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ON THE CONTINUITY OF THE SOLUTION OF THE SINGULAR EQUATION $(\beta(u))_t = \mathcal{L}u$

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Dedicated to Professor Sergio Campanato on his 70th birthday

We extend some result of [2] proving the continuity of bounded solutions of the singular equation $(\beta(u))_t = \mathcal{L}u$ where $\mathcal{L}$ is a more general operator of second order.

1. Introduction.

Let $\beta(s)$ be a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ and such that:

$$\beta(s_1) - \beta(s_2) \geq \gamma_0(s_1 - s_2) \quad \forall s_i \in \mathbb{R}, \quad \gamma_0 > 0$$

and $\sup_{-M \leq s \leq M} |\beta(s)| < \infty$.

Let Ω be a domain in $\mathbb{R}^N$ of class $C^{1,1}$ and Ω_T will denote the cross product

$$\Omega_T = \Omega \times (0, T).$$

We are concerned with the local continuity of local bounded solutions of the problem:

$$(1.1)\quad (\beta(u))_t = \sum_{ij} D_i(a_{ij}(x,t)D_j u + a_i(x,t)u) + b_i(x,t)D_i u + c(x,t)u\,,$$

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where a_{ij} are measurable functions satisfying:

$$\nu|\xi|^2 \leq \sum_{ij} a_{ij}(x,t)\xi_i\xi_j \leq M|\xi|^2,$$

$$\Big\| \sum a_i^2, \sum b_i^2, c \Big\|_{q,r,\Omega_T} \leq \mu_1 ,$$

where q and r are such that

$$\frac{1}{r} + \frac{N}{2q} = 1 - \kappa_1$$

and $q \in \left[\frac{N}{2(1-\kappa_1)}, \infty\right]$, $r \in \left[\frac{1}{1-\kappa_1}, \infty\right]$, $0 < \kappa_1 < 1$, $N \geq 2$.

In the case $\beta(u) = u$ it is a classical result [4] that locally weak solutions are locally Hölder continuous.

The local continuity for local bounded solutions has been settled in the case of laplacian by Di Benedetto and Vespri [2]. They prove the continuity at a point $P \in \Omega_T$ showing that the oscillation of u in a sequence of shrinking boxes about P tends to zero as the size of such neighborhoods tend to zero.

We will follow the same lines of proof, with suitable changes due to the lack of radial simmetry for general coefficients.

We first examine the case of coefficients independent of the time. Then we achieve the general case using a fixed point theorem for coefficients continuous in t.

In dimension $n = 2$ one can consider a maximal monotone graph $\beta = \beta_{AC} + \beta_s$ of bounded variation, with β_{AC} strictly increasing and $\beta_s \geq 0$ [3].

2. Local Energy estimates.

By weak solutions of equation (1.1) we mean a function $u \in L^2(0, T; W^{1,2}(\Omega))$ such that for all $t \in (0, T)$ satisfies:

$$\int_\Omega \xi\phi\Big|_0^t + \int_\Omega \int_0^t \Big\{ -\beta(u)\phi_t + \sum_{ij}(a_{ij}D_j u + a_i u)D_i\phi +$$

$$+ b_i D_i u\phi + cu\phi\Big\}\, dxdt = 0$$

for all $\phi \in W^{1,2}(0, T; L^2(\Omega)) \bigcap L^2(0, T; W_0^{1,2}(\Omega))$

We use the same notation of [2], that we recall for sake of completeness.

For $\rho > 0$ we denote by K_ρ the cube of wedge 2ρ centered at the origin, i.e.

$$K_\rho = \{x \in \mathbb{R}^N \mid \max_{1 \le i \le N} |x_i| < \rho\}$$

and by $[y + K_\rho]$ the cube centered at y and congruent to K_ρ. For $\theta > 0$ denote by $Q(\rho, \theta\rho^2)$ the cylinder of cross section K_ρ, height $\theta\rho^2$, and vertex at the origin, i.e.

$$Q(\rho, \theta\rho^2) = K_\rho \times (-\theta\rho^2, 0)$$

and for a point $(y, s) \in \mathbb{R}^{N+1}$ we let $[(y, s) + Q(\rho, \theta\rho^2)]$ be the cylinder of vertex at (y, s) and congruent to $Q(\rho, \theta\rho^2)$.
The truncations $(u - k)_+$ and $(u - k)_-$, for $k \in \mathbb{R}$, are defined by:

$$(u - k)_+ = \max\{u - k; 0\}\ ,\ (u - k)_- = \max\{k - u; 0\},$$

Next, define

$$\mathcal{A}^{\pm}_{k,\rho}(t) = \{x \in K_\rho \mid (u(x, t) - k)_\pm > 0\}$$

introduce the numbers

$$\mathcal{H}^{\pm}_k = \left\|(u - k)_\pm\right\|_{\infty,[(y,s)+Q(\rho,\theta\rho^2)]} \quad , \quad \hat{q} = \frac{2q(1 + \kappa)}{q - 1},$$

$$\hat{r} = \frac{2r(1 + \kappa)}{r - 1} \quad , \quad \kappa = \frac{2}{N}\kappa_1$$

and the function

$$\Psi(\mathcal{H}^{\pm}_k, (u - k)_\pm, c) = \ln^+\left\{\frac{\mathcal{H}^{\pm}_k}{\mathcal{H}^{\pm}_k - (u - k)_\pm + c}\right\}, \quad 0 < c < \mathcal{H}^{\pm}_k .$$

Proposition 1. *There exists a constant* $\gamma = \gamma(\text{data})$ *such that for every cylinder* $[(y, s) + Q(\sigma\rho, \theta\sigma\rho^2)] \subset [(y, s) + Q(\rho, \theta\rho^2)]$, $\sigma \in (0, 1)$, *we get*

$$\text{(2.1)} \qquad \sup_{s-\theta\rho^2 \le t \le s} \int_{y+K_\rho} (u - k)^2_\pm(x, t)\, dx +$$

$$+ \iint_{(y,s)+Q(\rho,\sigma\theta\rho^2)} |D(u - k)_\pm|^2\, dxdt \le$$

$$\le \gamma \iint_{(y,s)+Q(\rho,\theta\rho^2)} (u - k)^2_\pm |D\zeta_1|^2\, dxdt + \gamma \iint_{(y,s)+Q(\rho,\theta\rho^2)} (u - k)_\pm\, dxdt +$$

$$+ \left\| \sum a_i^2 + \sum b_i^2 + c \right\|_{q,r} \left(\int_{-\theta\rho^2}^{0} |\mathcal{A}_{k,\rho}^{\pm}(\tau)|^{\frac{\hat{r}}{\hat{q}}} \, d\tau \right)^{\frac{2(1+\kappa)}{\hat{r}}},$$

(2.2)
$$\sup_{s-\theta\rho^2 \le t \le s} \int \Psi^2(\mathcal{H}_k^{\pm}, (u-k)_{\pm}, c)(x,t)\, dx \le$$

$$\le \frac{\gamma(\text{data})}{(1-\sigma^2)\rho^2} \iint \Psi(\mathcal{H}_k^{\pm}, (u-k)_{\pm}, c)\, dx d\tau +$$

$$+ \frac{\gamma(\text{data})}{c} \int \Psi(\mathcal{H}_k^{\pm}, (u-k)_{\pm}, c)(x, s-\theta\rho^2) +$$

$$+ \frac{\gamma}{c^2}\left(1 + \frac{\ln \mathcal{H}_k^{\pm}}{c}\right) \left\{ \int_{s-\theta\rho^2}^{s} |\mathcal{A}_{k,\rho}^{\pm}(\tau)|^{\frac{\hat{r}}{\hat{q}}} \, d\tau \right\}^{\frac{2(1+\kappa)}{\hat{r}}}.$$

Proof. We may assume that (y, s) coincides with the origin. Let $x \to \zeta_1(x)$ be a nonnegative cut-off function in K_ρ such that

$$\begin{cases} \zeta_1 \equiv 1 & \text{on } K_{\sigma\rho}, \sigma \in (0,1) \\ \zeta_1(x) = 0 & \text{for } x \in \partial K_\rho \\ |D\zeta_1| \le \dfrac{1}{(1-\sigma)\rho} \end{cases}$$

and $t \to \zeta_2(t)$ the cut-off function

$$\zeta_2(t) = \begin{cases} 0 & \text{for } t \in (-\infty, -\theta\rho^2) \\ \dfrac{t+\theta\rho^2}{(1-\sigma)\theta\rho^2} & \text{for } t \in (-\theta\rho^2, -\sigma\theta\rho^2) \\ 1 & \text{for } t \ge -\sigma\theta\rho^2. \end{cases}$$

We multiply (1.1) by the test function

$$\pm (u-k)_{\pm} \zeta_1^2 \zeta_2^2$$

and integrate by parts over $K_\rho \times (-\theta\rho^2, t)$. For simplicity we indicate by ζ the product $\zeta_1\zeta_2$.

$$I = \pm \iint \sum_{ij} (a_{ij} D_j u + a_i u) D_i[(u-k)_{\pm} \zeta^2]\, dx d\tau \ge$$

$$\ge c \iint |D(u-k)_{\pm}|^2 \zeta^2 \, dx d\tau - \iint \sum a_i^2 \zeta^2 \chi[(u-k)_{\pm} > 0]\, dx d\tau -$$

$$-2C_1 \iint |D(u-k)_\pm|(u-k)_\pm \zeta |D\zeta| \, dx d\tau -$$

$$-2 \iint (\sum a_i^2)^{\frac{1}{2}} (u-k)_\pm \zeta |D\zeta| \, dx d\tau \, .$$

We use twice Young's inequality:

$$2C_1 \iint |D(u-k)_\pm|(u-k)_\pm \zeta |D\zeta| \, dx d\tau \leq \tag{2.3}$$

$$\leq C_0 \iint |D(u-k)_\pm|^2 \zeta^2 \, dx d\tau + \gamma(C_0) \iint (u-k)_\pm^2 |D\zeta|^2 \, dx d\tau \, ,$$

$$2 \iint (\sum a_i^2)^{\frac{1}{2}} (u-k)_\pm \zeta |D\zeta| \, dx d\tau \leq \tag{2.4}$$

$$\leq \iint (u-k)_\pm^2 |D\zeta|^2 \, dx d\tau + \gamma \iint \sum a_i^2 \zeta^2 \chi[(u-k)_\pm > 0] \, dx d\tau \, .$$

Therefore we get:

$$I \geq C_0 \iint |D(u-k)_\pm \zeta|^2 \, dx d\tau - \tag{2.5}$$

$$-\gamma \iint (u-k)_\pm^2 |D\zeta|^2 \, dx d\tau - \gamma \iint \sum a_i^2 \zeta^2 \chi[(u-k)_\pm > 0] \, dx d\tau \, ,$$

$$II = \iint \left| [b_i D_i u + cu](u-k)_\pm \zeta^2 \right| dx dt \leq$$

$$\leq c_2 \iint |D(u-k)_\pm|^2 (u-k)_\pm \zeta^2 \, dx d\tau + \iint \left| \sum b_i^2 + c \right| (u-k)_\pm \zeta^2 \, dx d\tau \, .$$

We impose to k the restrictions

$$\operatorname*{ess\,sup}_{Q(\theta\rho)} |(u-k)_\pm| \leq \delta_0 = \frac{C_0}{4C_2}$$

Then:

$$II \leq \frac{C_0}{4} \iint \left| D(u-k)_\pm \zeta \right|^2 dx d\tau + \iint \left| \sum b_i^2 + c \right| (u-k)_\pm \zeta^2 \, dx d\tau \leq$$

$$\leq \frac{C_0}{4}\iint |D(u-k)_\pm \zeta|^2\, dxd\tau + \delta_0 \iint \Big|\sum b_i^2 + c\Big|\chi[(u-k)_\pm]\, dxd\tau .$$

In conclusion

$$\sup_{-\theta\rho^2\leq t\leq 0}\int_{K_\rho}\left(\int_0^{(u-k)_\pm}\beta'(k\pm s)s\, ds\right)\zeta_1^2(x)\zeta_2^2(t)\, dx +$$

$$+\iint_{Q(\rho,\sigma\theta\rho^2)} |D(u-k)_\pm|^2\, dxdt \leq$$

$$\leq \gamma\iint_{Q(\rho,\theta\rho^2)}(u-k)_\pm^2|D\zeta_1|^2\, dxdt +$$

$$+\gamma\iint_{Q(\rho,\theta\rho^2)}\left(\int_0^{(u-k)_\pm}\beta'(k\pm s)\, ds\right)\zeta_1^2(x)\zeta_{2,t}(t)\, dxdt +$$

$$+\iint \Big|\sum a_i^2 + \sum b_i^2 + c\Big|\chi[(u-k)_\pm > 0]\, dxd\tau .$$

Taking into account that

$$\int_0^{(u-k)_\pm}\beta'(k\pm s)s\, ds \geq \frac{\gamma_0}{2}(u-k)_\pm^2 ,$$

$$\int_0^{(u-k)_\pm}\beta'(k\pm s)s\, ds \leq \sup|\beta(s)|(u-k)_\pm$$

and using Hölder inequality, we reach the assertion (2.1).

To prove (2.2), take as test function $\pm\Psi\Psi'\zeta_1^2(x)$ and write

$$\pm\beta'(u)u_t\Psi\Psi' = \frac{\partial}{\partial t}\int_0^{(u-k)_\pm}\beta'(k\pm s)\Psi\Psi'\, ds$$

for $t\in(-\theta\rho^2, 0)$

$$\sup_{-\theta\rho^2\leq t\leq 0}\int_{K_{\sigma\rho}}\left(\int_0^{(u-k)_\pm}\beta'(k\pm s)\Psi\Psi'\, ds\right)\zeta_1^2(x)\, dx +$$

$$+\iint_{Q(\rho,\theta\rho^2)}(1+\Psi)\Psi'^2|D(u-k)_\pm|^2\zeta_1^2(x)\, dxdt \leq$$

$$\leq \int_{K_{\sigma\rho}} \left(\int_0^{(u-k)_\pm} \beta'(k \pm s)\Psi\Psi' \, ds \right)(x - \theta\rho^2) \, dx +$$

$$+ 2 \iint \phi_0(x,t)\zeta^2(1+\Psi)\Psi'^2 \, dxdt +$$

$$+ \iint \Psi\Psi'^2\phi_1\zeta^2 \, dxdt + \frac{c_0}{2} \iint |Du|^2(1+\Psi)\Psi'^2\zeta^2 \, dxdt +$$

$$+ \frac{1}{c}\log\left(\frac{\mathcal{H}_k^\pm}{c}\right) \iint \phi_2 \chi[(u-k)_\pm > 0] \, dxdt \, .$$

Using that

$$\int_0^{(u-k)_\pm} \beta'(k \pm s)\Psi\Psi' \, ds \geq \frac{\gamma_0}{2}\Psi^2,$$

we get

$$\sup_{-\theta\rho^2 \leq t \leq 0} \int_{K_{\sigma\rho}} \int \Psi^2(\mathcal{H}_k^\pm, (u-k)_\pm, c)(x,t) \, dx \leq$$

$$\leq \frac{\gamma(\text{data})}{(1-\sigma^2)\rho^2} \iint \Psi(\mathcal{H}_k^\pm, (u-k)_\pm, c)(x,t) \, dxdt +$$

$$+ \frac{\gamma(\text{data})}{c} \iint \Psi(\mathcal{H}_k^\pm, (u-k)_\pm, c)(x, s - \theta\rho^2) \, dx +$$

$$+ \frac{\gamma}{c^2}\left(1 + \log\left(\frac{\mathcal{H}_k^\pm}{c}\right)\right) \iint (\phi_0 + \phi_1^2 + \phi_2)\chi[(u-k)_\pm > 0] \, dxdt$$

and this last term is substituted by

$$\frac{\gamma}{c^2}\left(1 + \log\left(\frac{\mathcal{H}_k^\pm}{c}\right)\right)\left\{ \int_{t_0-\theta}^{t_0} |\mathcal{A}_{k,\rho}^\pm(\tau)|^{\frac{\hat{r}}{\hat{q}}} \, d\tau \right\}^{\frac{2(1+\kappa)}{r}} . \qquad \square$$

Remark 1. *The estimate* (2.1) *holds true even in more general hypotheses*

a) $\operatorname{div}(a(x,t,u,Du)Du \geq C_0|Du|^2 - \phi_0(x,t)$,
b) $|a(x,t,u,Du)| \leq C_1|Du| + \phi_1(x,t)$,
c) $|b(x,t,u,Du)| \leq C_2|Du|^2 + \phi_2$,

where $\phi_0, \phi_1^2, \phi_2 \in L^{q,r}(\Omega_T)$.

Remark 2. *Along the proofs we will encounter quantities of the type $A_i \rho^{N\kappa} \omega^{-2}$, where A_i are constants that can be determined a priori only in terms of the data and are independent of ω and ρ. We may assume, without loss of generality, that they satisfy $A_i \rho^{N\kappa} \omega^{-2} \leq 1$. Indeed, if not, we would have $\omega \leq C\rho^{\varepsilon_0}$ for $C = \max A_i$ and $\varepsilon_0 = \frac{Nk}{2}$ and the first iterative step would be trivial.*

Fix $\theta > 0$ and consider $[(y, s) + Q(2\rho, 2\theta\rho^2)] \subset \Omega_T$, we put

$$\mu^+ = \sup_{[(y,s)+Q(2\rho,2\theta\rho^2)]} u \quad , \quad \mu^- = \inf_{[(y,s)+Q(2\rho,2\theta\rho^2)]} u$$

and

$$\omega = \operatorname*{osc}_{[(y,s)+Q(2\rho,2\theta\rho^2)]} u = \mu^+ - \mu^-.$$

We define the following level sets

$$\mathcal{A}^+_{\xi^+,\rho} = \{(x, t) \in [(y, s) + Q(\rho, \theta\rho^2)] : u(x, t) > \mu^+ - \xi^+\omega\},$$

$$\mathcal{A}^-_{\xi^-,\rho} = \{(x, t) \in [(y, s) + Q(\rho, \theta\rho^2)] : u(x, t) < \mu^- + \xi^-\omega\}.$$

We have the following estimates

Proposition 2. *There exists a number ν^+ depending on the structure of $\beta, \lambda, \xi^+, \omega$ such that*

$$\operatorname{meas} \mathcal{A}^+_{\xi^+,\rho} < \nu^+|Q(\rho, \theta\rho^2)| \Longrightarrow u(x, t) < \mu^+ - \lambda\xi^+\omega,$$

$$\forall (x, t) \in [(y, s) + Q(\frac{\rho}{2}, \frac{\theta\rho^2}{2})].$$

Proposition 3. *There exists a number ν^- depending on the structure of β, λ, ξ^-, ω such that*

$$\operatorname{meas} \mathcal{A}^-_{\xi^-,\rho} < \nu^-|Q(\rho, \theta\rho^2)| \Longrightarrow u(x, t) > \mu^- + \lambda\xi^-\omega,$$

$$\forall (x, t) \in [(y, s) + Q(\frac{\rho}{2}, \frac{\theta\rho^2}{2})].$$

Proof. We only prove the case $+$. The numbers $\nu^{\pm}$ are given by the formula:

$$\nu^{\pm} = \frac{c}{\theta}\Big(\frac{\theta\xi^{\pm}w}{1+\theta\xi^{\pm}w}\Big)^{\frac{1+\kappa}{\sigma}}, \quad \sigma = \min\{\frac{2}{N}\kappa_1; \frac{2}{N+2}\} \quad c = c(\text{data}, \omega).$$

Without loss of generality, we may suppose $(y, s) = (0, 0)$ and $\xi^{+} = \xi$. For $n = 0, 1, 2, \ldots$, we consider the sequences of radii

$$\rho_n = \frac{\rho}{2} + \frac{\rho}{2^{n+1}}, \quad \tilde{\rho}_n = \frac{\rho_n + \rho_{n+1}}{2}$$

and the sequence

$$\xi_n = \lambda\xi + (1-\lambda)\frac{\xi}{2^n}, \quad k_n = \mu^{+} - \xi_n\omega,$$

$$\tilde{Q}_n = K_{\tilde{\rho}_n} \times (-\theta\tilde{\rho}_n, 0), \quad Q_n = K_{\rho_n} \times (-\theta\rho_n, 0).$$

Finally, we get

$$\text{(2.6)} \quad \sup_{-\theta\tilde{\rho}_n^2 \le t \le 0} \int_{K_{\tilde{\rho}_n}} (u-k_n)_+^2\, dx + \iint_{Q(\tilde{\rho}_n, \sigma\theta\tilde{\rho}_n^2)} |D(u-k_n)_+|^2 1_{K_n}\, dxdt \le$$

$$\le \gamma\frac{4^n\xi^2\omega^2}{\rho^2}[1 + (\theta\xi\omega)^{-1}]|\mathcal{A}_{\xi_n,\rho_n}| +$$

$$+ \Big\|\sum a_i^2 + \sum b_i^2 + c\Big\|_{q,r}\Big\{\int_{-\theta\tilde{\rho}_n}^{0} |\mathcal{A}^{+}_{\xi_n,\rho_n}(\tau)|^{\frac{q-1}{q}\frac{r}{r-1}}\, d\tau\Big\}^{\frac{r-1}{r}}.$$

Now, we are in the position to repeat the same argument as in [2]. Let $\tilde{\zeta}_n(x)$ be a cut-off function in $K_{\tilde{\rho}_n}$, $\tilde{\zeta}_n(x) = 1$ on $K_{\rho_{n+1}}$ and $|D\tilde{\zeta}_n| \le \frac{2^{n+3}}{\rho}$. The function $(u-k_n)_+\tilde{\zeta}_n$ belongs to

$$L^{\infty}(-\theta\tilde{\rho}_n^2, 0; L^2(K_{\tilde{\rho}_n})) \cap L^2(-\theta\tilde{\rho}_n^2, 0; W_0^{1,2}(K_{\tilde{\rho}_n})),$$

we apply the embedding theorem:

$$\text{(2.7)} \quad (\xi_n - \xi_{n+1})^2\omega^2|\mathcal{A}_{\xi_{n+1},\rho_{n+1}}| \le \iint_{Q_{n+1}} (u-k_n)_+^2\, dxdt \le$$

$$\le \iint_{\tilde{Q}_n} (u-k_n)_+^2\zeta_n^2\, dxdt \le \Big(\iint_{\tilde{Q}_n} [(u-k_n)_+\zeta_n]^{2\frac{N+2}{N}}\, dxdt\Big)^{\frac{N}{N+2}} |\mathcal{A}_{\xi_n,\rho_n}|^{\frac{2}{N+2}} \le$$

$$\leq \gamma \left(\iint_{\tilde{Q}_n} |D(u-k_n)_+ \tilde{\zeta}_n|^2 \, dxdt \right)^{\frac{N}{N+2}} \cdot$$

$$\cdot \left(\sup_{-\theta \tilde{\rho}_n^2 \leq t \leq 0} \int_{K_{\tilde{\rho}_n}} (u-k_n)_+{}^2 \, dx \right)^{\frac{2}{N+2}} |\mathcal{A}_{\xi_n,\rho_n}|^{\frac{2}{N+2}}.$$

Now we compute

$$\iint_{\tilde{Q}_n} |D(u-k_n)_+ \tilde{\zeta}_n|^2 \, dxdt \leq$$

$$\leq \iint_{\tilde{Q}_n} |D(u-k_n)_+|^2 \, dxdt + \iint_{\tilde{Q}_n} |D\tilde{\zeta}_n|^2 (u-k_n)_+^2 \, dxdt .$$

Taking into account inequality (2.6), yields

$$\text{(2.8)} \qquad \iint_{\tilde{Q}_n} |D(u-k_n)_+ \tilde{\zeta}_n|^2 \, dxdt \leq \gamma \frac{4^n \xi^2 \omega^2}{\rho^2} [1 + (\theta \xi \omega^{-1})] |\mathcal{A}_{\xi_n,\rho_n}| +$$

$$+ \left\| \sum a_i^2 + \sum b_i^2 + c \right\|_{q,r} \left\{ \int_{-\theta \tilde{\rho}_n}^0 |\mathcal{A}^+_{\xi_n,\rho_n}(\tau)|^{\frac{\hat{q}}{\hat{r}}} \, d\tau \right\}^{\frac{2}{\hat{r}}(1+\kappa)},$$

where $\hat{q}, \hat{r}, \kappa = \frac{2}{N}\kappa_1$ have been introduced in Section 2.

Also the second factor of the right-hand side in (2.7) can be estimated analogously, combining (2.6) and (2.8), finally we get:

$$\text{(2.9)} \qquad |\mathcal{A}_{\xi_{n+1},\rho_{n+1}}| \leq \gamma \frac{16^n}{\rho^2} [1 + (\theta \xi \omega)^{-1}] |\mathcal{A}_{\xi_n,\rho_n}|^{1+\frac{2}{N+2}} +$$

$$+ \gamma 16^n \| \sum a_i^2 + \sum b_i^2 + c \|_{q,r} |\mathcal{A}_{\xi_n,\rho_n}|^{\frac{2}{N+2}} \left\{ \int_{-\theta \tilde{\rho}_n}^0 |\mathcal{A}^+_{\xi_n,\rho_n}(\tau)|^{\frac{\hat{q}}{\hat{r}}} d\tau \right\}^{\frac{2}{\hat{r}}(1+\kappa)}.$$

We put

$$Y_n = \frac{|\mathcal{A}_{\xi_n,\rho_n}|}{|Q_n|} \quad , \quad Z_n = \frac{1}{K_{R_n}} \left\{ \int_{-\theta \tilde{\rho}_n}^0 |\mathcal{A}^+_{\xi_n,\rho_n}(\tau)|^{\frac{\hat{q}}{\hat{r}}} \, d\tau \right\}^{\frac{2}{\hat{r}}}$$

and divide (2.9) by $|Q_n|$

$$Y_{n+1} \leq \gamma 16^n \theta^{\frac{2}{N+2}} [1 + (\theta \xi \omega)^{-1}] \{ Y_n^{1+\frac{2}{N+2}} + Y_n^{\frac{2}{N+2}} Z_n^{1+\kappa} \},$$

$$Z_{n+1}(k_n - k_{n+1})^2 \le |K_{R_n}|^{-1} \|(u-k_n)\|^2_{q,r,Q_{n+1}} \le$$
$$\le |K_{R_n}|^{-1} \|(u-k_n)\tilde{\zeta}_n\|^2_{q,r,Q_n} \le \gamma R^{-N} \|(u-k_n)\tilde{\zeta}_n\|^2_{V^2(Q_n)},$$
$$Z_{n+1} \le \gamma 16^n \theta^{\frac{2}{N+2}} [1 + (\theta\xi\omega)^{-1}]\{Y_n + Z_n^{1+\kappa}\}.$$

the sequences $\{Y_n\}$ and $\{Z_n\}$ tend to zero, provided

$$Y_0 + Z_0^{1+\kappa} \le \nu_0$$

(see Lemma 4.2 in [1]). □

Fix $\theta > 0$ and consider the cylinder $[(y,s) + Q(\rho, \theta\rho^2)]$; for $\xi \in (0,1)$ we set

$$\mathcal{A}_{\xi_+,\rho} = \{x \in K_\rho : u(x,t) > \mu^+ - \xi^+\omega\}.$$

We assume that the function $u(\cdot, s - \theta\rho^2)$ does not exceed the value $\mu^+ - \xi_0\omega$ for some $\xi_0 \in (0,1)$ at the bottom of the cylinder, i.e.

$$u(x, s-\theta\rho^2) \le \mu^+ - \xi_0^+\omega, \quad \forall x \in [y + K_\rho].$$

Proposition 4. *For every $\theta \in (0,1)$ there exists a number $\xi^+ \in (0, \frac{1}{4}\xi_0^+)$ depending only upon the data and the numbers ξ_0^+ and θ such that*

$$|\mathcal{A}_{\xi_+,\frac{1}{2}\rho}| \le \nu^+ |K_{\frac{1}{2}\rho}|, \quad \forall t \in (s - \theta\rho^2, s).$$

The number ξ^+ is chosen to satisfy $\nu^+ = \frac{\gamma\theta}{\ln(\xi_0^+/2\xi^+)}$.

Proof. Without loss of generality we can assume that $(y,s) = (0,0)$. Consider the logarithmic estimate (2.2) for $(u-k)_+$ with $k = \mu^+ - \xi_0\omega$, $\sigma = \frac{1}{2}$, $c = \xi_+\omega$, where ξ_+ need to be chosen. We first observe that the integral on the right at the time level $-\theta\rho^2$ is zero. The first integral on the right is majorised by $2\gamma\theta \left| \log\left(\frac{\xi_0^+}{2\xi^+}\right) \right| |K_\rho|$. The integral on the left hand side is minorised extending the integration over a smaller set $\mathcal{A}^+_{\xi^+,\frac{1}{2}\rho}$. On such a set $\Psi \ge \log\left(\frac{\xi_0^+}{2\xi^+}\right)$; therefore the logarithmic inequality (2.2) reads as follows:

$$\left(\log\left(\frac{\xi_0^+}{2\xi^+}\right)\right)^2 |\mathcal{A}^+_{\frac{1}{2}\rho}(t)| \le \gamma\theta \left| \log\left(\frac{\xi_0^+}{2\xi^+}\right) \right| |K_{\frac{1}{2}\rho}|.$$

The last term is estimated by

$$\gamma\left(\frac{1}{\xi^+\omega}\right)^2 \left(1 + \log\left(\frac{\xi_0^+}{\xi^+}\right)\right) \rho^{N\kappa} |K_{\frac{\rho}{2}}|$$

and we can choose the parameters in order to make it ≤ 1. □

We state a proposition that can be found in a more general way in [2].

Proposition 5. *Let* $v \in W^{1,2}(K_\rho)$, *satisfying*

$$\int_{K_\rho} |Dv|^2 \, dx \leq \gamma$$

for a given constant γ *and* $\text{meas}\{x \in K_\rho : v(x) < 1\} \geq \alpha |K_\rho|$ *for a given* $\alpha \in (0,1)$. *Then, for every* $\eta \in (0,1)$, *and* $\lambda > 1$, *there exists* $x^* \in K_\rho$ *and a number* $\delta \in (0,1)$ *such that, within the cube* $K_{\delta\rho}(x^*)$ *centered in* x^* *with wedge* $2\delta\rho$, *there holds:*

$$\text{meas}\{x \in K_{\delta\rho}(x^*) : v(x) < \lambda\} > (1-\eta)|K_{\delta\rho}|.$$

3. On the sets where u is near μ^+ or near μ^-.

Define $\mathcal{A}^+_{\xi^+,\rho}(t) = \{x \in K_\rho : u(x,t) > \mu^+ - \xi^+\omega\}$ and $\mathcal{A}^-_{\xi^-,\rho}(t) = \{x \in K_\rho : u(x,t) < \mu^- - \xi^-\omega\}$, we have

$$\text{meas}\, \mathcal{A}^\pm_{\xi^\pm,\rho} = \int_{-\rho^2}^{0} |\mathcal{A}^\pm_{\xi^\pm,\rho}(t)| \, dt \, .$$

Observe that the numbers $\nu^\pm$ are the ones introduced in Proposition 2 and Proposition 3.

Proposition 6. *If*

$$\text{meas}\, \mathcal{A}^+_{\xi^+,\rho} = \int_{-\rho^2}^{0} |\mathcal{A}^+_{\xi^+,\rho}(t)| \, dt > \nu^+ |Q(\rho,\rho^2)| \tag{3.1}$$

holds, for every $\lambda > 1$ *and* $\eta \in (0,1)$ *there exist a point* $(y^*_+, s^*_+) \in [(y,s) + Q(\delta_+\rho, \delta^2_+\rho^2)] \subset [(y,s) + Q(\rho,\rho^2)]$ *such that*

$$\text{meas}\{(x,t) \in [(y^*_+, s^*_+) + Q(\delta_+\rho; \delta^2_+\rho^2)] : u(x,t) > \mu^*_+ - \lambda\xi^+\omega^*\} > \tag{3.2}$$

$$> (1-\eta)|[(y^*_+, s^*_+) + Q(\delta_+\rho, \delta^2_+\rho^2)]|.$$

The number δ^+ *depends upon the data and the numbers* λ, η, ξ^+ *and* ω.

Proposition 7. *If*

$$\text{meas}\, \mathcal{A}^-_{\xi^-,\rho} = \int_{-\rho^2}^{0} |\mathcal{A}^-_{\xi^-,\rho}(t)|\, dt > \nu^- |Q(\rho, \rho^2)| \tag{3.3}$$

holds, for every $\lambda > 1$ *and* $\eta \in (0, 1)$ *there exist a point* $(y^*_-, s^*_-) \in [(y, s) + Q(\delta_- \rho, \delta^2_- \rho^2)] \subset [(y, s) + Q(\rho, \rho^2)]$ *such that*

$$\text{meas}\{(x, t) \in [(y^*_-, s^*_-) + Q(\delta_+ \rho, \delta^2_- \rho^2)] : u(x, t) < \mu^*_- + \lambda \xi^- \omega^*\} > \tag{3.4}$$

$$> (1 - \eta) |[(y^*_-, s^*_-) + Q(\delta_- \rho, \delta^2_- \rho^2)]|.$$

The number δ^- *depends upon the data and the numbers* λ, η, ξ^- *and* ω.

Proof. We write (2.1) on $Q(\rho, \rho^2)$ and $Q(2\rho, 2\rho^2)$ respectively for the functions $(u - k^+)_+$ with $k^+ = \mu^*_+ - \xi^+ \omega^*$ and $(u - k^-)_-$ with $k^- = \mu^*_- + \xi^- \omega^*$ and take into account that the term

$$\left\{ \int_{-\theta \tilde{\rho}_n}^{0} |\mathcal{A}^+_{\xi_n, \rho_n}(\tau)|^{\frac{\hat{q}}{\hat{r}}}\, d\tau \right\}^{\frac{2}{\hat{r}}(1+\kappa)}$$

is controlled by $\gamma \omega \rho^N$.

$$\iint_{Q(\rho,\rho^2)} |D(\mu^+ - u)|^2\, dx dt \le \gamma \omega^* \rho^N, \tag{3.5}$$

$$\iint_{Q(\rho,\rho^2)} |D(u - \mu^-)|^2\, dx dt \le \gamma \omega^* \rho^N. \tag{3.6}$$

We rewrite (3.5) and (3.2) in terms of $v^+ = \frac{\mu^*_+ - u}{\omega^* \xi^+}$, (3.6) and (3.4) in terms of $v^- = \frac{u - \mu^*_-}{\omega^* \xi^-}$

$$\text{meas}\{(x, t) \in Q(\rho, \rho^2) : v^\pm < 1\} > \nu^*_\pm |Q(\rho, \rho^2)|, \tag{3.7}$$

$$\iint_{Q(\rho,\rho^2)} |Dv^\pm|^2\, dx dt \le \frac{\gamma}{\omega^* \xi^{\pm 2}} \rho^N. \tag{3.8}$$

For $t \in (-\rho^2, 0)$ we put

$$\mathcal{A}^\pm(t) = \{x \in K_\rho : v^\pm(x, t) < 1\},$$

$$\mathcal{T}_\pm = \{t \in (-\rho^2, 0) : \text{meas}\, \mathcal{A}^\pm(t) > \frac{1}{2} \nu^*_\pm |K_\rho|\},$$

$\mathcal{T}_\pm^c$ the complement of $\mathcal{T}_\pm$ with respect to $(-\rho^2, 0)$

$$\nu_\pm^* |Q(\rho, \rho^2)| \le \int_{-\rho^2}^0 \int_{K_\rho} \text{meas}\, \mathcal{A}^\pm(t)\, dt = \int_{\mathcal{T}_\pm} \int_{K_\rho} \text{meas}\, \mathcal{A}^\pm(t)\, dt +$$

$$+ \int_{\mathcal{T}_\pm^c} \int_{K_\rho} \text{meas}\, \mathcal{A}^\pm(t)\, dt \le \frac{\text{meas}\, \mathcal{T}_\pm}{\rho^2} |Q(\rho, \rho^2)| + \frac{1}{2} \nu_\pm^* |Q(\rho, \rho^2)|.$$

From which we get

$$\text{meas}\, \mathcal{T}_\pm \ge \frac{1}{2} \nu_\pm^* \rho^2.$$

From this and (3.7) we get:

$$\frac{1}{2} \nu_\pm^* \rho^2 \inf_{t \in \mathcal{T}_\pm} \int_{K_\rho} |Dv^\pm(x, t)|^2\, dxdt \le \int_{\mathcal{T}_\pm} \int_{K_\rho} |Dv^\pm(x, t)|^2\, dxdt \le$$

$$\le \iint_{Q(\rho, \rho^2)} |Dv^\pm(x, t)|^2\, dxdt \le \frac{\gamma}{\xi^{\pm 2} \omega^*} \rho^N .$$

Therefore there exist time levels s_+^* and s_-^* for which

$$\int_{K_\rho} |Dv^\pm(x, s_\pm^*)|^2\, dx \le \frac{\gamma}{\omega^* \xi^{\pm 2}} \rho^{N-2}$$

and

$$\text{meas}\{x \in K_\rho : v^\pm(x, s_\pm^*) < 1\} \ge \frac{1}{2} \nu_\pm^* |K_\rho|.$$

Now we apply Proposition 5 to conclude. □

Let the cylinder $[(y, s) + Q(\rho, \rho^2)]$ be fixed and consider coaxial boxes of the type

$$[(y, \tau) + Q(r, r^2)], \quad 0 < r \le \rho . \tag{3.9}$$

The time-location of the vertices ranges over

$$\tau \in [s - (\rho^2 - r^2), s] \tag{3.10}$$

and r is a positive parameter ranging over

$$r \in [\delta\rho, \rho], \quad \text{where } \delta \in (0, 1) \text{ is to be chosen.} \tag{3.11}$$

We assume that conditions (3.1) and (3.3) are violated for all cylinders of the type previously defined. In such a case, we will identify , within $[(y, s) + Q(\rho, \rho^2)]$ two disjoint subcylinders such that in one of this u is all near μ^+ and in the other u is all near μ^-.

Proposition 8. *Let* (3.1) *and* (3.3) *both hold for all coaxial cylinders of the type previously defined. There exist two points* (y_1^*, s^*), (y_2^*, s^*), *at the same time level* s^*, *a number* $\delta \in (0, 1)$ *and two cylinders*

$$[(y_1^*, s^*) + Q(r, r^2)], \quad [(y_2^*, s^*) + Q(r, r^2)], \quad r = \delta_0 \rho,$$

contained in $[(y, s) + Q(\rho, \rho^2)]$, *such that*

$$u(x, t) > \mu^- + \frac{2}{3}(1 - \lambda\xi^+)\omega, \quad \forall (x, t) \in [(y_1^*, s^*) + Q(r, r^2)]$$

and

$$u(x, t) < \mu^+ - \frac{2}{3}(1 - \lambda\xi^-)\omega, \quad \forall (x, t) \in [(y_2^*, s^*) + Q(r, r^2)].$$

The proof of this Proposition is the same of Proposition 8.1 in [2]. Using Proposition 8 it is possible to derive a local estimate for the gradient Du.

Proposition 9. *Let* (3.1) *and* (3.3) *both hold for all coaxial cylinders previously defined and choice* $\xi^+ = \xi^- = \frac{1}{12}$ *and* $\lambda = \frac{3}{2}$. *There exists a constant* γ *depending only upon the data and* ω *such that*

$$\rho^N \omega^2 \gamma \leq \int_{s-\rho^2}^{s} \int_{\delta\rho < \|x-y\| < \rho} |Du|^2 \, dx dt \, .$$

4. Comparison function.

We consider first the case of coefficients independent of the time t. We consider an auxiliary function constructed with the difference of the parabolic problem in a circular cylindric section and an elliptic problem in a circular anulus. We consider ζ solution of the elliptic equation in a circular cross section $\mathcal{A}_{\varepsilon_0,4d}$

$$\text{(4.1)} \qquad \begin{cases} \mathcal{L}\zeta = 0 & \text{on } \mathcal{A}_{\varepsilon_0,4d} \\ \zeta(x) = 0 & \text{in } |x| = 4d \\ \zeta(x) = 1 & \text{in } |x| = \varepsilon_0 \end{cases}$$

and v solution of

$$\text{(4.2)} \qquad \begin{cases} (\tilde{\beta}(v))_t = \mathcal{L}v & \text{on } \mathcal{A}_{\varepsilon_0,4d} \times (0, k) \\ v(x, t) = 0 & \text{in } |x| = 4d \\ v(x, t) = 1 & \text{in } |x| = \varepsilon_0 \\ v(x, 0) = 0. & \end{cases}$$

Notice that ζ is locally Hölder continuous and v satisfies the maximum principle of Moser. Put $z = \zeta - v$, it satisfies:

$$(4.3)\qquad \begin{cases} (\tilde{\beta}(z))_t = \mathcal{L}z & \text{on } \mathcal{A}_{\varepsilon_0,4d} \times (0,k) \\ z(x,t) = 0 & \text{in } |x| = 4d \\ z(x,t) = 1 & \text{in } |x| = \varepsilon_0 \\ z(x,0) = 1 & \text{in } |x| = \varepsilon_0 \,. \end{cases}$$

We observe that z is positive and satisfies Harnack inequality. Here $\tilde{\beta}$ is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ satisfying:

$$\frac{\tilde{\beta}(s_1) - \tilde{\beta}(s_2)}{s_1 - s_2} \geq \tilde{\gamma}_0\,, \quad \sup |\tilde{\beta}(s)| \leq \tilde{\gamma}_1$$

for some given constants $\tilde{\gamma}_i$. The quantities $\varepsilon_0, d, \tilde{\gamma}_i$ will be chosen in dependance of the local oscillation of u.

We will prove the continuity of z in the cylindrical domain $C(1, 2d; k)$:

Proposition 10. *The function $z \to z(x,t)$ is continuous in the cylinder with anular cross section $C(1, 2d; k)$. Moreover there exists a nonnegative, continuous, increasing function $\rho \to h(\rho)$ such that:*

$$\forall x_0 \in \{1 < |x| < 2d\}, \quad \forall t \in (0,k), \quad \forall \rho \in (0, \frac{1}{2}),$$

$$\sup_{|x-x_0|<\rho} |z(x,t) - z(x_0,t)| \leq h(\rho).$$

We will indicate with ω a number such that

$$\omega \geq \operatorname*{osc}_{C(\varepsilon_0,4d;k)} u\,.$$

We will be working in a cylindrical domain slightly larger than $C(\varepsilon_0, 4d; k)$, say, for example:

$$\tilde{C}(1, 2d; k) = \{\frac{1}{2} < |x| < 2d\} \times (0,k).$$

For a cylinder $[(y,s) + Q(2\rho, 2\theta\rho^2)] \subset \tilde{C}(1, 2d; k)$ we set

$$z^+ = \sup_{[(y,s)+Q(2\rho,2\theta\rho^2)]} z \quad , \quad z^- = \inf_{[(y,s)+Q(2\rho,2\theta\rho^2)]} z$$

and denote by w a number satisfying:

$$w \geq z^+ - z^- = \operatorname*{osc}_{[(y,s)+Q(2\rho,2\theta\rho^2)]} z\,.$$

The function z satisfies the energy estimates of Section 2. For the logarithmic estimate, Proposition 4 continue to hold for the function z. In this context, having fixed $\nu^\pm \in (0,1)$, and $\xi_0^\pm \in (0,1)$, the numbers $\xi^\pm$ for which the analogues of Proposition 2 and 3 hold are chosen from the formulae

$$\nu^\pm = \frac{\gamma(\text{data}, w)}{\ln\left(\frac{\xi_0^\pm}{\xi^\pm}\right)}\,.$$

Putting z as test function in the equation, we obtain:

$$\int_0^k \int_{\mathcal{A}_{\varepsilon_0,4d}} \left(\frac{\partial}{\partial t}\int_0^z \beta'(s)s\,ds\right)dxdt + \frac{\nu}{2}\iint |Dz|^2\,dxdt \leq$$

$$\leq \int_0^k \frac{\nu}{2}\int |Dz|^2 + \left(\frac{1}{\nu}\sum a_i^2 + \frac{1}{\nu}\sum b_i^2 + |c|\right)z^2\,dxdt\,,$$

$$\int_{\mathcal{A}_{\varepsilon_0,4d}} z^2(x,t)|_0^k + \nu\iint |Dz|^2\,dxdt \leq c\iint \mathcal{D}z^2\,dxdt\,,$$

where $\mathcal{D} = \frac{1}{\nu}\sum a_i^2 + \frac{1}{\nu}\sum b_i^2 + |c|$. Using Hölder's inequality, we obtain:

$$\iint \mathcal{D}Z^2\,dxdt \leq 2\beta^2\|\mathcal{D}\|_{q,r}\left\{\sup_{0\leq t\leq k} z^2 + \iint |Dz|^2\,dxdt\right\},$$

$$\min\{1,\nu\}\left\{\sup_{0<t\leq k} z^2 + \iint |Dz|^2\,dxdt\right\} \leq$$

$$\leq \int z^2(x,0) + 2\beta^2\|\mathcal{D}\|_{q,r}\left\{\sup_{0\leq t\leq k} z^2 + \iint |Dz|^2\,dxdt\right\}.$$

If $2\beta^2\|\mathcal{D}\|_{q,r}$ is less than $\min\{1,\nu\}$ it is possible to estimate

$$\sup_{0\leq t\leq k} z^2 + \iint |Dz|^2\,dxdt$$

in terms of $\int z^2(x,0)$. We argue as in [4]; we consider a partition of $(0,k)$ in a finite number of intervals in such a way that $2\beta^2\|\mathcal{D}\|_{q,r} \leq \frac{\nu}{2}$ and we get

$$\sup_{0\leq t\leq k} z^2 + \iint |Dz|^2\,dxdt \leq c\int z^2(x,0)\,dx\,. \tag{4.4}$$

We notice that z is strictly positive. We extend $z(\cdot,t)$ for $|x| < \varepsilon_0$ by 1 and continue to denote by z such an extension, then, denoting by B_d the ball of radius d about the origin, we have:

$$z(\cdot,t)\in W_0^{1,0}(B_{4d}) \quad \text{for a.e.} \quad t\in(0,k).$$

In particular, we consider cylindric domains with anular cross section $\{\delta\rho < \|x-y\| < \rho\}\times\{s-\rho^2, s\}$. If (4.1) and (4.3) hold for all coaxial cylinders and for all choice of $\xi^\pm$, there exists $\gamma=\gamma(\omega,w)$ such that :

$$\rho^N w^2 \leq \gamma\int_{s-\rho^2}^{s}\int_{\{\delta\rho<\|x-y\|<\rho\}} |Dz|^2\,dxdt\,. \tag{4.5}$$

We suppose that δ^{-1} is a positive integer, (y,s) coincident with the origin and we iterate estimate (4.5) on cylinders

$$Q_n^j = [(0,t_n^j) + Q(\delta^n\rho, \delta^{2n}\rho^2)]$$

$$\delta^{2N-2}\rho^N\delta^{2n}w^2 \leq \gamma\int_{t_n^j}^{t_n^{j+1}}\int_{\{\delta^n\rho<\|x-y\|<\rho\}} |Dz|^2\,dxdt\,.$$

Summation with respect to j, $j=0,1,\ldots,\delta^{-2n}-1$

$$\delta^{2N-2}\rho^N w^2 \leq \gamma\int_{-\rho^2}^{0}\int_{\{\delta^n\rho<\|x-y\|<\rho\}} |Dz|^2\,dxdt\,.$$

Summation with respect to $n=0,1,\cdots,n_0-1$

$$n_0\delta^{2N-2}\rho^N w^2 \leq \gamma\int_{-\rho^2}^{0}\int_{\{\delta^n\rho<\|x-y\|<\rho\}} |Dz|^2\,dxdt\,.$$

Taking into account estimate (4.4), we get

$$n_0\delta^{2N-2}\rho^N w^2 \leq \gamma\rho^N$$

and then $n_0\delta^{2N-2} < \gamma'$, therefore we can choose n_0 in such a way that $\frac{\gamma'}{n_0\delta^{2N-2}}$ lies in $(\frac{1}{2},1)$. Therefore we have proved that

Proposition 11. *There exists a positive integer n_0 that can be determined only in terms of the data ω and w such that for $0 \le n \le n_0 - 1$, $0 \le j \le \delta^{-n} - 1$ we have the following alternative:*

$$\left|\{(x,t) \in Q_n^j : z(x,t) > z^+ - \frac{1}{12}w\}\right| < \nu^+ |Q(\delta^n\rho, \delta^{2n}\rho^2)| \tag{4.6}$$

or

$$\left|\{(x,t) \in Q_n^j : z(x,t) < z^- + \frac{1}{12}w\}\right| < \nu^- |Q(\delta^n\rho, \delta^{2n}\rho^2)|. \tag{4.7}$$

5. Reducing the oscillation.

Let $[(y,s) + Q(\rho,\rho^2)] \subset C(1, 2d; k)$ be fixed and consider coaxial boxes of the type $[(y,\tau) + Q(r, r^2)]$, where r ranges over $[\delta\rho, \rho]$, $\delta = \delta(\omega, w)$. The time location of their vertices ranges over $\tau \in [s - (\rho^2 - r^2), s]$. We assume that both (4.1) and (4.3) hold for z for all cylinders of the type specified above i.e.

$$\text{meas}\left\{(x,t) \in [(y,\tau) + Q(r,r^2)] : z(x,t) > z^+ - \frac{1}{12}w\right\} > \nu^+ |Q(r,r^2)|$$

and

$$\text{meas}\left\{(x,t) \in [(y,\tau) + Q(r,r^2)] : z(x,t) < z^- + \frac{1}{12}w\right\} > \nu^- |Q(r,r^2)|,$$

where $\nu^\pm$ are chosen as before. By Proposition 9

$$\rho^N w^2 \le \gamma(\omega, w) \int_{s-\rho^2}^{s} \int_{\{\delta\rho < \|x-y\| < \rho\}} |Dz|^2 \, dx dt \,.$$

Let Q_n^j be a cube for which, say for example, (4.6) holds. Then , by the analog for z of Proposition 2 we have

$$z(x,t) < z^+ - \frac{1}{18}w\,, \quad \forall (x,t) \in [(y, t_n^j) + Q(\frac{\delta^n\rho}{2}, \frac{\delta^{2n}\rho^2}{2})].$$

We return to the starting cylinder $[(y,s) + Q(\rho,\rho^2)]$. and , within it, consider the box $[y + K_{\frac{1}{2}\delta^n\rho}] \times \{t_n^j, s\}$; since $t_n^j - s = j\delta^{2n}\rho^2$, $1 \le \theta \le \delta^{-2n_0}$, we get by Proposition 11:

$$z(x, s - \theta\rho^2) < z^+ - \frac{1}{18}w\,, \quad \forall x \in [y + K_{4r}].$$

Then, by logarithmic estimate for z we have: $\forall \nu^+ \in (0,1)$ there exists a number ξ^+ such that

$$\text{meas}\{(x,t) \in [(y,s) + Q(2r, 4\theta r^2)] : z(x,t) > z^+ - \xi^+ w\} < \nu^+ |Q(2r, 4\theta r^2)|.$$

Now we fix ν^+ as in Proposition 2 for the largest choice of θ in the range indicated and select ξ^+ from Proposition 4 for z. Then, by the analogous of Proposition 2

$$z(x,t) < z^+ - \frac{2}{3}\xi^+ w, \quad \forall (x,t) \in [(y,s) + Q(r, 2\theta r^2)],$$

in particular

$$z(x,t) < z^+ - \frac{2}{3}\xi^+ w, \quad \forall (x,t) \in [(y,s) + Q(\frac{1}{2}\delta^{n_0}\rho; \frac{1}{2}\delta^{2n_0}\rho^2)].$$

6. Continuity for $t > 0$.

Set $C^{-1}(\omega, w) = \frac{1}{4}[\delta(\omega, w)]^{n_0(\omega,w)}$ The oscillation of z decreased of a factor $[1-\xi(\omega, w)]$ going down to a smaller box. The argument can be repeated for ω fixed and continued over a sequence of nested shrinking cylinders with a common vertex at (y,s). Introduce the sequences: $w_0 = 1$ and $\rho_0 = 1$ and, for $n = 1, 2, \ldots$, $w_{n+1} = [1 - \xi(\omega, w_n)]w_n$, $\rho_{n+1} = C^{-1}(\omega, w_n)\rho_n$ and set $Q_n = [(y,s) + Q(\rho_n, \rho_n^2)]$. Then $w_n \to 0$ and $\text{osc}_{Q_n} \le w_n$.

7. Continuity for $t = 0$.

Due to the information of the initial datum, the continuity within $\mathcal{A}_{1,2d}$ is simpler to establish. Fix $y \in \mathcal{A}_{1,2d}$ and consider $Q(y, 0; \rho) = [y+K_\rho]\times\{0, \rho^2\}$. This box lies with its bottom at $t = 0$. We let z^+ be defined by $z^+ = \sup_{Q(y,0;\rho)} z$ and observe that for all $\xi_0^+ \in (0,1)$

$$(z - (z^+ - \xi_0)_+)(x, 0) = 0, \quad \forall x \in [y + K_\rho].$$

For the logarithmic estimate Proposition 4 for every $\nu^+ \in (0,1)$ there exists $\xi^+ \in (0,1)$ such that

$$\text{meas}\{(x,t) \in Q(y, 0; \frac{1}{2}\rho) : z(x,t) > z^+ - \xi^+ z^+\} < \nu^+ |Q(y, 0; \frac{1}{2}\rho)|.$$

Then we select ν^+ from the logarithmic estimate (Proposition 4) with $\theta = 1$ and choose ξ^+. By the analog for z of Proposition 2 it follows that

$$z(x,t) \leq z^+ - \frac{2}{3}\xi^+ z^+, \quad \forall (x,t) \in Q(y, 0; \frac{1}{4}\rho).$$

Observe that there is no shrinkage in the time variable; we can consider a sequence of nested cylinders

$$Q(y, 0; 4^{-n}\rho)$$

and the sequences:

$$z_0 \equiv 1 \quad z_{n+1} \equiv (1 - \frac{2}{3}\xi^+) z_n, \quad n = 0, 1, 2, \dots .$$

The procedure can be continued in each pair of boxes $Q_{n+1} \subset Q_n$ to yield a Hölder modulus of continuity near $t = 0$:

$$\sup_{Q_n} z \leq (1 - \frac{2}{3}\xi^+)^n.$$

8. Coming back to v.

We observe that v is continuous, being the difference of ζ and z that are continuous. We prove

Proposition 12. *For every $\nu_0 \in (0,1)$, there exists a number $\sigma_0 \in (0,1)$ and $k > 1$ that can be determined a priori only in terms of $\varepsilon_0, d, \tilde{\gamma}_i$ and ν_0 such that:*

$$\text{meas}\{(x,t) \in B_{2d} \times (0,k) : v(x,t) < \sigma_0\} < \nu_0 |B_{2d} \times (0,k)|.$$

Proof. For a positive integer s, consider the truncated function

$$(v - 2^{-s})_- \equiv \max\{(2^{-s} - v); 0\}.$$

We regard $v(\cdot, t)$ as an element of $W^{1,0}(B_{4d})$, so that $(v - 2^{-s})_- \in W^{1,0}(B_{4d})$. Moreover, since $(v - 2^{-s})_-$ vanishes for $|x| = \epsilon_0$ we have for all $s \in N$

$$\text{meas}\{x \in B_{2d} : v(x,t) > 2^{-s}\} \geq |B_{\epsilon_0}|.$$

Therefore for all $t \in (0, k)$, the truncations $(v - 2^{-s})_-$ satisfy the Poincaré type inequality:

$$(2^{-s} - 2^{-(s+1)})\left|[v(\cdot, t) < 2^{-(s+1)}] \cap B_{2d}\right| \leq$$

$$\leq \gamma(N)\frac{d^{N+1}}{\epsilon_0^N}\int_{2^{-(s+1)}<v<2^{-s}} |Dv(x,t)|\, dx\,.$$

Let indicate with $A_s(t)$ the set $\{x \in B_{2d} : v(x,t) < 2^{-s}\}$. We integrate in dt over $(\frac{1}{2}k, k)$ majorising the result integral on the right hand side by means of Hölder inequality and square both sides and rewrite it as:

$$4^{-s}|A_{s+1}|^2 \leq \gamma(\text{data}, \omega)d^{2N+2}\int_{1/2k}^{k}\int_{A_s(t)} |Dv|^2\, dxdt\; |A_s - A_{s+1}|.$$

Let $(x,t) \to \zeta(x,t)$ be a nonnegative piecewise smooth cut-off function in $B_{4d} \times (0, k)$ such that

$$\begin{cases} \zeta(x,t) \equiv 1 & \text{on } B_{2d} \times \{\frac{1}{2}k, k\} \\ \zeta(x,t) = 0 & \text{for } |x| = 4d \text{ and for } t = 0 \\ |D\zeta| \leq \frac{1}{2d}\,, \quad 0 \leq \zeta_t \leq \frac{2}{k}\,. \end{cases}$$

In the weak formulation take the testing function $(v - 2^{-s})_-\zeta^2$ and integrate by parts over the cylinder with anular cross section $\{\epsilon_0 < |x| < 4d\} \times (0, k)$

$$\int_0^k\int_{\epsilon_0<|x|<2d} |D(v - 2^{-s})_-|^2\, dxdt \leq \frac{\gamma}{d^2}\iint (v - 2^{-s})_-^2\, dxdt +$$

$$+\frac{\gamma}{k}\iint (v-2^{-s})_-\, dxdt + \left\|\sum a_i^2 + \sum b_i^2 + c\right\|_{q,r}\left(\int_0^k |A^-_{2^{-s},2d}(\tau)|^{\frac{\hat r}{\hat q}}\right)^{\frac{r-1}{r}} \leq$$

$$\leq \gamma(\text{data}, \omega)\{4^{-s} + k^{-1}2^{-s}\}|B_{4d} \times (0,k)| +$$

$$+\left\|\sum a_i^2 + \sum b_i^2 + c\right\|_{q,r}\left(\int_0^k |A^-_{2^{-s},2d}(\tau)|^{\frac{\hat r}{\hat q}}\right)^{\frac{r-1}{r}},$$

$$|A_{s+1}(t)|^2 \leq \gamma(\text{data}, \omega)(1 + k^{-1}2^s)|B_{4d} \times (0,k)|\; |A_s - A_{s+1}| +$$

$$+\left\|\sum a_i^2 + \sum b_i^2 + c\right\|_{q,r}\left(\int_0^k |A^-_{2^{-s},2d}(\tau)|^{\frac{\hat r}{\hat q}}\right)^{\frac{r-1}{r}} |A_s - A_{s+1}|.$$

We add these inequalities for $s \equiv 1, 2, \ldots, s_0 - 1$, where s_0 is to be chosen. We have:

$$(s_0 - 1)|A_{s_0}|^2 \leq \sum_1^{s_0-1} |A_{s+1}|^2 \leq$$

$$\leq \gamma(\text{data}, \omega)(1 + k^{-1}2^{s_0})|B_{4d} \times (0, k)| \sum_1^{s_0-1} |A_s - A_{s+1}|.$$

We choose $k = 2^{s_0}$ and then choose s_0 so large that the right hand side is less than ν_0. □

Proposition 13. *There exists numbers σ_0 and k so that for every $1 < r < d$ there exists at least one point (y, t) of the cylindrical surface $|y| = r, \quad t \in (0, k)$, such that*

$$v(y, t) > \sigma_0 .$$

Proof. Suppose, by contradiction that

$$v(y, t) \leq \sigma_0 . \quad \forall\{|y| = r\} \times (0, k),$$

where σ_0 need to be chosen. Then v satisfies (4.2) in the cylindrical domain with anular cross section

$$\{r < |x| < 4d\} \times (0, k)$$

and it is nonnegative and less than σ_0 on the parabolic boundary of such domain. In particular

$$0 \leq v(x, t) \leq \sigma_0 , \quad \forall (x, t) \in \{d < |x| < 2d\} \times (\frac{1}{2}k, k)$$

This implies that

$$\text{meas}\{(x, t) \in B_{2d} \times (\frac{1}{2}k, k) : v(x, t) < \sigma_0\} >$$

$$> \frac{2^{N-1}}{2^N}|B_{2d} \times (\frac{1}{2}k, k)| \geq \frac{7}{8}|B_{2d} \times (\frac{1}{2}k, k)|.$$

In the previous proposition fix $\nu_0 = \frac{7}{8}$ and determines k and σ_0 accordingly. This generates a contradiction that proves the proposition. □

9. Proof of the Theorem for $N \geq 3$.

To prove the continuity of u at a point $(y, s) \in \Omega_T$, first assume that such point coincides with the origin and work within a cylinder $Q(\rho; \theta\rho^2)$, with θ positive number to be chosen. The number θ can be chosen as an integer, so the starting cylinder will be partitioned, up to a set of measure zero, into disjoint layers of the type

$$(9.1) \qquad [(0, t_i) + Q(\rho, \rho^2)], \quad t_i \doteq -i\rho^2, i = 0, 1, \dots, \theta - 1.$$

The numbers $\mu^{\pm}$ and ω are defined in Section 2. We will show that within such a layer, we can locate a small set where u is quantitatively bounded away, either from μ^+ or from μ^-.

We let $\xi^{\pm}$ and λ be defined as in Proposition 9 and δ be determined by Proposition 8. Notice that the number δ depends upon ω and is independent of ρ.

Fix any box of the type (9.1) and assume , after a translation, that his vertex coincides with the origin, so we can rewrite as $[Q(\rho, \rho^2]$. We partition the cylinder in two steps. First we partition the cube K_ρ, up to a set of measure zero, into m^N pairwise disjoint subcubes of wedge $\frac{2}{m}\rho$, with m positive integer to be chosen.

$$\bar{K}_\rho = \bigcup_{\ell=1}^{m^N} [x_\ell + \bar{K}_{\frac{\rho}{m}}],$$

where x_ℓ are their centres. Secondly, we partition the cylinder into $m^N m^2$ pairwise disjoint cylinders. We denote by (x_ℓ, t_h) their vertices:

$$[(x_\ell, t_h) + Q(\frac{1}{m}\rho, \frac{1}{m^2}\rho^2)]$$

where for each ℓ in the range $\ell = 1, \dots, m^N$ we have $t_h = (1-h)\frac{\rho^2}{m^2}$. Therefore

$$\bar{Q}(\rho, \rho^2) = \bigcup_{h=1}^{m^2} \bigcup_{\ell=1}^{m^N} \big[(x_\ell, t_h) + Q(\frac{1}{m}\rho, \frac{1}{m^2}\rho^2)\big].$$

Within each $[(x_\ell, t_h) + Q(\frac{1}{m}\rho, \frac{1}{m^2}\rho^2)]$ consider coaxial cylinders of the type $[(x_\ell, \tau) + Q(r, r^2)]$. The time location of their vertices ranges over $\tau \in [t_h - (\frac{1}{m^2}\rho^2 - r^2), t_h]$ and r is a positive parameter ranging over the interval $[\delta\frac{1}{m}\rho, \frac{1}{m}\rho]$. These are cylinders of the type (4.9), (4.10), (4.11), where ρ has been replaced by $\frac{1}{m}\rho$. For each of these cylinders Propositions 2 and 3 hold true for $\theta = 1$. Since we are choosing $\xi^+ = \xi^- = \frac{1}{12}$ we denote by ν the common value of $\nu^{\pm}$.

Proposition 14. *There exists a positive integer m than can be determined a priori only in terms of ω and the data, such that for some cylinder $[(x_\ell, t_h) + Q(\frac{1}{m}\rho, \frac{1}{m^2}\rho^2)]$ and for some cylinder $[(x_\ell, \tau) + Q(r, r^2)] \subset [(x_\ell, t_h) + Q(\frac{1}{m}\rho, \frac{1}{m^2}\rho^2)]$ either*

$$\text{meas}\left\{(x,t) \in [(x_\ell, \tau) + Q(r; r^2)] : u(x,t) > \mu^+ - \frac{1}{12}\omega\right\} < \nu |Q(r, r^2)|$$

or

$$\text{meas}\left\{(x,t) \in [(x_\ell, \tau) + Q(r, r^2)] : u(x,t) < \mu^- + \frac{1}{12}\omega\right\} < \nu |Q(r, r^2)|.$$

If both are violated for every cylinder of the type $[(x_\ell, \tau) + Q(r, r^2)]$ and for every $[(x_\ell, t_h) + Q(\frac{1}{m}\rho, \frac{1}{m^2}\rho^2)]$, making up the partition of $Q(\rho, \rho^2)$ by virtue of Proposition 9 there exists a constant that can be determined in terms of the data and ω and independent of ρ and m such that:

$$\left(\frac{1}{m}\rho\right)^N \omega^2 \leq \gamma \int_{t_h - (\frac{\rho}{m})^2}^{\rho_h} \int_{[x_\ell + K_{\frac{\rho}{m}}]} |Du|^2\, dxdt\,,$$

$$\forall \ell = 1, \ldots, m^N,\ \forall h = 1, \ldots, m^2.$$

Adding over such indices:

$$m^2 \rho^N \omega^2 \leq \gamma \int_{Q(\rho,\rho^2)} |Du|^2\, dxdt\,.$$

We combine this with Proposition 6 and 7 and rewrite the resulting inequality as:

$$1 < \frac{\gamma(\text{data}, \omega)}{\omega m^2}\,.$$

The proposition follows by choosing m so large that the right hand side does not exceed 1. It follows also that $\omega \to m(\omega)$ is a decreasing function of ω and $\lim\limits_{\omega \to 0} m(\omega) = \infty$.

10. The approach to continuity.

Let $[(x_\ell, \tau) + Q(\rho, \rho^2)]$ be a cylinder for which the alternative $-$ holds. Then, by Propositions 2 and 3 we have :

$$u(x,t) > \mu^- - \frac{1}{18}\omega, \quad \forall (x,t) \in [(x_\ell, \tau) + Q(\frac{\rho}{2}, \frac{\rho^2}{2})].$$

If instead the alternative $+$ holds true within $[(x_\ell, \tau) + Q(r, r^2)]$ then $u(x,t) < \mu^+ - \frac{1}{18}\omega, \quad \forall (x,t) \in [(x_\ell, \tau) + Q(\frac{\delta}{2m}\rho, \frac{\delta^2}{2m}\rho^2)]$. We may assume that δ^{-1} is an integer. Then we further partition the starting cube K_ρ up to a set of measure zero into

$$q(\omega) = \left(\frac{4m(\omega)}{\delta(\omega)}\right)^N$$

disjoint cubes of wedge

$$\frac{\delta(\omega)}{2m(\omega)}\rho = 2\delta_0\rho.$$

We let x_ℓ, $\ell = 1, 2, \ldots, q(\omega)$ denote their centres so that

$$\bar{K}_\rho = \bigcup_{\ell=1}^{q} [x_\ell + \bar{K}_{\frac{\delta}{2}n_0\rho}].$$

Analogously, we subdivide the cube $Q(\rho, \rho^2)$ into

$$p(\omega) = \left(\frac{4m(\omega)}{\delta(\omega)}\right)^N \left(\frac{4m(\omega)}{\delta(\omega)}\right)^2$$

pairwise disjoint cylinders. If we denote by (x_ℓ, t_h) their vertices, they take the form:

$$[(x_\ell, t_h) + Q(\frac{\delta}{2m}\rho, (\frac{\delta}{2m})^2\rho^2)], \tag{10.1}$$

where for each $\ell = 1, 2, \ldots, q(\omega)$

$$t_h = (1-h)\left(\frac{\delta}{2m}\right)^2 \rho^2, \quad h = 1, 2, \cdots, p(\omega).$$

Proposition 15. *For each boxes of the type* (9.1) *there exists a subcylinder of the type* (10.1) *for which either*

$$u(x,t) < \mu^+ - \frac{1}{18}\omega, \quad \forall (x,t) \in [(x_\ell, t_h) + Q(\frac{\delta}{2m}\rho, (\frac{\delta}{2m})^2\rho^2)] \tag{10.2}$$

or

$$u(x,t) > \mu^- + \frac{1}{18}\omega, \quad \forall (x,t) \in [(x_\ell, t_h) + Q(\frac{\delta}{2m}\rho, (\frac{\delta}{2m})^2\rho^2)]. \tag{10.3}$$

Let us concentrate on the lower half of the cylinder $Q(\rho, \theta\rho^2)$ i.e. $[(0, -\frac{1}{2}\theta) + Q(\rho, \frac{1}{2}\theta\rho^2)]$. We assume that the number θ has been chosen and let

$$(x_\ell, \tau) + Q(\delta_0\rho, \delta_0^2\rho^2)] \subset [(0, -\frac{1}{2}\theta) + Q(\rho, \frac{1}{2}\theta\rho^2)]$$

be a cylinder for which (10.3) holds. We start from such a box and construct a long, thin cylinder with vertex at the top of $Q(\rho, \theta\rho^2)$ i.e.

$$[x_\ell + K_{4r}] \times [t, 0], \quad 4r \equiv \delta_0\rho.$$

We rewrite this as

$$[(x_\ell, 0) + Q(4r, 4\bar{\theta}r^2)],$$

where

$$2\delta_0^{-2}(\theta - 1) \leq \bar{\theta} \leq 4\delta_0^{-2}\theta.$$

We have

$$u(x, -4\theta r^2) > \mu^- - \frac{1}{18}\omega, \quad \forall x \in [x_\ell + K_{4r}]. \tag{10.4}$$

Proposition 16. *There exists a number $\xi \in (0, \frac{1}{18})$ that can be determined a priori only in terms of the data and ω such that*

$$u(x, t) > \mu^- + \xi\omega, \quad \forall (x, t) \in [(x_\ell, 0) + Q(r, \bar{\theta}r^2)].$$

The proof uses the logarithmic estimate to expand the information to the top of the cylinder. Thus we have isolated a long, thin cylinder where u is bounded below. The abscissa of the vertex of such cylinder is not known; if $x_\ell \equiv 0$ then it would imply a decreasing of the oscillation of a factor $(1 - \xi)$. Since the location of $x_\ell \in K_\rho$ is not known, there is a necessity to establish that a version of (10.4) holds within a small thin cylinder with vertex at the origin. This is achieved into two stages. The first stage is some sort of spreading of positivity. If the alternative (10.3) holds, there exists a time level $t_0 \in (-\theta\rho^2, \frac{1}{2}\theta\rho^2)$ such that u is quantitatively bounded below in the full cube $[x_\ell + K_{\delta_0\rho}]$. Such positivity spreads sidewise to a full smaller cube about the origin, after a sufficiently long time.

Proposition 17. *There exists numbers $\xi_0, \delta_* \in (0, 1)$ and $\theta > 1$, that can be determined a priori in terms of the data and ω, and a time level*

$$-\theta\rho^2 \leq t \leq -\frac{1}{2}\theta\rho^2$$

such that either

$$u(x, t_0) < \mu^+ - \xi_0\omega, \quad \forall x \in [y + K_{\delta_*\rho}] \tag{10.5}$$

or

$$u(x, t_0) > \mu^- + \xi_0\omega, \quad \forall x \in [y + K_{\delta_*\rho}]. \tag{10.6}$$

The essential tool is a sequential selection of blocks of positivity (see [2]). The number θ will be a product of a finite, increasing sequence of positive integers $\theta = \prod k_j$ that determine a partition of $Q(\rho, \theta\rho^2)$ into disjoint stacks. There are two alternatives: either among the stacks there exists one where the bound $-$ is verified for the same abscissa x_ℓ for at least one cube within a smaller stack or the same with $+$. In fact one cannot present the case of neither of the two being verified because otherwise it will be in contradiction with Proposition 15. In the proof the key point is the use of the local comparison function, used to consider a suitable rescaled parabolic equation in a cylinder of the form $(x_\ell, t_{j+1}) + Q(r, k_{j+1}\delta_0^{-1}\rho^2)$. The change of variables that maps the ball of center x_ℓ in a ball with center in the origin is the following:

$$x \to \frac{4x - x_\ell}{|x_\ell|}, \quad t \to \frac{(t_{j+1} - t) + 16k_{j+1}\rho^2}{16\rho^2}.$$

It is convenient to introduce a new function $\tilde{u}$ normalized in order to satisfy

$$\tilde{u}(x, t) \geq 1.$$

The function $\tilde{u}$ satisfies the differential equation

$$(\tilde{\beta}(\tilde{u}))_t = \tilde{\mathcal{L}}(\tilde{u})$$

in an annular contained in $\{1 < |x| < 2d\}$. Let v the local comparison function introduced in Section 4

$$\begin{cases} (\tilde{\beta}(v))_t = \mathcal{L}v & \text{on } \mathcal{A}_{\varepsilon_0,4d} \times (0, k_{j+1}) \\ v(x,t) = 0 & \text{in } |x| = 4d \\ v(x,t) = 1 & \text{in } |x| = \varepsilon_0 \\ v(x,0) = 0. & \end{cases}$$

We use Proposition 13 and choose numbers $\sigma_{0,j}$ and k_{j+1} so that

$$v(y, t) > \sigma_{0,j}, \quad \forall 1 < |y| < d, \quad \text{and for some} \quad t \in (0, k_{j+1}).$$

Returnig to the original coordinates, there exists a time level t_0 such that

$$u(x, t_0) > \mu_- + \xi_{0,j}\omega, \quad \forall x \in K_{\delta_{*,j}\rho}.$$

The same holds if we start from the alternative with $+$. This procedure determines k_{j+1} from $k_1, k_2, \ldots, k_j$.

The second step is the reduction of the oscillation of u near the top of the starting box $Q(\rho, \theta\rho^2)$.

Proposition 18. *There exist numbers* $\xi_*, \delta_* \in (0, 1)$ *and a number* $\theta > 1$ *that can be determined a priori in terms of the data and* ω*, such that either*

$$u(x,t) < \mu^+ - \xi_*\omega, \quad \forall (x,t) \in Q(\delta_*\rho, \theta\delta_*\rho^2) \tag{10.7}$$

or

$$u(x,t) > \mu^- + \xi_*\omega, \quad \forall (x,t) \in Q(\delta_*\rho, \theta\delta_*\rho^2). \tag{10.8}$$

The proof starts from (10.6) and uses the logarithmic estimate (Proposition 4) and Propositions 3 and 2.

The argument of continuity consists in showing the existence of a family of nested shrinking cylinders with the same vertex , for each of them the oscillation is controlled by a sequence ω_n that tends to zero. We have determined the functions $\omega \to \xi_*(\omega), \delta_*(\omega), \theta(\omega)$ from the previous Proposition. Consider a cylinder with vertex at the origin, contained in Ω_T of the form $Q(2\rho, 2\theta(\omega)\rho^2)$, where ω is any number satisfying osc $u \leq \omega$; applying the previous Proposition, we get

$$\operatorname*{osc}_{Q[\delta_*\rho, \theta\delta_*^2\rho^2]} u \leq (1 - \xi_*(\omega))\omega.$$

Consider the sequence

$$\omega_0 = 2M, \quad \omega_{n+1} = (1 - \xi_*(\omega_n))\omega_n \,.$$

By induction, one constructs a sequence of cylinders Q_n with radii ρ_n and heights $\theta(\omega_n)\rho_n^2$, for each of them

$$\operatorname*{osc}_{Q_n} u \leq \omega_n$$

These cylinders form a family of nested shrinking cylinders with the same vertex at the origin.

11. Coefficients that depend on t.

Let us consider a compact subset $\mathcal{K} \subset \Omega_T$ and denote by B the Banach space of functions $u \in L^2(0, T; W^{1,2}(\Omega))$ such that

$$\|u\|_{\infty,\Omega_T} \leq K$$

and the modulus of continuity of u is bounded in $\mathcal{K}$; that is there exists a continuous increasing function ω, $\omega(0) = 0$ such that:

$$|u(x_1, t_1) - u(x_2, t_2)| \leq \omega(|x_1 - x_2| + |t_1 - t_2|^{\frac{1}{2}}).$$

We will apply the Schauder Theorem to the following family of nonlinear operators:

$$w \to v = \Phi(w, \tau),$$

where v is the solution of the linear equation:

$$\beta'(w)v_t = \Big[\tau \sum_{ij} D_i(a_{ij}(x,s)D_j v + a_i(x,s)v) +$$

$$+ (1-\tau) \sum_{ij} D_i(a_{ij}(x,t_0)D_j v + a_i(x,t_0)v)\Big] +$$

$$+ \tau b_i(x,s)D_i v + (1-\tau)b_i(x,t_0)D_i v + \tau c(x,s)v + (1-\tau)c(x,t_0)v,$$

with $s = t_0 + \tau h$.

Under the assumptions of Hölder-continuity on the coefficients in x and t with respect to the parabolic metric, one can prove the equicontinuity in w and in τ.

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