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ON SOME REGULARITY AND NONREGULARITY RESULTS FOR SOLUTIONS TO PARABOLIC SYSTEMS

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Dedicated to Professor Sergio Campanato on his 70th birthday

A short survey of recent results on smoothness of weak solutions to parabolic systems with nonsmooth coefficients in plane is given. Moreover, for space dimension $n \geq 3$ and any closed subset F in $\mathbb{R}^n$ we construct a linear parabolic system with bounded measurable coefficients and its solution which is essentially discontinuous on F and essentially continuous on $\mathbb{R}^n \setminus F$.

In this paper, in addition to surveying of several recent results concerning smoothness and discontinuities of weak solutions to parabolic systems, we present a new example which indicates how large the singular set of a solution can be.

Paper is organized as follows: In Section 1 a short comparison of results about elliptic and parabolic systems is given. Some open problems are mentioned here as well. Because of the extensive quantity of the results in the field we concentrate here on Hölder continuity in plane case and integrability of the time derivative, only. We apologize for being far from any kind of completeness

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of references. In Section 2 some known examples of parabolic systems with nonsmooth solutions are listed and a new one is explained in details. Partial positive results valid under additional assumptions on coefficients of the system are in Section 3 while the last section deals with applications to fluid mechanics.

For the sake of simplicity we shall deal throughout the paper mostly with the interior regularity.

1. Elliptic and parabolic systems: similarities and differences.

In elliptic case, we consider in general quasilinear elliptic systems of the form

$$(1.1) \qquad D_\alpha(A_{ij}^{\alpha\beta}(x,u)D_\beta u^j) = 0, \quad i = 1,\dots,N \text{ on } \Omega.$$

(The summation convention is used throughout the paper.) Domain Ω is considered to be a nonempty open subset of $\mathbb{R}^n$, $A_{ij}^{\alpha\beta}$ $(i, j = 1,\dots,N,$ $\alpha, \beta = 1,\dots,n)$ are uniformly bounded Carathéodory functions. Denoting by $\langle u, v\rangle$ scalar product in any finite dimensional space $\mathbb{R}^p$, $p \in \mathbb{N}$, $|u| = \langle u, u\rangle^{\frac{1}{2}}$, we suppose the ellipticity condition in the form

$$(1.2) \qquad \begin{aligned} &\exists\ \lambda_0, \lambda_1 \in (0,\infty),\ \forall\ \xi \in \mathbb{R}^{nN},\ \forall\ u \in \mathbb{R}^N, \text{ for a. e. } x \in \mathbb{R}^n \\ &\lambda_0|\xi|^2 \le \langle A(x,u)\xi, \xi\rangle \le \lambda_1|\xi|^2. \end{aligned}$$

If $A_{ij}^{\alpha\beta}$ depend only on x and are continuous on their domain, then according to classical results of C. B. Morrey [18], A. Douglis, L. Nirenberg [4] every weak solution of (1.1) is locally Hölder continuous. The proof of Theorem 3.1 in [5] indicates that the continuity of coefficients in one point implies the Hölder continuity of any weak solution in a neighbourhood of this point.

On the other hand, for $n \ge 3$ it was proved by E. De Giorgi that the discontinuity of coefficients in one point can cause the discontinuity (even unboundedness) of a solution (see [2]). The counter example of J. Souček (see [24]) gives a solution of (1.1) which is discontinuous on a dense countable subset. Moreover, for every set $F \subset \mathbb{R}^n$ of the type F_σ there is a system (1.1) and its solution u which is bounded, essentially discontinuous at all points of F and essentially continuous at all points of $\mathbb{R}^n \setminus F$. (See [10].)

E. De Giorgi observed already in 1968 (see [2]) that each such solution is the unique (non smooth) point of minimum of a quadratic functional with nonsmooth coefficients. Smoothness of minimizers in the scalar case $(N = 1)$ was proved in the fundamental work of E. De Giorgi [3] and J. Nash [18], however their method of proof cannot be used in vector case $(N > 1)$. The first example of a nonsmooth minimizer for a smooth strongly convex functional (depending only on ∇u) was given by J. Nečas in higher dimensions (see [22]). Later W. Hao, J. Nečas and S. Leonardi [9] modified the construction to work for any dimension $n \geq 5$. Recently V. Šverák and X. Yan constructed a nonsmooth minimizer of a smooth strongly convex functional of the same type for $n = 3, N = 5$ (see [29]) or $n = 4, N = 3$ (see [30]). Minimizers in [29], [30] have discontinuous or even unbounded first derivatives. As the minimizers of functionals considered belong to $W^{2,p}_{loc}(\Omega)$ for a $p > 2$ they are Hölder continuous on Ω for $n = 3, 4$.

In any dimension n solutions of (1.1) belong to $W^{1,p}_{loc}(\Omega)$ for some $p > 2$ (see e.g. [17], [5]). If $n = 2$, this estimate and embedding theorems guarantee interior Hölder continuity of any solution to (1.1).

In parabolic case we consider systems

$$u^i_t = D_\alpha(A^{\alpha\beta}_{ij}(x, t, u) D_\beta u^j), \quad i = 1, \dots, N \text{ on } Q. \tag{1.3}$$

Here $Q = \Omega \times (0, T)$ for a positive T, $A^{\alpha\beta}_{ij}$ $(i, j = 1, \dots, N, \alpha, \beta = 1, \dots, n)$ are uniformly bounded Carathéodory functions satisfying ellipticity condition

$$\begin{aligned} &\exists\ \lambda_0, \lambda_1 \in (0, \infty),\ \forall\ \xi \in \mathbb{R}^{nN},\ \forall\ u \in \mathbb{R}^N, \\ &\text{for almost every } x \in \mathbb{R}^n \text{ and for almost every } t \in (0, T) \\ &\lambda_0 |\xi|^2 \leq \langle A(x, t, u)\xi, \xi \rangle \leq \lambda_1 |\xi|^2. \end{aligned} \tag{1.4}$$

In this case, too, if $A^{\alpha\beta}_{ij}$ depend only on x and t, any weak solution (i.e., any locally square integrable function with locally square integrable space gradient) is Hölder continuous on a neighbourhood of any point of continuity of coefficients (see [6], [27]).

Any weak solution of an elliptic system (1.1) can be considered as a stationary solution to a parabolic system (1.2). Thus elliptic examples can be interpreted as stationary parabolic problems on Q. It would indicate singularities of solutions appearing on cylindrical subsets of $\mathbb{R}^{n+1}$. It is more interesting to ask whether a weak solution of a parabolic system can develop a singularity in the interior of space-time cylinder starting from smooth initial data. If $n \geq 3$ this situation can occur (see [25], [28], and part 2 of this paper) eventhough less is known about possible structure of a singular set. We give here a counterexample of a solution having as its singular set an arbitrary closed subset in $\mathbb{R}^{n+1}$ and it remains open whether there can be solutions with an arbitrary set of the type F_σ as their singular sets.

For parabolic systems L_p estimates of space gradient for sufficiently small $p > 2$ hold, too (see [1], [23], [12], part 3 of this paper), however even for

$n = 2$ they do not imply Hölder continuity of solutions. As far as we know, the question whether any weak solution to a linear parabolic system with L_∞ coefficients in plane domain is locally Hölder continuous is open.

2. Examples.

Theorem 2.1. (see [25]). *Let* $n \geq 3$, $k \in (0, 2(n-1)(n-2))$; *for* $x \in \mathbb{R}^n$, $t \in (-\infty, 1)$ *put*

$$u(x,t) = \frac{x}{\sqrt{k(1-t) + |x|^2}}\,.$$

Then u *is real analytic on* $\mathbb{R}_n \times (-\infty, 1)$ *and solves a quasilinear parabolic system*

$$u_t^i = D_\alpha(A_{ij}^{\alpha\beta}(u) D_\beta u^j), \quad i = 1, \dots, n \tag{2.1}$$

with coefficients $A_{ij}^{\alpha\beta}(u)$ *which are real analytic on a neighbourhood of* $\overline{B(0,1)}$.

The coefficients are given by the formula

$$A_{ij}^{\alpha\beta}(u) = \theta\delta_{ij}\delta_{\alpha\beta} + A_{i\alpha}(u)A_{j\beta}(u)$$

with

$$A_{i\alpha}(u) = \frac{\{n-1-\theta-|u|^2(1+\frac{k}{2(n-1)})\}\delta_{i\alpha} + (1+\theta+\frac{k}{2(n-1)})u^i u^\alpha}{\sqrt{n(n-1-\theta) - \{2(n-1-\theta)+\frac{k}{2}\}|u|^2 - \theta|u|^4}}\,. \tag{2.2}$$

For $\theta \in (0, n-2-\frac{k}{2(n-1)})$ the expression under the square root in the denominator of (2.2) is positive on $\overline{B(0,r)}$ for a $r > 1$. The coefficients are then real analytic on the same set and satisfy ellipticity condition

$$\lambda_0|\xi|^2 \leq \langle A(u)\xi, \xi\rangle \leq \lambda_1|\xi|^2$$

where

$$\lambda_0 = \theta\,,\ \lambda_1 = \frac{(n-1)^2 - \theta(n-2-\frac{k}{2(n-1)})}{n-2-\frac{k}{2(n-1)}-\theta}.$$

Let $\Phi \in C^\infty(\mathbb{R})$ be any function such that $0 \leq \Phi \leq 1$ on $\mathbb{R}$, $\Phi(s) = 0$ for $s \geq r^2$, $\Phi(s) = 1$ for $|s| < \frac{1+r^2}{2}$. Put

$$\tilde{A}_{ij}^{\alpha\beta}(u) = \begin{cases} \theta\delta_{ij}\delta_{\alpha\beta} + \Phi(|u|^2)A_{i\alpha}(u)A_{j\beta}(u) & \text{for } |u| < r \\ \theta\delta_{ij}\delta_{\alpha\beta} & \text{otherwise.} \end{cases}$$

Then $\tilde{A}_{ij}^{\alpha\beta}$ are infinitely differentiable on $\mathbb{R}^n$, system with these coefficients satisfies ellipticity condition with the same λ_0, λ_1 and admits the same solution u.

Inserting values of u in $\tilde{A}_{ij}^{\alpha\beta}(u)$ we see that u solves also a linear parabolic system with coefficients which are bounded, real analytic on $(\mathbb{R}^n \setminus \{0\}) \times (-\infty, 1)$ and can be extended by different ways on $\mathbb{R}^{n+1}$ as bounded and measurable functions. Thus the discontinuity of a solution can disappear for $t > 1$ or can survive for any time interval.

By analogous procedure as in the first quasilinear case we obtain examples of L_∞ blow up for linear parabolic system.

Theorem 2.2. (see [25]). *Let* $n \geq 3$, $\gamma \in (0, \min(\sqrt{n-1} - 1, \frac{1}{2}))$, $k \in (0, 2(n-1)(n-2-2\gamma))$; *for* $x \in \mathbb{R}^n$, $t \in (-\infty, 1)$ *put*

$$u(x,t) = \frac{x}{|x|^\gamma \sqrt{k(1-t) + |x|^2}}. \tag{2.3}$$

Then u is Hölder continuous on $\mathbb{R}^n \times (-\infty, 1)$ *and it is a weak solution of a linear parabolic system*

$$u_t^i = D_\alpha(A_{ij}^{\alpha\beta}(x,t) D_\beta u^j), \quad i = 1, \ldots, n, \tag{2.4}$$

with $A_{ij}^{\alpha\beta} \in L_\infty(\mathbb{R}^n \times (-\infty, 1))$ *satisfying uniform ellipticity condition*

$$\begin{aligned} &\exists\, \lambda_0, \lambda_1 \in (0,\infty) : \ \forall\, \xi \in \mathbb{R}^{n^2},\ \forall\, x \in \mathbb{R}^n,\ \forall\, t \in (-\infty, 1) \\ &\lambda_0|\xi|^2 \leq \langle A(x,t)\xi, \xi\rangle \leq \lambda_1|\xi|^2. \end{aligned} \tag{2.5}$$

Nevertheless,

$$\lim_{t \to 1^-} ||u(\cdot, t)||_{L_\infty(\mathbb{R}^n)} = \infty. \tag{2.6}$$

The question of how "large" the sets of singular points of a solution to nonsmooth parabolic system can be is not completely solved.

In what follows we shall describe the construction of a parabolic system and its solution with a given closed singular set F. During the construction we work with the standard parabolic metric on $\mathbb{R}^{n+1}$ and a corresponding measure.

Step 1. Construction of a counterexample.

Let $\{z_p = [x_p, t_p]\}_{p=1}^\infty$ be a sequence of points in $\mathbb{R}^{n+1}$. Denote

$$r_p = r_p(x) = |x - x_p|, \quad \nu_p = \nu_p(x) = \frac{x - x_p}{|x - x_p|}, \tag{2.7}$$

$$\xi_p = \xi_p(z) = \frac{r_p}{\sqrt{|t-t_p|}}, \quad \varphi = \varphi(\xi), \quad G_p = G_p(r),$$

$$f_1^p = f_1^p(z) = \frac{G_p(r_p)}{r_p}\varphi(\xi_p), \quad f_2^p = f_2^p(z) = \frac{G_p(r_p)}{r_p}\varphi'(\xi_p)\xi_p,$$

$$f_3^p = f_3^p(z) = \frac{G_p(r_p)}{r_p}\varphi'(\xi_p)\xi_p^3\frac{\operatorname{sgn}(t-t_p)}{2(n-1)}, \quad g^p = g^p(z) = G_p'(r_p)\varphi(\xi_p),$$

$$u_p = u_p(z) = \nu_p r_p f_1^p.$$

In this notation

(2.8) $$D_\alpha u_p^i = f_1^p(\delta_{i\alpha} - \nu_p^i\nu_p^\alpha) + (f_2^p + g^p)\nu_p^i\nu_p^\alpha,$$

$$\frac{\partial u_p}{\partial t} = (u_p)_t = -\nu_p\frac{\operatorname{sgn}(t-t_p)}{2}\frac{G_p}{r_p^2}\varphi_p'\xi_p^3 = -(n-1)\frac{\nu_p}{r_p}f_3^p.$$

Define

(2.9) $$(b_\alpha^i)_p = f_1^p[(n-2)\delta_{i\alpha} + \nu_p^i\nu_p^\alpha] + (f_p^2 + f_p^3 + g_p)(\delta_{i\alpha} - \nu_p^i\nu_p^\alpha).$$

An easy computation gives

(2.10) $$D_\alpha(b_\alpha^i)_p = (u_p^i)_t.$$

Fix now $p, q \in \mathbb{N}$ and denote $\Theta_{pq} = \langle\nu_p, \nu_q\rangle$. Then we have

(2.11) $$\langle Du_p, Du_q\rangle = f_p^1 f_q^1(n-2+\Theta_{pq}^2) +$$
$$+ (f_1^p f_2^q + f_2^p f_1^q)(1-\Theta_{pq}^2) + f_2^p f_2^q\Theta_{pq}^2 +$$
$$+ (f_1^p g^q + f_1^q g^p)(1-\Theta_{pq}^2) + (f_2^p g^q + f_2^q g^p + g^p g^q)\Theta_{pq}^2,$$

(2.12) $$\langle b_p, Du_q\rangle = f_p^1 f_q^1(n^2-3n+3-\Theta_{pq}^2) +$$
$$+ (f_1^p f_2^q + [f_2^p + f_3^p]f_1^q)(n-2+\Theta_{pq}^2) + (f_2^p + f_3^p)f_2^q(1-\Theta_{pq}^2) +$$
$$+ (f_1^p g^q + f_1^q g^p)(n-2+\Theta_{pq}^2) + ([f_2^p + f_3^p]g^q + f_2^q g^p + g^p g^q)(1-\Theta_{pq}^2),$$

(2.13) $$\langle b_p, b_q\rangle = f_p^1 f_q^1(n^3-4n^2+6n-4+\Theta_{pq}^2) + (f_1^p[f_2^q + f_3^q] +$$

$$+[f_2^p + f_3^p]f_1^q)(n^2 - 3n + 3 - \Theta_{pq}^2) + (f_2^p + f_3^p)(f_2^q + f_3^q)(n - 2 + \Theta_{pq}^2) +$$
$$+ (f_1^p g^q + f_1^q g^p) + (n^2 - 3n + 3 - \Theta_{pq}^2) +$$
$$+ ([f_2^p + f_3^p]g^q + [f_2^q + f_3^q]g^p + g^p g^q)(n - 2 + \Theta_{pq}^2).$$

In what follows we shall suppose

$$\varphi(\xi) = \frac{\xi}{\sqrt{1 + a^2\xi^2}}, \quad a > 0,$$
$$G_p(r) = \frac{1}{(1 + \omega_p r)^\tau}, \ \omega_p > 0, \ \tau \in (0, \ \min\{0.001, \ \frac{1}{n-1}\}). \tag{2.14}$$

It implies that g^p in (2.7) is negative and

$$0 \le |g^p| \le \tau f_1^p, \tag{2.15}$$

while f_2^p is nonnegative and

$$0 \le f_2^p \le f_1^p. \tag{2.16}$$

Eventhough the sign of f_3^p depends on the sign of $t - t_p$, we can sum

$$f_2^p + f_3^p = f_1^p\Big(\frac{1}{1 + a^2\xi_p^2} + \frac{\xi_p^2 \mathrm{sign}(t - t_p)}{2(n-1)(1 + a^2\xi_p^2)}\Big).$$

For $n > 2, a \ge 1$ the expression in parenthesis is a decreasing function of ξ_p attending its maximum value 1 at $\xi = 0$ and tending to $\frac{\mathrm{sign}(t-t_p)}{2(n-1)a^2}$ for $\xi_p \to \infty$. It implies that

$$-\frac{1}{2(n-1)a^2} f_1^p \le f_2^p + f_3^p \le f_1^p. \tag{2.17}$$

Thus from (2.11) – (2.13) we get

$$f_1^p f_1^q (n + \tau) \ge \langle Du_p, Du_q\rangle \ge f_1^p f_1^q (n - 2 - 4\tau),$$
$$\langle b_p, Du_q\rangle \ge f_p^1 f_q^1 (n^2 - 3n + 2 - \frac{1}{2a^2} - 2(n-1)\tau), \tag{2.18}$$
$$\langle b_p, b_q\rangle \le f_p^1 f_q^1 (n^3 - 2n^2 + n + 2\tau)$$

for sufficiently small $\tau (\tau < \frac{1}{n-1})$.

From (2.18) it follows that

$$\begin{aligned} &\langle b_p, Du_q \rangle \geq \lambda \langle Du_p, Du_q \rangle, \\ &\langle b_p, b_q \rangle \leq \mu^2 \langle Du_p, Du_q \rangle \end{aligned} \tag{2.19}$$

with

$$\lambda = \frac{n^2 - 3n + 2 - \frac{1}{2a^2} - 2(n-1)\tau}{n+\tau}, \qquad \mu^2 = \frac{n^3 - 2n^2 + n + 2\tau}{n - 2 - 4\tau}. \tag{2.20}$$

Putting now

$$u = \sum_{p=1}^{\infty} u_p, \quad b_i^\alpha = \sum_{p=1}^{\infty} (b_i^\alpha)_p, \tag{2.21}$$

we obtain from (2.19) that

$$\langle b, Du \rangle \geq \lambda \langle Du, Du \rangle, \quad \langle b, b \rangle \leq \mu^2 \langle Du, Du \rangle \tag{2.22}$$

with the same constants λ, μ as in (2.20).

The construction of coefficients $A_{ij}^{\alpha\beta}$ in the system (2.4) satisfying boundedness and ellipticity conditions follows by the same procedure as in [10], [25]. Putting for $\theta \in (0, \lambda)$ and $c = Du$

$$A_{ij}^{\alpha\beta}(z) = \theta \delta_{ij} \delta_{\alpha\beta} + \langle b - \theta c, c \rangle^{-1} (b_i^\alpha - \theta c_i^\alpha)(b_j^\beta - \theta c_j^\beta), \tag{2.23}$$

we obtain that u given by (2.21) is a solution to the system (2.4) with coefficients given by (2.23) and the ellipticity condition (2.5) is satisfied with

$$\lambda_0 = \frac{\mu}{\lambda}(\mu - \sqrt{\mu^2 - \lambda^2}), \quad \lambda_1 = \frac{\mu}{\lambda}(\mu + \sqrt{\mu^2 - \lambda^2}).$$

For each $R, T > 0$, $p \in \mathbb{N}$ we get

$$\begin{aligned} \|f_1^p\|_{L^2(B_R(0)\times(-T,T))} &\leq \left(2T\sigma_n \int_0^R G_p^2(r) r^{n-3} dr \right)^{1/2} \leq \\ &\leq \omega_p^{-\tau} \left(2T\sigma_n \frac{R^{n-2(1+\tau)}}{n - 2(1+\tau)} \right)^{1/2}. \end{aligned} \tag{2.24}$$

(σ_n stays here for the $(n-1)$ dimensional measure of unit sphere in R^n.) Hence if the sequence $(\omega_p)_{p=1}^{\infty}$ tends to infinity quickly enough, u and its space gradient are locally square integrable on R^{n+1}.

Step 2. Essential continuity and discontinuity.

As it was established in [10], for any F_σ-set F can be constructed a linear elliptic system with bounded measurable coefficients possessing a weak solution whose singular set equals F. In the parabolic case we prove a weaker assertion. Prescribing any closed set $F \subset \mathbb{R}^{n+1}$ we adjust the construction described in Step 1 in such a way that the set F will coincide with the singular set of the solution u.

This fact has two interesting consequences :

Smooth initial (and/or boundary) data cannot guarantee continuity of solutions to linear parabolic systems with nonsmooth coefficients in space dimension $n \geq 3$.

No "partial regularity" results can be expected for such systems without taking further in consideration either the smoothness of coefficients with respect to x, t or some structural conditions.

Since we are dealing with weakly differentiable functions, it is more meaningful to speak in terms of essential continuity and discontinuity. A measurable function u is said to be essentially continuous at a point $z_0 \in R^{n+1}$ if

$$\operatorname*{osc\,ess}_{z\to z_0} u(z) = 0,$$

where

$$\operatorname*{osc\,ess}_{z\to z_0} u(z) = \inf_{\delta>0} \inf_{\substack{Z\subset R^{n+1}\\ \operatorname{meas} Z=0}} \sup_{z_1,z_2\in B_\delta(z_0)\setminus Z} |u(z_1) - u(z_2|.$$

Points of essential continuity of u (u is defined up to a set of measure zero) are exactly the points of continuity of essential limsup of u, which is a representative of u defined everywhere.

Let F be a closed subset in $\mathbb{R}^{n+1}$. Find points $z_p \in \mathbb{R}^{n+1}$, ($p \in \mathbb{N}$) so that the set $\{z_p\}_{p\in\mathbb{N}}$ is dense in F.

Further, find compact sets K_p, $p \in \mathbb{N}$ so that each K_p has its Lebesgue density at z_p equal 1 and

$$K_p \cap \{z_q; q \neq p\} = \emptyset. \tag{2.25}$$

We proceed by induction. Having already found the compact sets $K_1, \dots, K_s$ we define K_{s+1} as follows: Choose $\Delta_{s+1} \in (0, 1/2^{s+1})$ so small that the closure of $B = B(z_{s+1}, \Delta_{s+1})$ does not meet the sets $K_1, \dots, K_s$. Taking $d_q \leq$ dist (z_q, z_{s+1}) sufficiently small we can put

$$K_{s+1} = \overline{B} \setminus \cup_{q\neq s+1, z_q\in B} B(z_q; d_q^2 2^{-q}). \tag{2.26}$$

Constructing a function u and coefficients b according to (2.21) we have

$$|u_p(z)| = r_p(z) f_1^p(z) = h_p(r_p(z), |t - t_p|) \tag{2.27}$$

with

$$h_p(v, w) = \frac{1}{(1 + \omega_p v)^\tau} \frac{v}{\sqrt{w + a^2 v^2}}, \qquad v, w \in [0, \infty).$$

Then

1) h is decreasing function of w,

2) $\dfrac{\partial h_p}{\partial v} = \dfrac{1}{(1 + \omega_p v)^{\tau+1} (w + a^2 v^2)^{3/2}} \{w + \omega_p v [w(1 - \tau) - \tau a^2 v^2]\}$ and it is positive for $\tau \in [0, \frac{1}{1+a^2}]$, $\quad w \geq \delta^2, \quad v \leq \delta, \quad \delta < 1$.

From 1) and 2) we obtain the following: If $v \geq \delta$, then $h_p(v, w) \leq h_p(v, 0) \leq h_p(\delta, 0)$. If $0 \leq v \leq \delta, w \geq \delta^2$, then $h_p(v, w) \leq h_p(\delta, w) \leq h_p(\delta, 0)$.

Having in mind the formula (2.27) we can conclude that

$$|u_p(z)| \leq \frac{1}{a(1 + \omega_p \delta)^\tau}, \tag{2.28}$$

on the complement of the neighbourhood

$$B(z_p, \delta) = \{z = [x, t]; r_p(z) < \delta, |t - t_p| < \delta^2\}.$$

Thus, taking

$$\delta_p = \Delta_p, \qquad \omega_p > \left(\frac{2^p}{a}\right)^{\frac{1}{\tau}} \frac{1}{\Delta_p} \quad \left(> \left(\frac{2^{2p}}{a}\right)^{\frac{1}{\tau}} \right), \tag{2.29}$$

we have

$$|u_p| \leq 2^{-p} \tag{2.30}$$

on the complement of $B(z_p, \Delta_p)$.

As it follows from the construction of K_q, (2.30) takes place in any K_q, $q < p$.

Theorem 2.3. *Let $u = \sum_{p=1}^\infty u_p$. Then u is a weak solution of a uniformly parabolic system* (2.4). *It is essentially discontinuous at all points of F and essentially continuous at all points of $\mathbb{R}^{n+1} \setminus F$.*

Proof. The first part of the assertion follows from the preceeding construction. So we have to prove essential discontinuity of u on F and essential continuity of u on the complement of F.

Essential discontinuity of u on F.

Consider first $z_0 = z_p$. All functions u_q with $q \neq p$ are continuous on K_p, and by (2.30) the sum converges uniformly on K_p. The function u_p behaves on parabolas $r_p^2 = |t - t_p|$ near z_p like

$$\frac{x - x_p}{|x - x_p|} \frac{1}{\sqrt{1 + a^2}}$$

and thus

$$\operatorname*{osc\,ess}_{z \to z_p} u_p(z) \geq \frac{1}{\sqrt{1 + a^2}}.$$

As Lebesgue density of K_p at z_p is 1 and the sum defining u converges uniformly on K_p we get that

$$\operatorname*{osc\,ess}_{z \to z_p} u(z) \geq \frac{1}{\sqrt{1 + a^2}}.$$

Last estimate is uniform on the set $\{z_p\}_{p \in \mathbb{N}}$ and this set is dense in F. It implies that

$$\operatorname*{osc\,ess}_{z \to z_0} u(z) \geq \frac{1}{\sqrt{1 + a^2}}$$

for all $z_0 \in F$. Hence u is essentially discontinuous at all points of F.

Essential continuity on $\mathbb{R}^{n+1} \setminus F$.

Choose $z \in \mathbb{R}^{n+1} \setminus F$ and a positive δ such that $\overline{B(z, 2\delta)}$ does not meet F. Then thanks to (2.28)

$$|u_p| \leq \frac{1}{a(1 + \omega_p \delta)^\tau}$$

on $\overline{B(z, \delta)}$, the sum defining u converges uniformly on $\overline{B(z, \delta)}$ and all functions u_p are continuous on $\overline{B(z, \delta)}$. Thus u is continuous on $\overline{B(z, \delta)}$.

3. Regularity.

In this part we concentrate mainly on two points, i.e., the existence of time derivative and a Hölder continuity of solutions.

J. Nečas and V. Šverák in [23] considered a nonlinear system

$$u_t^i = D_\alpha(a_i^\alpha(\nabla u)), \quad i = 1, \dots, N \text{ on } Q, \tag{3.1}$$

with continuously differentiable coefficients a_i^α and proved that $u \in C_{loc}^{1,\mu}(Q)$ if $n = 2$ and $u \in C_{loc}^{0,\mu}(Q)$ if $n \le 4$.

In 1995 K. Gröger and M. Rehberg (see [8]) considered the system

$$u_t^i - D_\alpha(A_{ij}^{\alpha\beta}(x, t, u) D_\beta u^j) = f^i, \quad i = 1, \dots, N \quad \text{on } Q. \tag{3.2}$$

They solved initial and boundary value problem for this system for sufficiently small time T in a space, which is for $n = 2$ embedded in $C^{0,\mu}(Q)$. Coefficients $A_{ij}^{\alpha\beta}$ are supposed to be uniformly continuous in t and bounded and measurable in space variables. Under these assumptions the time derivative belongs to $L_q((0, T); W^{-1,p})$ for a $p > 2$.

In 1997, J. Naumann, J. Wolf and M. Wolff (see [20]) proved that if we suppose coefficients in (3.2) to be μ-Hölder continuous with $\mu > \frac{1}{2}$ (sufficiently near to 1) and $n = 2$, then $u \in C^{0,\mu}(Q)$ and there is a $p > 2$ such that $u_t \in L_p((0, T); L_2(\Omega))$.

In 1996, the authors proved in [12] that if coefficients a_i^α are Lipschitz continuous in t and bounded and measurable in space variables and $n = 2$, then all solutions of

$$u_t^i - D_\alpha(a_i^\alpha(x, t, u, \nabla u)) = f^i, \quad i = 1, \dots, N \text{ on } Q, \tag{3.3}$$

are Hölder continuous and there is a $p > 2$ such that $u_t \in L_\infty((0, T); L_p(\Omega))$.

If we drop the assumption $n = 2$ we get a result slightly generalizing [21] in assumptions on f:

Theorem 3.1. (see [11]). *Let $f^i \in L_2(Q)$, coefficients $a_i^\alpha(x, t, u, p)$ be Carathéodory functions continuously differentiable in u, p and satisfy on their domains*

(i) growth conditions:

$$|a_i^\alpha(x, t, u, p)| \le M(1 + |u| + |p|),$$
$$\left|\frac{\partial a_i^\alpha}{\partial u^j}(x, t, u, p)\right| + \left|\frac{\partial a_i^\alpha}{\partial p_\beta^j}(x, t, u, p)\right| \le M,$$

(ii) ellipticity condition:

$$\frac{\partial a_i^\alpha}{\partial p_\beta^j}(x,t,u,p)\xi_\alpha^i\xi_\beta^j \geq \lambda_0|\xi|^2,$$

(iii) Hölder continuity in t: for $\gamma \in (\frac{1}{2}, 1]$

$$|a_i^\alpha(x,t_1,u,p) - a_i^\alpha(x,t_2,u,p)| \leq L|t_1 - t_2|^\gamma(1+|u|+|p|).$$

Then for every weak solution u to (3.3) *u_t belongs to $L_{2,loc}(Q)$.*

4. Higher integrability results.

Classical results guarantee that any weak solution of a linear parabolic system with nonsmooth uniformly elliptic coefficients has its space gradient integrable with a power $p > 2$. These estimates were proved by perturbation methods (see e.g. [1]) or by Gehring's reverse Hölder inequality and used for quasilinear parabolic systems in (see e.g. [1], [6], [27]).

Another kind of higher integrability results was proved in [23] and slightly generalized in [12]. It states that under natural assumptions on the right hand side there is an exponent $p > 2$ such that $u \in L_\infty(0, T; L_p(\Omega))$; namely

Lemma 4.1. *Let u be a weak solution of*

$$u_t^i(z) - D_\alpha(A_{ij}^{\alpha\beta}(z)D_\beta u^j(z)) = f^i(z) + D_\alpha g_\alpha^i(z), \quad i = 1,\dots,N, \tag{4.1}$$

on Q with $A_{ij}^{\alpha\beta}$ satisfying ellipticity condition

$$\lambda_0|\xi|^2 \leq \langle A\xi, \xi\rangle \tag{4.2}$$

and the estimate

$$\|A_{ij}^{\alpha\beta}\|_{L_\infty(Q)} \leq \lambda_1, \quad i,j = 1,\dots,N, \quad \alpha,\beta = 1,\dots,n. \tag{4.3}$$

Put $p_0 = 2 + \frac{1}{\sqrt{n}}\frac{\lambda_0}{\lambda_1}$ and suppose

$$f^i \in L_p(Q), \quad g_\alpha^i \in L_p(Q) \quad \textit{for} \quad p \in [2, p_0). \tag{4.4}$$

Then $u \in L_{\infty,\mathrm{loc}}(0, T; L_{p,\mathrm{loc}}(\Omega))$ and $\nabla u|u|^{\frac{p}{2}-1} \in L_{2,loc}(Q)$.

For the proof see [12].

An analogous result is proved in [14] for evolution version of generalized Stokes problem for $n = 2$ by the following procedure.

We start with the time dependent Stokes problem, i.e. we study existence and qualitative properties of L-periodic functions $v = (v_1, v_2) : \mathbb{R}^2 \times (0, T) \to \mathbb{R}^2$ and $\sigma : \mathbb{R}^2 \times (0, T) \to \mathbb{R}$ with zero mean value over $\Omega = (0, L) \times (0, L)$ solving for a given $G = (G_{ij})_{i,j=1}^2$ the system

$$\begin{aligned} v_t - \Delta v + \nabla\sigma &= \operatorname{div} G \\ \operatorname{div} v &= 0 \end{aligned} \tag{4.5}$$

on $Q = \Omega \times (0, T)$ with zero initial conditions. By analogous procedure to [31] where the existence and uniqueness result was proved for Dirichlet boundary value problem, we can obtain L_2-existence and uniqueness. The L_p-theory for the Laplace equation and heat equation then implies better estimates of (v, σ).

Lemma 4.2. *If $G \in L_2(Q)$, then there exists a uniquely defined weak solution (v, σ) to (4.5) such that*

$$v \in L_2(0, T; W_{1,2}(\Omega))\,, \quad v_t \in L_2(0, T; W^*_{1,2}(\Omega)), \quad \sigma \in L_2(Q)$$

and

$$\|\nabla v\|_{L_2(Q)} \le \|G\|_{L_2(Q)}\,, \quad \|\sigma\|_{L_2(Q)} \le C\|G\|_{L_2(Q)}. \tag{4.6}$$

Moreover, if $r > 2$, $G \in L_r(Q)$, then there exists a constant $K > 1$ such that

$$v \in L_r(0, T; W_{1,r}(\Omega))\,, \quad \sigma \in L_r(Q)$$

and

$$\|\nabla v\|_{L_r(Q)} \le K\|G\|_{L_r(Q)}\,, \quad \|\sigma\|_{L_r(Q)} \le C_r\|G\|_{L_r(Q)}. \tag{4.7}$$

Perturbation arguments enable us to generalize this result for an analogous system with $-\Delta$ replaced by a general linear elliptic operator with nonsmooth coefficients.

Consider coefficients $A_{ij}^{\alpha\beta}$ bounded and measurable, satisfying symmetry conditions

$$A_{ij}^{kl} = A_{kl}^{ij}, \quad i, j, k, l = 1, 2\,, \tag{4.8}$$

and ellipticity condition

$$\forall [x,t] \in Q,\ \forall\, \xi \in \mathbb{S},\quad \gamma_1 |\xi|^2 \le \langle A(x,t)\xi, \xi\rangle \le \gamma_2 |\xi|^2, \tag{4.9}$$

where $\mathbb{S}$ stands for the space of symmetric 2×2 matrices.

We want to find L-periodic functions v, σ with zero mean values over Ω solving generalized Stokes system with zero initial conditions

$$\begin{aligned} &v_t - \operatorname{div}\,(A(x,t)Dv) + \nabla\sigma = F, \\ &\operatorname{div}\, v = 0 \end{aligned} \tag{4.10}$$

on Q. (Dv denotes the symmetrized gradient of v, i.e. $D_{ij}v = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$.)

Combining Lemma 4.2 with the methods of the proof of Lemma 4.1 we obtain

Lemma 4.3. *Let $r > 2$ and let K be the constant from* (4.7). *Then if*

$$q \le 2\Big(1 - \ln\Big[\frac{1 - \frac{\gamma_1}{2\gamma_2}}{1 - \frac{\gamma_1}{\gamma_2}}\Big] / \ln 2K\Big)^{-1} \tag{4.11}$$

(with γ_1, γ_2 from (4.9)*), there is a constant $C > 0$ such that for $F \in L_q(0,T;\, W_{-1,q}(\Omega))$ the corresponding weak solution (v, σ) of* (4.10) *satisfies*

$$\begin{aligned} &v \in L_q(0,T;\, W_{1,q}(\Omega)),\ \sigma \in L_q(Q), \\ &\|\nabla v\|_{L_q(Q)} \le \frac{C}{\gamma_1} \|F\|_{L_q(0,T;\,W_{-1,q}(\Omega))}\,, \\ &\|\sigma\|_{L_q(Q)} \le C \frac{\gamma_2}{\gamma_1} \|F\|_{L_q(0,T;\,W_{-1,q}(\Omega))}\,. \end{aligned} \tag{4.12}$$

Moreover, for such q's we have

$$\begin{aligned} &v \in L_\infty(0,T;\, L_q(\Omega)), \\ &\|v\|_{L_\infty(0,T;L_q(\Omega))} \le \frac{C\gamma_2^{1/q}}{\gamma_1} \|F\|_{L_q(0,T;\,W_{-1,q}(\Omega))}\,. \end{aligned}$$

For the proof see [14]. (Different stationary variants of Lemma 4.3 for generalized Stokes problem under both Dirichlet and periodic boundary conditions are given in [13], [14], [15], [16].)

These estimates allow to show the existence of a $C^{1,\alpha}$ solutions to the following fluid model:

Find $u : \mathbb{R}^2 \times (0, T) \to \mathbb{R}^2$, $\pi : \mathbb{R}^2 \to \mathbb{R}$ which are L-periodic in each space variable, have zero mean value over Ω and solve the equations

$$\begin{aligned} u_t + \operatorname{div}(u \otimes u) - \operatorname{div}(\mathcal{T}(Du)) + \nabla\pi &= f \\ \operatorname{div} u &= 0 \end{aligned} \tag{4.13}$$

on Q, $u(\cdot, 0) = 0$ on $\mathbb{R}^2$. Here $f : \mathbb{R}^2 \times (0, T) \to \mathbb{R}^2$ is a given space periodic vector field with zero mean value over Ω. For simplicity reasons we suppose dist (supp $f; \mathbb{R}^2 \times \{0\}) > 0$.

$\mathcal{T}$ is a stress tensor satisfying following conditions

1) $\mathcal{T}$ is a potential tensor field, i.e., there exists a nonnegative function $U \in C^2([0, \infty))$ so that for all $i, j = 1, 2,\ \eta \in \mathbb{S}$

$$\mathcal{T}_{ij}(\eta) = \frac{\partial U(|\eta|^2)}{\partial \eta_{ij}}, \quad U(0) = 0, \quad \frac{\partial U(0)}{\partial \eta_{ij}} = 0. \tag{4.14}$$

2) $U(|\eta|^2)$ satisfies growth condition with some $p \in (0, \infty)$, i.e.

$$C_1(1 + |\eta|^2)^{\frac{p-2}{2}} |\xi|^2 \le \frac{\partial^2 U(|\eta|^2)}{\partial \eta_{ij} \partial \eta_{kl}} \xi_{ij} \xi_{kl} \le C_2(1 + |\eta|^2)^{\frac{p-2}{2}} |\xi|^2, \tag{4.15}$$

for all $\eta, \xi \in \mathbb{S}$.

These assumptions involve stress tensor forms used in various engineering areas for modelling a flow of a class of non Newtonian fluids, so called fluids with shear dependent viscosity.

We are going to formulate the main result of [14]:

Theorem 4.4. *Let* $p > \frac{4}{3}$ *and let* (4.14), (4.15) *be satisfied. Assume that*

$$f \in L_\infty(0, T; L_q(\Omega)), \quad f_t \in L_2(0, T; L_q(\Omega)) \cap L_{\overline{q}}(0, T; W_{-1,\overline{q}}(\Omega))$$

with $\overline{q} > 2$ *and* q *large enough, i.e.* $q = p/(p-1)$ *for* $p \in (1, 2)$ *and* $q > 2$ *for* $p \ge 2$. *Then there exists* $\alpha > 0$ *and a solution* (u, π) *of the problem* (4.13) *such that*

$$u \in C^{1,\alpha}(Q), \quad \pi \in C^{0,\alpha}(Q).$$

Moreover, this solution is unique in the class of weak solutions that satisfy the energy inequality.

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