1). Then there holds

$$\text{(2.12)}\quad \begin{cases} \displaystyle\int_{B_R} \frac{\partial u_\lambda^i}{\partial t}(x,t)\psi^i(x)\,dx + \int_{B_R} (a_i^\alpha(x, Dv))_\lambda(t) D_\alpha \psi^i(x)\,dx = 0 \\ \text{for a.e. } t \in (t_0 - R^2, t_1), \text{ for all } \lambda \in (0, t_0 - t_1) \\ \text{and all } \psi \in W_2^1(B_R; \mathbb{R}^N) \text{ with } \psi = 0 \text{ a.e. on } \partial B_R \end{cases}$$

(cf. [8], [9]).

The integral identity in (2.12) forms the basis for deriving estimates on t-differences of v and Dv. These estimates will provide (2.7) and (2.8).

2) We need the following technical

Lemma. *Let σ be a nonnegative bounded function on the interval $[a,b]$ $(-\infty < a < b < +\infty)$. Assume that*

$$\sigma(r) \le \frac{A}{(R-r)^\theta} + B + \frac{1}{2}\sigma(R)$$

for all r, R with $a < r < R \le b$, where A, B and θ are (fixed) nonnegative constants.

Then there exists a constant $C = C(\theta)$ such that

$$\sigma(r) \le C\Big(\frac{A}{(R-r)^\theta} + B\Big) \quad \forall a < r < R \le b.$$

A proof of this result may be found in [4]. □

Proof of Theorem 2.2. Define the t-difference of a function $f = f(x,t)$ by

$$\Delta_h f(x,t) = f(x, t+h) - f(x,t), \qquad h > 0.$$

Let $0 < r < R$. Let $\zeta \in C_c^\infty(B_R)$ be a cut-off function such that $\zeta(x) = 1$ for all $x \in B_r$, $0 \le \zeta(x) \le 1$ and $|D\zeta(x)| \le \dfrac{c_0}{R-r}$ for all $x \in B_R$ ($c_0 = \text{const}$), and let $\tau \in C^\infty(\mathbb{R})$ be a function satisfying $\tau(t) = 0$ for all $t \in (-\infty, t_0 - R^2]$, $\tau(t) = 1$ for all $t \in [t_0 - r^2, +\infty)$ and $0 \le \tau(t) \le 1$, $0 \le \tau'(t) \le \dfrac{c_0}{(R-r)^2}$ for all $t \in \mathbb{R}$.

Let $t_1 \in (t_0 - \tau^2, t_0)$ be arbitrary. The function

$$\psi(x) = (\Delta_h v)(x,t)\zeta^2(x) r^2(t), \quad x \in B_R,\ t \in (t_0 - R^2, t_1),\ h \in (0, t_0 - t_1)$$

is admissible in (2.12) (with $\lambda = h$). By (2.11) (with $\lambda = h$), $\frac{\partial v_h}{\partial t}(x,t) = \frac{1}{h}(\Delta_h v)(x,t)$ for a.e. $\{x,t\} \in B_R \times (t_0 - R^2, t_1)$. Integrating the integral identity (2.12) over the interval $(t_0 - R^2, T_1)$ and using (2.3) gives

$$\int_{t_0-R^2}^{t_1}\int_{B_R} |\Delta_h v(x,t)|^2 \zeta^2(x)\tau^2(t)\, dxdt =$$

$$= -h \int_{t_0-R^2}^{t_1}\int_{B_R} (a_i^\alpha(x, Dv))_h(t)\Big[(\Delta_h D_\alpha v^i(x,t))\zeta^2(x)\tau^2(t) + $$
$$+ 2(\Delta_h v^i(x,t))\zeta(x) D_\alpha \zeta(x)\tau^2(t)\Big]\, dxdt \le$$

$$\le ch \int_{t_0-R^2}^{t_1}\int_{B_R} (1 + |Dv(x,\cdot)|)_h(t)\Big(|\Delta_h Dv(x,t)|\zeta^2(x)\tau^2(t) + $$
$$+ |\Delta_h(x,t)||\zeta(x)||D\zeta(x)|\tau^2(t)\Big)\, dxdt.$$

Observing (2.9) and employing Young's inequality we obtain for all $\epsilon > 0$

$$\int_{t_0-R^2}^{t_1}\int_{B_R} |\Delta_h v|^2 \zeta^2 \tau^2\, dxdt \le \tag{2.13}$$

$$\le \varepsilon \int_{t_0-R^2}^{t_1}\int_{B_R} \Big(|\Delta_h Dv|^2 \zeta^4\tau^4 + |\Delta_h v|^2 |D\zeta|^2 \zeta^2 \tau^4\Big)\, dxdt +$$
$$+ \frac{ch}{\varepsilon}\int_{Q_R} (1 + |Dv|^2)\, dxdt\,.$$

Here the constant c is independent of r, R, h and ε.

Next, as above let $h \in (0, t_0 - t_1)$. We consider the integral identity in (2.12) for $\lambda \in (0, t_0 - t_1 - h)$ and form the t-difference Δ_h therein. Observing that $\Delta_h v_\lambda = (\Delta_h v)_\lambda$ and $\Delta_h \frac{\partial v_\lambda}{\partial t} = \frac{\partial}{\partial t}(\Delta_h v)$ we obtain

$$\int_{B_R} \frac{\partial}{\partial t}(\Delta_h v^i(x,\cdot))_\lambda(t)\psi^i(x)\, dx + \int_{B_R} [\Delta_h a_i^\alpha(x, Dv)]_\lambda(t) D_\alpha \psi^i(x)\, dx = 0$$

for a.e. $t \in (t_0 - R^2, t_1)$. Here we insert $\psi(x) = (\Delta_h v(x,\cdot))_\lambda(t)\zeta^2(x)\tau^2(t)$, where ζ and τ are cut-off functions as above. Since

$$\int_{B_R} \Big[\frac{\partial}{\partial t}(\Delta_h v^i(x,\cdot))_\lambda(t)\Big](\Delta_h v^i(x,\cdot))_\lambda \zeta^2(x)\tau^2(t)\, dx =$$

$$= \frac{1}{2}\frac{d}{dt}\int_{B_R} |(\Delta_h v(x,\cdot))_\lambda|^2 \zeta^2(x)\tau^2(t)\,dx -$$

$$- \int_{B_R} |(\Delta_h v(x,\cdot))_\lambda|^2 \zeta^2(x)\tau(t)\tau'(t)\,dx$$

for a.e. $t \in (t_0 - R^2, t_1)$, it follows by integration over the interval $(t_0 - R^2, t_1)$ that

$$\int_{t_0-R^2}^{t_1}\int_{B_R} [\Delta_h a_i^\alpha(x, Dv)]_\lambda(t)(\Delta_h D_\alpha v^i(x,\cdot))_\lambda(t)\zeta^2(x)\tau^2(t)\,dxdt \leq$$

$$\leq 2\int_{t_0-R^2}^{t_1}\int_{B_R} [\Delta_h a_i^\alpha(x, Dv)]_\lambda(t)(\Delta_h v^i(x,\cdot))_\lambda \zeta(x) D_\alpha\zeta(x)\tau^2(t)\,dxdt +$$

$$+ \int_{t_0-R^2}^{t_1}\int_{B_R} |(\Delta_h v^i(x,\cdot))_\lambda|^2 \zeta^2(x)\tau(t)\tau'(t)\,dxdt\,.$$

Letting tend $\lambda \to 0$ (cf. (2.10)) and using then (2.4) and (2.5) gives

$$\int_{t_0-R^2}^{t_1}\int_{B_R} |\Delta_h Dv(x,t)|^2 \zeta^2(x)\tau^2(t)\,dxdt \leq \tag{2.14}$$

$$\leq c\int_{t_0-R^2}^{t_1}\int_{B_R} |\Delta_h v(x,t)|^2 (|D\zeta(x)|^2\tau(t) + \zeta^2(x)\tau(t)\tau'(t))\,dxdt \leq$$

$$\leq \frac{c}{(R-r)^2}\int_{t_0-R^2}^{t_1}\int_{B_R} |\Delta_h v(x,t)|^2\,dxdt,$$

the constant c being independent of r, R and h.

We insert this estimate into the right hand side of (2.13) to obtain

$$\int_{t_0-r^2}^{t_1}\int_{B_r} |\Delta_h v|^2\,dxdt \leq$$

$$\leq \frac{c\varepsilon}{(R-r)^2}\int_{t_0-R^2}^{t_1}\int_{B_R} |\Delta_h v|^2\,dxdt + \frac{ch^2}{\varepsilon}\int_{Q_R} (1 + |Dv|^2)\,dxdt\,.$$

Choosing $\varepsilon = \dfrac{(R-r)^2}{2c}$ and employing the above technical lemma with

$$\sigma(r) = \int_{t_0-r^2}^{t_1}\int_{B_r} |\Delta_h v|^2\,dxdt, \quad 0 < r \leq R,$$

we find

$$\int_{t_0-r^2}^{t_1}\int_{B_r} |\Delta_h v|^2\,dxdt \le \frac{ch^2}{(R-r)^2}\int_{Q_R}(1+|Dv|^2)\,dxdt \tag{2.15}$$

for all $h \in (0, t_0 - t_1)$. By a standard argument, (2.15) implies the existence of the weak derivate $\frac{\partial v}{\partial t} \in L^2(B_r \times (t_0 - r^2, t_1); \mathbb{R}^N)$, which satisfies

$$\int_{t_0-r^2}^{t_1}\int_{B_r} \left|\frac{\partial v}{\partial t}\right|^2 dxdt \le \frac{c}{(R-r)^2}\int_{Q_R}(1+|Dv|^2)\,dxdt.$$

Hence $\frac{\partial v}{\partial t}$ is defined a.e. on $B_r \times (t_0 - r^2, t_0)$ and measurable. Taking into account the monotone convergence theorem, we may let tend $t_1 \to t_0$ in the latter inequality and obtain (2.7).

Finally, dividing (2.14) by h^2 (where $h \in (0, t_0 - t_1)$) gives

$$\int_{t_0-r^2}^{t_1}\int_{B_r}\left|\frac{1}{h}\Delta_h Dv\right|^2 dxdt \le \frac{c}{(R-r)^2}\int_{t_0-R^2}^{t_1}\int_{B_R}\left|\frac{1}{h}\Delta_h v\right|^2 dxdt \le$$

$$\le \frac{c}{(R-r)^2}\int_{Q_R}\left|\frac{\partial v}{\partial t}\right|^2 dxdt.$$

It follows that Dv possesses the weak derivative

$$\frac{\partial}{\partial t}Dv \in L^2(B_r \times (t_0 - r^2, t_1); \mathbb{R}^{nN}).$$

As above, (2.8) is readily seen. □

Remark. A different method for proving the existence of weak t-derivative of weak solutions to a class of nonlinear parabolic systems has been developed in [6].

Local higher integrability of Dv. We have

Theorem 2.3. *There exists a $q > 2$ such that: for every weak solution $v \in W_2^{1,0}(Q; \mathbb{R}^N)$ to* (2.1) *there holds*

$$|Dv| \in L^q(Q_\rho(x,s)) \quad \forall\, \overline{Q_\rho(y,s)} \subset B_R \times (t_0 - R^2, t_0]\ (^5).$$

In particular, there holds

$$\int_{Q_{R/2}}(1+|Dv|^q)\,dxdt \le cR^{(n+2)(1-q/2)}\left(\int_{Q_R}(1+|Dv|^2)\,dxdt\right)^{q/2}, \tag{2.16}$$

where the constant c does not depend on R.

(^{5}) Recall $Q_\rho(y,s) := B_\rho(y) \times (s - \rho^2, s)$.

The method of proving higher integrability of the gradient of weak solutions to nonlinear elliptic systems has been developed by M. Giaquinta and G. Modica. A proof of the analogous result for parabolic systems can be found in [8]. Theorem 2.3 is a special case of the latter result.

Fundamental estimate. The following result is crucial for our proof of the interior Hölder continuity of weak solutions to (1.1).

Theorem 2.4. *Let $n = 2$. There exists $\lambda \in (0, 1)$ such that: for every weak solution $v \in L^4(Q_R; \mathbb{R}^N) \cap W_2^{1,0}(Q_R; \mathbb{R}^N)$ to (2.1), there holds*

$$\int_{Q_r} (1 + |v|^4 + |Dv|^2)\, dxdt \le c\Big(\frac{r}{R}\Big)^{2+2\lambda} \int_{Q_R} (1 + |v|^4 + |Dv|^2)\, dxdt \tag{2.17}$$

for all $r \in (0, R]$, where the constant c is independent of both r and R.

Proof. It suffices to prove (2.17) for all $r \in \left(0, \frac{R}{2}\right]$.

Let be $q > 2$ the power of integrability of $|Dv|$ obtained in Theorem 2.3. Define

$$p := \frac{8q}{3q+2}, \quad \lambda := 1 - \frac{1}{p}.$$

It follows

$$2 < p < q\,, \ \frac{1}{p} = \frac{1-\theta}{2} + \frac{\theta}{q} \quad \text{with} \quad \theta = \frac{1}{4}\,, \ 0 < \lambda > 1.$$

Employing Hölder's inequality, Theorem A.4 (with $E = B_{R/2}$, $a = t_0$, $b = t_0 - R^2$, $r = \frac{R}{2}$ there) and (2.8) (with $r = \frac{R}{2}$ there) we obtain for all $r \in (0, \frac{R}{2}]$

$$\int_{Q_r} |Dv|^2\, dxdt \le |Q_r|^{1-2/p} \Big(\int_{Q_r} |Dv|^p\, dxdt \Big)^{2/p} \le \tag{2.18}$$

$$\le cr^{2+2\lambda} \operatorname*{ess\,sup}_{(t_0-R^2/4, t_0)} \Big(\int_{B_{R/2}} |Dv|^p\, dxdt \Big)^{2/p} \le$$

$$\le cr^{2+2\lambda} \Big(R \|Dv\|^{1/2}_{L^2(t_0-R^2/4, t_0; L^q(B_{R/2}))} \Big\| \frac{\partial}{\partial t} Dv \Big\|^{3/2}_{L^2(t_0-R^2/4, t_0; L^2(B_{R/2}))} +$$

$$+ \frac{1}{R^2} \|Dv\|^{1/2}_{L^2(t_0-R^2/4, t_0; L^q(B_{R/2}))} \|Dv\|^{3/2}_{L^2(t_0-R^2/4, t_0; L^2(B_{R/2}))} \Big) \le$$

$$\leq \frac{cr^{2+2\lambda}}{R^2}\Big(\int_{Q_r}(1+|Dv|^2)\,dxdt\Big)^{3/4}\|Dv\|^{1/2}_{L^2(t_0-R^2/4,t_0;L^q(B_{R/2}))}.$$

To estimate $\|Dv\|_{L^2(t_0-R^2/4,t_0;L^q(B_{R/2}))}$ we make use of Hölder's inequality and (2.16) to obtain

$$\|Dv\|^2_{L^2(t_0-R^2/4,t_0;L^q(B_{R/2}))} \leq \Big(\frac{R^2}{4}\Big)^{1-2/q}\bigg(\int_{Q_{R/2}}|Dv|^q\,dxdt\bigg)^{2/q} \leq$$

$$\leq cR^{2(2/q-1)}\int_{Q_R}(1+|Dv|^2)\,dxdt.$$

Inserting this estimate into (2.18) and observing that

$$-2+\frac{1}{2}\Big(\frac{2}{q}-1\Big) = -2-2\lambda,$$

it follows

$$\int_{Q_r}|Dv|^2\,dxdt \leq c\Big(\frac{r}{R}\Big)^{2+2\lambda}\int_{Q_R}(1+|Dv|^2)\,dxdt. \tag{2.19}$$

It remains to estimate the integral $\int_{Q_r}|v|^4\,dxdt$ for $r\in(0,\frac{R}{2}]$. We do this for $r\in(0,\frac{R}{4}]$ (the desired estimate for $r\in(\frac{R}{4},\frac{R}{2}]$ is readily seen). Let λ be as above. By Hölder's inequality,

$$\int_{Q_r}|v|^4\,dxdt \leq cr^{2+2\lambda}\bigg(\int_{Q_{R/4}}|v|^{8/(1-\lambda)}\,dxdt\bigg)^{(1-\lambda)/2} \leq$$

$$\leq cr^{2+2\lambda}R^{1-\lambda}\operatorname*{ess\,sup}_{(t_0-R^2/16,t_0)}\bigg(\int_{B_{R/4}}|v|^{8/(1-\lambda)}\,dx\bigg)^{(1-\lambda)/2}.$$

Next, by Sobolev's imbedding theorem,

$$\bigg(\int_{B_{R/4}}|v(x,t)|^{8/(1-\lambda)}\,dx\bigg)^{(1-\lambda)/2} \leq$$

$$\leq c\bigg\{R^{-(3+\lambda)}\bigg(\int_{B_{R/4}}|v(x,t)|^2\,dx\bigg)^2 + R^{1-\lambda}\bigg(\int_{B_{R/4}}|Dv(x,t)|^2\,dx\bigg)^2\bigg\}$$

for a.e. $t \in \left(t_0 - \frac{R^2}{16}, t_0\right)$. Here the constant c does not depent on R. This can be established by a homothetical argument. It follows

$$\int_{Q_r} |v|^4 \, dxdt \le c\left(\frac{r}{R}\right)^{2+2\lambda} \left\{ \operatorname*{ess\,sup}_{(t_0-R^2/16, t_0)} \left(\int_{B_{R/4}} |v|^2 \, dx \right)^2 + \right. \tag{2.20}$$

$$\left. + R^4 \operatorname*{ess\,sup}_{(t_0-R^2/16, t_0)} \left(\int_{B_{R/4}} |Dv|^2 \, dx \right)^2 \right\}.$$

We estimate the integrals on the right of (2.20). First, observing that $\dfrac{\partial v}{\partial t} \in L^2(Q_{R/4}; \mathbb{R}^N)$ (cf. Theorem 2.2), we find

$$\int_{B_{R/4}} |v(x,t)|^2 \, dx \le c\left(\frac{1}{R^2} \int_{Q_{R/4}} |v|^2 \, dxds + R^2 \int_{Q_{R/4}} \left| \frac{\partial v}{\partial t} \right|^2 dxds \right)$$

for all $t \in \left[t_0 - \frac{R^2}{16}, t_0\right]$, where the constant c dependes neither on t nor on R. Thus, by Hölder's inequality and (2.7) (with $\frac{R}{2}$ in place of R and $r = \frac{R}{4}$ there)

$$\operatorname*{ess\,sup}_{(t_0-R^2/16, t_0)} \int_{B_{R/4}} |v|^2 \, dx \le \tag{2.21}$$

$$\le c\left\{ \int_{Q_R} |v|^4 \, dxdt + \left(\int_{Q_{R/2}} (1 + |Dv|^2) \, dxdt \right)^2 \right\}.$$

Secondly, since $\dfrac{\partial}{\partial t} Dv \in L^2(Q_{R/4}; \mathbb{R}^{2N})$ (cf. Theorem 2.2), we obtain by (2.8) (with $\frac{R}{2}$ in place of R and $r = \frac{R}{4}$ there)

$$\operatorname*{ess\,sup}_{(t_0-R^2/16, t_0)} \left(\int_{B_{R/4}} |Dv|^2 \, dx \right)^2 \le \tag{2.22}$$

$$\le c\left(\frac{1}{R^2} \int_{Q_{R/4}} |Dv|^2 \, dxdt + R^2 \int_{Q_{R/4}} \left| \frac{\partial}{\partial t} Dv \right|^2 dxdt \right)^2 \le$$

$$\le \frac{c}{R^4} \left(\int_{Q_{R/2}} (1 + |Dv|^2) \, dxdt \right)^2.$$

Inserting (2.21) and (2.22) into (2.20) gives

$$\int_{Q_r} |v|^4\,dxdt \leq \tag{2.23}$$

$$\leq c\Big(\frac{r}{R}\Big)^{2+2\lambda}\left\{\int_{Q_R}(1+|v|^4)\,dxdt+\left(\int_{Q_{R/2}}|Dv|^2\,dxdt\right)^2\right\}$$

for all $r \in \left[0, \frac{R}{4}\right]$.

To conclude the proof, we employ the Caccioppoli inequality for weak solutions to (2.1) (cf. [2], [8]) and the Schwarz inequality. Thus

$$\int_{Q_{R/2}}|Dv|^2\,dxdt \leq \frac{c}{R^2}\int_{Q_R}|v|^2\,dxdt \leq c\left(\int_{Q_R}|v|^4\,dxdt\right)^{1/2}.$$

Inserting this estimate into (2.23) and adding the resulting inequality to (2.19) gives (2.17). □

3. Proof of the theorem.

We begin with noting the following technical

Lemma. *Let* $\sigma : (0, R_0] \to [0, +\infty)$ *be a nondecreasing function such that*

$$\sigma(r) \leq A\left[\Big(\frac{r}{R}\Big)^{\alpha} + (2A)^{2\alpha/(\beta-\alpha)}\right]\sigma(R) + BR^{\beta}$$

for all $0 < r \leq R \leq R_0$, *where* A, B *and* α, β *are (fixed) constants satisfying* $A > \frac{1}{2}$, $B \geq 0$ *and* $\alpha > \beta > 0$. *Then*

$$\sigma(r) \leq C\big(AR_0^{-\beta}\sigma(R_0) + B\big)r^{\beta} \quad \forall r \in (0, R_0],$$

where

$$C := \max\left\{\max\left\{2, \frac{1}{\tau_0^{\beta}}\right\}, \frac{1}{\tau_0^{\beta}}\max\left\{1, \frac{1}{\tau_0^{\beta} - \tau_0^{(\alpha+\beta)/2}}\right\}\right\},$$

$$\tau_0 := (2A)^{2/(\beta-\alpha)}.$$

A proof of this result can be found in [3]. □

Let $u \in V_2^{1,0}(Q;\mathbb{R}^N)$ be any weak solution to (1.1). We divide the proof of the theorem into four steps.

1. Let $Q_R = Q_R(x_0,t_0) = B_R(x_0)\times(t_0-R^2,t_0)$ be any cylinder such that $\overline{Q}_R \subset \Omega\times(0,T]$.

The following existence result is a straightforward generalization of [2], Lemma 2.XI, p. 98:

There exists exactly one $w \in W_2^{1,0}(Q_R;\mathbb{R}^N)$ *such that*

$$(3.1)\qquad \begin{cases} -\displaystyle\int_{Q_R} w^i \frac{\partial\varphi^i}{\partial t}\,dxdt + \int_{Q_R} a_i^\alpha(x,t_0,Dw+Du)D_\alpha\varphi^i\,dxdt = \\ \quad = \displaystyle\int_{Q_R}\Big(-b_i(x,t,u,Du)\varphi^i + a_i^\alpha(x,t,Du)D_\alpha\varphi^i\Big)\,dxdt\,, \\ \textit{for all } \varphi\in W_2^{1,1}(Q_R;\mathbb{R}^N) \textit{ with } \varphi = 0 \textit{ a.e. on} \\ \quad (\partial B_R\times(t_0-R^2,t_0))\cup(B_R\times\{t_0\}), \end{cases}$$

$$(3.2)\qquad w = 0 \ \text{a.e. on}\ \partial B_R\times(t_0-R^2,t_0).$$

Moreover, the function w *possesses the following additional properties*:

$$(3.3)\qquad \operatorname*{ess\,sup}_{t\in(t_0-R^2,t_0)}\int_{B_R}|w(x,t)|^2\,dx < +\infty,$$

$$(3.4)\qquad \frac{1}{2}\int_{B_R}|w(x,t)|^2\,dx + \int_{t_0-R^2}^{t}\int_{B_R} a_i^\alpha(x,t_0,Dw+Du)D_\alpha w^i\,dxds \le$$

$$\le \int_{t_0-R^2}^{t}\int_{B_R}\Big(-b_i(x,s,u,Du)w^i + a_i^\alpha(x,s,Du)D_\alpha w^i\Big)\,dxds$$

for a.e. $t\in(t_0-R^2,t_0)$.

Observing (3.2) and (3.3), we have the well-known estimate

$$(3.5)\qquad \left(\int_{Q_R}|w|^4\,dxdt\right)^{1/2} \le$$

$$\leq C_0\left(\operatorname*{ess\,sup}_{t\in(t_0-R^2,t_0)} \int_{B_R} |w(x,t)|^2\,dx + \int_{Q_R} |Dw|^2\,dxdt \right),$$

where the constant C_0 deos not depend on R.

To procede further, we note that (3.4) is equivalent to

$$\frac{1}{2}\int_{B_R} |w(x,t)|^2\,dx +$$

$$+\int_{t_0-R^2}^{t}\int_{B_R}\Big[a_i^\alpha(x,t_0,Dw+Du)-a_i^\alpha(x,t_0,Du)\Big]D_\alpha w^i\,dxds \leq$$

$$\leq -\int_{t_0-R^2}^{t}\int_{B_R} b_i(x,s,u,Du)w^i\,dxds +$$

$$+\int_{t_0-R^2}^{t}\int_{B_R}\Big[a_i^\alpha(x,s,Du)-a_i^\alpha(x,t_0,Du)\Big]D_\alpha w^i\,dxds$$

for a.e. $t\in(t_0-R^2,t_0)$. From this inequality we obtain by the aid of (1.3), (1.5) and (1.7)

$$\frac{1}{2}\int_{B_R} |w(x,t)|^2\,dx + \nu_0\int_{t_0-R^2}^{t}\int_{B_R}|Dw|^2\,dxds \leq$$

$$\leq c\int_{t_0-R^2}^{t}\int_{B_R}\omega(|s-t_0|)(1+|Du|)|Dw|\,dxds +$$

$$+\int_{t_0-R^2}^{t}\int_{B_R}(1+|u|^3+|Dw|^{3/2})|w|\,dxds$$

for a.e. $t\in(t_0-R^2,t_0)$, and therefore with the help of (3.5) by a routine argument

$$\int_{Q_R}(|w|^4+|Dw|^2)\,dxdt \leq \chi(R)\int_{Q_R}(1+|u|^4+|Du|^2)\,dxdt \tag{3.6}$$

with

$$\chi(R) := c\left[\omega(R)+\left(\int_{Q_R}(1+|u|^4+|Du|^2)\,dxdt\right)^{1/2}\right]\cdot$$

$$\cdot\left\{1+\left[\omega(R)+\left(\int_{Q_R}(1+|u|^4+|Du|^2)\,dxdt\right)^{1/2}\right]\int_{Q_R}(1+|u|^4+|Du|^2)\,dxdt\right\},$$

where the constant c does not depend on R. Clearly, $\chi(R) \to 0$ as $R \to 0$.

2. Define $v := u + w$. Then $v \in V_2^{1,0}(Q_R; \mathbb{R}^N)$. By (3.1),

$$-\int_{Q_R} v^i \frac{\partial \varphi^i}{\partial t}\, dxdt + \int_{Q_R} a_i^\alpha(x, t_0, Dv) D_\alpha \varphi^i \, dxdt =$$

$$= -\int_{Q_R} u^i \frac{\partial \varphi^i}{\partial t}\, dxdt - \int_{Q_R} w^i \frac{\partial \varphi^i}{\partial t}\, dxdt +$$

$$+ \int_{Q_R} a_i^\alpha(x, t_0, Du + Dw) D_\alpha \varphi^i \, dxdt = 0$$

for all $\varphi \in W_2^{1,1}(Q_R; \mathbb{R}^N)$ with $\varphi = 0$ a.e. on $(\partial B_R \times (t_0 - R^2)) \cup (B_R \times \{t_0\})$. Here the functions $a_i^\alpha = a_i^\alpha(x, t_0, \xi)$ satisfy conditions (2.2)–(2.5). Then v is a weak solution to (2.1) (with $a_i^\alpha(x, t_0, \xi)$ in place of $a_i^\alpha(x, \xi)$ there). Thus the fundamental estimate (2.17) holds.

3. Let $0 < r \leq R$. Observing that $u = v - w$, from (2.17) and (3.6) it follows

$$\int_{Q_r} (1 + |u|^4 + |Du|^2)\, dxdt \leq$$

$$\leq c \int_{Q_r} (1 + |v|^4 + |Dv|^2)\, dxdt + c \int_{Q_r} (1 + |w|^4 + |Dw|^2)\, dxdt \leq$$

$$\leq c \Big(\frac{r}{R}\Big)^{2+2\lambda} \int_{Q_R} (1 + |v|^4 + |Dv|^2)\, dxdt +$$

$$+ \chi(R) \int_{Q_R} (1 + |u|^4 + |Du|^2)\, dxdt \leq$$

$$\leq c \Big[\Big(\frac{r}{R}\Big)^{2+2\lambda} + \chi(R)\Big] \int_{Q_R} (1 + |u|^4 + |Du|^2)\, dxdt.$$

Since $\chi(R) \to 0$ as $R \to 0$, for each $0 < \mu < \lambda$ there exists $R_0 > 0$ such that the above technical lemma applies (with λ in place of α and μ in place of β). We obtain

$$\text{(3.7)} \quad \int_{Q_r} (1 + |u|^4 + |Du|^2)\, dxdt \leq c \Big(\frac{r}{R_0}\Big)^{2+2\mu} \int_{Q_{R_0}} (1 + |u|^4 + |Du|^2)\, dxdt$$

for all $0 < r \leq R_0$.

4. In [8], the following Poincaré inequality has been established:

$$\int_{Q_r} |u - u_{Q_r}|^2 \, dxdt \le cr^2 \int_{Q_r} (1 + |u|^4 + |Du|^2) \, dxdt, \tag{3.8}$$

where u is any weak solution to (1.1) and

$$u_{Q_r} = \frac{1}{|Q_r|} \int_{Q_r} u(y, s) \, dyds$$

($\overline{Q}_r \subset \Omega \times (0, T]$, $c =$ const independent of r). Inserting (3.7) into (3.8) we obtain

$$\int_{Q_r} |u - u_{Q_r}|^2 \, dxdt \le c\Big(\frac{r}{R_0}\Big)^{4+2\mu} \int_{Q_{R_0}} (1 + |u|^4 + |Du|^2) \, dxdt \tag{3.9}$$

for all $0 < r \le R_0$. Here the constants c and R_0 are independent of r. The constant c depends only on the constants in (1.3), (1.4), (1.5) and (1.7), while R_0 depends on these constants and on $t' \in (0, T)$ and $\mathrm{dist}(\Omega', \partial\Omega)$, too, where $\overline{\Omega'} \subset \Omega$ (both c and R_0 are independent of $\{x_0, t_0\} \in \Omega' \times (t', T)$) $(0 < R_0 < \min\{\mathrm{dist}(\Omega', \partial\Omega), \sqrt{T - t'}\})$.

From (3.9) it finally follows that u is Hölder continuous on any subcylinder $\Omega' \times (t', T)$ (cf. [1]). □

4. Appendix.

For the reader's convenience, we note some abstract results which have been applied in Sect. 2. Their proofs may be found in [11].

Let X be a normed space with norm $\|\cdot\|_X$. Let $a, b \in \mathbb{R}, a < b$. For $u \in L^1(a, b; X)$, define

$$u_{t_0,\rho} = \frac{1}{\rho} \int_{t_0-\rho}^{t_0} u(\sigma) \, d\sigma, \quad t_0 \in (a, b,], \ \rho \in (0, t_0 - a).$$

A1. *Let $u \in L^2(a, b; X)$. Assume there exists $\delta \in (0, 1)$ such that*

$$\int_{t_0-\rho}^{t_0} \|u(t) - u_{t_0,\rho}\|_X^2 \, dt \le K_0 \rho^{1+2\delta}$$

for all $t_0 \in (a, b]$ and all $\rho \in (0, t_0 - a)$ ($K_0 =$ const independent of both t_0 and ρ).

Then there exists a constant $c > 0$ such that

$$\|u\|_{C^\delta([a,b];X)} := \max_{t\in[a,b]} \|u(t)\|_X + \sup_{\substack{s,t\in[a,b]\\ s\neq t}} \frac{\|u(s)-u(t)\|_X}{|s-t|^\delta} \leq c\sqrt{K_0}.$$

A2. *Let $u \in L^2(a,b;X)$ have the distributional derivative $u' \in L^2(a,b;X)$. Then*

$$\int_{t_0-\rho}^{t_0} \|u(t) - u_{t_0,\rho}\|_X^2 \, dt \leq \rho^2 \int_{t_0-\rho}^{t_0} \|u'(\sigma)\|_X^2 \, d\sigma$$

for all $t_0 \in (a,b]$ and all $\rho \in (0, t_0 - a)$.

Next, let Y be a normed space with norm $\|\cdot\|_Y$. Suppose

$X \subset Y$ continuously;

$$\begin{cases} \text{for every } \theta \in (0,1) \text{ there is a normed space } X_\theta \text{ with} \\ \text{norm } \|\cdot\|_{X_\theta} \text{ such that } X \subset X_\theta \subset Y \text{ and} \\ \|z\|_{X_\theta} \leq c\|z\|_X^\theta \|z\|_Y^{1-\theta} \quad \forall z \in Y \ (c = \text{const}). \end{cases}$$

A3. *Let $u \in L^2(a,b;X)$ have the distributional derivative $u' \in L^2(a,b;Y)$. Let $a_1 \in (a,b)$, $\theta \in (0, \frac{1}{2})$.*

Then there exists a constant $c > 0$ such that

$$\operatorname*{ess\,sup}_{(t_0-r^2,t_0)} \|u(t)\|_{X_\theta}^2 \leq cr^{2(1-2\theta)} \|u\|_{L^2(t_0-r^2,t_0;X)}^{2\theta} \|u'\|_{L^2(t_0-r^2,t_0;Y)}^{2(1-\theta)} +$$

$$+ \frac{c}{r^2} \|u\|_{L^2(t_0-r^2,t_0;X)}^{2\theta} \|u\|_{L^2(t_0-r^2,t_0;Y)}^{2(1-\theta)}$$

for all $t_0 \in (a_1, b)$ and all $r \in (0, \sqrt{t_0 - a_1})$.

The latter statement will be applied as follows. Let $E \subset \mathbb{R}^n$ be measurable. Let $1 \leq p_0 < p_1 < +\infty$ and $0 < \theta < 1$. Define

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

By Hölder's inequality,

$$\|f\|_{L^p(E)} \leq \|f\|_{L^{p_0}(E)}^{1-\theta} \|f\|_{L^{p_1}(E)}^\theta \quad \forall f \in L^{p_0}(E) \cap L^{p_1}(E).$$

To be more specific, let $|E|$ (=Lebesgue measure) $< +\infty$, and put $p_0 = 2$, $p_1 = q > 2$ and $\theta = \frac{1}{4}$. Then from A3 we conclude

A4. *Let $u \in L^2(a, b; L^q(E))$ have the distributional derivative*

$$u' \in L^2(a, b; L^2(E)).$$

Let $a_1 \in (a, b)$.

Then there exists a constant $c > 0$ such that

$$\operatorname*{ess\,sup}_{(t_0-r^2,t_0)} \|u(t)\|^2_{L^p(E)} \le cr\|u\|^{1/2}_{L^2(t_0-r^2,t_0;L^q(E))}\|u'\|^{3/2}_{L^2(t_0-r^2,t_0;L^2(E))} +$$

$$+ \frac{c}{r^2}\|u\|^{1/2}_{L^2(t_0-r^2,t_0;L^q(E))}\|u\|^{3/2}_{L^2(t_0-r^2,t_0;L^2(E))}$$

for all $t_0 \in (a_1, b)$ and all $r \in (0, \sqrt{t_0 - a_1})$.

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